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U $3j$ 的量子 U_q符号

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摘要

我们提出了一个代数表达式，用于表示对称表示张量积分解中出现的 $U_q(\mathfrak{sl}_3)$ 量子 $3j$ 符号（量子 Clebsch-Gordan 系数）。这一紧凑形式有助于写出谱参数相关的 R 矩阵元，适用于边上承载对称表示态的任意二部顶点模型。

1 引言

在两个角动量相加的量子理论背景下，我们已经看到克莱布施–戈丹系数（Wigner $3j$-符号）的自然出现。尤其是，这些克莱布施–戈丹（CG）系数决定了由任意两个 $SU(2)$ 表示的张量积所得到的不可约表示的态，并在核物理和原子物理中有广泛应用 [1, 2]。

在量子反散射方法的发展过程中，人们发现，对经典系统进行量子化会导致系统某些结构发生量子变形。在 Kulish 和 Reshetikhin 的工作中， [3], 杨–巴克斯特方程的解需要对 $\mathfrak{sl}_2$ (over $\mathbb{C}$) 的包络代数进行变形。Drinfel'd 的工作 [4], 通过对李代数的包络代数进行量子化，成功刻画了这些变形的代数结构。满足杨–巴克斯特方程的 $U_q(\mathfrak{sl}_2)$ 通用 $\mathcal{R}$ 矩阵，将可积格点模型与 Wess-Zumino-Novikov-Witten（WZNW）模型等不同领域联系起来[5]。事实上，只要知道$U_q(\mathfrak{sl}_N)$ 量子 CG（q-CG）系数[7, 8]，就能确定支配顶点模型等精确可解模型的谱参数相关R-矩阵 [6] 。这些 q-CG 系数在核物理中也有应用[9, 10]。

$U_q(\mathfrak{sl}_2)$ 的 q-CG 系数闭式表达式已知[11]。因此，确定边上携带任意自旋 $n/2$ 的态的顶点模型的R-矩阵元并不困难。然而，要将这些结果推广到边上携带属于对称表示的态的顶点模型，

$$\underbrace{(n,0,0,\ldots 0)}_{N-1} \equiv \underbrace{\square\cdots\square}_{n} \in \mathfrak{sl}_N,$$

需要 $U_q(\mathfrak{sl}_N)$ 的 q-CG 系数。要确定 $U_q(\mathfrak{sl}_N)$ q-CG 系数的代数表达式，必须纳入一个新特征；当我们转向 $\mathfrak{sl}_{N\geq 3}$ 时，这一特征便会出现。也就是说，在对称表示的张量积中，属于不可约表示 (n_1+n_2-2s,s) 的线性无关态会多次出现，

$$\underbrace{(n_1,0,0,0,\ldots)}_{N-1}\otimes\underbrace{(n_2,0,0,0,\ldots)}_{N-1}=\oplus_{s=0}^{Min[n_1,n_2]}\underbrace{(n_1+n_2-2s,s,0,0,\ldots)}_{N-1}, \tag{1}$$

其中 $(n_1+n_2-2s=2,s=4,0,0,..) \equiv$ 表$\square\square\square\square$由 n_1+n_2-2s 个单格和 s 个竖直双格组成的两行杨图所对应的不可约表示。请注意，对于任意 $\mathfrak{sl}_N$，张量积分解中的不可约表示数量都相同。本文首先着重于为上述张量积分解求得 $U_q(\mathfrak{sl}_3)$ q-CG 系数。

本文结构安排如下：第2节简要介绍用于构造 q-CG 系数的 $U_q(\mathfrak{sl}_2)$ 代数。第3 节介绍处理 $U_q(\mathfrak{sl}_3)$ 的重数及态的分类的方法。第3.1节提出 q-CG 系数的一个代数表达式。第4节总结我们的工作，并指出未来可能的研究方向。然后，附录 A 讨论获得

线性无关态的方法；附录 B则给出了使用 Wolfram Mathematica 12.1 根据我们的表达式计算得到的一些 q-CG 系数数据。

2 U_q（$\mathfrak{sl}_2$）量子 $3j$-符号

$U_q(\mathfrak{sl}_2)$表示 $U(\mathfrak{sl}_2)$代数的量子变形，其三个生成元 $E_{\pm\alpha_1}$、H_{α_1}满足[12]

$$[H_{\alpha_1}, E^{\pm}_{\alpha_1}] = \pm E^{\pm}_{\alpha_1},\ [E^{+}_{\alpha_1}, E^{-}_{\alpha_1}] = [2H_{\alpha_1}], \tag{2}$$

where α_1 表示 $\mathfrak{sl}_2$ 只有一个正根，而方括号是量子 deformation of ordinary operator H_{α_1}。$[n]$ 定义为

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}\ , \tag{3}$$

这称为 q-数，因为它以下述方式化为经典数 n：

$$\text{taking } q = e^{\hbar},\ \text{limit } \hbar \to 0\ [n] = n. \tag{4}$$

我们将 $U_q(\mathfrak{sl}_2)$ 中自旋-j 表示的最高权态记为 $|j, j\rangle \equiv |\lambda, \lambda\rangle$，其最高权 $\lambda = 2j (j \in \frac{1}{2}\mathbb{Z})$。随后，升降算符作用于态 $|j, j\rangle$ 将生成自旋 j 表示的所有态：

$$(E^{-}_{\alpha_1})^{\ell}|j, j\rangle = \prod_{r=0}^{\ell-1} \sqrt{[j+j-r][j-j+1+r]}|j, j-\ell\rangle\ , \tag{5}$$

其中 $\ell \le 2j$，方括号由表达式（3）给出。对于一般的态，

$$|\lambda = 2j, \lambda - (j-m)\rangle \equiv |j, m\rangle,$$

与上述代数一致的阶梯算符为[12]

$$E^{\pm}_{\alpha_1}|j, m\rangle = \sqrt{[j \mp m][j \pm m + 1]}|j, m \pm 1\rangle\ . \tag{6}$$

两个不同自旋态的张量积上的三个生成元:

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

遵循以下余乘法则[12]:

$$\Delta H_{\alpha_1} = \mathbb{I} \otimes H_{\alpha_1} + H_{\alpha_1} \otimes \mathbb{I} \tag{7}$$

$$\Delta E^{\pm}_{\alpha_1} = E^{\pm}_{\alpha_1} \otimes q^{H_{\alpha_1}/2} + q^{-H_{\alpha_1}/2} \otimes E^{\pm}_{\alpha_1}\ . \tag{8}$$

假设变形参数 q 是 k次单位根，且 $k \gg j_1,\ j_2$，不可约表示（由张量积 $j_1 \otimes j_2$ 的分解得到）的自旋 J 取值为 $|j_1 - j_2|, |j_1 - j_2| + 1 \ldots j_1 + j_2$。此外，态 $|J\ M\rangle$，可以通过降阶梯算符作用于最高权态得到:

$$(E^{-}_{\alpha_1})^{\ell}|J = j_1 + j_2, M = j_1 + j_2\rangle = (\Delta E_{\alpha_1})^{\ell}|j_1, j_1\rangle|j_2, j_2\rangle\ , \tag{9}$$

从而得到

$$|J = j_1 + j_2, M = j_1 + j_2 - \ell\rangle = \sum_{m_1, m_2} C^{j_1, j_2}_{m_1, m_2}(J, M)\ |j_1, m_1\rangle \otimes |j_2, m_2\rangle\ ,$$

其中 $m_1 + m_2 = M$ 给出非零的 q-CG 系数

$$C^{j_1, j_2}_{m_1, m_2}(J, M) = \{\langle j_1 m_2| \otimes \langle j_2, m_2|\}|J, M\rangle. \tag{10}$$

还可以确定其他态

$$|J = j_1 + j_2 - 1, M\rangle, |J = j_1 + j_2 - 2, M\rangle \ldots$$

通过施加正交性

$$\langle J, M|J', M\rangle = \delta_{J, J'}\ ,$$

并由此推导出 $J < j_1 + j_2$ 的 q-CG 系数。若不经过上述一系列步骤，直接提出 q-CG 系数的闭式表达式，并非显而易见。有趣的是， [11] 中提出的闭式表达式给出了 $U_q(\mathfrak{sl}_2)$ q-CG 系数:

$$\begin{aligned} C^{j_1 \cdot j_2}_{m_1,m_2}(j,m) &= \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \\ &= (-1)^{j_1-m_1}\delta_{m,m_1+m_2} q^{\frac{1}{4}[j_2(j_2+1)-j_1(j_1+1)-j(j+1)]+\frac{m+1}{2}m_1} \\ &\quad \times \left\{ \frac{[2j+1][j_1+j_2-j]![j_1-m_1]![j_2-m_2]![j+m]![j-m]!}{[j+j_1+j_2+1]![j+j_1-j_2]![j+j_2-j_1]![j_1+m_1]![j_2+m_2]!} \right\}^{\frac{1}{2}} \\ &\quad \times \sum_{k=0}^{j-m} \frac{(-1)^k q^{\frac{k}{2}(j+m+1)}[j_2+j-m_1-k]![j_1+m_1+k]!}{[k]![j-m-k]![j_2-j+m_1+k]![j_1-m_1-k]!} \ , \end{aligned} \tag{11}$$

其中 q 数的阶乘（3）为 $[n]! = \prod_{m=1}^{n}[m]$。这些 q-CG 系数满足正交关系 [13]:

$$\begin{aligned} \sum_{m_1,m_2} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j' \\ m_1 & m_2 & m' \end{bmatrix} &= \delta_{j,j'}\delta_{m,m'}\delta(j_1 j_2 j) \\ \sum_{j,m} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} j_1 & j_2 & j \\ m_1' & m_2' & m \end{bmatrix} &= \delta_{m_1,m_1'}\delta_{m_2,m_2'} \ , \end{aligned} \tag{12}$$

其中

$$\delta(j_1 j_2 j) = \begin{cases} 1, & \text{if } |j_1 - j_2| \le j \le j_1 + j_2 \\ 0, & \text{otherwise} \ . \end{cases}$$

出现在表达式（ k 中的求和（11）可以用超几何级数 ${}_3\varphi_2$简洁地表示。我们首先回顾一下，广义超几何级数可以用 q-Pochhammer 符号表示为:

$$(x;q)_n = \prod_{k=0}^{n-1}(1-xq^k), \ n \ge 0, \tag{13}$$

如下:

$${}_r\varphi_s\left(\begin{matrix} a_1, a_2, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{n(n-1)/2} \right]^{1+s-r} z^n \ . \tag{14}$$

用超几何级数表示的 q-CG 系数为[14]

$$\begin{aligned} C^{j_1 \cdot j_2}_{m_1,m_2}(j,m) &= (-1)^{j_1-m_1}\delta_{m,m_1+m_2} q^{\frac{1}{4}[j_2(j_2+1)-j_1(j_1+1)-j(j+1)]+\frac{m+1}{2}m_1} \\ &\quad \times \left\{ \frac{[2j+1][j_1+j_2-j]![j+m]![j_1+m_1]![j_2-m_2]!}{[j+j_1+j_2+1]![j+j_1-j_2]![j-j_1+j_2]![j-m]![j_1-m_1]![j_2+m_2]!} \right\}^{\frac{1}{2}} \\ &\quad \times \frac{[j_2+j-m_1]!}{[j_2-j+m_1]!} {}_3\varphi_2\left(\begin{matrix} q^{m-j}, q^{m_1-j_1}, q^{j_1+m_1+1} \\ q^{m_1-j_2-j}, q^{j_2-j+m_1+1} \end{matrix} ; q, q \right) \ . \end{aligned} \tag{15}$$

回顾了最简单的 $U_q(\mathfrak{sl}_2)$ 后，我们将在下一节中将这一方法推广到 $U_q(\mathfrak{sl}_3)$ q-CG 系数。

3 $U_q(\mathfrak{sl}_3)$q-CG 系数

我们知道， $\mathfrak{sl}_3$ 群的秩为二，也就是说有两个 Cartan 生成元 $\mathbf{H} = (H_1、 H_2)$ 和两个简单根向量： $\boldsymbol{\alpha}_1 = (\alpha_1^{(1)}、 \alpha_1^{(2)})、 \boldsymbol{\alpha}_2 = (\alpha_2^{(1)}、 \alpha_2^{(2)})$。此外，还有一个额外的正根 $\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2$. 与这三个正根相关联的是三个 $U_q(\mathfrak{sl}_2)$ 子代数，其生成元为

$$\{E^{\pm}_{\boldsymbol{\alpha}_i}, H_{\boldsymbol{\alpha}_i}\} \text{ where } H_{\boldsymbol{\alpha}_i} = \boldsymbol{\alpha}_i.\mathbf{H}.$$

我们采用简单根 $\boldsymbol{\alpha}_1、 \boldsymbol{\alpha}_2$ 的标准归一化:

$$\boldsymbol{\alpha}_i.\boldsymbol{\alpha}_j = \delta_{ij} - \frac{1}{2}\delta_{i,j=i\pm1} \ . \tag{16}$$

请注意， $U_q(\mathfrak{sl}_3)$的根是二维向量，而 $U_q(\mathfrak{sl}_2)$的根 $\alpha_1 = 1$ 是一个数。现在我们将讨论属于 $U_q(\mathfrak{sl}_3)$的表示的各个态:

$$(5,2) \equiv \ydiagram{6,1} \ , \tag{17}$$

如图 1所示，这称为权图。该表示的最高权也是一个二维向量，其形式为

$$\boldsymbol{\lambda} = 5\boldsymbol{\mu}_1 + 2\boldsymbol{\mu}_2 \ ,$$

其中， $\boldsymbol{\mu}_1$， $\boldsymbol{\mu}_2$ 是满足以下关系的基本权重向量

$$\boldsymbol{\mu}_i.\boldsymbol{\alpha}_j = \frac{1}{2}\delta_{ij} \ , \tag{18}$$

其中简单根为 $\boldsymbol{\alpha}_1$， $\boldsymbol{\alpha}_2$。属于 (5，2) 权图的态以黑点表示外壳层中的态，以带红色同心圆的黑点表示内壳层中的态，并在第二内壳层中增加绿色圆，依此类推。这说明了内壳层中重数态的出现[15]。对于 (5，2) 权图，从第二内壳层开始出现三角结构。因此，从第二壳层开始，这些态始终具有重数3。对于对称表示 (3，0) =，权图呈三角形，因此这些态没有重数。对于任意表示 $(n，\ m)$， $\boldsymbol{\lambda} = n\boldsymbol{\mu}_1 + m\boldsymbol{\mu}_2$，三角形壳层从 $min(m，\ n)$-内壳层开始出现。

