Hans Riess

多智能体系统中的格理论

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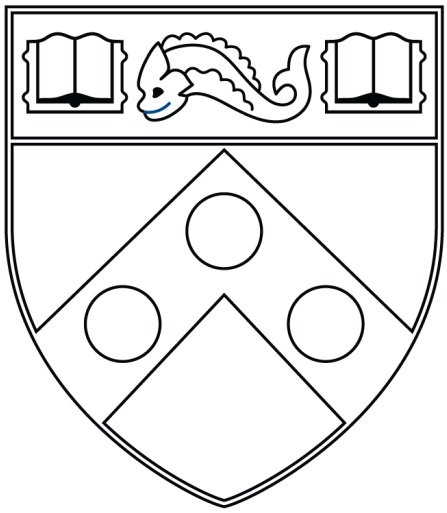

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Hans Riess：多智能体系统中的格论，塔斯基拉普拉斯算子及其应用，
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献给我的爸爸，他会在
餐巾纸上和飞机上教我代数

摘要

SIn this thesis, we argue that (order-) lattice-based multi-agent information systems constitute a broad class of networked multi-agent systems in which relational data is passed between nodes. Mathematically modeled as lattice-valued sheaves, we initiate a discrete Hodge theory with a Laplace operator, analogous to the graph Laplacian and the graph connection Laplacian, acting on assignments of data to the nodes of a Tarski sheaf. The Hodge-Tarski theorem (the main theorem) relates the fixed point theory of this operator, called the Tarski Laplacian in deference to the Tarski Fixed Point Theorem, to the global sections (consistent global states) of the sheaf. We present novel applications to signal processing and multi-agent semantics and supply a plethora of examples throughout.

致谢

当我循环播放 Plini 的“ElectricSunrise”时，我想首先感谢我的导师 RobGhrist，感谢您的无私坚忍、悉心指导，以及前卫却又富于思辨的音乐品味。

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第一部分

数学背景

1

引言

在本论文中，我们从格论的角度提出了一种理解多智能体系统的新方法，这与依赖谱图理论的现代方法形成了鲜明对比 [36],动力系统 [62], 和控制理论 [34]。

1.1 多智能体系统

A system is something so fundamental it is difficult to define. While we will even not attempt to define systems here, we will define multi-agent systems as a special class of systems consisting of many interacting components called *agents*. Agents are differentiated from ordinary components of systems such as the environment because agents are typically assumed to possess some degree of autonomy, intelligence, or communication capabilities, likely all three. Consider just a few examples of multi-agent systems.

1. 无线通信。通信系统本质上是多智能体系统。智能体发送/接收消息。更复杂的多智能体系统通常依赖通信子系统，以便从系统中的其他智能体获取信息。

2. *Swarm robotics*. Multiple (usually simple) robots (e.g. quadrotor UAVs) collaborate to perform tasks such as surveillance, payload transportation, and more [1]. Swarms, it is widely held, outperforms smaller groups of more sophisticated robots due to advantages of scalability and robustness.

3.传感器网络。多个配备一个或多个传感器的智能体，收集系统内外环境或其他智能体的信息。应用包括可穿戴设备、威胁检测、工业监测和环境状况。

4. 企业。企业与其他企业直接竞争，通过价格以及串谋、收购或合并等其他方式交换信息。

在多智能体系统中，各种异构的信息流可能编码在不同类型的数据结构中，并在系统中的各个智能体之间进行收集、处理和交换。请考虑以下由多智能体系统执行的信息处理任务示例。

1. 共识。 共识是一类广泛的信息处理过程，旨在使每个智能体最终就某个特定状态或感兴趣的量达成一致，而该状态或量取决于所有智能体的状态。例如，在群集机器人系统中包括群集 [157]和分布式编队控制 [54]，同步，以及空间中的会合 [40]和时间 [122]。共识算法，也称为共识协议，是智能体之间的交互规则，规定给定智能体与网络中其所有邻居之间的信息交换 [124]。

2. *Collaborative Filtering*. Collaborative filtering is a general technique of filtering information from multiple sources facilitated by "interactions" between agents. The most popular use-case of collaborative filtering is recommendation systems which take into account both the preferences of an individual agent as well as similar agents, neighbors, in a graph. To this end, graph signal processing [126] has shown to be an effective tool for rating prediction [84].

3. *Information Fusion*. Information fusion, also known as *data fusion*, is the task of integrating data from multiple sources in heterogeneous formats. In one view [91], data fusion is the study of the transformation of various sources of information into a representation that can be interpreted by a machine or human.

在上述每一项信息处理任务中，局部信息会被聚合，以获得全局信息。层论[24, 42], 在我们看来，是研究此类数据关系的恰当语境。层是为诸如局部数据是否一致？是否确实存在全局一致的数据赋值？之类的软问题提供数学严谨性的数据结构。在受农业启发的命名法中，层是将茎（例如向量空间、集合范畴、格等）固定到基底（例如图、超图、单纯复形、流形）上的结构。网络层（定义5.1）是一个层 $\mathcal{F}$ 基于无向图$\mathsf{G} = (\mathcal{V}_G, \mathcal{E}_G)$，由节点和边上的对象（例如向量空间）数据指定，这些对象称为茎；并由保持对象结构的映射（例如线性变换）指定，这些映射称为限制映射。如果i 和j 是节点，且 ij是一条边，则茎表示为 $\mathcal{F}(\mathrm{i})$、 $\mathcal{F}(\mathrm{j})$ 和 $\mathcal{F}(\mathrm{ij})$，限制映射表示为 $\mathcal{F}(\mathrm{i} \trianglelefteq \mathrm{ij}) : \mathcal{F}(\mathrm{i}) \to \mathcal{F}(\mathrm{ij})$。

层有助于识别称为整体截面（定义 5。3）。我们论证整体截面

$$\Gamma(G;\underline{\mathcal{F}}) = \Big\{ \underbrace{\mathbf{x} \in \prod_{i \in \mathcal{V}_G}}_{\text{assignments of data}} \; | \underbrace{\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) = \underline{\mathcal{F}}(j \trianglelefteq ij)(x_j)}_{\text{agreement along restriction maps}} \quad \forall ij \in \mathcal{E}_G \Big\}$$

是共识的自然延伸或推广。[1] 在第9章中，我们发现了多智能体认识逻辑中的一种共识概念，可以称之为语义共识。

1.2 格理论

Lattices are a class of ordered sets with two "merging" operations called *meet* and *join*. As ordered sets, finite lattices can be visualized with *Hasse diagrams*, directed acyclic graphs (DAGs). The nodes of a Hasse diagram are the elements of the ordered set. If $x \preccurlyeq y$, and there is no other element between x and y, we draw a directed edge from x to y. Orders are elementary, but lattices are less known. There may be a historical explanation.

格论 [18, 45, 70, 140] 是数学中一个多面向的领域，历史悠久 [17]。然而，作为一个领域，格论始终未能完全成熟，或许这并不公平 [142]:

> Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperations as lattice theory. Dedekind, Jonsson, Kurosh, Malcev, Ore, von Neumann, Tarski, and most prominently Garrett Birkhoff have contributed a new vision of mathematics, a vision that has been cursed by a conjunction of misunderstanding, resentment, and raw prejudice.

Lattices, as a mathematical concept, is a relaxation of a boolean algebra [21]. To describe these structures, it was Klein who originally coined a term, *verband*, which loosely translates from German as "association." However, Garret Birkhoff, popularized the English nomenclature [17], "lattice," due to the fact (with some speculation) that the Hasse diagrams of lattices often resemble lattices of the garden variety (pardon the pun!).

We recall this obscure history to highlight the fact that "lattice" has an unfortunate ambiguous meaning. To a physicist, a lattice is (likely) a discreet subset of $\mathbb{R}^d$ (Figure 1.1). To a network-scientist [162], a lattice is (likely) an undirected graph whose nodes uniformly follow a grid pattern and whose edges are drawn according to (unique!) nearest neighbors. We call these objects *integer lattice* or *lattice graphs* to differentiate them from an *order lattices*.

