arXiv:2304.00523v1 [math.DG] 2 Apr 2023

谱爱因斯坦泛函与 Kastler-Kalau-Walze 型定理

Yuchen Yang, Tong Wu*

School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, *China*

摘要

本文在定义与带边流形上的狄拉克算子相关的谱爱因斯坦泛函的基础上，证明了低维带边流形上与狄拉克算子相关的谱爱因斯坦泛函的 Kastler-Kalau-Walze 型定理。

关键词 主题：谱爱因斯坦泛函；狄拉克算子；Kastler-Kalau-Walze 型定理 ;

2010年数学主题分类：53C40；53C42。

1. 引言

近年来，许多几何学家对非交换留数进行了更深入的研究。在 [6],Guillemin 发现，非交换留数对于非交换几何的研究具有重要意义。在 20 世纪 90 年代，学者 Connes 利用非交换留数构造了一个与共形四维 Polyakov 作用量类似的泛函[2], ；他还指出，在紧致流形上，非交换留数等于阶数为 $-\dim M$ 的伪微分算子的 Dixmier 迹 [3], ，因此非交换留数可用作非交换积分。Connes 猜想，狄拉克算子的 -2 次幂的非交换留数与爱因斯坦-希尔伯特作用量成正比。1995 年，Kastler[8] 证明了这一猜想。与此同时，Kalau 和 Walze 也在 [7] 的正规坐标系中证明了这一猜想。1996 年，Ackermann 发现，狄拉克算子的逆平方的非交换留数 $\widetilde{\mathrm{Wres}}(D^{-2})$ 本质上是 [1] 中狄拉克算子平方的热核展开的第二个系数。同年，Fedosov 等人提出了带边流形上的非交换留数的概念。他们在 Boutet de Monvel 代数上定义了非交换留数，并证明它是一个唯一的连续迹 [5]。

另一方面，Wang 提供了一种研究带边流形上 Kastler-Kalau-Walze 型定理的方法。21 世纪初，中国学者 Wang Yong 将 Connes 的结果推广到带边流形，研究了带边流形的共形不变量问题和引力作用量，并证明了低维带边流形上狄拉克算子和符号算子的 Kastler-Kalau-Walze 型定理 [10, 11], [12]。在 [13], , Wang 定义了带边自旋流形的低维体积，并计算了低维体积 $Vol_6^{(2,2)}$ ，得到了 6 维带边自旋流形及其相关狄拉克算子 D 上的 Kastler-Kalau-Walze 型定理。他们计算了低维体积 $Vol_6^{(1,3)}$ ，并得到了 6 维带边自旋流形及其相关狄拉克算子 D 和 D^3在 [9]上的 Kastler-Kalau-Walze 型定理。在 [4], , Dabrowski L、Sitarz A 和 Zalecki P 定义了向量场和微分形式的双线性泛函，其密度给出了偶数维黎曼流形上的度量和爱因斯坦张量；他们还证明，对于非交换二维环面的共形缩放几何，爱因斯坦泛函为零。受 [4], 启发，我们定义了与狄拉克

*通讯作者。*Email addresses:* yangyc580@nenu.edu.cn (Yuchen Yang), wut977@nenu.edu.cn (Tong Wu)

算子相关的带边流形谱爱因斯坦泛函。本文旨在计算 4 维带边自旋流形上的非交换留数 $\widetilde{\text{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla^{S(TM)}{}_Y \quad D^{-1}) \circ \pi^+(D^{-3})]$，以及 6 维带边自旋流形上的 $\widetilde{\text{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla^{S(TM)}{}_Y \quad D^{-3}) \circ \pi^+(D^{-3})]$。

本文结构如下。第2节中，我们定义与狄拉克算子相关的谱爱因斯坦泛函，并计算无边界流形上的非交换留数。第3节中，我们证明带边界的四维流形上与狄拉克算子相关的谱爱因斯坦泛函的 Kastler-Kalau-Walze 型定理；第4节中，我们得到与三维自旋流形相关的 Kastler-Kalau-Walze 型定理。第5节中，我们计算带边界的六维流形上与狄拉克算子相关的谱爱因斯坦泛函的 Kastler-Kalau-Walze 型定理。

2. 与狄拉克算子相关的谱爱因斯坦泛函

首先，我们回顾狄拉克算子的相关知识。设 M 是一个 n 维（$n \geq 3$）定向紧致黎曼流形，其黎曼度量为 g^M，并令 ∇^L 为关于 g^M 的列维-奇维塔联络。在固定的标准正交标架 $\{e_1, \cdots, e_n\}$ 下，联络矩阵 $(\omega_{s,t})$ 定义为

$$\nabla^L(e_1, \cdots, e_n) = (e_1, \cdots, e_n)(\omega_{s,t}). \tag{2.1}$$

令 $\epsilon(e_j^*)$，$\iota(e_j^*)$ 分别为外乘法和内乘法，其中 $e_j^* = g^{TM}(e_j, \cdot)$。令

$$\widehat{c}(e_j) = \epsilon(e_j^*) + \iota(e_j^*); \quad c(e_j) = \epsilon(e_j^*) - \iota(e_j^*), \tag{2.2}$$

其满足

$$\begin{aligned}
&\widehat{c}(e_i)\widehat{c}(e_j) + \widehat{c}(e_j)\widehat{c}(e_i) = 2g^M(e_i, e_j);\\
&c(e_i)c(e_j) + c(e_j)c(e_i) = -2g^M(e_i, e_j);\\
&c(e_i)\widehat{c}(e_j) + \widehat{c}(e_j)c(e_i) = 0.
\end{aligned} \tag{2.3}$$

由 [15], ，我们得到狄拉克算子

$$D = \sum_{i=1}^{n} c(e_i)[e_i - \frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)c(e_s)c(e_t)]. \tag{2.4}$$

我们将自旋联络定义为 $\nabla^{S(TM)}{}_X := X + \frac{1}{4}\Sigma_{ij}\langle\nabla_X^L e_i, e_j\rangle c(e_i)c(e_j)$。令 $A(X) = \frac{1}{4}\Sigma_{ij}\langle\nabla_X^L e_i, e_j\rangle c(e_i)c(e_j)$，于是有

$$\begin{aligned}
\nabla_X^{S(TM)}\nabla_Y^{S(TM)} &= [X + A(X)][Y + A(Y)]\\
&= XY + X \cdot A(Y) + A(X)Y + A(X)A(Y)\\
&= XY + X[A(Y)] + A(Y)X + A(X)Y + A(X)A(Y),
\end{aligned} \tag{2.5}$$

其中 $X = \Sigma_{j=1}^n X_j\partial_{x_j}$、$Y = \Sigma_{l=1}^n Y_l\partial_{x_l}$ 和 $XY = \Sigma_{j,l=1}^n(X_jY_l\partial_{x_j}\partial_{x_l} + X_j\frac{\partial_{Y_l}}{\partial_{x_j}}\partial_{x_l})$。

设 $g^{ij} = g(dx_i, dx_j)$、$\xi = \sum_k \xi_j dx_j$ 和 $\nabla_{\partial_i}^L\partial_j = \sum_k \Gamma_{ij}^k\partial_k$，我们记为

$$\sigma_i = -\frac{1}{4}\sum_{s,t}\omega_{s,t}(e_i)c(e_s)c(e_t); \quad \xi^j = g^{ij}\xi_i; \quad \Gamma^k = g^{ij}\Gamma_{ij}^k; \quad \sigma^j = g^{ij}\sigma_i; \quad a^j = g^{ij}a_i. \tag{2.6}$$

根据 $\partial_{x_j} = -\sqrt{-1}\xi_j$，我们有以下引理。

引理 2.1。以下恒等式成立:

$$\begin{aligned}
&\sigma_0(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}) = X[A(Y)] + A(X)A(Y);\\
&\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}) = \sqrt{-1}\Sigma_{j,l=1}^n X_j\frac{\partial_{Y_l}}{\partial_{x_j}}\partial_{x_l} + \sqrt{-1}\Sigma_j A(Y)X_j\xi_j + \sqrt{-1}\Sigma_l A(Y)Y_l\xi_l;\\
&\sigma_2(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}) = -\Sigma_{j,l=1}^n X_jY_l\xi_j\xi_l.
\end{aligned} \tag{2.7}$$

引理 2.2. 由*[9],*可得以下恒等式:

$$\begin{aligned}
&\sigma_{-1}(D^{-1}) = \frac{\sqrt{-1}c(\xi)}{|\xi|^2};\\
&\sigma_{-2}(D^{-1}) = \frac{c(\xi)\sigma_0(D)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\Sigma_j c(\mathrm{d}x_j)(\partial_{x_j}[c(\xi)]|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2));\\
&\sigma_{-3}(D^{-3}) = -\sqrt{-1}|\xi|^{-4}c(\xi);\\
&\sigma_{-4}(D^{-3}) = \frac{c(\xi)\sigma_2(D^3)c(\xi)}{|\xi|^8} + \frac{\sqrt{-1}c(\xi)}{|\xi|^8}\bigg(|\xi|^4c(dx_n)\partial_{x_n}c(\xi') - 2h'(0)c(\mathrm{d}x_n)c(\xi)\\
&\quad + 2\xi_n c(\xi)\partial x_n c(\xi') + 4\xi_n h'(0)\bigg);
\end{aligned} \tag{2.8}$$

记

$$D_x^\alpha \quad = (-i)^{|\alpha|}\partial_x^\alpha;\ \sigma(D_t^3) = p_3 + p_2 + p_1 + p_0;\ (\sigma(D_t)^{-3}) = \textstyle\sum_{j=3}^\infty q_{-j}. \tag{2.9}$$

根据伪微分算子的复合公式，我们有

$$\begin{aligned}
1 = \sigma(D^3\circ D^{-3}) &= \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha[\sigma(D]D_x^\alpha[\sigma(D^{-3})]\\
&= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots)\\
&\quad + \sum_j(\partial_{\xi_j}p_3 + \partial_{\xi_j}p_2 + +\partial_{\xi_j}p_1 + \partial_{\xi_j}p_0)(D_{x_j}q_{-3} + D_{x_j}q_{-4} + D_{x_j}q_{-5} + \cdots)\\
&= p_3q_{-3} + (p_3q_{-4} + p_2q_{-3} + \sum_j \partial_{\xi_j}p_3D_{x_j}q_{-3}) + \cdots,
\end{aligned} \tag{2.10}$$

so

$$q_{-3} = p_3^{-1};\ q_{-4} = -p_3^{-1}[p_2p_3^{-1} + \sum_j \partial_{\xi_j}p_3D_{x_j}(p_3^{-1})]. \tag{2.11}$$