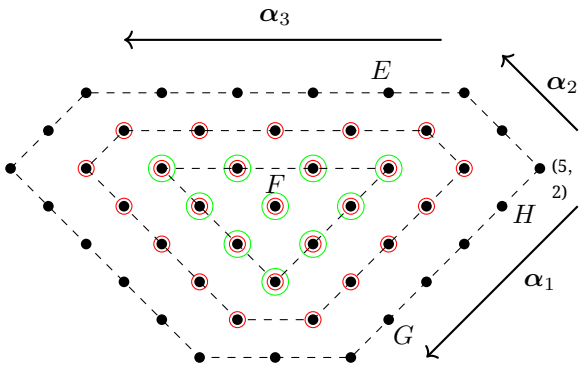


图1： $\mathfrak{sl}_3$ 表示(5，2)的权图

对于图 1 中标记点“F”，三个线性无关态可以通过降算符作用于最高权态 $|\boldsymbol{\lambda}，\ \boldsymbol{\lambda}\rangle$，其中 $\boldsymbol{\lambda} = 5\boldsymbol{\mu}_1 + 2\boldsymbol{\mu}_2$：

$$|\boldsymbol{\lambda}, \boldsymbol{\omega} = \boldsymbol{\lambda} - 4\boldsymbol{\alpha}_1 - 3\boldsymbol{\alpha}_2; \text{black}\rangle = \left(E^-_{\boldsymbol{\alpha}_3}\right)^3 \left(E^-_{\boldsymbol{\alpha}_1}\right)^1 |\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \ , \tag{19}$$

$$|\boldsymbol{\lambda}, \boldsymbol{\omega} = \boldsymbol{\lambda} - 4\boldsymbol{\alpha}_1 - 3\boldsymbol{\alpha}_2; \text{red}\rangle = \left(E^-_{\boldsymbol{\alpha}_3}\right)^2 \left(E^-_{\boldsymbol{\alpha}_1}\right)^1 \left(E^-_{\boldsymbol{\alpha}_2}\right)^1 \left(E^-_{\boldsymbol{\alpha}_1}\right)^1 |\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \ ,$$

$$|\boldsymbol{\lambda}, \boldsymbol{\omega} = \boldsymbol{\lambda} - 4\boldsymbol{\alpha}_1 - 3\boldsymbol{\alpha}_2; \text{green}\rangle = \left(E^-_{\boldsymbol{\alpha}_3}\right)^1 \left(E^-_{\boldsymbol{\alpha}_1}\right)^1 \left(E^-_{\boldsymbol{\alpha}_2}\right)^2 \left(E^-_{\boldsymbol{\alpha}_1}\right)^2 |\boldsymbol{\lambda}, \boldsymbol{\lambda}\rangle \ .$$

事实上，与黑点、红色圆、绿色圆.....相对应的所有线性无关态，都可以通过降算符的系统序列获得，正如附录A中所述，该序列作用于最高权态 $|\boldsymbol{\lambda} = n\boldsymbol{\mu}_1 + m\boldsymbol{\mu}_2，\ \boldsymbol{\lambda}\rangle$。每个 $U_q(\mathfrak{sl}_2)$ 子代数阶梯算符（6）的作用都需要定义

$$|\boldsymbol{\lambda}, \boldsymbol{\omega} = \boldsymbol{\lambda} - A\boldsymbol{\alpha}_1 - B\boldsymbol{\alpha}_2\rangle \equiv |\vec{j}(\boldsymbol{\omega}), \vec{m}(\boldsymbol{\omega})\rangle \ , \tag{20}$$

其中， $\vec{j}(\boldsymbol{\omega}) = \{j^{(1)}(\boldsymbol{\omega})\ j^{(2)}(\boldsymbol{\omega})\rangle，\ j^{(3)}(\boldsymbol{\omega})\}$ 的显式形式为

$$\begin{aligned} j^{(1)}(\boldsymbol{\omega}) &= (\boldsymbol{\lambda} - \max(B - m, 0)\boldsymbol{\alpha}_1 - B\boldsymbol{\alpha}_2) \cdot \boldsymbol{\alpha}_1 \\ j^{(2)}(\boldsymbol{\omega}) &= (\boldsymbol{\lambda} - A\boldsymbol{\alpha}_1 - \max(A - n, 0)\boldsymbol{\alpha}_2) \cdot \boldsymbol{\alpha}_2 \\ j^{(3)}(\boldsymbol{\omega}) &= (\boldsymbol{\lambda} - (A - B)\boldsymbol{\alpha}_1) \cdot \boldsymbol{\alpha_3} \quad \text{when } A \geq B \\ &= (\boldsymbol{\lambda} - (B - A)\boldsymbol{\alpha}_2) \cdot \boldsymbol{\alpha}_3 \quad \text{when } B \geq A \end{aligned} \tag{21}$$

相应的 $\vec{m}(\boldsymbol{\omega}) = \{m^{(1)}(\boldsymbol{\omega})\ m^{(2)}(\boldsymbol{\omega})\ m^{(3)}(\boldsymbol{\omega})\}$ 将为

$$m^{(i)}(\boldsymbol{\omega}) = (\boldsymbol{\lambda} - A\boldsymbol{\alpha}_1 - B\boldsymbol{\alpha}_2) \cdot \boldsymbol{\alpha}_i. \tag{22}$$

这些 $j^{(i)}$可以从由 $U_q(\mathfrak{sl}_2)$ 子代数生成的 $\{E^{\pm}_{\boldsymbol{\alpha}_i}, H_{\boldsymbol{\alpha}_i}\}$的角度来理解。例如，对于图1中标记点“F”，其由态（19）给出， $j^{(1)}$ 对应于对点“E”处权态的投影，其中 $\boldsymbol{\alpha}_1$。因此，

$$j^{(1)}(\boldsymbol{\omega} = \boldsymbol{\lambda} - 4\boldsymbol{\alpha}_1 - 3\boldsymbol{\alpha}_2) = (5\boldsymbol{\mu}_1 + 2\boldsymbol{\mu}_2 - \boldsymbol{\alpha}_1 - 3\boldsymbol{\alpha}_2) \cdot \boldsymbol{\alpha}_1 \ . \tag{23}$$

$\boldsymbol{\alpha}_2$ $\boldsymbol{\omega} =$

同样，很容易看出，标记为‘G’的权态的投影与 $\boldsymbol{\lambda} - 4\boldsymbol{\alpha}_1 - 3\boldsymbol{\alpha}_2$ $\boldsymbol{\alpha}_3$ $j^{(3)}$ $\boldsymbol{\omega} = \boldsymbol{\lambda} - 4\boldsymbol{\alpha}_1 - 3\boldsymbol{\alpha}_2$ 一起将给出 $j^{(2)}$()，而标记为‘H’的权态将给出 ()。

3.1 U_q（$\mathfrak{sl}_3$）q-CG 系数的代数表达式

由于两个对称表示的张量积分解(1) 对于 $U_q(\mathfrak{sl}_3)$ 与 $U_q(\mathfrak{sl}_2)$相同，不可约表示的最高权态可如下得到:

$$|\mathbf{\Lambda}^{(s)},\mathbf{\Omega}=\mathbf{\Lambda}^{(s)}\rangle=\sum_{k_1+k_2=s}C^{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(\mathbf{\Lambda}^{(s)},\mathbf{\Omega})|\boldsymbol{\lambda}_1,\boldsymbol{\omega}_1=\boldsymbol{\lambda}_1-k_1\boldsymbol{\alpha}_1\rangle|\boldsymbol{\lambda}_2,\boldsymbol{\omega}_2=\boldsymbol{\lambda}_2-k_2\boldsymbol{\alpha}_1\rangle\ , \tag{24}$$

其中 $\mathbf{\Lambda}^{(s)}=(n_1+n_2-2s)\boldsymbol{\mu}_1+s\boldsymbol{\mu}_2$, $\boldsymbol{\lambda}_1=n_1\boldsymbol{\mu}_1$ 和 $\boldsymbol{\lambda}_2=n_2\boldsymbol{\mu}_1$。注意，求和 k_1, k_2满足条件 $k_1+k_2=s$ 的变量，与权向量相同

$$\mathbf{\Lambda}^{(s)}=\mathbf{\Omega}=\boldsymbol{\omega}_1+\boldsymbol{\omega}_2.$$

与 $U_q(\mathfrak{sl}_2)$ q-CG 系数相比，这些 $U_q(\mathfrak{sl}_3)$ q-CG 系数必须满足:

$$C^{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(\mathbf{\Lambda}^{(s)},\ \mathbf{\Omega}=\mathbf{\Lambda}^{(s)})=\begin{bmatrix}\boldsymbol{\lambda}_1 & \boldsymbol{\lambda}_2 & \mathbf{\Lambda}^{(s)}\\ \boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 & \mathbf{\Omega}\end{bmatrix}_{U_q(\mathfrak{sl}_3)}=\begin{bmatrix}\frac{n_1}{2} & \frac{n_2}{2} & \frac{n_1+n_2}{2}\\ \frac{n_1}{2}-k_1 & \frac{n_2}{2}-k_2 & \frac{n_1+n_2}{2}-s\end{bmatrix}_{U_q(\mathfrak{sl}_2)}\ . \tag{25}$$

事实上，上述 q-CG 表达式对于 U_q（$\mathfrak{sl}_N$）也应当相同。现在，我们可以对最高权态（A）依次应用阶梯算符（参见附录24），从而得到每个不可约表示（n_1+n_2-2s, s）中的所有态。

对最高权态（$\left(E^-_{\boldsymbol{\alpha}_2}\right)^{l_2}\left(E^-_{\boldsymbol{\alpha}_1}\right)^{l_1}$ 应用阶梯算符（24）：

$$\left(E^-_{\boldsymbol{\alpha}_2}\right)^{l_2}\left(E^-_{\boldsymbol{\alpha}_1}\right)^{l_1}|\mathbf{\Lambda}^{(s)},\mathbf{\Omega}=\mathbf{\Lambda}^{(s)}\rangle=H_{\boldsymbol{\alpha}_1,l_1}(\mathbf{\Omega})H_{\boldsymbol{\alpha}_2,l_2}(\mathbf{\Omega}-\boldsymbol{l_1\alpha_1})|\mathbf{\Lambda}^{(s)},\mathbf{\Omega}-\sum_{i=1}^{2}l_i\boldsymbol{\alpha}_i\rangle\ , \tag{26}$$

其中，$H_{\boldsymbol{\alpha}_i,\ l}$（$\mathbf{\Omega}$）用 $\vec{j}$（21）表示为

$$H_{\boldsymbol{\alpha}_i,l}(\mathbf{\Omega})=\left(\frac{[j^{(i)}(\mathbf{\Omega})+m^{(i)}(\mathbf{\Omega})]![j^{(i)}(\mathbf{\Omega})-m^{(i)}(\mathbf{\Omega})+l]!}{[j^{(i)}(\mathbf{\Omega})+m^{(i)}(\mathbf{\Omega})-l]![j^{(i)}(\mathbf{\Omega})-m^{(i)}(\mathbf{\Omega})]!}\right)^{1/2}\ . \tag{27}$$

类似地，张量积态在方程24右端的余乘作用为:

$$\begin{aligned}\left(\Delta E^-_{\boldsymbol{\alpha}_1}\right)^{l_2}\left(\Delta E^-_{\boldsymbol{\alpha}_1}\right)^{l_1}|\boldsymbol{\lambda}_1,\boldsymbol{\omega}_1=\boldsymbol{\lambda}_1-k_1\boldsymbol{\alpha}_1\rangle|\boldsymbol{\lambda}_2,\boldsymbol{\omega}_2=\boldsymbol{\lambda}_1-k_2\boldsymbol{\alpha}_1\rangle=\\ \sum_{x_i+y_i=l_i}F^{x_1,y_1}_{\boldsymbol{\alpha}_1,l_1}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)F^{x_2,y_2}_{\boldsymbol{\alpha}_2,l_2}(\boldsymbol{\omega}_1-x_1\boldsymbol{\alpha}_1,\boldsymbol{\omega}_2-y_1\boldsymbol{\alpha}_1)\\ |\boldsymbol{\lambda}_1,\boldsymbol{\omega}_1-\sum_{i=1}^{2}x_i\boldsymbol{\alpha}_i\rangle|\boldsymbol{\lambda}_2,\boldsymbol{\omega}_2-\sum_{i=1}^{2}y_i\boldsymbol{\alpha}_i\rangle\ ,\end{aligned} \tag{28}$$

其中 $F^{(x,y)}_{\boldsymbol{\alpha}_i\ l}$,（$\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$）用 $U_q(\mathfrak{sl}_2)$ 变量（21）表示为

$$\begin{aligned}F^{(x,y)}_{\boldsymbol{\alpha}_i,l}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)\ &=\ q^{\frac{1}{2}(xm^{(i)}(\boldsymbol{\omega}_2)-ym^{(i)}(\boldsymbol{\omega}_1))}\frac{[x+y]!}{[x]![y]!}\times\\ &\times\left(\frac{[j^{(i)}(\boldsymbol{\omega}_2)+m^{(i)}(\boldsymbol{\omega}_2)]![j^{(i)}(\boldsymbol{\omega}_1)+m^{(i)}((\boldsymbol{\omega}_1)]!}{[j^{(i)}(\boldsymbol{\omega}_2)-m^{(i)}(\boldsymbol{\omega}_2)]![j^{(i)}(\boldsymbol{\omega}_1)-m^{(i)}(\boldsymbol{\omega}_1)]!}\right)^{1/2}\\ &\times\left(\frac{[j^{(i)}(\boldsymbol{\omega}_2)-m^{(i)}(\boldsymbol{\omega}_2)+y]![j^{(i)}(\boldsymbol{\omega}_1)-m^{(i)}(\boldsymbol{\omega}_1)+x]!}{[j^{(i)}(\boldsymbol{\omega}_2)+m^{(i)}(\boldsymbol{\omega}_2)-y]![j^{(i)}(\boldsymbol{\omega}_1)+m^{(i)}(\boldsymbol{\omega}_1)-x]!}\right)^{1/2}\ .\end{aligned} \tag{29}$$