因此，作为一门学科，格论的身份确实存在矛盾。一方面，格论源于对布尔逻辑公理的弱化（例如，不再要求否定和分配律），因而属于逻辑。也可以认为，格论属于组合数学 [141]。在我们看来，格论在建立上述领域之间的连接的同时，更 [142], 属于代数，即对符号操作的一般性研究。首先，

1 如果每个限制映射都是恒等映射，那么我们就能准确地恢复共识。

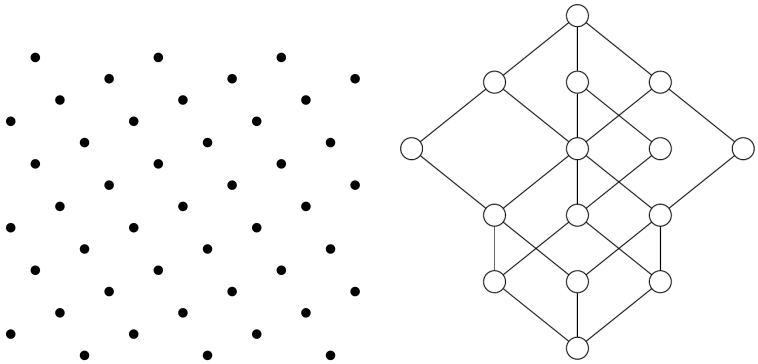


图 1.1：（左）整数格；（右）Order 格。

格是一种泛代数 [144]，粗略地说，集合范畴配备了一组自然数集元运算，并满足等式恒等式。强调格论的代数方面的另一个有力理由是，群（一个对称变换的集合）的子群格或许是格论中最早研究的例子。作用于自然数集个元素上的置换构成一个群。你可以通过将 j 映射到 i，而不是将 i 映射到 j，来逆转置换。此外，你可以重复应用置换，而且括号放在哪里并不重要。4 个元素上的置换群这一看似无害的结构，却具有相当复杂的子群格（图1。2）。即使在这样一个简单的例子中，子结构之间也存在非平凡的关系。

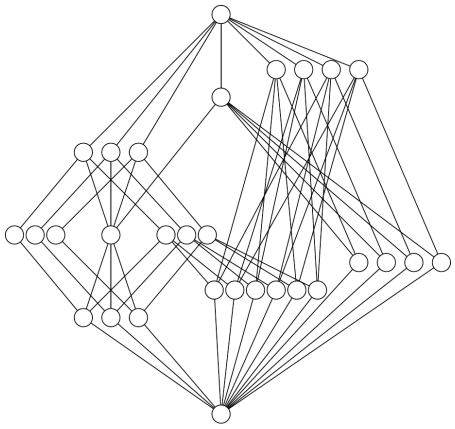


图1.2：S_4 子群格的哈斯图。

x

格既可作为有序集，也可作为代数结构。假设 x、y $\in \mathbf{L}$，一个带有 $\preccurlyeq$，那么 $x \vee y$（并）是最小上界（如果它存在），而 $x \wedge y$（交）是最大下界（同样地，它存在）。一方面，格是这样一种有序集：其最小上界和最大下界都保证存在（定义 2。6）。另一方面，格是一个带有二元运算 $\vee$ 和 $\wedge$ 并满足公理的集合范畴（定义2。5）。格的基本例子包括以下内容。

例 1.1(格)。

1. 任意集合的子集，运算为并集和交集。或者，带有包含关系的子集。2. 群（向量空间）的子群（子空间），运算为积（和）与交集（图1。2）。或者，带有包含关系的子群（子空间）。3. 任意集合的划分，在细化关系下具有最粗公共细化。或者，带有细化关系的划分。4. 真值 $\mathbf{B} = \{0, 1\}$ 带有 或/ 与。或者，$\mathbf{B}$ 带有 Order $0 < 1$。

Other quite general examples are embedded in other lattices such as powersets and partitions, including subpartitions (Example 2.14), and lattices characterizing formal concepts [165] and fuzzy logic [16]. Less known is a "lattice theory of information" in which the information content (relative entropy) of a random variable establishes a lattice order. Joins and meets have interesting interpretations (Example 2.15).

This reoccurring duality between order and algebraicity inspires a new outlook on multi-agent systems. Suppose a system consists of $\{1, 2, \ldots, n\}$ agents whose communication patterns are modeled by a graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ with $\mathcal{V}_G = \{1, 2, \ldots, n\}$. Data collected in a lattice by individual agents models relational information about the system. On the other hand, because of the duality, this data is amalgamated with data stored by neighboring agents via the binary operations, meet and join.

As a point of clarification, it is not our goal to develop a general theory of multi-agent systems, but to make a case that certain mathematical tools are fundamental in modeling multi-agent information systems. To this end, it is our hope that the theory of lattice-valued sheaves (Chapter 6) and sheaf Laplacians (Chapter 7) posited in this manuscript inspire novel information and control systems.

1.3 贡献概述

该塔斯基拉普拉斯算子是关键的构造和贡献。塔斯基拉普拉斯算子最初被提出 [64]，旨在建立霍奇理论，从而定义取值于格中的胞腔层的上同调。定义上同调的困难在于，由 Shepard 发起的胞腔层上链复形构造 [152] 解体了。长话短说，交和并缺乏逆元。尽管如此，在同调代数 [69]中，仍然可以进行一些关键的计算和理论构造。诸如单纯复形、拓扑空间或群等数学对象的上同调是

通常有助于对某个结构的性质进行分类。我们预计，格值层的上同调会与其所建模的信息系统的行为相关。在胞腔层理论中[42],层上同调 H^0 对整体截面进行分类。通过塔斯基拉普拉斯算子，整体截面被识别为一个不动点格（霍奇-塔斯基定理，定理7.1）。

Homology of chain complexes in semiexact categories (the category of lattices and Galois connections is semiexact) as well as exact sequences and the snake lemma were established Marco Grandis [69]. We suspect a proper theory of sheaf cohomology of $\mathcal{S}up$-valued sheaves would require either a modified notion of projective or injective resolutions of sheaves [24], and, thus, a notion of projective or injective (complete) lattices. With the program to develop a cohomology theory for lattice-valued sheaves supplanted by novel applications of the Tarski Laplacian, our efforts are contained in the following manuscript [64].

It has been suggested that sheaf theory simultaneously straddles both the axes of algebra/geometry and structure/obstruction [61]. Lattice-valued sheaf theory, we place in the second quadrant of this helpful philosophical model (Figure 1.3). While lattices are pure algebraic structures, in Chapter 7, we introduce geometric notations such as Laplace operators and even parallel transport.

从范畴论的角度来看，定义在（关联）偏序集上的格值层是 [8]取值于阿贝尔范畴的层的一种范畴化。例如，阿贝尔群被子群格（范畴）取代，而同态被伽罗瓦连接（伴随）取代。将经典层论提升到格论这一巧合并非毫无代价。当一个从偏序集到完备格与伽罗瓦连接范畴的函子，经过将对象映射到其（正规）子对象格的函子进行分解时，我们通常无法从格值层中恢复阿贝尔层的同调 [64, 第4节3]。

塔斯基拉普拉斯算子是一个局部算子，作用于赋给格层的数据。塔斯基拉普拉斯算子之所以被称为“拉普拉斯算子”，有多种原因。首先，霍奇-塔斯基定理将格值层的整体截面与不动点对应起来。我们基于近期提出的网络层局部截面之间的平行移动概念，建立了与图联络拉普拉斯算子的深刻联系 [153] 。我们希望塔斯基拉普拉斯算子能够建立与信息几何 [103] 和表示论 [95]的联系 [20]。

以希尔伯特空间为系数的网络层上的动力系统（例如 $\mathbb{R}^d$ 或 $L_2(\mathbb{R}^d)$）由 Hansen [72, 75, 78] 开创。我们引入一个动力系统。

$$\mathbf{x}[t+1] = \qquad\qquad L\mathbf{x}[t] \wedge \mathbf{x}[t]$$

称为 热方程 其时变版本称为 *Gossip* 方程 (3)。除了扩散动力学之外，鞍点动力学以及基于简单聚合规则的其他局部动力学也值得关注。在常值层近似这一特例中（定义7.2），共识算法是由塔斯基拉普拉斯算子驱动的扩散动力学的副产物。

我们在第 8 章的大部分篇幅中比较格信号处理的各种方法，并提出我们自己的方法。我们将 Püschel 定义的卷积 [132] 视为一种代数信号模型，并描述了两种对应于格上信号的代数信号模型之间的关系。在最后一章中，我们讨论了一类网络模型检验问题，并介绍塔斯基拉普拉斯算子在语义中的应用，提出受认识逻辑启发的新型知识扩散/共识模型 [52]。

1.4 相关工作

我们不打算全面回顾文献，而是聚焦于少数相关主题。

1.4.1 量子逻辑

在物理学中，Birkhoff 和 von Neumann 提出将格论作为量子力学的逻辑模型 [19]。他们的工作基于以下观察：假设 H 是一个希尔伯特空间，那么 H 的闭子空间集合在子空间和的闭包（并）与交集（交）运算下构成一个（正交补）格。这些子空间与投影一一对应，von Neumann 据此推断，投影可以视为量子可观测量[161]。然而，由于贝尔定理后来被实验结果证伪 [6]，他们的方法未能经受时间的考验。因此，如今幺半群范畴/弦图 [37] 以及线性逻辑 [66], ——量子信息论的逻辑——更为流行。

1.4.2 超模博弈

经济学家将格论应用于一类特殊的n人博弈（多智能体系统）。格上的实值函数称为超模，如果

$$f(x \vee y) + f(x \wedge y) \geqslant f(x) + f(y) \quad \forall x, y \in \mathbf{L}.$$

那么，一个超模博弈由参与者$i \in \{1, 2, ...、n\}$组成，每位参与者都有一个可能策略构成的格$\mathbf{L}_i$，以及一个效用函数

$$u_i(x_i, \mathbf{x}_{-i}) : \prod_{i=1}^{n} \mathbf{L}_i \to \mathbb{R},$$
$$x_i \in \mathbf{L}_i, \quad \mathbf{x}_{-i} \in \prod_{j \neq i} \mathbf{L}_i \quad i \in \{1, 2, \ldots, n\}$$

such that $u_i(-, \mathbf{x}_{-i})$ is supermodular and u_i satisfies an additional property of *increasing differences*. Then, every n-person supermodular game has a Nash equilibrium [159] (see also [160, Chapter 7]). It was later shown by Zhou that if each strategy lattice is complete, the set of Nash equilibrium forms a complete lattice [172]. While pertinent to problems in economics such as Bertrand competition [50], supermodular games have found practical use in wireless communication networks, with applications to energy-efficient power allocation [105], power control [4], and interference compensation [83]. Submodular functions on a lattice, real-valued functions satisfying

$$f(x \vee y) + f(x \wedge y) \leqslant \qquad f(x) + f(y) \quad \forall x, y \in \mathbf{L},$$

and algorithms to minimize them were introduced by Topkis [159]. Unsupervised learning problems such as principal component analysis (PCA) and generalized PCA, have been reformulated as constrained submodular maximization problems on the lattice of spaces of $\mathbb{R}^d$ [119], as well as meta-learning [3].