于是,

$$\begin{aligned}
\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) &= -\sqrt{-1}\Sigma_{j,l=1}^n X_jY_l\xi_j\xi_l c(\xi)|\xi|^{-2};\\
\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3}) &= \sqrt{-1}\Sigma_{j,l=1}^n X_jY_l\xi_j\xi_l c(\xi)|\xi|^{-4}.
\end{aligned} \tag{2.12}$$

根据 [4], , 我们有以下定理

定理 2.3. 若 M 是一个 n维无边界紧致定向流形，且 n为偶数，则有以下等式:

$$\mathrm{Wres}[\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-n}] = \frac{(2\pi)^{\frac{n}{2}}}{3(\frac{n}{2}-1)!}\int_M[Ric(X,Y) - \frac{1}{2}sg(X,Y)]d\mathrm{Vol_M} + \frac{(2\pi)^{\frac{n}{2}}}{4(\frac{n}{2}-1)!}\int_M sg(X,Y)d\mathrm{Vol_M}, \tag{2.13}$$

其中， s 是标量曲率，$\mathrm{Vol_M}$ 是 M 的体积， Ric 表示 M上的里奇张量。

3. 带边界的四维流形上的非交换留数

本节计算四维带边定向紧流形上的非交换留数 $\widetilde{\mathrm{Wres}}[\pi^+(\nabla^{S(TM}{}_X \quad \nabla_Y^{S(TM)}D^{-1}) \circ \pi^+(D^{-3})]$。我们回顾 Boutet de Monvel 演算的一些基本事实与公式，以及带边流形上非交换留数的定义，这些内容将在下文中使用。详细内容见 [12] 中的第2节。

设 M 是一个带边界 ∂M 的四维紧致定向流形。我们假设度量 g^M 在边界 M 附近具有如下形式，

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2, \tag{3.1}$$

其中 $g^{\partial M}$ 是 ∂M 上的度量，且 $h(x_n) \in C^\infty([0,1)) := \{\widehat{h}|_{[0,1)} | \widehat{h} \in C^\infty((-\varepsilon, 1))\}$，其中 $\varepsilon > 0$，并且 $h(x_n)$ 满足 $h(x_n) > 0$， $h(0) = 1$；其中 x_n 表示法向坐标。

类似于 [12], ，我们可以计算非交换留数

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \circ \pi^+(D^{-3})] \\ &= \int_M \int_{|\xi|=1} \mathrm{trace}_{\wedge^* T^*M \otimes \mathbb{C}}[\sigma_{-4}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1} \circ D^{-3})]\sigma(\xi)dx + \int_{\partial M}\Phi,\end{aligned} \tag{3.2}$$

其中

$$\begin{aligned}\Phi = &\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \mathrm{trace}_{\wedge^* T^*M \otimes \mathbb{C}}[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})(x', 0, \xi', \xi_n) \\ &\times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l(D^{-3})(x', 0, \xi', \xi_n)]d\xi_n \sigma(\xi')dx',\end{aligned} \tag{3.3}$$

求和遍历 $r + l - k - j - |\alpha| = -4$、 $r \leq 1$ 和 $l \leq -3$。

由定理 2.3，我们可以计算 $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \circ \pi^+(D^{-3})]$, 的内部，得到

$$\mathrm{Wres}[\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-4}] = \frac{4\pi^2}{3}\int_M [Ric(X,Y) - \frac{1}{2}sg(X,Y)]d\mathrm{Vol}_M + \pi^2 \int_M sg(X,Y)d\mathrm{Vol}_M. \tag{3.4}$$

现在我们需要计算 $\int_{\partial M}\Phi$。当 $n = 4$ 时， $\mathrm{tr}_{S(TM)}[\mathrm{id}] = \dim(\wedge^*(\mathbb{R}^2)) = 4$， 求和遍历 $r + l - k - j - |\alpha| = -3$、 $r \leq 0$、 $l \leq -2$，于是得到以下五种情形:

情形 a) I) $r = 1$, $l = -2$, $k = j = 0$, $|\alpha| = 1$。
由 (3.3)，得到

$$\Phi_1 = -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.5}$$

由 [12], 中的引理 2.2，对于 $i < n$，则有

$$\partial_{x_i}\sigma_{-3}(D^{-3})(x_0) = \partial_{x_i}(\sqrt{-1}c(\xi)|\xi|^{-4})(x_0) = \sqrt{-1}\frac{\partial_{x_i}c(\xi)}{|\xi|^4}(x_0) + \sqrt{-1}\frac{c(\xi)\partial_{x_i}(|\xi|^4)}{|\xi|^8}(x_0) = 0, \tag{3.6}$$

因此 $\Phi_1 = 0$。

情形 a) II) $r = 1$, $l = -3$, $k = |\alpha| = 0$, $j = 1$。
由 (3.3)，我们得到

$$\Phi_2 = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi_{\xi_n}^+\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \times \partial_{\xi_n}^2\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.7}$$

由引理 2.1 和引理 2.2 可得

$$\partial^2_{\xi_n}\sigma_{-3}((D^{-3}))(x_0)=\partial^2_{\xi_n}(c(\xi)|\xi|^{-4})(x_0)=\sqrt{-1}\frac{(20\xi_n^2-4)c(\xi')+12(\xi^3-\xi)c(\mathrm{d}x_n)}{(1+\xi_n^2)^4}, \tag{3.8}$$

and

$$\begin{aligned}\partial_{x_n}\sigma_1(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})&=\partial_{x_n}(-\sqrt{-1}\Sigma^n_{j,l=1}X_jY_l\xi_j\xi_l c(\xi)|\xi|^{-2})\\&=\Sigma^n_{j,l=1}X_jY_l\xi_j\xi_l\left[\frac{\partial_{x_n}c(\xi')}{1+\xi_n^2}+\frac{c(\xi)h'(0)|\xi'|^2}{(1+\xi_n^2)^2}\right].\end{aligned} \tag{3.9}$$

于是，我们有

$$\begin{aligned}\pi^+_{\xi_n}\partial_{x_n}\sigma_1(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})&=\partial_{x_n}\pi^+_{\xi_n}\sigma_1(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})\\&=\sqrt{-1}\Sigma^{n-1}_{j,l=1}X_jY_l\xi_j\xi_l h'(0)|\xi'|^2\left[\frac{ic(\xi')}{4(\xi_n-i)}+\frac{c(\xi')+ic(\mathrm{d}x_n)}{4(\xi_n-i)^2}\right]\\&-\Sigma^{n-1}_{j,l=1}X_jY_l\xi_j\xi_l\frac{\partial_{x_n}c(\xi')}{2(\xi_n-i)}\\&-\sqrt{-1}X_nY_n\left\{\frac{\partial_{x_n}c(\xi')}{2(\xi_n-i)}+h'(0)|\xi'|\left[-\frac{2ic(\xi')-3c(\mathrm{d}x_n)}{4(\xi_n-i)}\right.\right.\\&\left.\left.+\frac{[c(\xi')+ic(\mathrm{d}x_n)][i(\xi_n-i)+1]}{4(\xi_n-i)^2}\right]\right\}\\&-\Sigma^{n-1}_{j=1}X_jY_n\xi_j\left[\sqrt{-1}\frac{\partial_{x_n}c(\xi')}{2(\xi_n-i)}-\frac{\sqrt{-1}h'(0)|\xi'|[c(\xi')+2ic(\mathrm{d}x_n)]}{4(\xi_n-i)}\right.\\&\left.-\frac{[ic(\xi')-c(\mathrm{d}x_n)][i(\xi_n-i)+1]}{(\xi_n-i)^2}\right]\\&-\Sigma^{n-1}_{l=1}X_nY_l\xi_l\left[\sqrt{-1}\frac{\partial_{x_n}c(\xi')}{2(\xi_n-i)}-\frac{\sqrt{-1}h'(0)|\xi'|[c(\xi')+2ic(\mathrm{d}x_n)]}{4(\xi_n-i)}\right.\\&\left.-\frac{[ic(\xi')-c(\mathrm{d}x_n)][i(\xi_n-i)+1]}{(\xi_n-i)^2}\right].\end{aligned} \tag{3.10}$$

注意，$i<n$，$\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$，因此我们省略了一些对计算情形 a) II) 没有贡献的项。

于是有如下公式

$$\begin{aligned}&\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_1(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})\times\partial^2_{\xi_n}\sigma_{-3}(D^{-3})](x_0)\\&=\Sigma^{n-1}_{j,l=1}X_jY_l\xi_j\xi_l h'(0)\left[8i\frac{5\xi_n^2-1}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{4(5\xi_n^2-1)+12i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right]\\&+X_nY_nh'(0)\left[\frac{(4i-4)(\xi^2-1)+48(\xi_n^3-\xi_n)}{(\xi_n-i)^5(\xi_n+i)^4}-\frac{4(5\xi_n^2-1)+12i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right]\\&+8\Sigma^{n-1}_{j=1}X_jY_n\xi_j\left[\frac{(6-3ih'(0))(\xi_n^3-\xi_n)-2i(5\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{2(5\xi_n^2-1)+6i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right]\\&+8\Sigma^{n-1}_{l=1}X_nY_l\xi_l\left[\frac{(6-3ih'(0))(\xi_n^3-\xi_n)-2i(5\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{2(5\xi_n^2-1)+6i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right].\end{aligned} \tag{3.11}$$