因此，结合方程(28,26)，表示的权图中的相应态(n_1+n_2-2s,s) 将为

$$\begin{aligned}|\mathbf{\Lambda}^{(s)},\mathbf{\Omega}-\sum_{i=1}^{2}l_i\boldsymbol{\alpha}_i\rangle\ &=\ \sum_{k_1,k_2}\sum_{x_i+y_i=l_i}(-1)^sC^{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(\mathbf{\Lambda}^{(s)},\mathbf{\Omega})\frac{F^{x_1,y_1}_{\boldsymbol{\alpha}_1,l_1}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)}{H_{\boldsymbol{\alpha}_1,l_1}(\mathbf{\Omega})}\\ &\frac{F^{x_2,y_2}_{\boldsymbol{\alpha}_2,l_2}(\boldsymbol{\omega}_1-x_1\boldsymbol{\alpha}_1,\boldsymbol{\omega}_2-x_2\boldsymbol{\alpha}_1)}{H_{\boldsymbol{\alpha}_2,l_2}(\mathbf{\Omega}-l_1\boldsymbol{\alpha}_1)}|\boldsymbol{\lambda}_1,\boldsymbol{\omega}_1-\sum_{i=1}^{2}x_i\boldsymbol{\alpha}_i\rangle|\boldsymbol{\lambda}_2,\boldsymbol{\omega}_2-\sum_{i=1}^{2}y_i\boldsymbol{\alpha}_i\rangle\ ,\end{aligned} \tag{30}$$

其中 $(-1)^s$ 只是一种符号选择。

我们可以从上述态分解中读出 $U_q(\mathfrak{sl}_3)$ q-CG 系数:

$$\begin{aligned}C^{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2}_{\boldsymbol{\omega}_1-\sum_{i=1}^{2}x_i\boldsymbol{\alpha}_i,\boldsymbol{\omega}_2-\sum_{i=1}^{2}y_i\boldsymbol{\alpha}_i}(\mathbf{\Lambda}^{(s)},\mathbf{\Omega}-l_1\boldsymbol{\alpha}_1-l_2\boldsymbol{\alpha}_2)=\\ (-1)^sC^{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(\mathbf{\Lambda}^{(s)},\mathbf{\Omega})\frac{F^{x_1,y_1}_{\boldsymbol{\alpha}_1,l_0}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)}{H_{\boldsymbol{\alpha}_1,l_1}(\mathbf{\Omega})}\ \frac{F^{x_2,y_2}_{\boldsymbol{\alpha}_2,l_2}(\boldsymbol{\omega}_1-x_1\boldsymbol{\alpha}_1,\boldsymbol{\omega}_2-y_1\boldsymbol{\alpha}_1)}{H_{\boldsymbol{\alpha}_2,l_2}(\mathbf{\Omega}-l_1\boldsymbol{\alpha}_1)}\ .\end{aligned} \tag{31}$$

应用 $(E^{-}_{\boldsymbol{\alpha}_3})^{l_3}$ 阶梯算符作用于最高权态（24），我们可以得到

$$|\boldsymbol{\Lambda}^{(s)}, \boldsymbol{\Omega} - l_3\boldsymbol{\alpha}_3\rangle \quad = \quad \sum_{k_1,k_2}\sum_{x_3+y_3=l_3} (-1)^s C^{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2}_{\boldsymbol{\omega}_1,\boldsymbol{\omega}_2}(\boldsymbol{\Lambda}^{(s)}, \boldsymbol{\Omega}) \frac{F^{x_3,y_3}_{\boldsymbol{\alpha}_3,l_3}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2)}{H_{\boldsymbol{\alpha}_3,l_3}(\boldsymbol{\Omega})} \\ |\boldsymbol{\lambda}_1, \boldsymbol{\omega}_1 - x_3\boldsymbol{\alpha}_3\rangle |\boldsymbol{\lambda}_2, \boldsymbol{\omega}_2 - y_3\boldsymbol{\alpha}_3\rangle \ . \tag{32}$$

注意，上述态与态(30)具有相同的权向量（当 $l_3 = l_1 = l_2 \neq 0$ 时），但它们是线性无关态，且彼此不正交。因此，我们可以采用常规的 Gram-Schmidt 正交化过程，得到一个与 $|\boldsymbol{\Lambda}^{(s)}、\boldsymbol{\Omega} - l_3\boldsymbol{\alpha}_3\rangle\big|_{l_1=l_2=l_3}$正交归一的态:

$$\mathcal{N}\left(|\boldsymbol{\Lambda}^{(s)}, \boldsymbol{\Omega} - \sum_{i=1}^{2} l_i\boldsymbol{\alpha}_i\rangle\big|_{l_1=l_2=l_3} - \langle\boldsymbol{\Lambda}^{(s)}, \boldsymbol{\Omega} - \sum_{i=1}^{2} l_i\boldsymbol{\alpha}_i|\boldsymbol{\Lambda}^{(s)}, \boldsymbol{\Omega} - l_3\boldsymbol{\alpha}_3\rangle|\boldsymbol{\Lambda}^{(s)}, \boldsymbol{\Omega} - l_3\boldsymbol{\alpha}_3\rangle\big|_{l_1=l_2=l_3}\right) \ . \tag{33}$$

归一化态所需的总因子为 $\mathcal{N}$。现在，我们可以提取 q-CG 系数 $C^{\boldsymbol{\lambda}_2}_{\boldsymbol{\lambda}_1,\ \boldsymbol{\omega}_1-x_3\boldsymbol{\alpha}_3,\ \boldsymbol{\omega}_2-y_3\boldsymbol{\alpha}_3}(\boldsymbol{\Lambda}^{(s)},\ \boldsymbol{\Omega} - l_3\boldsymbol{\alpha}_3)$。

我们提出的方法可推广到任意态(19)，并已在 Mathematica 程序中实现，用于确定 $U_q(sl_3)$ 这一类对称表示张量积分解的 q-CG 系数(1)。

在共轭下，$U_q(\mathfrak{sl}_3)$ 表示 $(n,\ m) \to (m,\ n)$。也就是说，$\boldsymbol{\mu}_1 \leftrightarrow \boldsymbol{\mu}_2$。我们提出，张量积分解的共轭(1)的 q-CG 系数满足

$$\begin{bmatrix} n_1\boldsymbol{\mu}_2 & n_2\boldsymbol{\mu}_2 & \bar{\boldsymbol{\Lambda}}^{(s)} \\ \bar{\boldsymbol{\omega}}_1 & \bar{\boldsymbol{\omega}}_2 & \bar{\boldsymbol{\Omega}} \end{bmatrix}_{U_q(\mathfrak{sl}_3)} = \begin{bmatrix} n_1\boldsymbol{\mu}_1 & n_2\boldsymbol{\mu}_1 & \boldsymbol{\Lambda}^{(s)} \\ \boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 & \boldsymbol{\Omega} \end{bmatrix}_{U_q(\mathfrak{sl}_3)} , \tag{34}$$

对于权图中所有图形发生镜像反射的点，使得根$\boldsymbol{\alpha}_1 \leftrightarrow \boldsymbol{\alpha}_2$。因此，对于$\boldsymbol{\omega} = n_1\boldsymbol{\mu}_1 - A\boldsymbol{\alpha}_1 - B\boldsymbol{\alpha}_2$，共轭的 $\bar{\boldsymbol{\Lambda}}^{(s)} = s\boldsymbol{\mu}_1 + (n_1 + n_2 - 2s)\ \boldsymbol{\mu}_2$和 $\bar{\boldsymbol{\omega}} = n_1\boldsymbol{\mu}_2 - A\boldsymbol{\alpha}_2 - B\boldsymbol{\alpha}_1$。

4 结论

本文重点研究了对称表示的张量积分解（1），并定义了每个不可约表示 $\boldsymbol{\Lambda}^{(s)}$中的线性无关权态。利用$U_q(\mathfrak{sl}_2)$ 最高权态的 q-CG 系数（24），我们能够系统地得到权图中态（黑色虚点） $U_q(\mathfrak{sl}_3)$ 的 q-CG 系数的代数表达式（32）。

此外，可以证明权图内壳层中的多个态线性无关但不正交。经过格拉姆–施密特正交化后，可以得到这些态（以红色、绿色.....同心圆表示）的 q-CG 系数。这些 q-CG 系数列于附录B 中，适用于杨图最多包含两个和三个方框的对称表示。此外，Mathematica 程序还可用于计算其他对称表示。这些汇编整理的 $U_q(\mathfrak{sl}_3)$ q-CG 系数有助于求得新型二部顶点模型的 R 矩阵元[7, 8]。

尝试为最一般的张量积分解构造 q-CG 仍然是一项具有挑战性的问题

$$(n_1, m_1) \otimes (n_2, m_2) = \oplus_i (N_i, M_i) \ ,$$

其中某些不可约表示可能出现不止一次。与每个态仅出现一次的 $(n, 0)$ 态不同，一般的 (n_1, m_1) 态会具有重数。我们希望未来继续研究这一问题。

致谢

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通过升降算符操作系统地获得黑色、红色···态

要为**黑色点状态**构造一组标准正交集，我们必须按照图2中的路径进行。首先，我们必须对最高权态$(E^-_{\boldsymbol{\alpha}_3})$作用不同次幂的$(5,2)$，以到达所有点“P”“Q”···，直至“L”。这条路径将权图分为上半部和下半部。

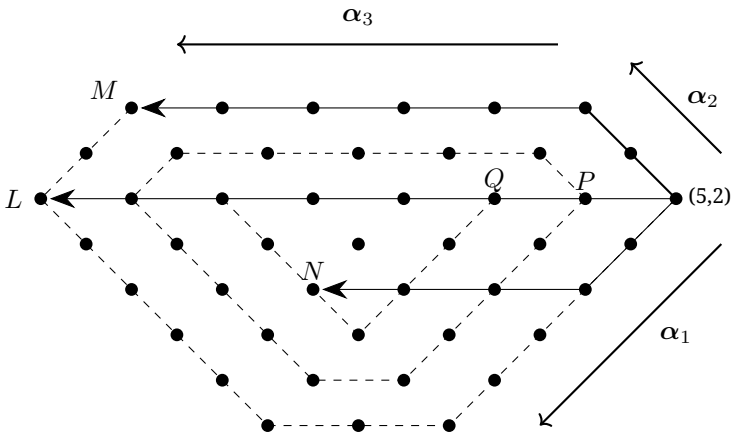


图2：黑色点状态的升降算符操作序列

现在，为了覆盖权图上半部的所有点，我们必须遵循如下升降算符序列：$(E^-_{\boldsymbol{\alpha}_3})^{l_3}(E^-_{\boldsymbol{\alpha}_2})^{l_2}$，作用于最高权$(5,2)$。同样，对于下半部态，我们必须遵循如下序列：$(E^-_{\boldsymbol{\alpha}_3})^{l_3}(E^-_{\boldsymbol{\alpha}_1})^{l_1}$，同样作用于最高权。因此，在图2中，与“L”、“M”和“N”相关的态如下：

$$\begin{aligned} |L;\text{black}\rangle &= (E^-_{\boldsymbol{\alpha}_3})^7|\boldsymbol{\lambda},\boldsymbol{\lambda}=5\boldsymbol{\mu}_1+2\boldsymbol{\mu}_2\rangle \\ |M;\text{black}\rangle &= (E^-_{\boldsymbol{\alpha}_3})^5(E^-_{\boldsymbol{\alpha}_2})^2|\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle \\ |N;\text{black}\rangle &= (E^-_{\boldsymbol{\alpha}_3})^3(E^-_{\boldsymbol{\alpha}_1})^2|\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle\ . \end{aligned} \tag{35}$$

现在，对于**红圈态**，首先要到达图3中标示为“P”的权态，我们必须选择一组不同于图2的升降算符序列，并且该序列应为$|P$提供一个线性无关态；红色$\rangle$。因此，这里的适当序列是$(E^-_{\boldsymbol{\alpha}_2})(E^-_{\boldsymbol{\alpha}_1})$，作用于最高权$(5,2)$，以到达“P”。

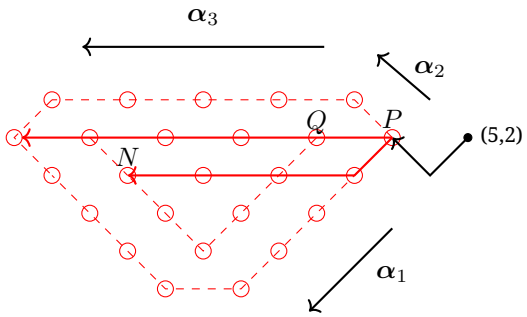


图3：红圈态的升降算符操作序列

接下来我们遵循与上述类似的步骤，即对权重“P”施加$(E^-_{\boldsymbol{\alpha}_3})^{l_3}(E^-_{\boldsymbol{\alpha}_1})^{l_1}$或$(E^-_{\boldsymbol{\alpha}_3})^{l_3}(E^-_{\boldsymbol{\alpha}_1})^{l_1}$。因此我们可以写出$|N;$红色$\rangle$，它与$|N;$黑色$\rangle$线性无关：

$$|N;\text{red}\rangle = (E^-_{\boldsymbol{\alpha}_3})^3(E^-_{\boldsymbol{\alpha}_1})|P;\text{red}\rangle = (E^-_{\boldsymbol{\alpha}_3})^3(E^-_{\boldsymbol{\alpha}_1})[(E^-_{\boldsymbol{\alpha}_2})(E^-_{\boldsymbol{\alpha}_1})]|\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle\ . \tag{36}$$

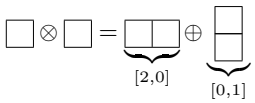
类似地，对于**绿圈态**，通过对$|\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle$执行$(E^-_{\boldsymbol{\alpha}_2})^2(E^-_{\boldsymbol{\alpha}_1})^2$，我们可以得到由“Q”标示的权态；之后，其他态也可以通过与上述相同类型的操作得到。因此，我们可以将其写为$|N;$绿色$\rangle$，它与$|N;$黑色$\rangle$和$|N;$红色$\rangle$线性无关，形式如下

$$|N,\text{green}\rangle = (E^-_{\boldsymbol{\alpha}_3})^2(E^-_{\boldsymbol{\alpha}_1})|Q;\text{green}\rangle = (E^-_{\boldsymbol{\alpha}_3})^2(E^-_{\boldsymbol{\alpha}_1})[(E^-_{\boldsymbol{\alpha}_2})^2(E^-_{\boldsymbol{\alpha}_1})^2]|\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle\ . \tag{37}$$

请注意，这种方法得到三个态： $|N;$ 黑色$\rangle$、$|N;$ 绿色$\rangle$ 和 $|N;$ 红色$\rangle$ 。它们仅线性无关，并非正交归一。为使它们正交归一，我们对 $|N;$ 绿色$\rangle$ 和 $|N;$ 红色$\rangle$进行Gram-Schmidt正交化。