1.4.3 离散事件系统

In discrete event systems [31], an area of control theory, Galois connections model the triggering of events. A discrete event system is a transition system (Chapter 9) with discrete states and transitions triggered by events. Linear systems [34] and their properties (e.g. controlability, observability), are well-understood. In general, discrete event systems are less tractable, however, in the past two decades, much headway has been made by representing discrete-event systems with square matrices with elements in the following lattice.

令 $\mathbb{R}_{\max}$ 表示扩展实数 $\mathbb{R} \cup \{-\infty, \infty\}$。$\mathbb{R}_{\max}$ 在我们记作$\vee$ 和 $\wedge$的最大值与最小值运算下构成一个（完备）格。$\mathbb{R}_{\max} \setminus \{-\infty, \infty\}$ 在加法下具有群的结构（定义 2。8）。在普通欧几里得空间中，将“乘法”替换为“加法”，并将“乘法”替换为“最大值”或“最小值”，即可得到一个极大加向量空间 $\mathbb{R}_{\max}^m$，它是一个 加权格 [110]。如同普通线性代数中一样，矩阵乘法定义了一个变换。

tion. 如果$A \in \mathbb{R}_{\max}^{m\times m}$是一个矩阵，且$\mathbf{x} \in \mathbb{R}_{\max}^{m}$ 是一个向量，那么矩阵乘法有两种对偶定义方式。

$$(A \otimes \mathbf{x})_i = \bigvee_{i=1}^{m} a_{ij} + x_j,$$

$$(A \oslash \mathbf{x})_i = \bigwedge_{i=1}^{m} a_{ij} + x_j.$$

举例来说，考虑以下离散事件系统。假设一组 $\{1,2,...m\}$ 个事件在时间 $x_i[t], i \in \{1,2,...m\}$ 轮后开始。假设事件 j 只能在另一个事件 j 结束后才能开始（例如，衬衫烘干后再熨烫）。那么，我们称 i 从 j 转移。考虑系统设计时，等待时间不仅取决于正在进行的事件i ，还取决于目标事件j。令a^{ij} 表示i 开始与j随后开始之间的时间间隔。若j 不依赖于过程i，则令$a^{ij} = -\infty$。于是，

$$x_i[t+1] = \bigvee_{j=1}^{m} x_i[t] + a_{ij} \quad i = 1,2,\ldots,m$$

是事件i的新开始时间。令$A \in \mathbb{R}_{\max}^{m}$ 为元素为a^{ij}的矩阵，则全局动态可写成以下形式

$$\mathbf{x}[t+1] = A \otimes \mathbf{x}[t].$$

假设两个相互作用的离散事件系统中的事件都来自同一集合 $\{1,2,...n\}$。若$\mathbf{x}[t], \mathbf{y}[t] \in \mathbb{R}_{\max}^{m}$ 分别是系统 1 和系统 2 经过t 轮后的事件时间向量，且A、$B \in \mathbb{R}^{m\times m}$ 是转移时间矩阵，那么同步问题 询问是否可以选择$(\mathbf{x}[0],\mathbf{y}[0])$ ，使得存在一个t_0，令$\mathbf{x}[t] = \mathbf{y}[t]$ 对所有$t \geqslant t_0$都成立。显然，这等价于求解系统

$$A \otimes \mathbf{x} = B \otimes \mathbf{y}$$

对于 $(\mathbf{x},\mathbf{y}) \in \mathbb{R}_{\max}^{m} \times \mathbb{R}_{\max}^{m}$。

Cunninghame-Green 和 Butkovic 提出了一种同步问题的解决方法，称为交替法 [41]。假设 $A^{\dagger} \in \mathbb{R}_{\max}^{m\times m}$的元素为 $[a_{ij}^{\dagger}] = -a^{ji}$。那么，映射

$$\begin{aligned} \mathbf{x} &\mapsto A \otimes \mathbf{x}, \\ A^{*} \oslash \mathbf{y} &\leftarrow\!\shortmid \mathbf{y} \end{aligned}$$

形成伽罗瓦连接（示例 3。1）。交替方法的每次迭代都等价于应用塔斯基拉普拉斯算子（第7章）

$$L(\mathbf{x},\mathbf{y}) = \left(A^{\dagger} \oslash (B \otimes \mathbf{y}), B^{\dagger} \oslash (A \otimes \mathbf{x})\right) \tag{1.1}$$

在图 $\bullet - \bullet$（图）的一个特定“层”上
同步等价于成为该层的一个截面。因此，根据霍奇-塔斯基定理（定理 7.1），同步也等价于条件

$$A^{\dagger} \oslash (B \otimes \mathbf{y}) \succcurlyeq \mathbf{x}$$
$$B^{\dagger} \oslash (A \otimes \mathbf{x}) \succcurlyeq \mathbf{y}$$

对于所有 $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^m_{\max} \times \mathbb{R}^m_{\max}$.

1.4.4 共识

Lattice consensus algorithms in the networked multi-agent setting (Algorithm 1, Algorithm 2, Algorithm 3) are novel, but consensus on lattices, while obscure, broadly construed, is not. While we have discussed several notions of consensus in multi-agent systems, consensus has other connotations in both computer science as well as economics.

Reaching agreement among remote processes (agents) is a fundamental problem in distributed computing. In a consensus protocol, each process $i \in \{1, 2, \ldots, n\}$ receives or seeds an input register x_i, sends and receives messages to other processes, and, eventually, must determine an output register y_i coinciding with the output registers of all other processes. Unfortunately, it has been shown that, roughly, no asynchronous consensus protocol can tolerate (i.e. not affect the output) a single fault. For instance, if a process is unexpectedly removed from the system [55], this would be considered a *fault*.

Now, suppose input and output registers are collected in a lattice (Definition 2.5). Then, lattice agreement is defined to be a decision $\{y_i\}_{i=1}^n$ such that $\{y_i\}_{i=1}^n$ is a chain and the following inequality holds for all $i \in \{1, 2, \ldots, n\}$

$$\bigwedge_{i=1}^{n} x_i \preccurlyeq \qquad\qquad y_i \preccurlyeq \qquad\qquad \bigvee_{i=1}^{n} x_i.$$

格协议、集中式分支定界算法已被证明在各种设置下都具备容错性 [169–171],与格共识不同。

在社会选择理论中， [5], 共识大致等同于“群体选择” [118]。在最一般的层面上，如果X 是一个有限的备选方案（选项）集合，那么共识规则就是一个映射。

$$\chi : \bigcup_{k>1} X^k \to \wp(X) \setminus \varnothing$$

将元组 $\boldsymbol{\pi}$ （由任意个人选择构成，称为偏好剖面）映射到非空的群体选择集合 $chi(\boldsymbol{\pi})$ [13]。假设我们固定这个数目

个选民（智能体）为 n 并将 X 替换为一个格 $\mathbf{L}$，那么一个 （格论）共识函数[2] [97] 或 聚合函数 [85, 98] 是一个映射

$$\chi : \mathbf{L}^n \to \mathbf{L}.$$

Lattices frequently arise in consensus settings in social choice. For instance, $\mathbf{L}$ could be the lattice of (transitive, reflexive) preference relations on X or a lattice of choice functions [114], maps $c : \wp(X) \to \wp(X)$ such that $c(A) \subseteq A$ for all $A \subseteq X$. In the nomenclature, a consensus function χ is a *meet-projection* if there is a subset $N \subseteq \{1, 2, \ldots, n\}$ such that $\chi(\boldsymbol{\pi}) = \bigwedge_{i \in N} \pi_i$ and, in particular, a *Pareto consensus function* if $N = \{1, 2, \ldots, n\}$.