因此，我们得到

$$\begin{aligned}
\Phi_2 &= \frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\left\{\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l h'(0)\left[8i\frac{5\xi_n^2-1}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{4(5\xi_n^2-1)+12i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right]\right.\\
&\left.+X_nY_nh'(0)\left[\frac{(4i-4)(\xi^2-1)+48(\xi_n^3-\xi_n)}{(\xi_n-i)^5(\xi_n+i)^4}-\frac{4(5\xi_n^2-1)+12i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right]\right\}d\xi_n\sigma(\xi')dx'\\
&=\Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\Omega_3\int_{\Gamma^+}\left[8i\frac{5\xi_n^2-1}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{4(5\xi_n^2-1)+12i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right]\xi_j\xi_l d\xi_n dx'\\
&+X_nY_nh'(0)\Omega_3\int_{\Gamma^+}\left[\frac{(4i-4)(\xi^2-1)+48(\xi_n^3-\xi_n)}{(\xi_n-i)^5(\xi_n+i)^4}-\frac{4(5\xi_n^2-1)+12i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right]d\xi_n dx'\\
&=\Sigma_{j=1}^{n-1}X_jY_j\frac{4\pi}{3}h'(0)\Omega_3\left\{\frac{2\pi i}{4!}\left[\frac{8i(5\xi_n^2-1)}{(\xi_n+i)^4}\right]^{(4)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{5!}\left[\frac{4(5\xi_n^2-1)}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'\right.\\
&\left.+\frac{2\pi i}{5!}\left[\frac{\xi_n^3-\xi_n}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'\right\}\\
&-X_nY_nh'(0)\Omega_3\left\{\frac{2\pi i}{4!}\left[\frac{(4i-4)(\xi_n^2-1)}{(\xi_n+i)^4}\right]^{(4)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{4!}\left[\frac{48(\xi_n^3-\xi_n)}{(\xi_n+i)^4}\right]^{(4)}\Big|_{\xi_n=i}dx'\right.\\
&\left.-\frac{2\pi i}{5!}\left[\frac{4(5\xi_n^2-1)}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{5!}\left[\frac{12i(\xi_n^3-\xi_n)}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'\right\}\\
&=-\left[\frac{592}{3}\pi\Sigma_{j=1}^{n-1}X_jY_j+\left(\frac{461}{4}+\frac{23}{4}i\right)X_nY_n\right]h'(0)\pi\Omega_3dx',
\end{aligned}\tag{3.12}$$

其中 Ω_3 是 S^3.的典范体积

情形 a) III) $r=1,\ l=-3,\ j=|\alpha|=0,\ k=1$。
由 (3.3)，我们得到

$$\begin{aligned}
\Phi_3&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\
&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}^2\pi_{\xi_n}^+\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\times\partial_{x_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'.
\end{aligned}\tag{3.13}$$

由引理 2.2，我们有

$$\partial_{x_n}\sigma_{-3}(D^{-3})(x_0)|_{|\xi'|=1}=\frac{\sqrt{-1}\partial_{x_n}[c(\xi')]}{(1+\xi_n^2)^4}-\frac{2\sqrt{-1}h'(0)c(\xi)|\xi'|^2_{g^{\partial M}}}{(1+\xi_n^2)^6},\tag{3.14}$$

此外，

$$\begin{aligned}
\pi_{\xi_n}^+\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})&=-\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)}\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l-\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)}X_nY_n\\
&-\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)}\Sigma_{j=1}^{n-1}X_jY_n\xi_j-\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)}\Sigma_{l=1}^{n-1}X_nY_l\xi_l.
\end{aligned}\tag{3.15}$$

于是，我们有

$$\partial_{\xi_n}^2\pi_{\xi_n}^+\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})=-\frac{c(\xi')+ic(\mathrm{d}x_n)}{(\xi_n-i)^3}\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l-\frac{c(\xi')+ic(\mathrm{d}x_n)}{(\xi_n-i)^3}X_nY_n.\tag{3.16}$$

注意， $i<n$、 $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma\left(\xi'\right)=0$，因此我们省略了计算情形 a) III) 时没有贡献的部分

$$\begin{aligned}&\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_1(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(D^{-3})](x_0)\\&=-2\frac{h'(0)}{(\xi_n-i)^5(\xi_n+i)^2}\Sigma^{n-1}_{j,l=1}X_jY_l\xi_j\xi_l-2\frac{h'(0)}{(\xi_n-i)^5(\xi_n+i)^2}X_nY_n.\end{aligned}\tag{3.17}$$

因此，我们得到

$$\begin{aligned}\Phi_3&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\bigg(-2\frac{h'(0)}{(\xi_n-i)^5(\xi_n+i)^2}\Sigma^{n-1}_{j,l=1}X_jY_l\xi_j\xi_l-2\frac{h'(0)}{(\xi_n-i)^5(\xi_n+i)^2}X_nY_n\bigg)d\xi_n\sigma(\xi')dx'\\&=-\Sigma^{n-1}_{j,l=1}X_jY_lh'(0)\Omega_3\int_{\Gamma^+}\frac{1}{(\xi_n-i)^5(\xi_n+i)^2}\xi_j\xi_l d\xi_n dx'-X_nY_nh'(0)\Omega_3\int_{\Gamma^+}\frac{1}{(\xi_n-i)^5(\xi_n+i)^2}d\xi_n dx'\\&=-\Sigma^{n-1}_{j,l=1}X_jY_lh'(0)\Omega_3\frac{2\pi i}{4!}\left[\frac{1}{(\xi_n+i)^2}\right]^{(4)}\bigg|_{\xi_n=i}dx'+2X_nY_nh'(0)\Omega_3\frac{2\pi i}{4!}\left[\frac{1}{(\xi_n+i)^2}\right]^{(4)}\bigg|_{\xi_n=i}dx'\\&=\left(\frac{5\pi i}{6}\Sigma^{n-1}_{j=1}X_jY_j+\frac{5i}{8}X_nY_n\right)h'(0)\pi\Omega_3dx'.\end{aligned}\tag{3.18}$$

情形 b) $r=0,\ l=-3,\ k=j=|\alpha|=0$。
由 (3.3)，我们得到

$$\begin{aligned}\Phi_4&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_0(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})\times\partial_{\xi_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_0(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})\times\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.19}$$

由引理 2.2，我们有

$$\partial_{\xi_n}\sigma_{-3}(D^{-3})(x_0)|_{|\xi'|=1}=\frac{ic(\mathrm{d}x_n)}{(1+\xi_n^2)^2}-\frac{4\sqrt{-1}\xi_nc(\xi)}{(1+\xi_n^2)^3}.\tag{3.20}$$

由引理 2.1，我们有

$$\begin{aligned}\sigma_0(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})=&-\Sigma^n_{j,l=1}X_jY_l\xi_j\xi_l\left[\frac{c(\xi)\sigma_0(D)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}\Sigma_jc(\mathrm{d}x_j)(\partial_{x_j}[c(\xi)]|\xi|^2-c(\xi)\partial_{x_j}(|\xi|^2))\right]\\&-i\Sigma^n_{j,l=1}X_j\frac{\partial_{Y_l}}{\partial_{x_j}}\xi_j\frac{c(\xi)}{|\xi|^2}-\Sigma_jA(Y)X_j\xi_j\frac{c(\xi)}{|\xi|^2}-\Sigma_lA(Y)Y_l\xi_l\frac{c(\xi)}{|\xi|^2},\end{aligned}\tag{3.21}$$

此外，

$$\begin{aligned}\partial_{\xi_n}\pi^+_{\xi_n}\sigma_0(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})=&-\Sigma^{n-1}_{j,l=1}X_jY_l\xi_j\xi_l[B_1-B_2]\\&-X_nY_n\bigg\{\sigma_0\left[\frac{[c(\xi')+ic(\mathrm{d}x_n)]c(\mathrm{d}x_n)}{4(\xi_n-i)}+\frac{[c(\xi')+ic(\mathrm{d}x_n)]^2}{4(\xi_n-i)^2}\right]\\&+c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')]\left[\frac{i[c(\xi')+ic(\mathrm{d}x_n)]c(\mathrm{d}x_n)}{-4(\xi_n-i)}+\frac{[c(\xi')+ic(\mathrm{d}x_n)]}{4(\xi_n-i)^2}\right]\\&+h'(0)|\xi'|^2\left[\frac{i[c(\xi')+ic(\mathrm{d}x_n)]^2-2c(\mathrm{d}x_n)[3c(\xi')+4ic(\mathrm{d}x_n)]}{16(\xi_n-i)}\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{[c(\xi')+ic(\mathrm{d}x_n)][c(\xi')+5ic(\mathrm{d}x_n)]}{16(\xi_n-i)^2}-\frac{i[c(\xi')+ic(\mathrm{d}x_n)]^2}{8(\xi_n-i)^3}\Bigg]\Bigg\}\\
&-\Sigma_{j=1}^{n-1}X_jY_n\xi_j\left\{\sigma_0\left[\frac{[-ic(\xi')+ic(\mathrm{d}x_n)]c(\mathrm{d}x_n)}{2(\xi_n-i)}-\frac{i[c(\xi')+ic(\mathrm{d}x_n)]^2}{4(\xi_n-i)^2}\right]\right.\\
&+c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')]\left[\frac{-ic(\mathrm{d}x_n)}{4(\xi_n-i)}-\frac{i[c(\xi')+ic(\mathrm{d}x_n)]}{4(\xi_n-i)^2}\right]\\
&-c(\mathrm{d}x_n)h'(0)|\xi'|^2\left[\frac{ic(\mathrm{d}x_n)c(\xi')}{8(\xi_n-i)}+\frac{[c(\xi')+ic(\mathrm{d}x_n)][c(\xi')+3ic(\mathrm{d}x_n)]}{8(\xi_n-i)^2}\right.\\
&\left.\left.-\frac{[c(\xi')+ic(\mathrm{d}x_n)]^2}{8(\xi_n-i)^3}\right]\right\}\\
&-\Sigma_{l=1}^{n-1}X_nY_l\xi_l\left\{\sigma_0\left[\frac{[-ic(\xi')+ic(\mathrm{d}x_n)]c(\mathrm{d}x_n)}{2(\xi_n-i)}-\frac{i[c(\xi')+ic(\mathrm{d}x_n)]^2}{4(\xi_n-i)^2}\right]\right.\\
&+c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')]\left[\frac{-ic(\mathrm{d}x_n)}{4(\xi_n-i)}-\frac{i[c(\xi')+ic(\mathrm{d}x_n)]}{4(\xi_n-i)^2}\right]\\
&-c(\mathrm{d}x_n)h'(0)|\xi'|^2\left[\frac{ic(\mathrm{d}x_n)c(\xi')}{8(\xi_n-i)}+\frac{[c(\xi')+ic(\mathrm{d}x_n)][c(\xi')+3ic(\mathrm{d}x_n)]}{8(\xi_n-i)^2}\right.\\
&\left.\left.-\frac{[c(\xi')+ic(\mathrm{d}x_n)]^2}{8(\xi_n-i)^3}\right]\right\}\\
&-\Sigma_{j,l=1}^{n}X_j\frac{\partial_{Y_l}}{\partial_{x_j}}\xi_j\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^2}-i\Sigma_jA(Y)X_j\xi_j\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^2}\\
&-i\Sigma_lA(Y)Y_l\xi_l\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^2}.
\end{aligned}
\tag{3.22}
$$