B 不同不可约表示的量子 Clebsch–Gordan 系数

• 张量积与 $U_q(\mathfrak{sl}_3)$ 群的 q-CG 系数



属于不可约表示 $\mathbf{\Lambda}^{(0)} = 2\boldsymbol{\mu}_1 =$ 的所有权态



(从这里开始，我们将最高权 $|\boldsymbol{\lambda},\boldsymbol{\lambda}\rangle$ 称为 $|\boldsymbol{\lambda}\rangle$，并将其他权重 $|\boldsymbol{\lambda},\boldsymbol{\omega}\rangle$ 称为 $|\boldsymbol{\omega}\rangle$。注意，第一批态始终是最高权态。)

$$
\begin{aligned}
&\left|2\,\mu_1\right\rangle = \left|\mu_1\right\rangle\,\left|\mu_1\right\rangle \\
&\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle = \frac{q^{1/4}}{\sqrt{[2]}}\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|\mu_1\right\rangle + \frac{q^{-1/4}}{\sqrt{[2]}}\,\left|\mu_1\right\rangle\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle \\
&\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle = \left|\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle \\
&\left|2\,\mu_1-\alpha_1\right\rangle = \frac{q^{1/4}}{\sqrt{[2]}}\,\left|\mu_1-\alpha_1\right\rangle\,\left|\mu_1\right\rangle + \frac{q^{-1/4}}{\sqrt{[2]}}\,\left|\mu_1\right\rangle\,\left|\mu_1-\alpha_1\right\rangle \\
&\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle = \frac{q^{1/4}}{\sqrt{[2]}}\,\left|\mu_1-\alpha_1\right\rangle\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle + \frac{q^{-1/4}}{\sqrt{[2]}}\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|\mu_1-\alpha_1\right\rangle \\
&\left|2\,\mu_1-2\,\alpha_1\right\rangle = \left|\mu_1-\alpha_1\right\rangle\,\left|\mu_1-\alpha_1\right\rangle
\end{aligned}
$$

属于不可约表示 $\mathbf{\Lambda}^{(1)} = \boldsymbol{\mu}_2 =$ 的所有权态



$$
\begin{aligned}
&\left|\mu_2\right\rangle = \frac{q^{-1/4}}{\sqrt{[2]}}\,\left|\mu_1-\alpha_1\right\rangle\,\left|\mu_1\right\rangle - \frac{q^{1/4}}{\sqrt{[2]}}\,\left|\mu_1\right\rangle\,\left|\mu_1-\alpha_1\right\rangle \\
&\left|\mu_2-\alpha_2\right\rangle = \frac{q^{-1/4}}{\sqrt{[2]}}\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|\mu_1\right\rangle - \frac{q^{1/4}}{\sqrt{[2]}}\,\left|\mu_1\right\rangle\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle \\
&\left|\mu_2-\alpha_1-\alpha_2\right\rangle = \frac{q^{-1/4}}{\sqrt{[2]}}\,\left|\mu_1-\alpha_1\right\rangle\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle - \frac{q^{1/4}}{\sqrt{[2]}}\,\left|\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|\mu_1-\alpha_1\right\rangle \qquad (38)
\end{aligned}
$$

由方程 (38) 可清楚看出，相应的 q-CG 系数为

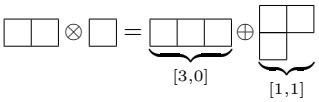
$$
\begin{bmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_1-\boldsymbol{\alpha}_1 & \boldsymbol{\mu}_1-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2 & \boldsymbol{\mu}_2-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2 \end{bmatrix}_{U_q(\mathfrak{sl}_3)} = \frac{q^{-1/4}}{\sqrt{[2]}}
$$

and

$$
\begin{bmatrix} \boldsymbol{\mu}_1 & \boldsymbol{\mu}_1 & \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_1-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2 & \boldsymbol{\mu}_1-\boldsymbol{\alpha}_1 & \boldsymbol{\mu}_2-\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2 \end{bmatrix}_{U_q(\mathfrak{sl}_3)} = -\frac{q^{1/4}}{\sqrt{[2]}}\,.
$$

同样，对于每个不可约表示，都可以从以下这组态中提取所有 q-CG 系数。

- **张量积的 q-CG 系数，以及 $U_q(\mathfrak{sl}_3)$ 群** □□ □



属于不可约表示的所有权态 

$$\left|3\,\mu_1\right\rangle = \left|2\,\mu_1\right\rangle \left|\mu_1\right\rangle$$

$$\left|3\,\mu_1-\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]}}\,q^{1/4}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|\mu_1\right\rangle + \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1\right\rangle \left|\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle = \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|\mu_1\right\rangle + \sqrt{\frac{[2]}{[3]}}\,q^{-1/4}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\right\rangle = \left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|3\,\mu_1-\alpha_1\right\rangle = \sqrt{\frac{[2]}{[3]}}\,q^{1/4}\left|2\,\mu_1-\alpha_1\right\rangle \left|\mu_1\right\rangle + \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1\right\rangle \left|\mu_1-\alpha_1\right\rangle$$

$$\left|3\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle = \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|\mu_1-\alpha_1\right\rangle + \frac{1}{\sqrt{[3]}}\left|2\,\mu_1-\alpha_1\right\rangle \left|\mu_1-\alpha_1-\alpha_2\right\rangle + \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|\mu_1\right\rangle$$

$$\left|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]}}\,q^{1/4}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|\mu_1-\alpha_1-\alpha_2\right\rangle + \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|\mu_1-\alpha_1\right\rangle$$

$$\left|3\,\mu_1-2\,\alpha_1\right\rangle = \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1\right\rangle \left|\mu_1\right\rangle + \sqrt{\frac{[2]}{[3]}}\,q^{-1/4}\left|2\,\mu_1-\alpha_1\right\rangle \left|\mu_1-\alpha_1\right\rangle$$

$$\left|3\,\mu_1-3\,\alpha_1-\alpha_2\right\rangle = \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1\right\rangle \left|\mu_1-\alpha_1-\alpha_2\right\rangle + \sqrt{\frac{[2]}{[3]}}\,q^{-1/4}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|\mu_1-\alpha_1\right\rangle$$

$$\left|3\,\mu_1-3\,\alpha_1\right\rangle = \left|2\,\mu_1-2\,\alpha_1\right\rangle \left|\mu_1-\alpha_1\right\rangle$$

属于不可约表示的所有黑色点状态 

$$\left|\mu_1+\mu_2\right\rangle = \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\mu_1-\alpha_1\right\rangle\left|\mu_1\right\rangle - \sqrt{\frac{[2]}{[3]}}\,q^{1/4}\left|2\mu_1\right\rangle\left|\mu_1-\alpha_1\right\rangle$$

$$\left|\mu_1+\mu_2-\alpha_1-\alpha_2\right\rangle = \frac{q^{-3/4}}{\sqrt{[2][3]}}\left|2\mu_1-\alpha_1\right\rangle\left|\mu_1-\alpha_1-\alpha_2\right\rangle + \frac{q^{-1/4}}{\sqrt{[2][3]}}\left|2\mu_1-2\alpha_1-\alpha_2\right\rangle\left|\mu_1\right\rangle - \sqrt{\frac{[2]}{[3]}}\,q^{1/4}\left|2\mu_1-\alpha_1-\alpha_2\right\rangle\left|\mu_1-\alpha_1\right\rangle$$

$$\left|\mu_1+\mu_2-2\alpha_1-2\alpha_2\right\rangle = \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\mu_1-2\alpha_1-\alpha_2\right\rangle\left|\mu_1-\alpha_1-\alpha_2\right\rangle - \sqrt{\frac{[2]}{[3]}}\,q^{1/4}\left|2\mu_1-2\alpha_1-2\alpha_2\right\rangle\left|\mu_1-\alpha_1\right\rangle$$

$$\left|\mu_1+\mu_2-\alpha_1\right\rangle = \sqrt{\frac{[2]}{[3]}}\,q^{-1/4}\left|2\mu_1-2\alpha_1\right\rangle\left|\mu_1\right\rangle - \frac{q^{1/2}}{\sqrt{[3]}}\left|2\mu_1-\alpha_1\right\rangle\left|\mu_1-\alpha_1\right\rangle$$

$$\left|\mu_1+\mu_2-2\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]}}\,q^{-1/4}\left|2\mu_1-2\alpha_1\right\rangle\left|\mu_1-\alpha_1-\alpha_2\right\rangle - \frac{q^{1/2}}{\sqrt{[3]}}\left|2\mu_1-2\alpha_1-\alpha_2\right\rangle\left|\mu_1-\alpha_1\right\rangle$$

$$\left|\mu_1+\mu_2-\alpha_2\right\rangle = \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\mu_1-\alpha_1-\alpha_2\right\rangle\left|\mu_1\right\rangle - \sqrt{\frac{[2]}{[3]}}\,q^{1/4}\left|2\mu_1\right\rangle\left|\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|\mu_1+\mu_2-\alpha_1-2\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]}}\,q^{-1/4}\left|2\mu_1-2\alpha_1-2\alpha_2\right\rangle\left|\mu_1\right\rangle - \frac{q^{1/2}}{\sqrt{[3]}}\left|2\mu_1-\alpha_1-\alpha_2\right\rangle\left|\mu_1-\alpha_1-\alpha_2\right\rangle$$

所有红圈标出的态都属于不可约表示（仅与权重相同的黑色点状态线性无关） 

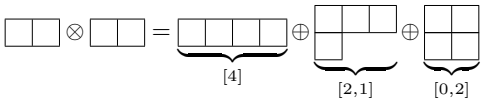
$$\left|\mu_1+\mu_2-\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]}}\,q^{-1/4}\left|2\mu_1-2\alpha_1-\alpha_2\right\rangle\left|\mu_1\right\rangle - \frac{q^{3/4}}{\sqrt{[2][3]}}\left|2\mu_1-\alpha_1-\alpha_2\right\rangle\left|\mu_1-\alpha_1\right\rangle - \frac{q^{1/4}}{\sqrt{[2][3]}}\left|2\mu_1-\alpha_1\right\rangle\left|\mu_1-\alpha_1-\alpha_2\right\rangle$$

表示的红圈态（经格拉姆–施密特正交化后），可由此提取 q-CG 系数 

$$\left|\mu_1+\mu_2-\alpha_1-\alpha_2,\ \mathrm{red}\right\rangle_\perp = \frac{1}{\sqrt{1-P^2}}\Bigg[\left(\sqrt{\frac{[2]}{[3]}}\,q^{-1/4} - P\,\frac{q^{-1/4}}{\sqrt{[2][3]}}\right)\left|2\mu_1-2\alpha_1-\alpha_2\right\rangle\left|\mu_1\right\rangle - \left(\frac{q^{3/4}}{\sqrt{[2][3]}} - P\sqrt{\frac{[2]}{[3]}}\,q^{1/4}\right)\left|2\mu_1-\alpha_1-\alpha_2\right\rangle\left|\mu_1-\alpha_1\right\rangle$$
$$- \left(\frac{q^{1/4}}{\sqrt{[2][3]}} + P\,\frac{q^{-3/4}}{\sqrt{[2][3]}}\right)\left|2\mu_1-\alpha_1\right\rangle\left|\mu_1-\alpha_1-\alpha_2\right\rangle\Bigg]$$

其中 $P = \left\langle \mu_1+\mu_2-\alpha_1-\alpha_2\text{、黑色}\quad \mu_1+\mu_2-\alpha_1-\alpha_2\text{、红色}\right\rangle$

- **张量积的 q-CG 系数，以及 U_q（$\mathfrak{sl}_3$）群**



属于不可约表示的所有权态

$$\left|4\,\mu_1\right\rangle = \left|2\,\mu_1\right\rangle \left|2\,\mu_1\right\rangle$$

$$\left|4\,\mu_1-\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1\right\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1\right\rangle \left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|4\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle =$$

$$\sqrt{\frac{[2]}{[3]\,[4]}}\,q\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|2\,\mu_1\right\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,[2]\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1}\,\left|2\,\mu_1\right\rangle \left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle$$

$$\left|4\,\mu_1-3\,\alpha_1-3\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle +$$

$$\left|4\,\mu_1-4\,\alpha_1-4\,\alpha_2\right\rangle = \left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle$$

$$\left|4\,\mu_1-\alpha_1\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-\alpha_1\right\rangle \left|2\,\mu_1\right\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1\right\rangle \left|2\,\mu_1-\alpha_1\right\rangle$$

$$\left|4\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]\,[4]}}\,q\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1\right\rangle + \frac{[2]}{\sqrt{[3]\,[4]}}\,q^{-1/4}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-\alpha_1\right\rangle +$$

$$\frac{[2]}{\sqrt{[3]\,[4]}}\,q^{1/4}\,\left|2\,\mu_1-\alpha_1\right\rangle \left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1}\,\left|2\,\mu_1\right\rangle \left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle$$

$$\left|4\,\mu_1-3\,\alpha_1-2\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]\,[4]}}\,\left|2\,\mu_1-\alpha_1\right\rangle \left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|2\,\mu_1-\alpha_1\right\rangle +$$

$$\frac{[2]}{\sqrt{[3]\,[4]}}\,q^{-3/4}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle + \frac{[2]}{\sqrt{[3]\,[4]}}\,q^{3/4}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|4\,\mu_1-4\,\alpha_1-3\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle$$

$$\left|4\,\mu_1-2\,\alpha_1\right\rangle = \sqrt{\frac{[2]}{[3]\,[4]}}\,q\,\left|2\,\mu_1-2\,\alpha_1\right\rangle \left|2\,\mu_1\right\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1}\,\left|2\,\mu_1\right\rangle \left|2\,\mu_1-2\,\alpha_1\right\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,[2]\,\left|2\,\mu_1-\alpha_1\right\rangle \left|2\,\mu_1-\alpha_1\right\rangle$$

$$\left|4\,\mu_1-3\,\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1\right\rangle + \frac{[2]}{\sqrt{[3]\,[4]}}\,q^{-1/4}\,\left|2\,\mu_1-\alpha_1\right\rangle \left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle +$$

$$\frac{[2]}{\sqrt{[3]\,[4]}}\,q^{1/4}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-\alpha_1\right\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,q\,\left|2\,\mu_1-2\,\alpha_1\right\rangle \left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|4\,\mu_1-4\,\alpha_1-2\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1}\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1\right\rangle +$$

$$\sqrt{\frac{[2]}{[3]\,[4]}}\,[2]\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,q\,\left|2\,\mu_1-2\,\alpha_1\right\rangle \left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle$$

$$\left|4\,\mu_1-3\,\alpha_1\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-\alpha_1\right\rangle \left|2\,\mu_1-2\,\alpha_1\right\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-2\,\alpha_1\right\rangle \left|2\,\mu_1-\alpha_1\right\rangle$$

$$\left|4\,\mu_1-4\,\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle \left|2\,\mu_1-2\,\alpha_1\right\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-2\,\alpha_1\right\rangle \left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle$$

$$\left|4\,\mu_1-4\,\alpha_1\right\rangle = \left|2\,\mu_1-2\,\alpha_1\right\rangle \left|2\,\mu_1-2\,\alpha_1\right\rangle$$