A third interpretation of lattice consensus comes about in the general science of classification/taxonomy. In evolutionary biology, phylogenetic trees can be inferred from DNA sequences [121]. Having several candidates for the evolutionary tree of a particular organism, consensus is one approach to determining an aggregate phylogenetic tree [26]. In the same spirit, when an ensemble of hierarchical clustering algorithms produces various candidate dendrograms, consensus on the outputs selects a single representative tree [123].

Last, but not least, max/min consensus aims to compute the aggregate max or min of a signal on the nodes of a network. In some settings, maximum and minimum are approximated [156], or algorithms for reaching consensus on general functions on the inputs were proposed [39]. Max consensus was also framed as a max-plus-linear system [122]. Useful applications of min/max consensus and, arguably, meet/join consensus, include decentralized leader election [22] and minimum-time rendezvous [122].

1.4.5 应用层论

Applied sheaf theory likely originated in the 1970s. Sheaves were argued to be an intuitive way of thinking about certain image segmentation problems [10]. In the 1990s, the following audacious claim was made by Goguen, who applied sheaf theory to electrical circuitry [67]:

> The sheaf condition appears to be satisfied by the behaviours of all naturally arising systems from computing science. This "Sheaf Hypothesis" is similar to the Church-Turing thesis, that all intuitively computable functions are computable in the precise sense of Turning machines.

层论作为应用数学的一个域，尚未成熟。一些成功案例包括将层作为整合传感器、滤波器设计、路径优化和路由的工具 [138], [137], [38, 116]。最伟大的

2 如果这样的函数是单调的，并且将顶端（底端）偏好剖面映射到$\mathbf{L}$的顶端（底端）元素，那么它就被称为聚合函数 [23], ，尽管"聚合函数"有时是一般共识函数的同义词。

应用层的胜利可能是拓扑数据分析基础中的一个持续项目 [60]。持久同调的理论 [42, 89, 108] 和Reeb图 [46] 从层论观点中受益匪浅。在大多数情况下，层的数据范畴（即“楼上”的内容）是有限维向量空间的范畴，而空间是图。在本手稿中，我们研究图上的层取值于 $\mathcal{S}up$，完备格的数据范畴（第 6章）。我们简要回顾取值于非阿贝尔范畴的层的相关工作。

In quantum mechanics, sheaves have been used to model non-locality: mysterious interactions between fundamental particles, presumably occurring faster than the speed of light [2]. Contextuality, a version of non-locality, is formulated as a sheaf of events, a sheaf in the data category of sets. Cohomology is undefinable here, hence, obstructions to locality, computed as cohomology in degree one, cannot be explicitly calculated from the sheaf of events. One approach is to extend the sheaf of events to a sheaf of abelian groups factoring through the free functor sending a set to its free abelian group. one disadvantage of this approach is that sections may appear that weren't there before. Alternatively, Čech cohomology was recently defined for sheaves of semimodules [115].

In our opinion, the most significant development in applied sheaf theory is the introduction of sheaf Laplacians [77]. Thus far, sheaf Laplacians have inspired applications in opinion dynamics [78], distributed optimization [75], sheaf learning [76], and graph neural networks [12, 20].

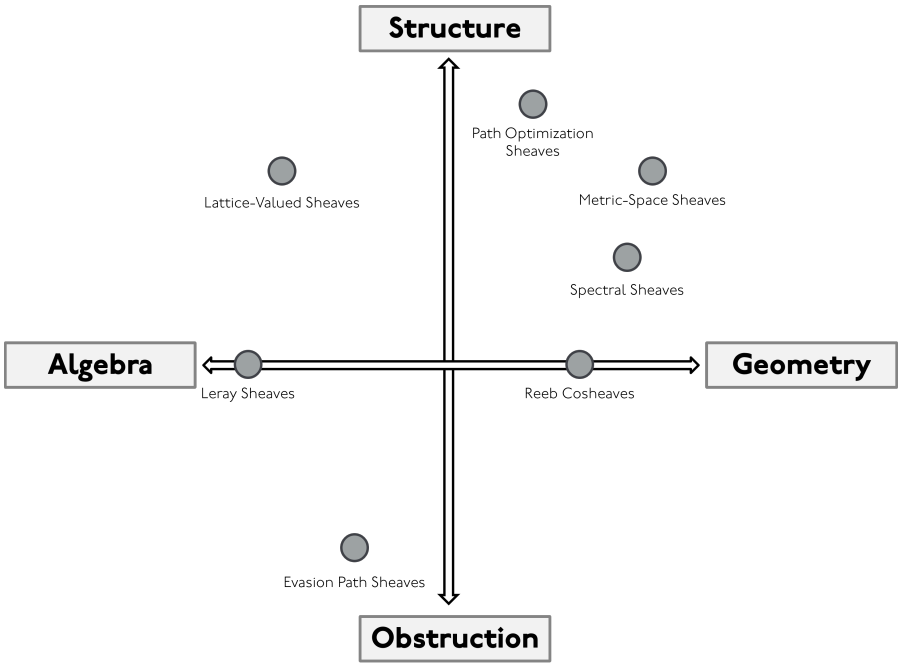


Figure 1.3: A philosophical model of applied sheaf theory: (from left to right) Leray sheaves [43], lattice-valued sheaves [64], evasion path sheaves [63], path optimization sheaves [116], Reeb cosheaves [46], spectral sheaves [77], and (pseudo)-metric space sheaves [139].

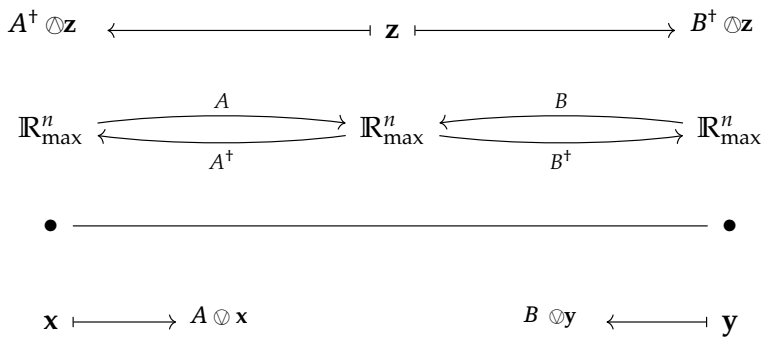


图 1.4：对离散事件系统同步进行建模的 max-plus 层 [41]。

2

格

2.1 从关系到偏序集

Given sets X and Y, a binary relation is simply a subset $\mathcal{R} \subseteq X \times Y$. Sometimes we write $x\mathcal{R}y$ whenever $(x, y) \in \mathcal{R}$. Suppose $\mathcal{R} \subseteq X \times Y$ and $\mathcal{S} \subseteq Y \times Z$. We can compose relations as follows:

$$\mathcal{S} \circ \mathcal{R} = \{(x, z) \mid \exists\, y \in Y \text{ such that } x\mathcal{R}y,\ y\mathcal{S}z\}.$$

We may also take unions $\mathcal{R} \cup \mathcal{R}'$ and intersections $\mathcal{R} \cap \mathcal{R}'$ of relations. Relations have duals. Suppose $\mathcal{R} \in X \times Y$ is a relation. Then $\mathcal{R}^\dagger$ is the relation $\{(y, x) \mid x\mathcal{R}y\}$.

Example 2.1 (Databases)**.** Suppose X is a set of instances and Y is a set of attributes. We set $x\mathcal{R}y$ if object x has attribute y. Most of the time, attributes are not binary labels. One way to accommodate this situation to convert these labels to integers. Then, a multi-valued relation is a map $\mathcal{R} : X \times Y \to \mathbb{N}$. Then, we define a filtration of relations

$$\mathcal{R}_{\leqslant m} = \{(x, y) \mid \mathcal{R}(x, y) \leqslant m\}. \tag{2.1}$$

这一系列关系构成一个滤过，也就是说，当且仅当$m \leqslant m'$时，
$\mathcal{R}_{\leqslant m} \subseteq \mathcal{R}_{\leqslant m'}$。

Motivated by the preceding example, it is convenient to have notation for the set of attributes that is related to a given object. Let $\mathcal{R} \subseteq X \times Y$ be a relation and $\sigma \subseteq X$. Then, the *intent* of σ, denoted $\sigma^\uparrow$, is the set $\{y \mid x\mathcal{R}y\ \forall\ x \in \sigma\}$. Conversely, the *extent* of a $\tau \subseteq Y$, denoted $\tau^\downarrow$, is the set $\{x \mid x\mathcal{R}y\ \forall\ y \in \tau\}$.

Whenever $X = Y$, we say $\mathcal{R}$ is an *endorelation* (or simply a relation from context). We denote the set of endorelations on X with $\operatorname{Rel} X$. Endorelations are the same as directed graphs. For an $x \in X$, the intent $x^\uparrow$ is sometimes called the *reachability set* or *children* of x. The extent $x^\downarrow$ is called the *predecessor set* or *parents* of x. Several properties of endorelations are of note.