与 [12], 类似，我们有

$$
B_1=-\frac{A_1}{4(\xi_n-i)}-\frac{A_2}{4(\xi_n-i)^2}
\tag{3.23}
$$

其中

$$
\begin{aligned}
A_1&=ic(\xi')p_0c(\xi')+ic(\mathrm{d}x_n)p_0c(\mathrm{d}x_n)+ic(\xi')c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')];\\
A_2&=[c(\xi')+ic(\mathrm{d}x_n)]p_0[c(\xi')+ic(\mathrm{d}x_n)]+c(\xi')c(\mathrm{d}x_n)\partial_{x_n}[c(\xi')]-i\partial_{x_n}[c(\xi')];
\end{aligned}
\tag{3.24}
$$

$$
B_2=\frac{h'(0)}{2}\left[\frac{c(\mathrm{d}x_n)}{4i(\xi_n-i)}+\frac{c(\mathrm{d}x_n)-ic(\xi')}{8(\xi_n-i)^2}+\frac{3\xi_n-7i}{8(\xi_n-i)^3}[ic(\xi')-c(\mathrm{d}x_n)]\right].
\tag{3.25}
$$

注意到 $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, 因此我们省略了对计算情形 b) 没有贡献的一些项。

于是，我们有

$$
\begin{aligned}
&\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_0(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\times\sigma_{-3}(D^{-3})](x_0)\\
&=\left[\frac{-2ic_0}{(1+\xi_n^2)^2}+h'(0)\frac{\xi_n^2-i\xi_n-2}{2(\xi_n-i)(1+\xi_n^2)^2}-\frac{\sqrt{-1}}{2}h'(0)\frac{-i\xi_n^2-\xi_n+4i}{4(\xi_n-i)^3(\xi_n+i)^2}tr[id]\right]\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\
&+\left[\frac{4-4i\xi_n-3(\xi_n^2+1)}{(\xi_n-i)^4(\xi_n+i)^3}+\frac{4i+4\xi_n-3i(\xi_n^2+1)}{2(\xi_n-i)^5(\xi_n+i)^3}\right]h'(0)X_nY_n.
\end{aligned}
\tag{3.26}
$$

因此，我们得到

$$
\Phi_4=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\left[\frac{-2ic_0}{(1+\xi_n^2)^2}+h'(0)\frac{\xi_n^2-i\xi_n-2}{2(\xi_n-i)(1+\xi_n^2)^2}\right.
$$

$$
\begin{aligned}
&- \frac{\sqrt{-1}}{2}h'(0)\frac{-i\xi_n^2-\xi_n+4i}{(\xi_n-i)^3(\xi_n+i)^2}\Big]\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\
&+\left[\frac{4-4i\xi_n-3(\xi_n^2+1)}{(\xi_n-i)^4(\xi_n+i)^3}+\frac{4i+4\xi_n-3i(\xi_n^2+1)}{2(\xi_n-i)^5(\xi_n+i)^3}\right]h'(0)X_nY_nd\xi_n\sigma(\xi')dx'\\
&=\Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\Omega_3\int_{\Gamma^+}\left[\frac{-2ic_0}{(1+\xi_n^2)^2}+h'(0)\frac{\xi_n^2-i\xi_n-2}{2(\xi_n-i)(1+\xi_n^2)^2}\right.\\
&-\frac{\sqrt{-1}}{2}h'(0)\frac{-i\xi_n^2-\xi_n+4i}{(\xi_n-i)^3(\xi_n+i)^2}\Big]\xi_j\xi_ld\xi_ndx'\\
&+X_nY_nh'(0)\Omega_3\int_{\Gamma^+}\left[\frac{4-4i\xi_n-3(\xi_n^2+1)}{(\xi_n-i)^4(\xi_n+i)^3}+\frac{4i+4\xi_n-3i(\xi_n^2+1)}{2(\xi_n-i)^5(\xi_n+i)^3}\right]d\xi_ndx'\\
&=\frac{55}{3}\Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\pi^2\Omega_3dx'+X_nY_nh'(0)\Omega_3i\left\{\frac{2\pi i}{3!}\left[\frac{1}{(\xi_n+i)^3}\right]^{(3)}\Big|_{\xi_n=i}dx'\right.\\
&-\frac{4i\times2\pi i}{3!}\left[\frac{\xi_n}{(\xi_n+i)^3}\right]^{(3)}\Big|_{\xi_n=i}dx'-\frac{3\times2\pi i}{2!}\left[\frac{1}{(\xi_n+i)^2}\right]^{(2)}\Big|_{\xi_n=i}dx'\\
&+\frac{4i\times2\pi i}{2\times4!}\left[\frac{1}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'+\frac{4\times2\pi i}{2\times4!}\left[\frac{\xi_n}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'\\
&-\left.\frac{3i\times2\pi i}{2\times3!}\left[\frac{1}{(\xi_n+i)^2}\right]^{(3)}\Big|_{\xi_n=i}dx'\right\}\\
&=\left(\frac{55\pi}{3}\Sigma_{j=1}^{n-1}X_jY_j-\frac{3}{8}X_nY_n\right)h'(0)\pi\Omega_3dx'.
\end{aligned}\tag{3.27}
$$

情况 c) $r=1,\ \ell=-4,\ k=j=|\alpha|=0$。

由 (3.3)，我们得到

$$
\begin{aligned}
\Phi_5&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\times\partial_{\xi_n}\sigma_{-4}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\
&=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\times\sigma_{-4}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'.
\end{aligned}\tag{3.28}
$$

由引理 2.1 和 2.2，我们得到

$$
\begin{aligned}
\sigma_{-4}(D^{-3})=&\frac{1}{(\xi_n^2+1)^4}\left[\left(\frac{11}{2}\xi_n(1+\xi_n^2)+8i\xi_n\right)h'(0)c(\xi')\right.\\
&+\left[-2i+6i\xi_n^2-\frac{7}{4}(1+\xi_n^2)+\frac{15}{4}\xi_n^2(1+\xi_n^2)\right]h'(0)c(\mathrm{d}x_n)\\
&\left.-3i\xi_n(1+\xi_n^2)\partial_{x_n}c(\xi')+i(1+\xi_n^2)c(\xi')c(\mathrm{d}x_n)\partial_{x_n}c(\xi')\right],
\end{aligned}\tag{3.29}
$$

and

$$
\begin{aligned}
\partial_{\xi_n}\pi^+_{\xi_n}\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})=&\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^2}\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l-\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^2}X_nY_n\\
&+\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)^2}\Sigma_{j=1}^{n}X_jY_n\xi_j+\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)^2}\Sigma_{l=1}^{n}X_nY_l\xi_l.
\end{aligned}\tag{3.30}
$$

注意，$i<n$，$\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$，因此我们省略了对计算情况 c) 没有贡献的部分。

我们有

$$
\mathrm{tr}[c(\xi')c(\xi')c(\mathrm{d}x_n)\partial_{x_n}c(\xi')]=0;
$$

$$\mathrm{tr}[c(\mathrm{d}x_n)c(\xi')c(\mathrm{d}x_n)\partial_{x_n}c(\xi')] = -2h'(0) \tag{3.31}$$

于是得到

$$\begin{aligned}
&\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\times\sigma_{-4}(D^{-3})](x_0)|_{|\xi'|=1}\\
&=\frac{h'(0)(7+6i-(20-15i)\xi_n-(7-6i)\xi_n^2+15i\xi_n^3)}{(\xi_n-i)^5(\xi_n+i)^4}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\
&+\frac{(3i-11)\xi_n(1-\xi_n^2)-16i\xi_n+(13+\frac{7}{2}i)(1+\xi_n^2)-16-\frac{15}{2}\xi_n^2(1+\xi_n^2)}{(\xi_n-i)^2(\xi_n+i)^4}X_nY_n.
\end{aligned} \tag{3.32}$$

因此，我们有

$$\begin{aligned}
\Phi_5&=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1})\times\sigma_{-4}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\
&=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{h'(0)(7+6i-(20-15i)\xi_n-(7-6i)\xi_n^2+15i\xi_n^3)}{(\xi_n-i)^5(\xi_n+i)^4}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\
&+\frac{(3i-11)\xi_n(1-\xi_n^2)-16i\xi_n+(13+\frac{7}{2}i)(1+\xi_n^2)-16-\frac{15}{2}\xi_n^2(1+\xi_n^2)}{(\xi_n-i)^2(1+\xi_n^2)^4}X_nY_nd\xi_n\sigma(\xi')dx'\\
&=\left(-\frac{35}{3}+\frac{50}{3}i\right)\sum_{j=1}^{n-1}X_jY_jh'(0)\pi^2\Omega_3dx'\\
&+h'(0)\Omega_3\int_{\Gamma^+}\frac{(3i-11)\xi_n(1-\xi_n^2)-16i\xi_n+(13+\frac{7}{2}i)(1+\xi_n^2)-16-\frac{15}{2}\xi_n^2(1+\xi_n^2)}{(\xi_n-i)^6(\xi_n+i)^4}\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_ld\xi_ndx'\\
&=\left(-\frac{35}{3}+\frac{50}{3}i\right)\sum_{j=1}^{n-1}X_jY_jh'(0)\pi^2\Omega_3dx'\\
&+h'(0)\Omega_3\frac{2\pi i}{5!}X_nY_n\left[\frac{(3i-11)\xi_n(1-\xi_n^2)-16i\xi_n+(13+\frac{7}{2}i)(1+\xi_n^2)-16-\frac{15}{2}\xi_n^2(1+\xi_n^2)}{(\xi_n+i)^4}\right]^{(5)}\Bigg|_{\xi_n=i}dx'\\
&=\left[\left(-\frac{35}{3}+\frac{50}{3}i\right)\sum_{j=1}^{n-1}X_jY_j\pi+\left(5-\frac{137}{32}i\right)X_nY_n\right]h'(0)\pi^2\Omega_3dx'.
\end{aligned} \tag{3.33}$$

令 $X=X^T+X_n\partial_n$, $Y=Y^T+Y_n\partial_n$, 则有 $\sum_{j=1}^{n-1}X_jY_j=g(X^T,Y^T)$。现在， Φ 是情形 (a)、(b) 和 (c) 的总和。因此，我们得到

$$\Phi=\sum_{i=1}^{5}\Phi_i=\left[\left(-\frac{1411}{12}+\frac{27i}{32}\right)X_nY_n+\left(-\frac{572}{3}+\frac{35i}{2}\right)\pi g(X^T,Y^T)\right]h'(0)\pi\Omega_3dx'. \tag{3.34}$$