属于不可约表示的所有黑色点状态

$$\left|2\,\mu_1+\mu_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-\alpha_1\right\rangle\,\left|2\,\mu_1\right\rangle - \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1\right\rangle\,\left|2\,\mu_1-\alpha_1\right\rangle$$

$$\left|2\,\mu_1+\mu_2-\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]\,[4]}}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1\right\rangle - \sqrt{\frac{[2]}{[3]\,[4]}}\,\left|2\,\mu_1\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle - \frac{[2]}{\sqrt{[3]\,[4]}}\,q^{3/4}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-\alpha_1\right\rangle + \frac{[2]}{\sqrt{[3]\,[4]}}\,q^{-3/4}\,\left|2\,\mu_1-\alpha_1\right\rangle\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|2\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1}\,\left|2\,\mu_1-\alpha_1\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle - \sqrt{\frac{[2]}{[3]\,[4]}}\,q\,\left|2\,\mu_1-2\,\alpha_2-2\,\alpha_1\right\rangle\,\left|2\,\mu_1-\alpha_1\right\rangle - \frac{[2]}{\sqrt{[3]\,[4]}}\,q^{1/4}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle + \frac{[2]}{\sqrt{[3]\,[4]}}\,q^{-1/4}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|2\,\mu_1+\mu_2-3\,\alpha_1-3\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle - \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle$$

$$\left|2\,\mu_1+\mu_2-\alpha_1\right\rangle = \sqrt{\frac{[2]}{[4]}}\,\left|2\,\mu_1-2\,\alpha_1\right\rangle\,\left|2\,\mu_1\right\rangle - \sqrt{\frac{[2]}{[4]}}\,\left(q^{1/2}-q^{-1/2}\right)\,\left|2\,\mu_1-\alpha_1\right\rangle\,\left|2\,\mu_1-\alpha_1\right\rangle - \sqrt{\frac{[2]}{[4]}}\,\left|2\,\mu_1\right\rangle\,\left|2\,\mu_1-2\,\alpha_1\right\rangle$$

$$\left|2\,\mu_1+\mu_2-2\,\alpha_1-\alpha_2\right\rangle = \left(q^{-1/2}-q^{1/2}\right)\,\frac{q^{-1/4}}{\sqrt{[4]}}\,\left|2\,\mu_1-\alpha_1\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle - \left(q^{1/2}-q^{-1/2}\right)\,\frac{q^{1/4}}{\sqrt{[4]}}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-\alpha_1\right\rangle + \sqrt{\frac{[2]}{[4]}}\,\left|2\,\mu_1-2\,\alpha_1\right\rangle\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle - \sqrt{\frac{[2]}{[4]}}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-2\,\alpha_1\right\rangle$$

$$\left|2\,\mu_1+\mu_2-3\,\alpha_1-2\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,\left|2\,\mu_1-2\,\alpha_1\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle - \sqrt{\frac{[2]}{[4]}}\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\,\left|2\,\mu_1-2\,\alpha_1\right\rangle - \sqrt{\frac{[2]}{[4]}}\,\left(q^{1/2}-q^{-1/2}\right)\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle$$

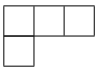
$$\left|2\,\mu_1+\mu_2-2\,\alpha_1\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-2\,\alpha_1\right\rangle\,\left|2\,\mu_1-\alpha_1\right\rangle - \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-\alpha_1\right\rangle\,\left|2\,\mu_1-2\,\alpha_1\right\rangle$$

$$\left|2\,\mu_1+\mu_2-3\,\alpha_1-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-2\,\alpha_1\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle - \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-2\,\alpha_1\right\rangle$$

$$\left|2\,\mu_1+\mu_2-\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1\right\rangle - \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1\right\rangle\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle$$

$$\left|2\,\mu_1+\mu_2-\alpha_1-2\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\,\left|2\,\mu_1\right\rangle - \sqrt{\frac{[2]}{[4]}}\,\left(q^{1/2}-q^{-1/2}\right)\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle - \sqrt{\frac{[2]}{[4]}}\,\left|2\,\mu_1\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle$$

$$\left|2\,\mu_1+\mu_2-2\,\alpha_1-3\,\alpha_2\right\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle - \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\,\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle$$

属于不可约表示的所有红圈态（仅与具有相同权重的黑色点状态线性无关） 

$$\left|2\,\mu_1+\mu_2-\alpha_1-\alpha_2\right\rangle = \left(q^{-1/2}-q^{1/2}\right)\frac{q^{-1/4}}{\sqrt{[4]}}\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle -$$
$$\left(q^{1/2}-q^{-1/2}\right)\frac{q^{1/4}}{\sqrt{[4]}}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle - \sqrt{\frac{[2]}{[4]}}\left|2\,\mu_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle + \sqrt{\frac{[2]}{[4]}}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1\right\rangle$$

$$\left|2\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2\right\rangle = \frac{\left(q^{-1}-1\right)}{\sqrt{[2]\,[4]}}\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle - \frac{(q-1)}{\sqrt{[2]\,[4]}}\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle +$$
$$\left(q^{-1}-1+[2]\right)\frac{q^{1/4}}{[2]\,\sqrt{[4]}}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle - (q-1+[2])\,\frac{q^{-1/4}}{[2]\,\sqrt{[4]}}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle$$

$$\left|2\,\mu_1+\mu_2-2\,\alpha_1-\alpha_2\right\rangle = \frac{q^{-1/2}}{\sqrt{[2]\,[4]}}\left(1-q^{1/2}-q\right)\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-2\,\alpha_1\right\rangle -$$
$$\frac{q^{1/2}}{\sqrt{[2]\,[4]}}\left(1-q^{-1}-q^{-1/2}\right)\left|2\,\mu_1-2\,\alpha_1\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle + \left(1-q+\sqrt{q}\,[2]\right)\frac{q^{-1/4}}{[2]\,\sqrt{[4]}}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle -$$
$$\left(1-q^{-1}+q^{-1/2}\,[2]\right)\frac{q^{1/4}}{[2]\,\sqrt{[4]}}\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle$$



表示中的红圈态（经过 Gram-Schmidt 正交归一化），可从中提取 q-CG 系数

$$\left|2\,\mu_1+\mu_2-\alpha_1-\alpha_2,\ \text{red}\right\rangle_{\perp}=\frac{1}{\sqrt{1-P_1^{\,2}}}\left[\left(\sqrt{\frac{[2]}{[4]}}-P_1\sqrt{\frac{[2]}{[3]\,[4]}}\right)\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1\right\rangle-\right.$$

$$\left(\sqrt{\frac{[2]}{[4]}}-P_1\sqrt{\frac{[2]}{[3]\,[4]}}\right)\left|2\,\mu_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle+\left(\left(q^{-1/2}-q^{1/2}\right)\frac{q^{-1/4}}{\sqrt{[4]}}-P_1\frac{[2]}{\sqrt{[3]\,[4]}}\,q^{-3/4}\right)\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle-$$

$$\left.\left(\left(q^{1/2}-q^{-1/2}\right)\frac{q^{1/4}}{\sqrt{[4]}}-P_1\frac{[2]}{\sqrt{[3]\,[4]}}\,q^{3/4}\right)\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle\right]$$

其中 $P_1=\left\langle 2\,\mu_1+\mu_2-\alpha_1-\alpha_2\text{、黑色}\ \ 2\,\mu_1+\mu_2-\alpha_1-\alpha_2\text{、红色}\right\rangle$

$$\left|2\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{red}\right\rangle_{\perp}=$$

$$\frac{1}{\sqrt{1-P_2^{\,2}}}\left[\left(N_2\frac{\left(q^{-1}-1\right)}{\sqrt{[2]\,[4]}}-P_2\sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1}\right)\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle-\left(N_2\frac{(q-1)}{\sqrt{[2]\,[4]}}-P_2\sqrt{\frac{[2]}{[3]\,[4]}}\,q\right)\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle+\right.$$

$$\left(N_2\left(q^{-1}-1+[2]\right)\frac{q^{1/4}}{[2]\,\sqrt{[4]}}-P_2\frac{[2]}{\sqrt{[3]\,[4]}}\,q^{-1/4}\right)\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle-$$

$$\left.\left(N_2\,(q-1+[2])\frac{q^{-1/4}}{[2]\,\sqrt{[4]}}-P_2\frac{[2]}{\sqrt{[3]\,[4]}}\,q^{1/4}\right)\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\right]$$

其中 $P_2=\left\langle 2\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{黑色}\ N_2\left(2\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2,\text{红色}\right)\right\rangle$ 和 $N_2=\dfrac{1}{\left\langle 2\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2\right|,\ \text{红色}\ \ 2\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{红色}\right\rangle^{1/2}}$

$$\left|2\,\mu_1+\mu_2-2\,\alpha_1-\alpha_2,\ \text{红色}\right\rangle_{\perp}=$$

$$\frac{1}{\sqrt{1-P_3^{\,2}}}\left[\left(-N_3\frac{q^{1/2}}{\sqrt{[2]\,[4]}}\left(1-q^{-1}-q^{-1/2}\right)-P_3\sqrt{\frac{[2]}{[4]}}\right)\left|2\,\mu_1-2\,\alpha_1\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle+\left(N_3\frac{q^{-1/2}}{\sqrt{[2]\,[4]}}\left(1-q^{1/2}-q\right)-P_3\sqrt{\frac{[2]}{[4]}}\right)\right.$$

$$\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-2\,\alpha_1\right\rangle+\left(-N_3\left(1-q^{-1}+q^{-1/2}\,[2]\right)\frac{q^{1/4}}{[2]\,\sqrt{[4]}}-P_3\left(q^{-1/2}-q^{1/2}\right)\frac{q^{-1/4}}{\sqrt{[4]}}\right)\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle+$$

$$\left.\left(N_3\left(1-q+\sqrt{q}\,[2]\right)\frac{q^{-1/4}}{[2]\,\sqrt{[4]}}-P_3\left(q^{1/2}-q^{-1/2}\right)\frac{q^{1/4}}{\sqrt{[4]}}\right)\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle\right]$$

其中 $P_3=\left\langle 2\,\mu_1+\mu_2-2\,\alpha_1-\alpha_2,\ \text{黑色}\ N_3\left(2\,\mu_1+\mu_2-2\,\alpha_1-\alpha_2,\text{红色}\right)\right\rangle$ 和 $N_3=\dfrac{1}{\left\langle 2\,\mu_1+\mu_2-2\,\alpha_1-\alpha_2\right|,\ \text{红色}\ \ 2\,\mu_1+\mu_2-2\,\alpha_1-\alpha_2,\ \text{红色}\right\rangle^{1/2}}$

属于不可约表示的所有权态



$$\left|2\,\mu_2\right\rangle = \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1\right\rangle\left|2\,\mu_1-2\,\alpha_1\right\rangle + \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1\right\rangle\left|2\,\mu_1\right\rangle - \frac{1}{\sqrt{[3]}}\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle$$

$$\left|2\,\mu_2-\alpha_1-\alpha_2\right\rangle = \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-2\,\alpha_1\right\rangle + \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle -$$
$$\frac{q^{-1/4}}{\sqrt{[2]\,[3]}}\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle - \frac{q^{1/4}}{\sqrt{[2]\,[3]}}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle$$

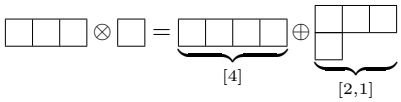
$$\left|2\,\mu_2-2\,\alpha_1-2\,\alpha_2\right\rangle = \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\left|2\,\mu_1-2\,\alpha_1\right\rangle +$$
$$\frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle - \frac{1}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle$$

$$\left|2\,\mu_2-\alpha_2\right\rangle = \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle + \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1\right\rangle -$$
$$\frac{q^{-1/4}}{\sqrt{[2]\,[3]}}\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle - \frac{q^{1/4}}{\sqrt{[2]\,[3]}}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle$$

$$\left|2\,\mu_2-\alpha_1-2\,\alpha_2\right\rangle = \frac{([2]-1)\,q^{1/4}}{\sqrt{[2]\,[3]}}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle + \frac{([2]-1)\,q^{-1/4}}{\sqrt{[2]\,[3]}}\left|2\,\mu_1-2\,\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle -$$
$$\frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1-\alpha_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle + \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1\right\rangle$$

$$\left|2\,\mu_2-2\,\alpha_2\right\rangle = \frac{q^{1/2}}{\sqrt{[3]}}\left|2\,\mu_1\right\rangle\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle + \frac{q^{-1/2}}{\sqrt{[3]}}\left|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\right\rangle\left|2\,\mu_1\right\rangle - \frac{1}{\sqrt{[3]}}\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle\left|2\,\mu_1-\alpha_1-\alpha_2\right\rangle$$

- **q-CG 系数，适用于和 $U_q(\mathfrak{sl}_3)$ 群**



属于不可约表示的所有权态 

$$|4\,\mu_1\rangle = |3\,\mu_1\rangle\,|\mu_1\rangle$$

$$|4\,\mu_1-\alpha_1-\alpha_2\rangle = \sqrt{\frac{[3]}{[4]}}\,q^{1/4}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|\mu_1\rangle + \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\,\mu_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$|4\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|\mu_1\rangle$$

$$|4\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle = \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\,|\mu_1\rangle + \sqrt{\frac{[3]}{[4]}}\,q^{-1/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$|4\,\mu_1-4\,\alpha_1-4\,\alpha_2\rangle = |3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$|4\,\mu_1-\alpha_1\rangle = \sqrt{\frac{[3]}{[4]}}\,q^{1/4}\,|3\,\mu_1-\alpha_1\rangle\,|\mu_1\rangle + \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\,\mu_1\rangle\,|\mu_1-\alpha_1\rangle$$

$$|4\,\mu_1-2\,\alpha_1-\alpha_2\rangle = \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|\mu_1\rangle + \frac{q^{-1/4}}{\sqrt{[4]}}\,|3\,\mu_1-\alpha_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$|4\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle = \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|\mu_1-\alpha_1\rangle + \frac{q^{3/4}}{\sqrt{[4]}}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\,|\mu_1\rangle + \sqrt{\frac{[2]}{[4]}}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$|4\,\mu_1-4\,\alpha_1-3\,\alpha_2\rangle = \sqrt{\frac{[3]}{[4]}}\,q^{1/4}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|4\,\mu_1-2\,\alpha_1\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\,\mu_1-\alpha_1\rangle\,|\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\,\mu_1-2\,\alpha_1\rangle\,|\mu_1\rangle$$