定义2.1。 假设 $\mathcal{R} \in \mathrm{Rel}X$。那么， $\mathcal{R}$ 是

1. 传递性如果 $x\mathcal{R}y$， $y\mathcal{R}z$ 则 $x\mathcal{R}z$，2. 自反性如果 $x\mathcal{R}x$，3. 反对称如果 $x\mathcal{R}y$ 且 $y\mathcal{R}x$ 则 $x = y$，
4. 对称性如果 $x\mathcal{R}y$ 且 $y\mathcal{R}x$，

5. 串行 对于每个 $x \in X$，都存在 $y \in X$ 使得 $x \mathcal{R} y$，

6. 欧几里得 若对于每个 x、y、$z \in X$，$x \mathcal{R} y$ 且 $x \mathcal{R} z$ 均蕴含 $y \mathcal{R} z$，

7. 全连通 若对于每个 x,$y \in X$, $x \mathcal{R} y$ 或 $y \mathcal{R} x$.

A *equivalence relation* is a relation $\sim \in \operatorname{Rel} X$ satisfying transitivity, reflexivity and symmetry. It is well known that equivalence relations are in one-to-one correspondence between equivalence relations and partitions. A partition of X is a collection of subsets $\pi = \{X_i\}_{i \in I}$ such that $\bigcup \pi = X$ and $X_i \cap X_j = \emptyset$ for all $i \neq j$. Let $\Pi(X)$ denote the set of partitions of X. It is often convenient to represent a partition e.g. $\pi \in \Pi(\{1,2,3,4\})$ as $\pi = \{12|3|4\}$.

示例 2.2（语义）。本示例为第9章作铺垫。假设 S 是一个系统的状态集合。例如，S可以是时间单位；或者，在一个多智能体系统中，每个智能体都可以有局部状态S_i，例如，在传感器网络中，检测状态为 {正、负}，或者消息状态为 {已发送、已接收}。在局部范式中，全局状态是局部状态$S = \prod_i S_i$的乘积。克里普克关系是一个自关系 $\mathcal{K} \in \mathrm{Rel}(S)$。克里普克关系定义了一个“可能世界模型”，各种多智能体逻辑 [52] 都建立于其上。例如，给定一个状态s，可达集 $s^{\uparrow} = \{t \mid s \mathcal{K} t\}$ 构成这样一个状态集合：对于这些状态，给定命题必须为真，才能使该命题在状态 s 处“特别”地为真。

定义 2。2。 一个偏序集 （偏序集） $(P, \preccurlyeq)$ 是一个集合范畴X，其上带有一个具有传递性、自反性和反对称性的二元关系 $\preccurlyeq$。

We often omit the order $\preccurlyeq$ in $(P, \preccurlyeq)$ when it is clear from context. If we remove the anti-symmetry requirement, we say P is a preordered set or *proset*. We write $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$. The ordering $\prec$ is called a *strict order*.

Before getting to far along, we describe a standard way to visualize posets which will require a definition. We say that an element y of P *covers* x written $x \lessdot y$ if there does not exist $z \in P$ with $x \prec z \prec y$. We form a covering (endo)relation $\lessdot$ depicting the elements that cover each other. We can draw the covering relation as a directed graph called a *Hasse diagram*. For historical reasons, we do not draw arrows because arrows always point from the bottom to the top of the page. Paths in the hasse diagram correspond to chains. Transitivity and reflexivity is implicit. We do not draw self-loops or arrows that realize transitivity, but they are there (Figure 2.1.)

2.1.1 偏序集的子集

P 的若干种子集值得关注。上集 是一个子集$U \subseteq P$，具有以下性质

$$x \in U,\ y \succcurlyeq x \ \Rightarrow\ y \in U.$$

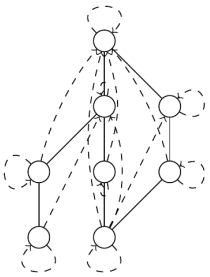


图2.1：覆盖关系（哈斯图）与完整偏序的比较。

同样，一个 下集 D 是子集 $D \subseteq \mathrm{P}$，具有以下性质

$$x \in D,\ y \preccurlyeq x,\ \Rightarrow\ y \in D.$$

主上集由单个$x \in \mathrm{P}$: $\uparrow x = \{y \in \mathrm{P} \mid y \succcurlyeq x\}$生成。主下集由单个$x \in \mathrm{P}$: $\downarrow x = \{y \in \mathrm{P} \mid y \preccurlyeq x\}$生成。

偏序集 P 中的区间是一个子集 $\mathrm{I} \subseteq \mathrm{P}$，满足凸性：对于任意 $\mathrm{x}, \mathrm{y} \in \mathrm{I}$，若在 P 中 $\mathrm{x} \preccurlyeq \mathrm{z} \preccurlyeq \mathrm{y}$，则 $\mathrm{z} \in \mathrm{P}$ 也成立。若区间具有 $[\mathrm{x}, \mathrm{y}] = \{\mathrm{z} \mid \mathrm{x} \preccurlyeq \mathrm{z} \preccurlyeq \mathrm{y}\}$ 的形式，则称其为闭区间。我们用 $\mathbb{I}\mathrm{P}$ 表示偏序集中的闭区间集合。链是偏序集 P 的子集 $\mathrm{C} \subseteq$，且 C 是全序的：对于任意 $\mathrm{x}, \mathrm{y} \in \mathrm{C}$，$\mathrm{x} \preccurlyeq \mathrm{y}$ 或 $\mathrm{y} \preccurlyeq \mathrm{x}$（或两者同时成立）。类似地，反链是偏序集 P 的子集 $\mathrm{A} \subseteq$，满足对于任意 $\mathrm{x},\mathrm{y} \in \mathrm{A}$，$\mathrm{x} \preccurlyeq \mathrm{y}$ 和 $\mathrm{y} \preccurlyeq \mathrm{x}$ 均不成立（记作 $\mathrm{x} \parallel \mathrm{y}$）。P 的高度是极大链的（路径）长度。P 的宽度是极大反链的基数。

A subset $V \subseteq \mathrm{P}$ is *directed* if for every $x, y \in \mathrm{P}$, there is a z such that $z \succcurlyeq x, z \succcurlyeq y$. A subset $F \subseteq \mathrm{P}$ is *filtered* if for every $x, y \in \mathrm{P}$, there is a z such that $z \preccurlyeq x, y$.

2.1.2 收敛

在格上进行计算时，定义链在有限和超限意义下“收敛”的概念很有用。

Definition 2.3. $(\mathrm{P}, \preccurlyeq)$ satisfies the descending chain condition (DCC) if every every chain $C \subseteq P$ has a minimum element. Similarly, P satisfies the ascending chain condition (ACC) if every chain C has a maximum element.

满足 DCC（对偶地，满足 ACC）的一个巧妙充分条件是存在分次。若存在一个地图，则偏序集 P 是分次的

$$r : \mathrm{P} \to \mathbb{N} \tag{2.2}$$

使得 $r(x) < r(y)$ 当 $x \preccurlyeq y$ (严格单调) 且$r(y) = r(x) + 1$ 当 $x \prec y$。偏序集上的分次是哈斯图的一种特别严格的拓扑排序。一般而言，一个 拓扑排序 [38]为一个图表的每个节点分配一个自然数

有向图，使得分配给箭头起点的数字小于分配给箭头终点的数字。

Examples 2.1 (Grading)**.** Suppose V is a vector space. The lattice of finite-dimensional subspaces is a lattice graded by subspace dimension. Similarly, if X is a set, the lattice of finite subsets is a lattice graded by cardinality. The lattice of (finite) partitions of X is graded by the number of parts.

命题 2.1. 假设 P 具有分次，分次为$r:\ \mathrm{P}\ \to \mathbb{N}$。那么，P 满足下降链条件。

证明。 设链$C\ \subseteq \mathrm{P}$。那么，$r(C)$ 是一个以下界 0 为界的递减自然数序列。因此，$r(C)$ 收敛，即存在一个最小值。

□

Some posets, for instance, the unit interval $[0,1] \subseteq \mathbb{R}$, do not satisfy the ascending/descending chain condition. For instance, the sequence $\{a\}_{n\in\mathbb{N}} = 1 - 1/n$ is an ascending chain in $[0,1]$, yet a_{n+1} is strictly greater than a_n for every $n \geqslant 0$. However, the least upper bound of $\{a_n\}$ provides an answer to the question: *what does a_n converge to?* In general, least upper bounds and greatest lower bounds equip posets with notions of convergence as well as algebraic properties.

2.1.3 完备性

设$S\ \subseteq$P 是任意子集。若它们存在，我们可以计算最小上界和最大下界，下文分别称为并 和 交：

$$\begin{aligned} \bigvee S = &\qquad \min\{y \mid y \succcurlyeq x\ \forall\ x \in S\}; \\ \bigwedge S = &\qquad \max\{y \mid y \preccurlyeq x\ \forall\ x \in S\}. \end{aligned}$$

We say that P is a *complete lattice* if for every subset $S \subseteq \mathrm{P}$, $\bigvee S$ exists. P is *directed complete* if $\bigsqcup V$ exists for every directed subset V.[1] In fact, it is enough that a poset contains either arbitrary meets or arbitrary joins in order to be a complete lattice, explaining the alternative nomenclature *suplattices*.