因此，由 (3.3)-(3.34)，我们得到如下定理

定理 3.1. 设 M 为带边界 ∂M 的4维定向紧致自旋流形，且度量g^M 定义如式（3.1），则有以下等式:

$$\begin{aligned}
&\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-2})\circ\pi^+(D^{-2})]\\
&=\frac{4\pi^2}{3}\int_M[Ric(X,Y)-\frac{1}{2}sg(X,Y)]d\mathrm{Vol_M}+\pi^2\int_M sg(X,Y)d\mathrm{Vol_M}\\
&+\int_{\partial M}\left[\left(-\frac{1411}{12}+\frac{27i}{32}\right)X_nY_n+\left(-\frac{572}{3}+\frac{35i}{2}\right)\pi g(X^T,Y^T)\right]h'(0)\pi\Omega_3d\mathrm{Vol_M}.
\end{aligned} \tag{3.35}$$

4. 带边界的三维流形的非交换留数

本节计算带边界的3维定向紧流形上的非交换留数 $\widetilde{\text{Wres}}[\pi^+(\nabla^{S(TM)}{}_X \quad \nabla_Y^{S(TM)}D^{-1}) \circ \pi^+(D^{-3})]$。与 [10], 中带边界的奇维流形的情形相同，我们有如下公式

$$\widetilde{\text{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \circ \pi^+(D^{-3})] = \int_{\partial M}\Phi \tag{4.1}$$

设 M 为带边界 ∂M的三维紧致定向流形。我们有

$$\begin{aligned} &\text{tr}[c^2(\xi')] = -2;\ \text{tr}[c(\xi')c(\mathrm{d}x_n))] = 0;\\ &\text{tr}[c(\mathrm{d}x_n))c(\mathrm{d}x_n))] = -2;\ \text{tr}[c(\xi') + ic(\mathrm{d}x_n)][c(\xi') + ic(\mathrm{d}x_n)] = -2 - 2i\xi_n. \end{aligned} \tag{4.2}$$

由 (3.3)，满足 $r + l - k - |\alpha| - j - 1 = -3$，因此得到 $r = 1$、 $\ell = -3$、 $k = j = |\alpha| = 0$，于是

$$\begin{aligned} \Phi &= -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\text{trace}[\pi^+_{\xi_n}\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \times \partial_{\xi_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\ &= \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\text{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \times \sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \end{aligned} \tag{4.3}$$

注意到 $i < n$， $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi') = 0$，因此我们省略了一些对计算没有贡献的项。由 (3.30) 和 (2.8)，我们有

$$\begin{aligned} &\text{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \times \sigma_{-3}(D^{-3})]\\ &= -\frac{i-\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l - \frac{\sqrt{-1}\xi_n^2(2+2i\xi_n)}{2(\xi_n-i)^3(\xi_n+i)^2}X_nY_n. \end{aligned} \tag{4.4}$$

于是有如下公式

$$\begin{aligned} \Phi &= -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\text{trace}[\pi^+_{\xi_n}\sigma_1(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \times \partial_{\xi_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\ &= \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{i-\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l + \frac{\sqrt{-1}\xi_n^2(2+2i\xi_n)}{2(\xi_n-i)^3(\xi_n+i)^2}X_nY_n(x_0)d\xi_n\sigma(\xi')dx'\\ &= \Omega_2\int_{\Gamma^+} -\frac{i-\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l + \frac{\sqrt{-1}\xi_n^2(2+2i\xi_n)}{2(\xi_n-i)^3(\xi_n+i)^2}X_nY_n(x_0)d\xi_n dx'\\ &= -\Omega_2\frac{2\pi i}{3!}\frac{4\pi}{3}\Sigma_{j,l=1}^{n-1}X_jY_l\left[\frac{i-\xi_n}{(\xi_n+i)^2}\right]^{(3)}\Big|_{(\xi_n=i)}\mathrm{d}x' + \Omega_2\frac{2\pi i}{3!}X_nY_n\left[\frac{\xi_n-i}{(\xi_n+i)^2}\right]^{(3)}\Big|_{(\xi_n=i)}\mathrm{d}x'\\ &= \left(\frac{i}{3}\Sigma_{j=1}^{n-1}X_jY_j\pi + \frac{3i}{4}X_nY_n\right)\pi\Omega_2\mathrm{d}x'. \end{aligned} \tag{4.5}$$

因此，我们得到以下定理

定理 4.1. 设 M 为一个带边界的三维定向紧致自旋流形，边界为 ∂M，且度量g^M 如第3节所定义，则有以下等式:

$$\widetilde{\text{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-1}) \circ \pi^+(D^{-3})] = \left(\frac{i}{3}\Sigma_{j=1}^{n-1}X_jY_j\pi + \frac{3i}{4}X_nY_n\right)\pi\Omega_2\mathrm{d}x'. \tag{4.6}$$

5. 六维带边流形上的非交换留数

本节中，我们计算带边界的六维定向紧流形上的非交换留数 $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\circ\pi^+(D^{-3})]$。更多详情请参见 [12] 中的第 2 节。

设 M 是一个带边的六维紧致定向流形， ∂M 上的度量与第3节中的相同。我们可以计算非交换留数

$$\begin{aligned}&\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\circ\pi^+(D^{-3})]\\&=\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\sigma_{-6}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3}\circ D^{-3})]\sigma(\xi)dx+\int_{\partial M}\Phi,\end{aligned}\tag{5.1}$$

其中

$$\begin{aligned}\Phi=&\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times\mathrm{trace}_{\wedge^*T^*M\otimes\mathbb{C}}[\partial_{x_n}^j\partial_{\xi'}^\alpha\partial_{\xi_n}^k\sigma_r^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})(x',0,\xi',\xi_n)\\&\times\partial_{x'}^\alpha\partial_{\xi_n}^{j+1}\partial_{x_n}^k\sigma_l(D^{-3})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx',\end{aligned}\tag{5.2}$$

求和遍及 $r+l-k-j-|\alpha|=-5$、 $r\leq-1$、 $l\leq-3$。

根据定理 2.3，我们可以计算 $\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\circ\pi^+(D^{-3})]$, 的内部项, 得到

$$\mathrm{Wres}[\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-6}]=\frac{4\pi^3}{3}\int_M[Ric(X,Y)-\frac{1}{2}sg(X,Y)]d\mathrm{Vol_M}+\pi^3\int_M sg(X,Y)d\mathrm{Vol_M}.\tag{5.3}$$

现在我们需要计算 $\int_{\partial M}\Phi$。当 $n=6$ 时，$\mathrm{tr}_{S(TM)}[\mathrm{id}]=\dim(\wedge^*(\mathbb{R}^3))=8$，求和遍及 $r+l-k-j-|\alpha|=-6$、 $r\leq-1$、 $l\leq-3$，于是得到以下五种情形:

情形 a) I) $r=-1$、 $l=-3$、 $k=j=0$、 $|\alpha|=1$。
由（5.2）可得

$$\Phi_1=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi'}^\alpha\pi_{\xi_n}^+\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\partial_{x'}^\alpha\partial_{\xi_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'.\tag{5.4}$$

根据 [12], 中关于 $i<n$ 的引理 2.2，有

$$\partial_{x_i}\sigma_{-3}(D^{-3})(x_0)=\partial_{x_i}(\sqrt{-1}c(\xi)|\xi|^{-4})(x_0)=\sqrt{-1}\frac{\partial_{x_i}c(\xi)}{|\xi|^4}(x_0)+\sqrt{-1}\frac{c(\xi)\partial_{x_i}(|\xi|^4)}{|\xi|^8}(x_0)=0,\tag{5.5}$$

因此 $\Phi_1=0$。

情形 a) II) $r=-1$、 $l=-3$、 $k=|\alpha|=0$、 $j=1$。
由（5.2）可得

$$\Phi_2=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\pi_{\xi_n}^+\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\partial_{\xi_n}^2\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'.\tag{5.6}$$

由引理 2.2，可得

$$\partial_{\xi_n}^2\sigma_{-3}((D^{-3}))(x_0)=\partial_{\xi_n}^2(c(\xi)|\xi|^{-4})(x_0)=\sqrt{-1}\frac{(20\xi_n^2-4)c(\xi')+12(\xi^3-\xi)c(\mathrm{d}x_n)}{(1+\xi_n^2)^4}.\tag{5.7}$$

$$\partial_{x_n}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})=\Sigma_{j,l=1}^{n}X_jY_l\xi_j\xi_l\left[\frac{\partial_{x_n}c(\xi')^{\,2}}{1+\xi_n^2}-\frac{2c(\xi)h'(0)|\xi'|^2}{(1+\xi_n^2)^3}\right]. \tag{5.8}$$

因此，可得

$$\begin{aligned}
\pi^+_{\xi_n}\partial_{x_n}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3}) &= \partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\\
&=\sqrt{-1}\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_lh'(0)|\xi'|^2\left[\frac{-3ic(\xi')}{4(\xi_n-i)}-\frac{3c(\xi')+ic(\mathrm{d}x_n)}{4(\xi_n-i)^2}\right.\\
&\left.+\frac{c(\xi')+ic(\mathrm{d}x_n)}{(\xi_n-i)^3}\right]-\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\frac{\partial_{x_n}c(\xi')[i(\xi_n-i)+1]}{4(\xi_n-i)^2}\\
&-\sqrt{-1}X_nY_n\left\{\frac{-3i\partial_{x_n}c(\xi')}{4(\xi_n-i)}+\frac{\partial_{x_n}c(\xi')}{4(\xi_n-i)^2}+\frac{c(\xi')+ic(\mathrm{d}x_n)}{2i(\xi_n-i)^3}\right.\\
&\left.+h'(0)|\xi'|\left[-\frac{-5c(\xi')+9ic(\mathrm{d}x_n)}{4(\xi_n-i)}+\frac{3c(\xi')+3ic(\mathrm{d}x_n)}{4(\xi_n-i)^2}\right]\right\}\\
&-\Sigma_{j=1}^{n-1}X_jY_n\xi_j\left\{-\frac{i\partial_{x_n}c(\xi')}{4(\xi_n-i)^2}+h'(0)|\xi'|\left[\frac{11ic(\mathrm{d}x_n)}{8(\xi_n-i)}\right.\right.\\
&\left.\left.-\frac{ic(\xi')+(1+4i)c(\mathrm{d}x_n)}{16(\xi_n-i)^2}-\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)^3}\right]\right\}\\
&-\Sigma_{l=1}^{n-1}X_nY_l\xi_l\left\{-\frac{i\partial_{x_n}c(\xi')}{4(\xi_n-i)^2}+h'(0)|\xi'|\left[\frac{11ic(\mathrm{d}x_n)}{8(\xi_n-i)}\right.\right.\\
&\left.\left.-\frac{ic(\xi')+(1+4i)c(\mathrm{d}x_n)}{16(\xi_n-i)^2}-\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)^3}\right]\right\}.
\end{aligned} \tag{5.9}$$