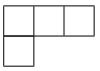
$$|4\,\mu_1-3\,\alpha_1-\alpha_2\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1\rangle + \frac{q^{1/4}}{\sqrt{[4]}}\,|3\,\mu_1-2\,\alpha_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \frac{q^{3/4}}{\sqrt{[4]}}\,|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\,|\mu_1\rangle$$

$$|4\,\mu_1-4\,\alpha_1-2\,\alpha_2\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|4\,\mu_1-3\,\alpha_1\rangle = \frac{q^{3/4}}{\sqrt{[4]}}\,|3\,\mu_1-3\,\alpha_1\rangle\,|\mu_1\rangle + \sqrt{\frac{[3]}{[4]}}\,q^{-1/4}\,|3\,\mu_1-2\,\alpha_1\rangle\,|\mu_1-\alpha_1\rangle$$

$$|4\,\mu_1-4\,\alpha_1-\alpha_2\rangle = \frac{q^{3/4}}{\sqrt{[4]}}\,|3\,\mu_1-3\,\alpha_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[3]}{[4]}}\,q^{-1/4}\,|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|4\,\mu_1-4\,\alpha_1\rangle = |3\,\mu_1-3\,\alpha_1\rangle\,|\mu_1-\alpha_1\rangle$$



属于不可约表示的所有黑色点状态

$$|2\mu_1+\mu_2\rangle = \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\mu_1-\alpha_1\rangle\,|\mu_1\rangle - \sqrt{\frac{[3]}{[4]}}\,q^{-1/4}\,|3\mu_1\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\mu_1+\mu_2-\alpha_1-\alpha_2\rangle = \sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1/2}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|\mu_1\rangle + \frac{q^{-5/4}}{\sqrt{[3]\,[4]}}\,|3\mu_1-\alpha_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle - \sqrt{\frac{[3]}{[4]}}\,q^{1/4}\,|3\mu_1-\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\mu_1+\mu_2-2\alpha_1-2\alpha_2\rangle = \frac{q^{-1/4}}{\sqrt{[3]\,[4]}}\,|3\mu_1-3\alpha_1-2\alpha_2\rangle\,|\mu_1\rangle + \sqrt{\frac{[2]}{[3]\,[4]}}\,q^{-1}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$- \sqrt{\frac{[3]}{[4]}}\,q^{1/4}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\mu_1+\mu_2-3\alpha_1-3\alpha_2\rangle = \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\mu_1-3\alpha_1-2\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle - \sqrt{\frac{[3]}{[4]}}\,q^{1/4}\,|3\mu_1-3\alpha_1-3\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\mu_1+\mu_2-\alpha_1\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\mu_1-2\alpha_1\rangle\,|\mu_1\rangle - \frac{\left([3]-q^{-1}\right)}{\sqrt{[2]\,[4]}}\,|3\mu_1-\alpha_1\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\mu_1+\mu_2-2\alpha_1-\alpha_2\rangle = \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\mu_1-2\alpha_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \frac{q^{-1/4}}{\sqrt{[4]}}\,|3\mu_1-3\alpha_1-\alpha_2\rangle\,|\mu_1\rangle - \frac{\left([3]-q^{-1}\right)}{\sqrt{[2]\,[4]}}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\mu_1+\mu_2-3\alpha_1-2\alpha_2\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\mu_1-3\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle - \frac{\left([3]-q^{-1}\right)}{\sqrt{[2]\,[4]}}\,|3\mu_1-3\alpha_1-2\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\mu_1+\mu_2-2\alpha_1\rangle = \sqrt{\frac{[3]}{[4]}}\,q^{-1/4}\,|3\mu_1-3\alpha_1\rangle\,|\mu_1\rangle + \frac{\left(q^{-3/4}[2]-q^{-1/4}[3]\right)}{\sqrt{[4]}}\,|3\mu_1-2\alpha_1\rangle\,|\mu_1-\alpha_1\rangle$$

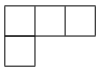
$$|2\mu_1+\mu_2-3\alpha_1-\alpha_2\rangle = \sqrt{\frac{[3]}{[4]}}\,q^{-1/4}\,|3\mu_1-3\alpha_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \frac{\left(q^{-3/4}[2]-q^{-1/4}[3]\right)}{\sqrt{[4]}}\,|3\mu_1-3\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\mu_1+\mu_2-\alpha_2\rangle = \frac{q^{-3/4}}{\sqrt{[4]}}\,|3\mu_1-\alpha_1-\alpha_2\rangle\,|\mu_1\rangle - \sqrt{\frac{[3]}{[4]}}\,q^{1/4}\,|3\mu_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$|2\mu_1+\mu_2-2\alpha_2-\alpha_1\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|\mu_1\rangle - \frac{\left([3]-q^{-1}\right)}{\sqrt{[2]\,[4]}}\,|3\mu_1-\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$|2\mu_1+\mu_2-2\alpha_1-3\alpha_2\rangle = \frac{[2]}{\sqrt{[4]}}\,q^{-3/4}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[3]}{[4]}}\,q^{1/4}\,|3\mu_1-3\alpha_1-3\alpha_2\rangle\,|\mu_1\rangle$$

$$- \frac{[3]}{\sqrt{[4]}}\,q^{-1/4}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

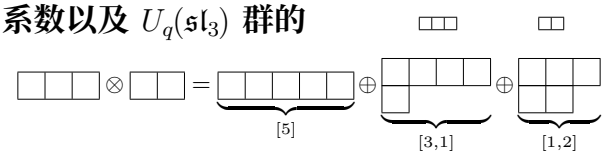
属于不可约表示的所有红圈态

（仅线
性独立于具有相同权重的黑色点状态）

$$|2\,\mu_1+\mu_2-\alpha_1-\alpha_2\rangle = \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|\mu_1\rangle - \frac{\left([3]-q^{-1}\right)}{[2]\,\sqrt{[4]}}\,q^{1/4}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1\rangle - \frac{\left([3]-q^{-1}\right)}{[2]\,\sqrt{[4]}}\,q^{-1/4}\,|3\,\mu_1-\alpha_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle$$

$$|2\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2\rangle = \frac{(1+q\,[2]-q\,[3])}{[2]\,\sqrt{[2]\,[4]}}\,q^{-3/2}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle + \frac{q^{-1/4}}{\sqrt{[4]}}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\,|\mu_1\rangle$$
$$+\frac{(1-q\,[3])}{[2]\,\sqrt{[4]}}\,q^{-3/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$

$$|2\,\mu_1+\mu_2-2\,\alpha_1-\alpha_2\rangle = \frac{q^{-1/4}}{\sqrt{[4]}}\,|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\,|\mu_1\rangle + \frac{q^{-1}}{[2]^{3/2}\,\sqrt{[4]}}\left(1+q^{1/2}\,[2]-q\,[3]\right)|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|\mu_1-\alpha_1\rangle$$
$$+|3\,\mu_1-2\,\alpha_1\rangle\,|\mu_1-\alpha_1-\alpha_2\rangle\,\frac{q^{-5/4}}{[2]\,\sqrt{[4]}}\left(q^{1/2}+1-q\,[3]\right)$$

- **张量积的 q-CG 系数以及 $U_q(\mathfrak{sl}_3)$ 群的**



属于不可约表示的所有权态



$$|5\,\mu_1\rangle = |3\,\mu_1\rangle\ |2\,\mu_1\rangle$$

$$|5\,\mu_1-\alpha_1-\alpha_2\rangle = \sqrt{\frac{[3]}{[5]}}\,q^{1/2}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\ |2\,\mu_1\rangle + \sqrt{\frac{[2]}{[5]}}\,q^{-3/4}\,|3\,\mu_1\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle$$

$$|5\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle = \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1\rangle + \sqrt{\frac{[3]}{[4]\,[5]}}\,[2]\,q^{-1/4}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[3]}{[4]\,[5]}}\,q^{-3/2}\,|3\,\mu_1\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

$$|5\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle = \sqrt{\frac{[2]}{[4]\,[5]}}\,q^{3/2}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\ |2\,\mu_1\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q^{-1}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle + \sqrt{\frac{[3]}{[4]\,[5]}}\,[2]\,q^{1/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle$$

$$|5\,\mu_1-4\,\alpha_1-4\,\alpha_2\rangle = \sqrt{\frac{[2]}{[5]}}\,q^{3/4}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[3]}{[5]}}\,q^{-1/2}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

$$|5\,\mu_1-5\,\alpha_1-5\,\alpha_2\rangle = |3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

$$|5\,\mu_1-\alpha_1\rangle = \sqrt{\frac{[3]}{[5]}}\,q^{1/2}\,|3\,\mu_1-\alpha_1\rangle\ |2\,\mu_1\rangle + \sqrt{\frac{[2]}{[5]}}\,q^{-3/4}\,|3\,\mu_1\rangle\ |2\,\mu_1-\alpha_1\rangle$$

$$|5\,\mu_1-2\,\alpha_1-\alpha_2\rangle =$$
$$\sqrt{\frac{[2]}{[4]\,[5]}}\,q^{-3/2}\,|3\,\mu_1\rangle\ |2\,\mu_1-2\,\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q^{-1/2}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\ |2\,\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,|3\,\mu_1-\alpha_1\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1\rangle$$

$$|5\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle = \sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{-5/4}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,q^{-1/2}\,|3\,\mu_1-\alpha_1\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle +$$
$$\sqrt{\frac{[2]}{[4]\,[5]}}\,q^{3/2}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1\rangle + \sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{-1/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,[2]\,q^{1/2}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle$$

$$|5\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle = \sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{-5/4}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,q^{-1/2}\,|3\,\mu_1-\alpha_1\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle +$$
$$\sqrt{\frac{[2]}{[4]\,[5]}}\,q^{3/2}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1\rangle + \sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{-1/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,[2]\,q^{1/2}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle$$

$$|5\,\mu_1-4\,\alpha_1-3\,\alpha_2\rangle = \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle +$$
$$\sqrt{\frac{[2]}{[4]\,[5]}}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\ |2\,\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q^{-1}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-\alpha_2\rangle$$

$$|5\,\mu_1-5\,\alpha_1-4\,\alpha_2\rangle = \sqrt{\frac{[3]}{[5]}}\,q^{1/2}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle + \sqrt{\frac{[2]}{[5]}}\,q^{-3/4}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-\alpha_2\rangle$$

$$|5\,\mu_1-2\,\alpha_1\rangle = \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q\,|3\,\mu_1-2\,\alpha_1\rangle\ |2\,\mu_1\rangle + \sqrt{\frac{[3]}{[4]\,[5]}}\,[2]\,q^{-1/4}\,|3\,\mu_1-\alpha_1\rangle\ |2\,\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,q^{-3/2}\,|3\,\mu_1\rangle\ |2\,\mu_1-2\,\alpha_1\rangle$$

$$|5\,\mu_1-3\,\alpha_1-\alpha_2\rangle = \sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{3/4}\,|3\,\mu_1-2\,\alpha_1\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{-3/4}\,|3\,\mu_1-\alpha_1\rangle\ |2\,\mu_1-2\,\alpha_1-\alpha_2\rangle +$$
$$\sqrt{\frac{[2]}{[4]\,[5]}}\,q^{-3/2}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,q^{3/2}\,|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,[2]\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1-\alpha_1\rangle$$

$$|5\,\mu_1-4\,\alpha_1-2\,\alpha_2\rangle = \sqrt{\frac{[2]}{[4]\,[5]}}\,q^{-3/2}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,q^{1/2}\,|3\,\mu_1-2\,\alpha_1\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle +$$
$$\sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{5/4}\,|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{1/4}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]\,[5]}}\,[2]\,q^{-1/2}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-\alpha_2\rangle$$

$$|5\,\mu_1-5\,\alpha_1-3\,\alpha_2\rangle = \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q\,|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle + \sqrt{\frac{[3]}{[4]\,[5]}}\,[2]\,q^{-1/4}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1-\alpha_2\rangle +$$
$$\sqrt{\frac{[2]}{[4]\,[5]}}\,q^{-3/2}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\ |2\,\mu_1-2\,\alpha_1\rangle$$

$$|5\,\mu_1-3\,\alpha_1\rangle = \sqrt{\frac{[2]}{[4]\,[5]}}\;q^{3/2}\;|3\,\mu_1-3\,\alpha_1\rangle\;|2\,\mu_1\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\;q^{-1}\;|3\,\mu_1-\alpha_1\rangle\;|2\,\mu_1-2\,\alpha_1\rangle + \sqrt{\frac{[3]}{[4]\,[5]}}\;[2]\;q^{1/4}\;|3\,\mu_1-2\,\alpha_1\rangle\;|2\,\mu_1-\alpha_1\rangle$$

$$|5\,\mu_1-4\,\alpha_1-\alpha_2\rangle = \sqrt{\frac{[2]}{[4]\,[5]}}\;q^{3/2}\;|3\,\mu_1-3\,\alpha_1\rangle\;|2\,\mu_1-\alpha_1-\alpha_2\rangle +$$
$$\sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\;q^{-1}\;|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\;|2\,\mu_1-2\,\alpha_1\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\;|3\,\mu_1-2\,\alpha_1\rangle\;|2\,\mu_1-2\,\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\;q^{1/2}\;|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\;|2\,\mu_1-\alpha_1\rangle$$

$$|5\,\mu_1-5\,\alpha_1-2\,\alpha_2\rangle = \sqrt{\frac{[2]}{[4]\,[5]}}\;q^{3/2}\;|3\,\mu_1-3\,\alpha_1\rangle\;|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle + \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\;q^{-1}\;|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\;|2\,\mu_1-2\,\alpha_1\rangle +$$
$$\sqrt{\frac{[3]}{[4]\,[5]}}\;[2]\;q^{1/4}\;|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\;|2\,\mu_1-2\,\alpha_1-\alpha_2\rangle$$

$$|5\,\mu_1-4\,\alpha_1\rangle = \sqrt{\frac{[2]}{[5]}}\;q^{3/4}\;|3\,\mu_1-3\,\alpha_1\rangle\;|2\,\mu_1-\alpha_1\rangle + \sqrt{\frac{[3]}{[5]}}\;q^{-1/2}\;|3\,\mu_1-2\,\alpha_1\rangle\;|2\,\mu_1-2\,\alpha_1\rangle$$

$$|5\,\mu_1-5\,\alpha_1-\alpha_2\rangle = \sqrt{\frac{[2]}{[5]}}\;q^{3/4}\;|3\,\mu_1-3\,\alpha_1\rangle\;|2\,\mu_1-2\,\alpha_1-\alpha_2\rangle + \sqrt{\frac{[3]}{[5]}}\;q^{-1/2}\;|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\;|2\,\mu_1-2\,\alpha_1\rangle$$

$$|5\,\mu_1-5\,\alpha_1\rangle = |3\,\mu_1-3\,\alpha_1\rangle\;|2\,\mu_1-2\,\alpha_1\rangle$$