命题 2。2。 设 $(\mathrm{P}, \preccurlyeq)$ 是一个偏序集，且 $S\ \subseteq \mathrm{P}$ 是任意子集。那么，$\bigvee S$ 存在，当且仅当 $\bigwedge S$ 存在。

证明。我们写作

$$\begin{aligned} \bigvee S = &\qquad \bigwedge\{y \in \mathrm{P} \mid y \succcurlyeq x,\ \forall\ x \in S\} \\ = &\qquad \bigwedge \bigcap_{x\in S} \uparrow x. \end{aligned}$$

1当然，也存在滤过完备的对偶概念，但出于传统原因，研究的是定向完备偏序集。此外，出于历史原因，定向集的并写作 $\bigsqcup$。

□

Warning 2.1. Going forward, we make the following abuse of notation. If P and Q are partial orders, we do not always distinguish between the order relation on P and Q, writing $\preccurlyeq$ for both of them. We also do not distinguish between joins/meets in P and joins/meets in Q. Both we write both with the symbols $\bigvee$ and $\bigwedge$.

2.1.4 偏序集之间的映射

A map f between posets $f : \mathrm{P} \to \mathrm{Q}$ is *order-preserving* or *monotone* if $x \preccurlyeq y$ implies $f(x) \preccurlyeq f(y)$. The set of order-preserving maps between P and Q is a poset with $f \preccurlyeq G$ if $f(x) \preccurlyeq g(x)$ for all $x \in \mathrm{P}$. f is an *order embedding* if $f(x) \preccurlyeq f(y)$ if and only if $x \preccurlyeq y$; an *order isomorphism* if f is a surjective order embedding.

以下示例与可解释人工智能相关。

Example 2.3 (Binary Classification)**.** Suppose $f_\theta : \mathbb{R}^d \to [0,1]$ is a statistical model with parameter θ that inputs financial data about an applicant (e.g. income, negative debt, credit score) and outputs the likelihood that an application will not default on a load. In a supervised training, $f_\theta(\mathbf{x}_i) = 1$ if i made loan payments and or $f_\theta(\mathbf{x}_i) = 0$ if i defaulted. In this scenario, one would desire for f_θ to be monotone to account for the fact that an applicant with a higher income, lower debt, and higher credit score will always be more likely to pay a loan back. This means that $f_\theta(\mathbf{x}) \leqslant f_\theta(\mathbf{x})$ whenever $\mathbf{x} \preccurlyeq \mathbf{y}$.

Posets, and all relations, have a duality baked into their definitions. Given $(\mathrm{P}, \preccurlyeq)$, let $(\mathrm{P}^{\mathrm{op}}, \sqsubseteq)$ be the poset called the *opposite poset* with the same base set P and order $x \sqsubseteq y$ if and only if $x \succcurlyeq y$. Often in order-theoretic arguments, we invoke the *Duality Prinicple* which loosely says *if a statement is true for* P*, then the dual statement obtained by dualizing all the order-theoretic definitions in the statment is true for* P^{op}. As an example, a map f is *order-reversing* or *antitone* if $x \preccurlyeq y$ implies $f(y) \preccurlyeq f(x)$. Alternatively, by the Duality Principle, we could say an antitone map between P and Q is a monotone map $f : \mathrm{P} \to \mathrm{Q}^{\mathrm{op}}$ or, equivalently, a monotone map $f : \mathrm{P}^{\mathrm{op}} \to \mathrm{Q}$.

地图$f : \mathrm{P} \to \mathrm{Q}$ 若对每个子集$S \subseteq \mathrm{P}$ 都保持并

$$f(\bigvee S) = \bigvee f(S).$$

地图$f : \mathrm{P} \to \mathrm{Q}$ 若对每个子集$S \subseteq \mathrm{P}$ 都保持交

$$f(\bigwedge S) = \bigwedge f(S).$$

如果 P 和 Q 是定向完备偏序，则称$f : \mathrm{P} \to \mathrm{Q}$ 为连续的，如果对于每个定向子集 $V \subseteq \mathrm{P}$，$f(\bigsqcup V) = \bigsqcup f(V)$。

Example 2.4 (Computer Vision)**.** Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty, +\infty\}$ denote the extended real numbers. Suppose $f : \Omega \to \mathbb{R}_{\max}$ is a grayscale image where Ω is the image domain (e.g. $\mathbb{Z}^2$). In image filtering, it is popular to define filters called erosions and dilations [79]. If is a second image called the *kernel*, define the *dilation* of f by

$$(f \oplus g)(x) = \bigvee_{y \in \Omega} f(x-y) + g(y).$$

定义 腐蚀f的腐蚀，按其定义为

$$(f \ominus g)(x) = \bigvee_{y \in \Omega} f(x+y) - g(y).$$

在像的格中

$$\mathbb{R}_{\max}^{\Omega} = \{f : \Omega \to \mathbb{R}_{\max}\},$$

这些算子

$$\begin{aligned} \delta_g &= f \oplus g \\ \varepsilon_g &= f \ominus g \\ \delta_g, \varepsilon_g : & \quad \mathbb{R}_{\max}^{\Omega} \to \mathbb{R}_{\max}^{\Omega} \end{aligned}$$

满足

$$\begin{aligned} \delta_g(\bigvee_i x_i) &= \bigvee_i \delta_g(x_i) \\ \varepsilon_g(\bigwedge_i x_i) &= \bigwedge_i \varepsilon_g(x_i) \end{aligned}$$

for any $g \in \mathbf{L}$. Hence, dilations, as the name suggests, are join-preserving, and erosions are meet-preserving. Other filters called *openings* and *closing* are defined by composing an erosions and dilations and are useful in denoising, contrast enhancement, reconstruction and edge detection [109].

2.2 不动点

Frequently, we study maps $f : \mathrm{P} \to \mathrm{P}$. Such a map is *inflationary* if $f(x) \succcurlyeq x$ for all $x \in \mathrm{P}$ and *deflationary* if $f(x) \preccurlyeq x$ for all $x \in \mathrm{P}$. We write $\mathrm{Pre}(f)$ and $\mathrm{Post}(f)$ for the subsets of deflationary and inflationary points, respectively. Let $\mathrm{Fix}(f)$ denote the set of fixed points $\mathrm{Pre}(f) \cap \mathrm{Post}(f)$ (anti-symmetry axiom).

2.2.1 （余）闭包算子

一个地图$f : \mathrm{P} \to \mathrm{P}$是 幂等 的，当且仅当$f^2 = f$。设 P 是一个偏序集，且$f : \mathrm{P} \to \mathrm{P}$是一个保序映射。根据 P 和$f$的性质，有多种方式刻画 $\mathrm{Fix}(f)$。

Definition 2.4 (Closure Operators)**.** Suppose $f : \mathrm{P} \to \mathrm{P}$ is a map and P is a poset. If f is order-preserving, inflationary, and idempotent, we say f is a *closure operator*. Conversely, if $g : \mathrm{P} \to \mathrm{P}$ is order-preserving, deflationary, and idempotent we say g is a *coclosure operator*. The fixed points of a (co)closure operator are called *(co)closed*.

（余）闭包算子理论 [140]使我们能够显式计算不动点格的并和交。

定理 2.1 （（余）闭包）**.** 设**L** 是一个完备格，且f :**L** $\to$ **L** 是完备格 (**L**上的闭包算子。那么，Fix(f) 是一个完备格，其交与并如下。

$$\begin{aligned} \bigsqcap_{i\in I} x_i &= \bigwedge_{i\in I} x_i \\ \bigsqcup_{i\in I} x_i &= f\left(\bigvee_{i\in I} x_i\right). \end{aligned}$$

对偶地，设 g 是 **L** 上的余闭包算子。那么，Fix(f) 是一个完备格，其交和并如下

$$\begin{aligned} \bigsqcap_{i\in I} x_i &= f\left(\bigwedge_{i\in I} x_i\right) \\ \bigsqcup_{i\in I} x_i &= \bigvee_{i\in I} x_i. \end{aligned}$$

证明。 参见 [140, 定理 3.8]. □

闭包算子在拓扑学中自然出现，这并不令人意外。

Example 2.5 (Closure Spaces)**.** A *closure space* is a pair (X, c) where X is an (arbitrary) set and c is a closure operator $c : \wp(X) \to \wp(X)$. Closed subsets in X are precisely subsets A with $c(A) = A$. A closure space is *topological* if

$$\begin{aligned} c(A \cup B) &= c(A) \cup c(B), \\ c(\varnothing) &= \varnothing. \end{aligned}$$