我们注意到 $i<n$，$\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$，因此省略了一些对计算情形 a) II) 没有贡献的项。

于是得到以下公式

$$\begin{aligned}
&\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\partial^2_{\xi_n}\sigma_{-3}(D^{-3})](x_0)\\
&=-\sqrt{-1}\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_lh'(0)\left[\frac{24i(5\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{[2i(\xi_n-i)+26](5\xi_n^2-1)+24i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right.\\
&\left.-\frac{32(5\xi_n^2-1)+96(\xi_n^3-\xi_n)}{(\xi_n-i)^7(\xi_n+i)^4}\right]\\
&+\sqrt{-1}X_nY_nh'(0)\left[\frac{46i(5\xi^2-1)-216(\xi_n^3-\xi_n)}{(\xi_n-i)^5(\xi_n+i)^4}-\frac{26(5\xi_n^2-1)+72i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right.\\
&\left.-\frac{16(5\xi_n^2-1)+48i(\xi_n^3-\xi_n)}{i(\xi_n-i)^7(\xi_n+i)^4}\right].
\end{aligned} \tag{5.10}$$

因此，有如下公式

$$\begin{aligned}
\Phi_2=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}&\left\{-i\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_lh'(0)\left[\frac{24i(5\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^4}\right.\right.\\
&\left.\left.+\frac{[2i(\xi_n-i)+26](5\xi_n^2-1)+24i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}-\frac{32(5\xi_n^2-1)+96(\xi_n^3-\xi_n)}{(\xi_n-i)^7(\xi_n+i)^4}\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{-1}X_nY_nh'(0)\left[\frac{46i(5\xi^2-1)-216(\xi_n^3-\xi_n)}{(\xi_n-i)^5(\xi_n+i)^4}-\frac{26(5\xi_n^2-1)+72i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right.\\
&\left.-\frac{16(5\xi_n^2-1)+48i(\xi_n^3-\xi_n)}{i(\xi_n-i)^7(\xi_n+i)^4}\right]\Big\}d\xi_n\sigma(\xi')dx'\\
&=-\frac{1}{2}\sqrt{-1}\Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\Omega_4\int_{\Gamma^+}\left[\frac{24i(5\xi_n^2-1)}{(\xi_n-i)^5(\xi_n+i)^4}+\frac{[2i(\xi_n-i)+26](5\xi_n^2-1)+24i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right.\\
&\left.-\frac{32(5\xi_n^2-1)+96(\xi_n^3-\xi_n)}{(\xi_n-i)^7(\xi_n+i)^4}\right]\xi_j\xi_ld\xi_ndx'\\
&+\frac{1}{2}\sqrt{-1}X_nY_nh'(0)\Omega_4\int_{\Gamma^+}\left[\frac{46i(5\xi^2-1)-216(\xi_n^3-\xi_n)}{(\xi_n-i)^5(\xi_n+i)^4}-\frac{26(5\xi_n^2-1)+72i(\xi_n^3-\xi_n)}{(\xi_n-i)^6(\xi_n+i)^4}\right.\\
&\left.-\frac{16(5\xi_n^2-1)+48i(\xi_n^3-\xi_n)}{i(\xi_n-i)^7(\xi_n+i)^4}\right]d\xi_ndx'\\
&=-\frac{1}{2}\sqrt{-1}\Sigma_{j=1}^{n-1}X_jY_j\frac{4\pi}{3}h'(0)\Omega_4\left\{\frac{2\pi i}{4!}\left[\frac{24i(5\xi_n^2-1)}{(\xi_n+i)^4}\right]^{(4)}\Big|_{\xi_n=i}dx'\right.\\
&+\frac{2\pi i}{5!}\left[\frac{[2i(\xi_n-i)+26](5\xi_n^2-1)+24i(\xi_n^3-\xi_n)}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'\\
&\left.+\frac{2\pi i}{6!}\left[\frac{32(5\xi_n^2-1)+96(\xi_n^3-\xi_n)}{(\xi_n+i)^4}\right]^{(6)}\Big|_{\xi_n=i}dx'\right\}\\
&+\frac{1}{2}\sqrt{-1}X_nY_nh'(0)\Omega_4\left\{\frac{2\pi i}{4!}\left[\frac{46i(5\xi^2-1)-216(\xi_n^3-\xi_n)}{(\xi_n+i)^4}\right]^{(4)}\Big|_{\xi_n=i}dx'\right.\\
&\left.+\frac{2\pi i}{5!}\left[\frac{26(5\xi_n^2-1)+72i(\xi_n^3-\xi_n)}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{6!}\left[\frac{16(5\xi_n^2-1)+48i(\xi_n^3-\xi_n)}{(\xi_n+i)^4}\right]^{(6)}\Big|_{\xi_n=i}dx'\right\}\\
&=\left[\left(-\frac{5925}{16}-\frac{877}{6}i\right)\pi\Sigma_{j=1}^{n-1}X_jY_j+(311i)\,X_nY_n\right]h'(0)\pi\Omega_4dx',
\end{aligned}
\tag{5.11}
$$

其中 Ω_4 是 S^4.的典范体积

情形 a) III) $r=-1,\ l=-3,\ j=|\alpha|=0,\ k=1$。由
(5.2) 可得

$$
\begin{aligned}
\Phi_3&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\
&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{\xi_n}^2\pi_{\xi_n}^{+}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\partial_{x_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'.
\end{aligned}
\tag{5.12}
$$

由 [8], 可得

$$
\begin{aligned}
\pi_{\xi_n}^{+}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})&=\left[\frac{c(\xi')}{4(\xi_n-i)}-\frac{ic(\xi')-c(\mathrm{d}x_n)}{4(\xi_n-i)^2}\right]\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\
&+\left[\frac{c(\xi')-2ic(\mathrm{d}x_n)}{4(\xi_n-i)}+\frac{ic(\xi')-c(\mathrm{d}x_n)}{4(\xi_n-i)^2}\right]X_nY_n\\
&+\left[\frac{c(\mathrm{d}x_n)}{4(\xi_n-i)}+\frac{c(\xi')+ic(\mathrm{d}x_n)}{4(\xi_n-i)^2}\right]\Sigma_{j=1}^{n-1}X_jY_n\xi_j\\
&+\left[\frac{c(\mathrm{d}x_n)}{4(\xi_n-i)}+\frac{c(\xi')+ic(\mathrm{d}x_n)}{4(\xi_n-i)^2}\right]\Sigma_{l=1}^{n-1}X_nY_l\xi_l.
\end{aligned}
\tag{5.13}
$$

于是,

$$\begin{aligned}\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-3}) &= \left[-\frac{ic(\xi')}{4(\xi_n-i)^2}+\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)^3}\right]\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\ &-\left[\frac{c(\xi')+2ic(\mathrm{d}x_n)}{4(\xi_n-i)^2}+\frac{ic(\xi')-c(\mathrm{d}x_n)}{2(\xi_n-i)^3}\right]X_nY_n\\ &.-\left[\frac{c(\mathrm{d}x_n)}{4(\xi_n-i)^2}+\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^3}\right]\Sigma_{j=1}^{n-1}X_jY_n\xi_j\\ &-\left[\frac{c(\mathrm{d}x_n)}{4(\xi_n-i)^2}+\frac{c(\xi')+ic(\mathrm{d}x_n)}{2(\xi_n-i)^3}\right]\Sigma_{l=1}^{n-1}X_nY_l\xi_l. \qquad (5.14)\end{aligned}$$

由式 (3.14) 和 (5.14)，可得

$$\begin{aligned}&\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-3})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}(D^{-3})](x_0)\\ &=\left[\frac{24i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^4}-\frac{4i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^3}+\frac{48\xi_n-8i(1-5\xi_n^2)}{(\xi_n-i)^3(1+\xi_n^2)^4}-\frac{8\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^3}\right]h'(0)\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\ &+\left[\frac{24i\xi_n+8(1_5\xi_n^2)}{(\xi_n-i)^2(1+\xi_n^2)^4}-\frac{4i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^3}-\frac{48\xi_n-8i(1-5\xi_n^2)}{(\xi_n-i)^3(1+\xi_n^2)^4}+\frac{8\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^3}\right]h'(0)X_nY_n.\end{aligned} \qquad (5.15)$$

因此，我们得到以下公式

$$\begin{aligned}\Phi_3 &= \frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\Bigg(\left[\frac{24i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^4}-\frac{4i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^3}+\frac{48\xi_n-8i(1-5\xi_n^2)}{(\xi_n-i)^3(1+\xi_n^2)^4}\right.\\ &\left.-\frac{8\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^3}\right]h'(0)\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\ &+\left[\frac{24i\xi_n+8(1_5\xi_n^2)}{(\xi_n-i)^2(1+\xi_n^2)^4}-\frac{4i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^3}-\frac{48\xi_n-8i(1-5\xi_n^2)}{(\xi_n-i)^3(1+\xi_n^2)^4}\right.\\ &\left.+\frac{8\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^3}\right]h'(0)X_nY_nbigg)d\xi_n\sigma(\xi')dx'\\ &=-\frac{1}{2}\Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\Omega_4\left\{\frac{2\pi i}{5!}\left[\frac{24\xi_n}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}-\frac{2\pi i}{4!}\left[\frac{4\xi_n}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}\right.\\ &\left.+\frac{2\pi i}{6!}\left[\frac{48\xi_n-8i(1-5\xi_n^2)}{(\xi_n+i)^4}\right]^{(6)}\Big|_{\xi_n=i}-\frac{2\pi i}{5!}\left[\frac{8\xi_n}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}\right\}dx'\\ &-\frac{1}{2}X_nY_nh'(0)\Omega_4\left\{\frac{2\pi i}{5!}\left[\frac{24\xi_n+8(1-5\xi_n^2)}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}-\frac{2\pi i}{4!}\left[\frac{4i\xi_n}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}\right.\\ &\left.-\frac{2\pi i}{6!}\left[\frac{48\xi_n-8i(1-5\xi_n^2)}{(\xi_n+i)^4}\right]^{(6)}\Big|_{\xi_n=i}+\frac{2\pi i}{5!}\left[\frac{8xi_n}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}\right\}dx'\\ &=\left[\left(\frac{49}{3}+86i\right)\pi\Sigma_{j=1}^{n-1}X_jY_j-\left(\frac{17}{4}-68i\right)X_nY_n\right]h'(0)\pi\Omega_3dx'. \qquad (5.16)\end{aligned}$$