属于不可约表示的所有黑色点状态

$$|3\,\mu_1+\mu_2\rangle = \sqrt{\frac{[2]}{[5]}}\,q^{-3/4}\,|3\,\mu_1-\alpha_1\rangle\,|2\,\mu_1\rangle \;-\sqrt{\frac{[3]}{[5]}}\,q^{1/2}\,|3\,\mu_1\rangle\,|2\,\mu_1-\alpha_1\rangle$$

$$|3\,\mu_1+\mu_2-\alpha_1-\alpha_2\rangle = \sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{-5/4}\,|3\,\mu_1-\alpha_1\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle +$$
$$\sqrt{\frac{1}{[4]\,[5]}}\,[2]\,q^{-1/4}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|2\,\mu_1\rangle \;-\sqrt{\frac{1}{[4]\,[5]}}\,[3]\,q^{3/4}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|2\,\mu_1-\alpha_1\rangle - \sqrt{\frac{[3]}{[4]\,[5]}}\,q^{-1/4}\,|3\,\mu_1\rangle\,|2\,\mu_1-2\,\alpha_1-\alpha_2\rangle$$

$$|3\,\mu_1+\mu_2-2\,\alpha_1-2\,\alpha_2\rangle = \frac{[2]}{\sqrt{[3]\,[4]\,[5]}}\,q^{1/4}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1\rangle \;+\frac{[2]^2}{\sqrt{[3]\,[4]\,[5]}}\,q^{-3/4}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle +$$
$$\frac{[2]}{\sqrt{[3]\,[4]\,[5]}}\,q^{-7/4}\,|3\,\mu_1-\alpha_1\rangle\,|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle - \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|2\,\mu_1-2\,\alpha_1-\alpha_2\rangle - \sqrt{\frac{[2]\,[3]}{[4]\,[5]}}\,q\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1-\alpha_1\rangle$$

$$|3\,\mu_1+\mu_2-3\,\alpha_1-3\,\alpha_2\rangle = \frac{[2]}{\sqrt{[4]\,[5]}}\,q^{-1/4}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle + \frac{[2]}{\sqrt{[4]\,[5]}}\,q^{-5/4}\,|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\,|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle \;-$$
$$\sqrt{\frac{[3]}{[4]\,[5]}}\,q^{5/4}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\,|2\,\mu_1-\alpha_1\rangle - \frac{[3]}{\sqrt{[4]\,[5]}}\,q^{1/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1-2\,\alpha_1-\alpha_2\rangle$$

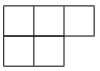
$$|3\,\mu_1+\mu_2-4\,\alpha_1-4\,\alpha_2\rangle = \sqrt{\frac{[2]}{[5]}}\,q^{-3/4}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle \;-\sqrt{\frac{[3]}{[5]}}\,q^{1/2}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\,|2\,\mu_1-2\,\alpha_1-\alpha_2\rangle$$

$$|3\,\mu_1+\mu_2-\alpha_2\rangle = \sqrt{\frac{[2]}{[5]}}\,q^{-3/4}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|2\,\mu_1\rangle \;-\sqrt{\frac{[3]}{[5]}}\,q^{1/2}\,|3\,\mu_1\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle$$

$$|3\,\mu_1+\mu_2-\alpha_1-2\,\alpha_2\rangle = \frac{[2]\;-q^{3/2}\,[3]}{q\,\sqrt{[3]}\,\sqrt{[5]}}\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle \;+\frac{[2]^{3/2}}{\sqrt{[3]\,[4]\,[5]}}\,q^{-1/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1\rangle \;-\sqrt{\frac{[2]}{[5]}}\,q^{-1/4}\,|3\,\mu_1\rangle\,|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

$$|3\,\mu_1+\mu_2-2\,\alpha_1-3\,\alpha_2\rangle = q^{-1/2}\,\frac{[2]^2}{\sqrt{[3]\,[4]\,[5]}}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle \;+\sqrt{\frac{[2]}{[5]}}\,q^{1/4}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\,|2\,\mu_1\rangle \;-$$
$$\sqrt{\frac{[3]}{[5]}}\,q^{1/2}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[3]\,[5]}}\,q^{-5/4}\,(1-q\,[3])\,|3\,\mu_1-\alpha_1-\alpha_2\rangle\,|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

$$|3\,\mu_1+\mu_2-3\,\alpha_1-4\,\alpha_2\rangle = \frac{[2]^{3/2}}{\sqrt{[5]}}\,q^{-3/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle \;+[2]\,\sqrt{\frac{[3]}{[5]}}\,|3\,\mu_1-3\,\alpha_1-2\,\alpha_2-\alpha_2\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle \;-$$
$$q^{1/2}\,\sqrt{\frac{[3]}{[5]}}\,|3\,\mu_1-3\,\alpha_1-3\,\alpha_2\rangle\,|2\,\mu_1-\alpha_1-\alpha_2\rangle - [3]\,\sqrt{\frac{[2]}{[5]}}\,q^{-1/4}\,|3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\,|2\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$



属于不可约表示的所有黑色点状态

$$|\mu_1+2\mu_2\rangle = \sqrt{\frac{[2]}{[3][4]}}\,q^{-1}\,|3\mu_1-2\alpha_1\rangle\,|2\mu_1\rangle - \frac{[2]}{\sqrt{[3][4]}}\,q^{-1/4}\,|3\mu_1-\alpha_1\rangle\,|2\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\mu_1\rangle\,|2\mu_1-2\alpha_1\rangle$$

$$|\mu_1+2\mu_2-\alpha_1-\alpha_2\rangle = \frac{1}{[3]}\sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\mu_1-3\alpha_1-\alpha_2\rangle\,|2\mu_1\rangle + \frac{[2]}{[3]\sqrt{[4]}}\,q^{-5/4}\,|3\mu_1-2\alpha_1\rangle\,|2\mu_1-\alpha_1-\alpha_2\rangle -$$
$$\frac{[2]}{[3]\sqrt{[4]}}\,q^{-3/4}\,|3\mu_1-\alpha_1\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle - \frac{[2]^{3/2}}{[3]\sqrt{[4]}}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|2\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\mu_1-\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1\rangle$$

$$|\mu_1+2\mu_2-2\alpha_1-2\alpha_2\rangle = \frac{1}{[3]}\sqrt{\frac{[2]}{[4]}}\,q^{-3/2}\,|3\mu_1-2\alpha_1\rangle\,|2\mu_1-2\alpha_1-2\alpha_2\rangle + \frac{[2]}{[3]\sqrt{[4]}}\,q^{-3/4}\,|3\mu_1-3\alpha_1-\alpha_2\rangle\,|2\mu_1-\alpha_1-\alpha_2\rangle -$$
$$\frac{[2]}{[3]\sqrt{[4]}}\,q^{1/4}\,|3\mu_1-3\alpha_1-2\alpha_2\rangle\,|2\mu_1-\alpha_1\rangle - \frac{[2]^{3/2}}{[3]\sqrt{[4]}}\,q^{-1/2}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle + \frac{1}{[3]}\sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|2\mu_1-2\alpha_1\rangle$$

$$|\mu_1+2\mu_2-3\alpha_1-3\alpha_2\rangle = \sqrt{\frac{[2]}{[3][4]}}\,q^{-1}\,|3\mu_1-3\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1-2\alpha_2\rangle -$$
$$\frac{[2]}{\sqrt{[3][4]}}\,q^{-1/4}\,|3\mu_1-3\alpha_1-2\alpha_2\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[4]}}\,q\,|3\mu_1-3\alpha_1-3\alpha_2\rangle\,|2\mu_1-2\alpha_1\rangle$$

$$|\mu_1+2\mu_2-\alpha_1\rangle = \frac{[2]\,\left(1-\sqrt{q}\,[2]\right)}{\sqrt{[3]}\,\sqrt{[4]}}\,q^{-3/4}\,|3\mu_1-2\alpha_1\rangle\,|2\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\mu_1-3\alpha_1\rangle\,|2\mu_1\rangle + \sqrt{\frac{[2]}{[3][4]}}\,\left(-[2]^{1/2}+\sqrt{q}\,[3]\right)q^{-1/2}\,|3\mu_1-\alpha_1\rangle\,|2\mu_1-2\alpha_1\rangle$$

$$|\mu_1+2\mu_2-2\alpha_1-\alpha_2\rangle = \sqrt{\frac{[2]}{[3][4]}}\,\left(1-\sqrt{q}\,[2]\right)q^{-1}\,|3\mu_1-2\alpha_1\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[3][4]}}\,\left(1-\sqrt{q}\,[2]\right)q^{-1/2}\,|3\mu_1-3\alpha_1-\alpha_2\rangle\,|2\mu_1-\alpha_1\rangle +$$
$$\sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\mu_1-3\alpha_1\rangle\,|2\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[3][4]}}\,\left(-[2]^{1/2}+\sqrt{q}\,[3]\right)q^{-1/2}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1\rangle$$

$$|\mu_1+2\mu_2-3\alpha_1-2\alpha_2\rangle = \frac{[2]\,\left(1-\sqrt{q}\,[2]\right)}{\sqrt{[3]}\,\sqrt{[4]}}\,q^{-3/4}\,|3\mu_1-3\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle +$$
$$\sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\mu_1-3\alpha_1\rangle\,|2\mu_1-2\alpha_1-2\alpha_2\rangle + \sqrt{\frac{[2]}{[3][4]}}\,\left(-[2]^{1/2}+\sqrt{q}\,[3]\right)q^{-1/2}\,|3\mu_1-3\alpha_1-2\alpha_2\rangle\,|2\mu_1-2\alpha_1\rangle$$

$$|\mu_1+2\mu_2-\alpha_2\rangle = -\sqrt{\frac{[2]}{[3][4]}}\,|3\mu_1-\alpha_1-\alpha_2\rangle\,|2\mu_1-\alpha_1\rangle + \sqrt{\frac{[2]}{[3][4]}}\,q^{-1}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|2\mu_1\rangle -$$
$$\sqrt{\frac{[2]}{[3][4]}}\,q^{-1/2}\,|3\mu_1-\alpha_1\rangle\,|2\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\mu_1\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle$$

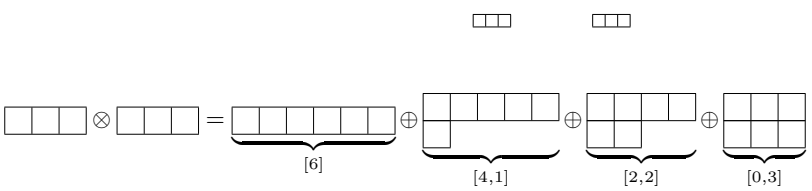
$$|\mu_1+2\mu_2-\alpha_1-2\alpha_2\rangle = -\sqrt{\frac{[2]}{[3][4]}}\,q^{-1}\,|3\mu_1-\alpha_1\rangle\,|2\mu_1-2\alpha_1-2\alpha_2\rangle + \sqrt{\frac{[2]}{[3][4]}}\,q^{-1/2}\,|3\mu_1-3\alpha_1-2\alpha_2\rangle\,|2\mu_1\rangle - \frac{[2]}{\sqrt{[3][4]}}\,q^{1/4}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|2\mu_1-\alpha_1\rangle -$$
$$\sqrt{\frac{[2]}{[3][4]}}\,\left(-1+\sqrt{q}\,\right)q^{-1}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|2\mu_1-\alpha_1-\alpha_2\rangle + \frac{q^{-1/4}}{\sqrt{[3][4]}}\,\left(-1+\sqrt{q}\,[3]\right)\,|3\mu_1-\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle$$

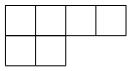
$$|\mu_1+2\mu_2-2\alpha_1-3\alpha_2\rangle = \sqrt{\frac{[2]}{[3][4]}}\,q^{-1}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1-2\alpha_2\rangle - [2]\sqrt{\frac{[2]}{[3][4]}}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle - [$$
$$2]\sqrt{\frac{[2]}{[3][4]}}\,q^{-1}\,|3\mu_1-2\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1-2\alpha_2\rangle - \sqrt{\frac{[2]}{[3][4]}}\,(1-[2])\,q^{-1/2}\,|3\mu_1-3\alpha_1-2\alpha_2\rangle\,|2\mu_1-\alpha_1-\alpha_2\rangle -$$
$$\sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\mu_1-3\alpha_1-2\alpha_2-\alpha_2\rangle\,|2\mu_1-\alpha_1\rangle + \sqrt{\frac{[3][2]}{[4]}}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|2\mu_1-2\alpha_1-\alpha_2\rangle$$

$$|\mu_1+2\mu_2-2\alpha_2\rangle = \sqrt{\frac{[2]}{[3][4]}}\,q^{-1}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|2\mu_1\rangle - \frac{[2]}{\sqrt{[3][4]}}\,q^{-1/4}\,|3\mu_1-\alpha_1-\alpha_2\rangle\,|2\mu_1-\alpha_1-\alpha_2\rangle + \sqrt{\frac{[2]}{[4]}}\,q^{1/2}\,|3\mu_1\rangle\,|2\mu_1-2\alpha_1-2\alpha_2\rangle$$

$$|\mu_1+2\mu_2-3\alpha_2-\alpha_1\rangle = \frac{[2]\,\left(1-\sqrt{q}\,[2]\right)}{\sqrt{[3][4]}}\,q^{-3/4}\,|3\mu_1-2\alpha_1-2\alpha_2\rangle\,|2\mu_1-\alpha_1-\alpha_2\rangle +$$
$$\sqrt{\frac{[2]}{[4]}}\,q^{-1/2}\,|3\mu_1-3\alpha_1-3\alpha_2\rangle\,|2\mu_1\rangle + \sqrt{\frac{[2]}{[3][4]}}\,\left(-[2]^{1/2}+\sqrt{q}\,[3]\right)q^{-1/2}\,|3\mu_1-\alpha_1-\alpha_2\rangle\,|2\mu_1-2\alpha_1-2\alpha_2\rangle$$