近期的研究利用闭包空间将持久同调扩展到新的领域（例如有向图）[27]。

虽然（余）闭包算子在数学的各个分支中经常出现，但在实际情况下，幂等性和扩张性的要求通常过于严格。以下不动点定理是我们研究格值层上算子的核心。

定理 2.2(Tarski不动点定理)。 假设 **L** 是完备格，且f: **L** $\to$ **L** 是保序映射。那么，Fix(f) 是完备格。

我们将需要几个引理。

引理2.1. 假设$f : \mathbf{L} \to \mathbf{L}$是保序的，且$\mathbf{L}$是完备的。假设$\{x_i\}_{i\in I}$是由任意集合$I$索引的$\mathbf{L}$的子集。那么，

$$f\left(\bigwedge_{i\in I} x_i\right) \preccurlyeq \bigwedge_{i\in I} f(x_i).$$

同样，

$$f\left(\bigvee_{i\in I} x_i\right) \succcurlyeq \bigvee_{i\in I} f(x_i).$$

Proof of Lemma 2.1. $f\left(\bigwedge_{i\in I} x_i\right)$ is *a* lower bound of each $f(x_i)$ because f is order-preserving and $\bigwedge_i x_i \preccurlyeq x_i$. Moreover, $\bigwedge_{i\in I} f(x_i)$ is *the* greatest lower bound of the subset $\{f(x_i)\}_{i\in I}$, implying $\bigwedge_{i\in I} f(x_i)$ precedes all other lower bounds. Hence,

$$f\left(\bigwedge_{i\in I} x_i\right) \preccurlyeq \bigwedge_{i\in I} f(x_i).$$

第二个陈述遵循对偶原理。 □

Lemma 2.2. Suppose $f : \mathrm{P} \to \mathrm{P}$ is order-preserving. Then, $f^{\mathrm{op}} : \mathrm{P}^{\mathrm{op}} \to \mathrm{P}^{\mathrm{op}}$ defined to be the same map on sets, but under the opposite order is order-preserving as well. If f is join-preserving, then f^{op} is meet preserving and vice versa. Furthermore, $\mathrm{Pre}(f^{\mathrm{op}}) = \mathrm{Post}(f)$ and $\mathrm{Post}(f^{\mathrm{op}}) = \mathrm{Pre}(f)$.

Proof of Lemma 2.2. $x \preccurlyeq y$ in P if and only if $x \succcurlyeq y$ in P^{op}. f is order-preserving implies $f(x) \preccurlyeq f(y)$ in P, or $f(x) \succcurlyeq f(y)$ in P^{op}. Hence, f^{op} is order-preserving. Meets in P^{op} are equivalent to joins in P. □

Proof of Theorem 2.2. The classical argument [158] that $\mathrm{Fix}(f)$ is complete has three parts. First, we show $\mathrm{Pre}\, f \subseteq \mathbf{L}$ is an invariant subset under f. Suppose $x \in \mathrm{Pre}(f)$. Then,

$$f(f(x)) \preccurlyeq f(x) \preccurlyeq x$$

which implies $f(x) \in \mathrm{Pre}(f)$ by monotonicity of f and transitivity. Next, we show that $\mathrm{Pre}(f)$ is complete. Let $\{x_i\}_{i\in I} \subseteq \mathrm{Pre}(f)$. Then, Lemma 2.1 implies that $\mathrm{Pre}(f)$ is $\bigwedge$-complete:

$$f\left(\bigwedge_{i\in I} f(x_i)\right) \preccurlyeq \bigwedge_{i\in I} f(x_i)$$

意味着$\bigwedge_{i\in}$如果$(x_i) \in \mathrm{Pre}(f)$。根据命题2。2，$\mathrm{Pre}(f)$也是$\bigvee$-完备的。最后，定义

$$f_{|\,\mathrm{Pre}(f)}{}^{\mathrm{op}} : \mathrm{Pre}(f)^{\mathrm{op}} \to \mathrm{Pre}(f)^{\mathrm{op}}.$$

Invariance and Lemma 2.2 imply $f_{|\operatorname{Pre}(f)}{}^{\mathrm{op}}$ is well-defined and order-preserving on $\operatorname{Pre}(f)^{\mathrm{op}}$ and that $\operatorname{Pre}(f_{|\operatorname{Pre}(f)}{}^{\mathrm{op}}) = \operatorname{Post}\left(f_{|\operatorname{Pre}(f)}\right)$. Now, the argument in the second part shows $\operatorname{Post}\left(f_{|\operatorname{Pre}(f)}\right)$ is complete. Furthermore,

$$\operatorname{Fix}(f) = \operatorname{Post}\left(f_{|\operatorname{Pre}(f)}\right)$$

这样就完成了。 □

定理 2.2的证明蕴含以下推论。

推论 2。1。 子集 $\operatorname{Pre}(f)$ 和 $\operatorname{Post}(f)$ 是完备的。

Proof. Let $f \wedge \mathrm{id}$ be the map defined $f(x) \wedge x$ on $x \in \mathbf{L}$. Then, $\operatorname{Fix}(f \wedge \mathrm{id}) = \operatorname{Post}(f)$ because $f(x) \succcurlyeq x$ if and only if $f(x) \wedge x = x$. Similarly, $\operatorname{Fix}(f \vee \mathrm{id}) = \operatorname{Pre}(f)$. □

特别是在某些应用领域，只需证明存在最小不动点。因此，Tarski 不动点定理有时被表述为以下推论。

推论2。2。 假设 $\mathbf{L}$ 是完备格，且 $f : \mathbf{L} \to \mathbf{L}$ 是保序的。那么，f 有一个最小不动点 x^*。

Proof. $x_* = \bigwedge \operatorname{Fix}(f)$. □

2.3 从偏序集到格

至此，我们已经看到，完备格是具有任意交和并的偏序集。现在，我们介绍一种格的替代代数观点，在这种观点中，我们只要求二元或零元（对于有界格）交和并。

定义 2。5 （代数定义）。 一个格 $\mathbf{L}$ 是一个集合范畴 L，具有一对二元运算

$$\vee, \wedge : \mathbf{L} \times \mathbf{L} \to \mathbf{L}$$

称为 并 和 交，使得对于每个 x、y、$z \in \mathbf{L}$

1. $x \vee (y \vee z) = (x \vee y) \vee z$; $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (Associativity),
2. $x \vee y = y \vee z$; $x \wedge y = y \wedge z$ (交换律),
3. $x \vee x = x$; $x \wedge x = x$ (幂等律),
4. $x \wedge (x \vee y) = x$; $x \vee (x \wedge y) = x$ (Absorption).

A 有界格 是一个具有元素 0、$1 \in \mathbf{L}$ 的格，并满足公理。

5. $x \wedge 1 = x$; $x \vee 0 = x$

一个 半格**S** 是一个集合范畴S ， 带有一个二元运算

$$\wedge : \mathbf{S} \times \mathbf{S} \to \mathbf{S}$$

使得对于所有 x,$y \in \mathbf{S}$。

1. $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (Associativity),
2. $x \wedge y = y \wedge z$ (Commutativity),
3. $x \wedge x = x$ (Idempotence).

A 有界半格 是具有元素 $1 \in \mathbf{S}$ 并满足公理的半格。

4. $x \wedge 1 = x$.

Example 2.6 (Nesting Poset)**.** Suppose $\mathbb{M}$ is a compact smooth 2-manifold (e.g. a closed and bounded surface in $\mathbb{R}^3$) diffeomorphic to the 2-sphere $\mathbb{S}^2$. Suppose $f : \mathbb{M} \to \mathbb{R}$ is a Morse function [112] factoring through an embedding and a projection onto the z-axis

$$i : \mathbb{M} \overset{i}{\hookrightarrow} \mathbb{R}^3 \overset{\pi}{\to} \mathbb{R}$$

By a version of the Implicit Function Theorem [112, Theorem 2.3], if $t \in \mathbb{R}$ is a regular value, then $f^{-1}(t)$ is a compact manifold of dimension 1. By the well-known classification of such manifolds, it follows there is a diffeomorphism

$$f^{-1}(t) \simeq \bigsqcup_{i=1}^{m} \mathbb{S}^1,$$

and, furthermore, f^{-1} is embedded in $\mathbb{R}^2$. Hence, f^{-1} is a disjoint union of Jordan curve [81], say, $\mathbb{Y} = \gamma_1, \gamma_2, \gamma_m\}$. The complement $\mathbb{R}^2 \setminus \mathbb{Y}$ consists of exactly $m + 1$ connected components, with 1 unbounded component and m bounded components. It follows that the set of (path) connected components $\pi_0(\mathbb{M} \setminus \mathbb{Y})$ have the structure of a join-semilattice under the following order [32] equivalent to the circle containment order [145].

We say two components in $\pi_0(\mathbb{M} \setminus \mathbb{Y})$ are adjacent if they they share a boundary. Then, for any pair of adjacent components $c, c' \in \pi_0(\mathbb{M} \setminus \mathbb{Y})$, then $\alpha_i \preccurlyeq \alpha_j$ if and only if the for the corresponding (unshared) boundaries $\gamma_i and \gamma_j$, we have an inclusion of interiors $\text{int}(\gamma_i) \subseteq \text{int}(\gamma_j)$ (existence, by the Jordan Curve Thoerem [81]) or if $\text{int}(\gamma_j)$ is undbounded.