情形 b) $r=-1,\ l=-4,\ k=j=|\alpha|=0$。
由 (5.2)，可得

$$\Phi_4=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_{-1}(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-3})\times\partial_{\xi_n}\sigma_{-4}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'$$

$$= i\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \mathrm{trace}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\sigma_{-4}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \tag{5.17}$$

注意到 $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, 因此我们省略了对计算情形 b) 没有贡献的一些项。由 (3.29) 和 (5.14), 可得

$$\begin{aligned}
&\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\sigma_{-3}(D^{-3})](x_0)\\
&= h'(0)\left[\frac{(-11+3i)\xi_n(1+\xi_n^2)-16i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^4}\right.\\
&-\left.\sqrt{-1}\frac{(-22+6i)\xi_n(1+\xi_n^2)-32i\xi_n+(26-7i)(1+\xi_n^2)-32-15i\xi_n^2(1+\xi_n^2)}{(\xi_n-i)^3(\xi_n+i)^4}\right]\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\
&-\sqrt{-1}\left[\frac{(-11-3i)\xi_n(1+\xi_n^2)+16\xi_n+(48i-14)(1+\xi_n^2)-32i+30\xi^2(1+\xi_n^2)}{(\xi_n-i)^6(\xi_n+i)^4}\right.\\
&+\left.\frac{(-22+6i)\xi_n(1+\xi_n^2)-32i\xi_n+(26-7i)(1+\xi_n^2)-32-15i\xi_n^2(1+\xi_n^2)}{(\xi_n-i)^3(\xi_n+i)^4}\right]h'(0)X_nY_n.
\end{aligned}\tag{5.18}$$

因此, 我们得到

$$\begin{aligned}
\Phi_4 &= i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}h'(0)\left[\frac{(-11+3i)\xi_n(1+\xi_n^2)-16i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^4}\right.\\
&-\left.\sqrt{-1}\frac{(-22+6i)\xi_n(1+\xi_n^2)-32i\xi_n+(26-7i)(1+\xi_n^2)-32-15i\xi_n^2(1+\xi_n^2)}{(\xi_n-i)^3(1+\xi_n^2)^4}\right]\Sigma_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\
&-\sqrt{-1}\left[\frac{(-11-3i)\xi_n(1+\xi_n^2)+16\xi_n+(48i-14)(1+\xi_n^2)-32i+30\xi^2(1+\xi_n^2)}{(\xi_n-i)^2(1+\xi_n^2)^4}\right.\\
&+\left.\frac{(-22+6i)\xi_n(1+\xi_n^2)-32i\xi_n+(26-7i)(1+\xi_n^2)-32-15i\xi_n^2(1+\xi_n^2)}{(\xi_n-i)^3(1+\xi_n^2)^4}\right]h'(0)X_nY_nd\xi_n\sigma(\xi')dx'\\
&=\Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\Omega_4\int_{\Gamma^+}\left[\frac{(-11+3i)\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^3}-\frac{16i\xi_n}{(\xi_n-i)^2(1+\xi_n^2)^4}-\frac{(-22+6i)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^3}\right.\\
&-\left.\frac{32\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^4}-\frac{(26-7i)i}{(\xi_n-i)^3(1+\xi_n^2)^3}+\frac{32i}{(\xi_n-i)^3(1+\xi_n^2)^4}-\frac{15\xi_n^2}{(\xi_n-i)^3(1+\xi_n^2)^3}\right]\xi_j\xi_ld\xi_ndx'\\
&+X_nY_nh'(0)\Omega_4\int_{\Gamma^+}\left[\frac{16\xi_n}{(\xi_n-i)^6(\xi_n+i)^4}-\frac{(3+11i)\xi_n}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{48i-14}{(\xi_n-i)^6(\xi_n+i)^4}\right.\\
&-\frac{32i}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{30\xi_n^2}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{(-22+6i)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^3}+\frac{32\xi_n}{(\xi_n-i)^3(1+\xi_n^2)^4}\\
&+\left.\frac{(26-7i)i}{(\xi_n-i)^3(1+\xi_n^2)^3}-\frac{32i}{(\xi_n-i)^3(1+\xi_n^2)^4}+\frac{15\xi_n^2}{(\xi_n-i)^3(1+\xi_n^2)^3}\right]d\xi_ndx'\\
&=\Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\pi^2\Omega_4\left\{\frac{2\pi i}{4!}\left[\frac{(-11+3i)\xi_n}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{5!}\left[\frac{16i\xi_n}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'\right.\\
&-\frac{2\pi i}{5!}\left[\frac{(-22+6i)i\xi_n}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{6!}\left[\frac{32\xi_n}{(\xi_n+i)^4}\right]^{(6)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{5!}\left[\frac{(26-7i)i}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'\\
&+\left.\frac{2\pi i}{6!}\left[\frac{32i}{(\xi_n+i)^4}\right]^{(6)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{5!}\left[\frac{15\xi_n^2}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'\right\}
\end{aligned}$$

$$
\begin{aligned}
&+X_nY_nh'(0)\Omega_4\left\{\frac{2\pi i}{5!}\left[\frac{16\xi_n}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{4!}\left[\frac{(3+11i)\xi_n}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'\right.\\
&-\frac{2\pi i}{5!}\left[\frac{48i-14}{(\xi_n+i)^4}\right]^{(5)}\Big|_{\xi_n=i}dx'-\frac{32i\times 2\pi i}{4!}\left[\frac{1}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{4!}\left[\frac{30\xi_n^2}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'\\
&+\frac{2\pi i}{5!}\left[\frac{(-22+6i)i\xi_n}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{6!}\left[\frac{32\xi_n}{(\xi_n+i)^4}\right]^{(6)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{5!}\left[\frac{(26-7i)i}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'\\
&\left.-\frac{2\pi i}{6!}\left[\frac{32i}{(\xi_n+i)^4}\right]^{(6)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{5!}\left[\frac{15\xi_n^2}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'dx'\right\}\\
&=\left[-\left(\frac{53}{3}+\frac{41}{6}i\right)\pi\Sigma_{j=1}^{n-1}X_jY_j+\left(\frac{173i}{4}+36\right)X_nY_n\right]h'(0)\pi\Omega_3dx'. \qquad (5.19)
\end{aligned}
$$

情况 c) $r=-2$、 $\ell=-3$、
$k=j=|\alpha|=0$。由 (5.2)，我们得到

$$
\Phi_5=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\pi^+_{\xi_n}\sigma_{-2}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\partial_{\xi_n}\sigma_{-3}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'. \qquad (5.20)
$$

由 [8], 我们有

$$
\begin{aligned}
\sigma_{-2}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})=&-\sum_j A(Y)X_j\xi_j\frac{c(\xi)}{|\xi|^4}-\sum_l A(X)Y_l\xi_l\frac{c(\xi)}{|\xi|^4}-\sum_{j,l=1}^{n-1}X_j\frac{\partial_{Y_l}}{\partial_{X_j}}\xi_l\frac{c(\xi)}{|\xi|^4}\\
&+\sum_{j,l=1}^{n}X_jY_l\xi_j\xi_l\frac{1}{(\xi_n^2+1)^4}\left\{\left(\frac{11}{2}\xi_n(1+\xi_n^2)+8i\xi_n\right)h'(0)c(\xi')\right.\\
&+\left[-2i+6i\xi_n^2-\frac{7}{4}(1+\xi_n^2)+\frac{15}{4}\xi_n^2(1+\xi_n^2)\right]h'(0)c(\mathrm{d}x_n)\\
&\left.-3i\xi_n(1+\xi_n^2)\partial_{x_n}c(\xi')+i(1+\xi_n^2)c(\xi')c(\mathrm{d}x_n)\partial_{x_n}c(\xi')\right\}. \qquad (5.21)
\end{aligned}
$$

注意到 $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\sigma(\xi')=0$, 因此我们省略了对计算情况 c) 没有贡献的一些项。

由（5.21）和（3.20），可得

$$
\begin{aligned}
&\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-2}(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\times\partial_{\xi_n}\sigma_{-3}(D^{-3})](x_0)|_{|\xi'|=1}\\
&=\left[\frac{1-9i}{(\xi_n-i)^4(\xi_n+i)^3}-\frac{6-27i}{4(\xi_n-i)^3(\xi_n+i)^2}-\frac{27+58i}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{(1+5i)\xi_n}{(\xi_n-i)^5(\xi_n+i)^3}\right.\\
&+\frac{81+174i}{4(\xi_n-i)^4(\xi_n+i)^2}-\frac{10-2i}{(\xi_n-i)^6(\xi_n+i)^3}+\frac{(14-i)\xi_n}{(\xi_n-i)^6(\xi_n+i)^3}+\frac{16-3i}{2(\xi_n-i)^5(\xi_n+i)^2}\\
&\left.+\frac{4i-12}{(\xi_n-i)^7(\xi_n+i)^3}-\frac{16i\xi_n}{(\xi_n-i)^7(\xi_n+i)^3}+\frac{9-3i}{(\xi_n-i)^6(\xi_n+i)^2}\right]h'(0)\sum_{j,l=1}^{n-1}X_jY_l\xi_j\xi_l\\
&+\left[-\frac{72+25i}{4(\xi_n-i)^4(\xi_n+i)^3}+\frac{54+8i}{(\xi_n-i)^3(\xi_n+i)^2}-\frac{7+43i}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{(44-9i)\xi_n}{2(\xi_n-i)^5(\xi_n+i)^3}\right.\\
&+\frac{21+129i}{4(\xi_n-i)^4(\xi_n+i)^2}+\frac{22-26i}{(\xi_n-i)^6(\xi_n+i)^3}+\frac{3\xi_n}{(\xi_n-i)^6(\xi_n+i)^3}-\frac{33-39i}{2(\xi_n-i)^6(\xi_n+i)^2}\\
&\left.+\frac{2i}{(\xi_n-i)^7(\xi_n+i)^3}+\frac{16\xi_n}{(\xi_n-i)^7(\xi_n+i)^3}-\frac{3i}{2(\xi_n-i)^6(\xi_n+i)^2}\right]h'(0)X_nY_n. \qquad (5.22)
\end{aligned}
$$