- **群中与和的张量积里，属于$(2,\ 2)$ 不可约表示的一个具有3重数的态示例 $U_q\ (\mathfrak{sl}_3)$**

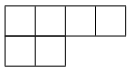


属于中心态的  **重数为3的表示（规范化且线性无关）**

$$|2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{black}\rangle = -\frac{[2]^2\,q^{-1}}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\ |3\,\mu_1-\alpha_1\rangle\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle - \frac{[2]^2\,q}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-\alpha_1\rangle -$$
$$\frac{[2]^3}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle + \frac{\sqrt{[3]}\ [2]\ q^{1/2}}{[4]\ \sqrt{[5]}}\ |3\,\mu_1-\alpha_1-\alpha_2\rangle\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle +$$
$$\frac{\sqrt{[3]}\ [2]\ q^{-1/2}}{[4]\ \sqrt{[5]}}\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-\alpha_1-\alpha_2\rangle + \frac{\sqrt{[3]}\ [2]\ q^{3/2}}{[4]\ \sqrt{[5]}}\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1\rangle + \frac{\sqrt{[3]}\ [2]\ q^{-3/2}}{[4]\ \sqrt{[5]}}\ |3\,\mu_1-2\,\alpha_1\rangle\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

$$|2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{red}\rangle \equiv \frac{q^{-1}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(q\ [3]-[2]^2\right)\ |3\,\mu_1-\alpha_1\rangle\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle +$$
$$\frac{q\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(q^{-1}\ [3]-[2]^2\right)\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-\alpha_1\rangle - \frac{q^{-1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2+q\ [2]^2-[3]-q\,[3]\right)\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle +$$
$$\frac{[2]^{-1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(-[2]^2+[3]\ (q+[3])\right)\ |3\,\mu_1-\alpha_1-\alpha_2\rangle\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle + \frac{[2]^{-1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(-[2]^2+[3]\ \left(q^{-1}+[3]\right)\right)\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-\alpha_1-\alpha_2\rangle -$$
$$\frac{q^{1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2-q\ [3]\right)\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1\rangle - \frac{q^{-1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2-q^{-1}\ [3]\right)\ |3\,\mu_1-2\,\alpha_1\rangle\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

$$|2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{green}\rangle \equiv \frac{\left(-q^{-1/2}\,[2]+[3]\right)}{q^{1/2}\,[4]\ \sqrt{[5]}}\,[2]\,[3]^{1/2}\ |3\,\mu_1-\alpha_1\rangle\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle +$$
$$\frac{\left(-q^{1/2}\,[2]+[3]\right)}{q^{-1/2}\,[4]\ \sqrt{[5]}}\,[2]\,[3]^{1/2}\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-\alpha_1\rangle + \frac{\left(-q^{1/2}\,[2]^3+(1+q)\,[3]\right)}{q^{1/2}\,[3]^{1/2}\,[4]\ \sqrt{[5]}}\,[2]^2\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle +$$
$$\frac{\left(-q^{-1/2}\,[2]+[3]\right)}{q^{-1/2}\,[4]\ \sqrt{[5]}}\,[2]\,[3]^{1/2}\ |3\,\mu_1-\alpha_1-\alpha_2\rangle\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle + \frac{\left(-q^{1/2}\,[2]+[3]\right)}{q^{1/2}\,[4]\ \sqrt{[5]}}\,[2]\,[3]^{1/2}\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-\alpha_1-\alpha_2\rangle +$$
$$\frac{\left(-q^{-1/2}\,[2]^3+\left(1+q^{-1}\right)\,[3]\right)}{q^{-3/2}\,[3]^{1/2}\,[4]\ \sqrt{[5]}}\,[2]\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1\rangle + \frac{\left(-q^{1/2}\,[2]^3+(1+q)\,[3]\right)}{q^{3/2}\,[3]^{1/2}\,[4]\ \sqrt{[5]}}\,[2]\ |3\,\mu_1-2\,\alpha_1\rangle\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

属于中心的3重数态  **表示（经过格拉姆-施密特正交归一化），可从中提取 q-CG 系数**

$$|2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{black}\rangle = -\frac{[2]^2\,q^{-1}}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\ |3\,\mu_1-\alpha_1\rangle\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle -$$
$$\frac{[2]^2\,q}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-\alpha_1\rangle - \frac{[2]^3}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle + \frac{\sqrt{[3]}\ [2]\ q^{1/2}}{[4]\ \sqrt{[5]}}\ |3\,\mu_1-\alpha_1-\alpha_2\rangle\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle +$$
$$\frac{\sqrt{[3]}\ [2]\ q^{-1/2}}{[4]\ \sqrt{[5]}}\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-\alpha_1-\alpha_2\rangle + \frac{\sqrt{[3]}\ [2]\ q^{3/2}}{[4]\ \sqrt{[5]}}\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1\rangle + \frac{\sqrt{[3]}\ [2]\ q^{-3/2}}{[4]\ \sqrt{[5]}}\ |3\,\mu_1-2\,\alpha_1\rangle\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle$$

$$|2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{red}\rangle_\perp \equiv \frac{1}{\sqrt{1-P_{br}^2}}\Big[\left(\frac{q^{-1}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(q\ [3]-[2]^2\right)+P_{br}\,\frac{[2]^2\,q^{-1}}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\right)\ |3\,\mu_1-\alpha_1\rangle\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle +$$
$$\left(\frac{q\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(q^{-1}\,[3]-[2]^2\right)+P_{br}\,\frac{[2]^2\,q}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\right)\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-\alpha_1\rangle -$$
$$\left(\frac{q^{-1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2+q\,[2]^2-[3]-q\,[3]\right)-P_{br}\,\frac{[2]^3}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\right)\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle +$$
$$\left(\frac{[2]^{-1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(-[2]^2+[3]\ (q+[3])\right)-P_{br}\,\frac{\sqrt{[3]}\ [2]\ q^{1/2}}{[4]\ \sqrt{[5]}}\right)\ |3\,\mu_1-\alpha_1-\alpha_2\rangle\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle +$$
$$\left(\frac{[2]^{-1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(-[2]^2+[3]\ \left(q^{-1}+[3]\right)\right)-P_{br}\,\frac{\sqrt{[3]}\ [2]\ q^{-1/2}}{[4]\ \sqrt{[5]}}\right)\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-\alpha_1-\alpha_2\rangle -$$
$$\left(\frac{q^{1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2-q\ [3]\right)+P_{br}\,\frac{\sqrt{[3]}\ [2]\ q^{3/2}}{[4]\ \sqrt{[5]}}\right)\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1\rangle -$$
$$\left(\frac{q^{-1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2-q^{-1}\,[3]\right)+P_{br}\,\frac{\sqrt{[3]}\ [2]\ q^{-3/2}}{[4]\ \sqrt{[5]}}\right)\ |3\,\mu_1-2\,\alpha_1\rangle\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\Big]$$

其中 $P_{br} = \langle 2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2$，黑色　$2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2$，红色$\rangle$

$$|2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2,\ \text{green}\rangle_\perp \equiv \frac{1}{\sqrt{1-P_{bg}^2-P_{rg}^2}}\Big[$$
$$\left(\frac{\left(-q^{-1/2}\,[2]+[3]\right)}{q^{1/2}\,[4]\ \sqrt{[5]}}\,[2]\,[3]^{1/2}+P_{bg}\,\frac{[2]^2\,q^{-1}}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}-\frac{P_{rg}}{\sqrt{1-P_{br}^2}}\left(\frac{q^{-1}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(q\ [3]-[2]^2\right)+P_{br}\,\frac{[2]^2\,q^{-1}}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\right)\right)\ |3\,\mu_1-\alpha_1\rangle\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle +$$
$$\left(\frac{\left(-q^{1/2}\,[2]+[3]\right)}{q^{-1/2}\,[4]\ \sqrt{[5]}}\,[2]\,[3]^{1/2}+P_{bg}\,\frac{[2]^2\,q}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}-\frac{P_{rg}}{\sqrt{1-P_{br}^2}}\left(\frac{q\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(q^{-1}\,[3]-[2]^2\right)+P_{br}\,\frac{[2]^2\,q}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\right)\right)\ |3\,\mu_1-3\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-\alpha_1\rangle +$$
$$\left(\frac{\left(-q^{1/2}\,[2]^3+(1+q)\,[3]\right)}{q^{1/2}\,[3]^{1/2}\,[4]\ \sqrt{[5]}}\,[2]^2+P_{bg}\,\frac{[2]^3}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}+\frac{P_{rg}}{\sqrt{1-P_{br}^2}}\left(\frac{q^{-1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2+q\,[2]^2-[3]-q\,[3]\right)-P_{br}\,\frac{[2]^3}{[4]\ \sqrt{[3]}\ \sqrt{[5]}}\right)\right)$$
$$|3\,\mu_1-2\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1-\alpha_2\rangle + \left(\frac{\left(-q^{-1/2}\,[2]+[3]\right)}{q^{-1/2}\,[4]\ \sqrt{[5]}}\,[2]\,[3]^{1/2}-\right.$$
$$P_{bg}\,\frac{\sqrt{[3]}\ [2]\ q^{1/2}}{[4]\ \sqrt{[5]}}-\frac{P_{rg}}{\sqrt{1-P_{br}^2}}\left(\frac{[2]^{-1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(-[2]^2+[3]\ (q+[3])\right)-P_{br}\,\frac{\sqrt{[3]}\ [2]\ q^{1/2}}{[4]\ \sqrt{[5]}}\right)\Big)\ |3\,\mu_1-\alpha_1-\alpha_2\rangle\ |3\,\mu_1-3\,\alpha_1-\alpha_2\rangle +$$
$$\left(\frac{\left(-q^{1/2}\,[2]+[3]\right)}{q^{1/2}\,[4]\ \sqrt{[5]}}\,[2]\,[3]^{1/2}-P_{bg}\,\frac{\sqrt{[3]}\ [2]\ q^{-1/2}}{[4]\ \sqrt{[5]}}-\frac{P_{rg}}{\sqrt{1-P_{br}^2}}\left(\frac{[2]^{-1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left(-[2]^2+[3]\ \left(q^{-1}+[3]\right)\right)-P_{br}\,\frac{\sqrt{[3]}\ [2]\ q^{-1/2}}{[4]\ \sqrt{[5]}}\right)\right)$$
$$|3\,\mu_1-3\,\alpha_1-\alpha_2\rangle\ |3\,\mu_1-\alpha_1-\alpha_2\rangle + \left(\frac{\left(-q^{-1/2}\,[2]^3+\left(1+q^{-1}\right)\,[3]\right)}{q^{-3/2}\,[3]^{1/2}\,[4]\ \sqrt{[5]}}\,[2]-P_{bg}\,\frac{\sqrt{[3]}\ [2]\ q^{3/2}}{[4]\ \sqrt{[5]}}\right.$$
$$+\frac{P_{rg}}{\sqrt{1-P_{br}^2}}\left(\frac{q^{1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2-q\ [3]\right)+P_{br}\,\frac{\sqrt{[3]}\ [2]\ q^{3/2}}{[4]\ \sqrt{[5]}}\right)\Big)\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\ |3\,\mu_1-2\,\alpha_1\rangle + \left(\frac{\left(-q^{1/2}\,[2]^3+(1+q)\,[3]\right)}{q^{3/2}\,[3]^{1/2}\,[4]\ \sqrt{[5]}}\,[2]-\right.$$
$$P_{bg}\,\frac{\sqrt{[3]}\ [2]\ q^{-3/2}}{[4]\ \sqrt{[5]}}+\frac{P_{rg}}{\sqrt{1-P_{br}^2}}\left(\frac{q^{-1/2}\,[2]^{1/2}}{[3]\ \sqrt{[4]}\ \sqrt{[5]}}\ \left([2]^2-q^{-1}\,[3]\right)+P_{br}\,\frac{\sqrt{[3]}\ [2]\ q^{-3/2}}{[4]\ \sqrt{[5]}}\right)\Big)\ |3\,\mu_1-2\,\alpha_1\rangle\ |3\,\mu_1-2\,\alpha_1-2\,\alpha_2\rangle\Big]$$

其中 $P_{bg} = \langle 2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2$、黑色　$2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2$、绿色$\rangle$和
$P_{rg} = \langle 2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2$、绿色　$2\,\mu_1+2\,\mu_2-2\,\alpha_1-2\,\alpha_2$、红色$\rangle_\perp$

参考文献

[1] L. C. Biedenharn 和 J. D. Louck。《量子物理中的角动量》。剑桥大学出版社，2009年。[2] L. C. Biedenharn 和 J. D. Louck。《量子理论中的 Racah-Wigner代数》。剑桥大学出版社，1984年。[3] Kulish, P.P.、Reshetikhin, N.Y.。用于正弦-戈登方程的量子线性问题与高阶表示。《数学科学杂志》23，2435–2441（1983年）。[4] Drinfel'd, V.G.。量子群。《数学科学杂志》41，898–915（1988年）[5] Alekseev, A.、Shatashvili, S.。量子群与WZNW模型。《数学物理通讯》133，353–368，（1990年）[6] Yasuhiro Akutsu 和 Miki Wadati，《统计力学中的结、链环、辫子与精确可解模型》，《数学物理通讯》117（2）：2432（1988年）。[7] Romesh K Kaul。陈-西蒙斯理论、结不变量、顶点模型与三维流形不变量。arXiv:hep-th/9804122，1998年。[8] Saswati Dhara、Romesh K. Kaul、P. Ramadevi 和 Vivek Kumar Singh。二部顶点模型与多色链环不变量。arXiv:1811。07538 [hep-th], 2018。[9] Dennis Bonatsos 和 C Daskaloyannis。量子群及其在核物理中的应用。《粒子与核物理进展》43（1999年）537-618，arXiv:nucl-th/9909003。[10] Raychev, P.《量子群：在核与分子光谱学中的应用》。《量子化学进展》第26卷，第239-357页，（1995年）。[11] A N Kirillov 和 N Yu Reshetikhin。代数 U_q（sl(2)）的表示、q-正交多项式与链环不变量。载于《结理论的新进展》，Toshitake Kohno 编，第202–256页。世界科学出版社，1990年。[12] Masao Nomura， su_q(2)中的克莱布施–戈丹系数与Racah系数的关系及杨–巴克斯特方程。《数学物理杂志》30，2397（1989年）[13] E. Ardonne、J. Slingerland，《秩2量子群的克莱布施–戈丹系数与6j系数》，《物理学杂志 A：数学与理论》43 395205，（2010年）[14] Atakishiev, N.M.、Suslov, S.K.。量子代数 su_q(2)的克莱布施-戈丹系数。《理论与数学物理》98，1–7（1994年）。[15] Howard Georgi。《粒子物理学中的李代数：从同位旋到统一理论》。CRC出版社，2018年。