因此可得

$$
\begin{aligned}
\Phi_5 &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_1(\nabla^{S(TM)}_X\nabla^{S(TM)}_Y D^{-1})\times\partial_{\xi_n}\sigma_{-4}(D^{-3})](x_0)d\xi_n\sigma(\xi')dx'\\
&= \Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\Omega_4\int_{\Gamma^+}\left[\frac{1-9i}{(\xi_n-i)^4(\xi_n+i)^3}-\frac{6-27i}{4(\xi_n-i)^3(\xi_n+i)^2}-\frac{27+58i}{(\xi_n-i)^5(\xi_n+i)^3}\right.\\
&+\frac{(1+5i)\xi_n}{(\xi_n-i)^5(\xi_n+i)^3}+\frac{81+174i}{4(\xi_n-i)^4(\xi_n+i)^2}-\frac{10-2i}{(\xi_n-i)^6(\xi_n+i)^3}+\frac{(14-i)\xi_n}{(\xi_n-i)^6(\xi_n+i)^3}\\
&\left.+\frac{16-3i}{2(\xi_n-i)^5(\xi_n+i)^2}+\frac{4i-12}{(\xi_n-i)^7(\xi_n+i)^3}-\frac{16i\xi_n}{(\xi_n-i)^7(\xi_n+i)^3}+\frac{9-3i}{(\xi_n-i)^6(\xi_n+i)^2}\right]\xi_j\xi_l d\xi_n dx'\\
&+X_nY_nh'(0)\Omega_4\int_{\Gamma^+}\left[-\frac{72+25i}{4(\xi_n-i)^4(\xi_n+i)^3}+\frac{54+8i}{(\xi_n-i)^3(\xi_n+i)^2}-\frac{7+43i}{(\xi_n-i)^5(\xi_n+i)^3}\right.\\
&+\frac{(44-9i)\xi_n}{2(\xi_n-i)^5(\xi_n+i)^3}+\frac{21+129i}{4(\xi_n-i)^4(\xi_n+i)^2}+\frac{22-26i}{(\xi_n-i)^6(\xi_n+i)^3}+\frac{3\xi_n}{(\xi_n-i)^6(\xi_n+i)^3}\\
&\left.-\frac{33-39i}{2(\xi_n-i)^6(\xi_n+i)^2}+\frac{2i}{(\xi_n-i)^7(\xi_n+i)^3}+\frac{16\xi_n}{(\xi_n-i)^7(\xi_n+i)^3}-\frac{3i}{2(\xi_n-i)^6(\xi_n+i)^2}\right]d\xi_n dx'\\
&= \Sigma_{j,l=1}^{n-1}X_jY_lh'(0)\pi\Omega_4\left\{\frac{2\pi i}{3!}\left[\frac{1-9i}{(\xi_n+i)^3}\right]^{(3)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{2!}\left[\frac{6-27i}{4(\xi_n+i)^2}\right]^{(2)}\Big|_{\xi_n=i}dx'\right.\\
&-\frac{2\pi i}{4!}\left[\frac{27+58i}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{4!}\left[\frac{(1+5i)\xi_n}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{3!}\left[\frac{81+174i}{4(\xi_n+i)^2}\right]^{(3)}\Big|_{\xi_n=i}dx'\\
&-\frac{2\pi i}{5!}\left[\frac{10-2i}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{5!}\left[\frac{(14-i)\xi_n}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{4!}\left[\frac{16-3i}{2(\xi_n+i)^2}\right]^{(4)}\Big|_{\xi_n=i}dx'\\
&\left.+\frac{2\pi i}{6!}\left[\frac{4i-12}{(\xi_n+i)^3}\right]^{(6)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{6!}\left[\frac{16i\xi_n}{(\xi_n+i)^3}\right]^{(6)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{5!}\left[\frac{9-3i}{(\xi_n+i)^2}\right]^{(5)}\Big|_{\xi_n=i}dx'\right\}\\
&+X_nY_nh'(0)\Omega_4\left\{\frac{2\pi i}{3!}\left[-\frac{72+25i}{4(\xi_n+i)^3}\right]^{(3)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{2!}\left[\frac{54+8i}{(\xi_n+i)^2}\right]^{(2)}\Big|_{\xi_n=i}dx'\right.\\
&-\frac{2\pi i}{4!}\left[\frac{7+43i}{(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{4!}\left[\frac{(44-9i)\xi_n}{2(\xi_n+i)^3}\right]^{(4)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{3!}\left[\frac{21+129i}{4(\xi_n+i)^2}\right]^{(3)}\Big|_{\xi_n=i}dx'\\
&+\frac{2\pi i}{5!}\left[\frac{22-26i}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{5!}\left[\frac{3\xi_n}{(\xi_n+i)^3}\right]^{(5)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{5!}\left[\frac{33-39i}{2(\xi_n+i)^2}\right]^{(5)}\Big|_{\xi_n=i}dx'\\
&\left.+\frac{2\pi i}{6!}\left[\frac{2i}{(\xi_n+i)^3}\right]^{(6)}\Big|_{\xi_n=i}dx'+\frac{2\pi i}{6!}\left[\frac{16\xi_n}{(\xi_n+i)^3}\right]^{(6)}\Big|_{\xi_n=i}dx'-\frac{2\pi i}{5!}\left[\frac{3i}{2(\xi_n+i)^2}\right]^{(5)}\Big|_{\xi_n=i}dx'\right\}\\
&= \left[\left(-\frac{775}{6}+113i\right)\sum_{j=1}^{n-1}X_jY_j\pi+\left(\frac{171}{2}+\frac{369}{8}i\right)X_nY_n\right]h'(0)\pi\Omega_4dx'. \qquad (5.23)
\end{aligned}
$$

令 $X=X^T+X_n\partial_n$, $Y=Y^T+Y_n\partial_n$, 则有 $\sum_{j=1}^{n-1}X_jY_j=g(X^T,Y^T)$。现在 Φ 是情形 (a)、(b) 和 (c) 之和。因此，我们得到

$$
\Phi=\sum_{i=1}^{5}\Phi_i=\left[\left(\frac{475}{4}+\frac{3747}{8}i\right)X_nY_n+\left(-\frac{8013}{16}+46i\right)\pi g(X^T,Y^T)\right]h'(0)\pi\Omega_4dx'. \qquad (5.24)
$$

然后，由 (5.4)-(5.24)，我们得到以下定理

定理 5.1. 设 M 为一个带边界 ∂M 的 6 维定向紧致自旋流形，且度量g^M 如第 (3) 节所定义，则有以下等式:

$$\begin{aligned}
&\widetilde{\mathrm{Wres}}[\pi^+(\nabla_X^{S(TM)}\nabla_Y^{S(TM)}D^{-3})\circ\pi^+(D^{-3})]\\
&=\frac{4\pi^3}{3}\int_M[Ric(X,Y)-\frac{1}{2}sg(X,Y)]d\mathrm{Vol_M}+\pi^3\int_M sg(X,Y)d\mathrm{Vol_M}\\
&+\int_{\partial M}\left[\left(\frac{475}{4}+\frac{3747}{8}i\right)X_nY_n+\left(-\frac{8013}{16}+46i\right)\pi g(X^T,Y^T)\right]h'(0)\pi\Omega_4 d\mathrm{Vol_M}. \qquad (5.25)
\end{aligned}$$

声明

- 伦理审批和参与同意 不适用。
- 发表同意 不适用。
- 数据和材料的可用性 作者确认，支持本研究结果的数据均可在本文中获取。
- 利益冲突 作者声明不存在利益冲突。
- 基金资助 本研究获国家自然科学基金资助：编号11771070。
- 作者贡献 所有作者均参与了研究构思与设计。YY 和 TW 负责材料准备、数据收集与分析。YY 撰写了稿件初稿，所有作者均对稿件的早期版本提出了意见。所有作者均已阅读并批准最终稿件。
- 致谢 本研究得到了国家自然科学基金（NSFC）项目（编号：11771070）的资助。作者感谢审稿人的认真审阅和有益意见。

参考文献

参考文献

[1] Ackermann T. 关于 Wodzicki余项 的一则注记。J. Geom.Phys. 20: 404-406, (1996). [2] Connes A. 量子化微积分及其应用。第十一届国际数学物理大会（巴黎，1994），Internat Press, Cambridge, MA, 15-36, (1995). [3] Connes A. 非交换几何中的作用泛函。数学物理通讯 117: 673-683, (1998). [4] Dabrowski L, Sitarz A, Zalecki P. 谱度量与爱因斯坦泛函。arXiv:2206.02587, (2022). [5] Fedosov B V, Golse F, Leichtnam E, Schrohe E. 带边流形上的非交换留数。泛函分析杂志 142, 1-31, (1996). [6] Guillemin V W. Weyl公式及特征值渐近分布的新证明。数学进展 55(2): 131-160, (1985). [7] Kalau W, Walze M. 引力、非交换几何与 Wodzicki余项。J. Geom. Physics. 16: 327-344,(1995). [8] Kastler D. 狄拉克算子与引力。数学物理通讯 166: 633-643, (1995). [9] Wang J, Wang Y.: 六维带边流形上的 Kastler-Kalau-Walze 型定理。J. Math. Phys. 56: 052501 (2015). [10] Wang Y. 带边流形上的微分形式与 Wodzicki余项。J. Geom. Physics. 56: 731-753, (2006). [11] Wang Y.: 非乘积情形下带边流形的微分形式与非交换留数。Lett. Math. Phys. 77: 41-51, (2006). [12] Wang Y. 带边流形上的引力与非交换留数。Lett. Math. Phys. 80: 37-56, (2007).

[13] Wang Y.《带边流形的低维体积与 Kastler-Kalau-Walze 型定理》。《理论物理通讯》54：38-42, （2010年）。[14] Wodzicki. M：谱不对称性的局部不变量。《数学发明》75(1)：143-178, （1995）。[15] Yu Y.《指标定理与热方程方法》, 《南开数学丛书》第2卷, World Scientific 出版社, （2001）。