另一方面，格是具有二元交、空交以及并的有序集。

定义 2.6 (组合). 格 是偏序集 $\mathbf{L}$，满足对每个 x、$y \in \mathbf{L}$

$$\begin{aligned} x \vee y = & \quad \min\{z \mid z \succcurlyeq x,\ z \succcurlyeq y\}, \\ x \wedge y = & \quad \max\{z \mid z \preccurlyeq x,\ z \preccurlyeq y\}. \end{aligned}$$

格是 有界 的，若还存在元素$0, 1 \in \mathbf{L}$ 使得$0 \preccurlyeq x \preccurlyeq 1$ 对所有 $x \in \mathbf{L}$。

以下命题允许你在格的序论定义和代数定义之间自由转换。

命题 2。3。 格的代数定义（定义2。5）与组合定义（定义2。6）等价。

证明。 参见 [140, 定理3。17]。 □

因此，（有界）格 $\mathbf{L}$ 同时具有（有界）交半格 $(\mathbf{L}, \wedge)$ 和（有界）并半格 $(\mathbf{L}, \vee)$的结构。

我们注意到格有几种特殊化形式。一个分配格 是具有如下性质的格：对于所有x、y、$z \in \mathbf{L}$

$$\begin{aligned} x \wedge (y \vee z) = & \quad (x \wedge y) \vee (x \wedge z), \\ x \vee (y \wedge z) = & \quad (x \vee y) \wedge (x \vee z). \end{aligned}$$

A bounded lattice is *complemented* if for every $x \in \mathbf{L}$ there exists an element $y \in \mathbf{L}$ such that $x \vee y = 1$ and $x \wedge y = 0$. Powersets (Example 2.12) are an example of a complemented distributive lattice, also know as a *Boolean lattice*.

A *sublattice* $\mathbf{K} \subseteq \mathbf{L}$ is a subset $K \subseteq \mathbf{L}$ closed under the $\vee$ and $\wedge$ operation in $\mathbf{L}$. A *quasisublattice* of $\mathbf{L}$ is a lattice $\mathbf{K}$ with an order embedding $i : \mathbf{K} \hookrightarrow \mathbf{L}$. Clearly, a sublattice is a quasi-sublattice, but not the other way around. If $\mathbf{K}$ and $\mathbf{L}$ are complete, then we say $\mathbf{K}$ is a *complete quasisublattice*.

例 2.7（子群格）。 假设 $(G, \ \star e)$ 是一个群**[dummitfoote]**。G 的（正规、任意）子群构成一个格 子群G ， 其具有

$$\begin{aligned} M \vee N = & \quad \{m \cdot n \mid m \in M,\ n \in N\}, \\ M \wedge N = & \quad M \cap N. \end{aligned}$$

格 Sub$G$ 序嵌入幂集 $\wp(G)$。然而，Sub$G$ 不是子格，因为M 与N 在 Sub$G$ 中的并不同于在 $\wp(G)$中的并，$M \cup N$。

格 同态 是一个地图 $f : \mathbf{K} \to \mathbf{L}$ ， 满足对所有 x、$y \in \mathbf{K}$

$$\begin{aligned} f(x \vee y) = & \quad f(x) \vee f(x), \\ f(x \wedge y) = & \quad f(x) \wedge f(y). \end{aligned}$$

A 有界格同态是一个格同态，具有

$$\begin{aligned} f(0) &= 0, \\ f(1) &= 1. \end{aligned}$$

A homomorphism f is an isomoprhimsm if there is another homomorphism $g : \mathbf{L} \to \mathbf{K}$ with $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. The reader may check that an order isomoprhism between (bounded) lattices is an isomorphism of (bounded) lattices.

Example 2.8 (Multidimensional Persistence)**.** This example was adapted from recent work on persistence modules over lattice [113]. Given a lattice $\mathbf{L}$ (e.g $\mathbb{N} \times \mathbb{N}$ representing a bifiltration), a *persistence module* is an assignment $F : \mathbf{L} \to \mathcal{V}ec$ of lattice elements to finite-dimensional $\Bbbk$-vector spaces such that $x \preccurlyeq y$ in $\mathbf{L}$ induces a linear transformation $F(x) \xrightarrow{F(x \preccurlyeq y)} F(y)$ (formally, a functor into the category of vector spaces). A "change in basis" of a persistence modules is a lattice homomorpism $f : \mathbf{K} \to \mathbf{L}$. This induces a lattice homomorphism $\hat{f} : \mathbb{I}\mathbf{K} \to \mathbb{I}\mathbf{L}$. Certain interesting functions describing births and deaths of i-dimensional topological features, e.g. in a metric space or filtration of simplicial complexes [60], are written as maps $f_i : \mathbb{I}\mathbf{L} \to \mathbb{Z}$ which map an interval $[a, b] \mapsto \dim\left(Z_i F(a) \cap B_i F(b)\right)$ where $Z_i F(a)$ and $B_i F(a)$ are the cycles and boundaries of homology [81].

2.3.1 带乘法的格

格通常还配备不同于二元交与并的代数结构。例如，实数 $\mathbb{R}$ 是一个（无界）格，同时也具有域的结构。下面简要回顾几个定义。

Definition 2.7 (Monoid)**.** A *monoid* is a tuple $(M, \star, e)$ such that M is an arbitrary set, $\cdot : M \times M \to M$ is an associative binary operation, and $e \in M$ is an element such that $m \star e = e \cdot m = m$ for all $m \in M$. A monoid is a *group* if for every $m \in M$, there exist a unique element $m^{-1} \in M$ such that $m^{-1} \star m = m \star m^{-1} = e$. A monoid is *commutative* if $m \star n = n \star m$ for all $m, n \in M$. A monoid is *idempotent* if $m \star m = m$ for all $m \in M$.

例 2.2 （幺半群）。

1. The natural numbers with addition $(\mathbb{N}, +, 0)$ is a monoid, but not a group. On the other hand, $(\mathbb{Z}, +, 0)$ is a group.

2. Suppose X is a set, with possible additional structure. Then, the set of maps $f : X \to X$ is a monoid denoted $\mathrm{End}(X)$. The identity e is the idenity map $\mathrm{id} : X \to X$. Multiplication is merely composition of endomorphisms.

3. 有界半格是一个幺半群，具有 $(\mathbf{S}, \wedge, 1)$。

命题 2.4. 假设 $(M, \star, e)$ 是交换幂等幺半群。那么，M 是半格。

Proof. Define a relation $m \preccurlyeq n$ if and only if $m \cdot n = m$. First check $(M, \preccurlyeq)$ is a partial order with top element e. Then, check $m \wedge n$ is in fact the element $m \cdot n$. □

剩余格是配备了相容幺半群结构的格。

Definition 2.8 (Residuated Lattice)**.** A *residuated lattice* $(\mathbf{L}, \wedge, \vee, 0, 1, \star, e, [-,-])$ is a bounded lattice $(\mathbf{M}, \wedge, \vee, 0, 1)$ such that $(\mathbf{L}, \cdot, e)$ is a commutative monoid with a second (non-commutative) binary operation $[-,-]$ on $\mathbf{M}$ with the property that

$$x \star y \preccurlyeq z \quad \text{if and only if} \qquad x \preccurlyeq [y, z] \tag{2.3}$$

对于所有 x、y、$z \in \mathbf{L}$。若 $\mathbf{L}$ 是完备格，则剩余格是完备的。

Example 2.9 (Fuzzy Logic)**.** Consider three residuated lattice structures on the unit interval with the lattice structure $([0,1], \min, \max, 0, 1)$, sometimes called *t-norms*. Let $r, s, t \in [0,1]$.

1. 假设$s \star t = s \cdot t$ 是普通乘法。那么，$[s,t]$ 可计算为

$$[s,t] = \begin{cases} t/s & s > t \\ 1 & s \leqslant t \end{cases}$$

from

$$r \cdot s \leqslant t \quad \text{if and only if} \qquad r \leqslant [s,t].$$

2. 定义 $s \star t = \min(s + t - 1, 0)$ 并检查

$$[s,t] = \begin{cases} 1 - s + t & s > t \\ 1 & s \leqslant t \end{cases}.$$

3. 定义 $s \star t = \min(s,t)$ 并检查

$$[s,t] = \begin{cases} t & s > t \\ 1 & s \leqslant t \end{cases}.$$

Given a residuated lattice structure on $[0,1]$ and a set X, a *fuzzy set* is a map $A : X \to [0,1]$. Viewing fuzzy sets as propositions, the logical operations of conjunction ($\wedge$), disjunction ($\vee$) and implication ($\to$) are defined pointwise according to the residuated lattice structure. For instance, if A and B are propositions, $[A, B]$ is a fuzzy set with $([A,B])(s) = A(s) \to B(s)$.

例 2.10 （海廷代数）。每个有界格在交运算下都是一个幺半群。然而，这个幺半群结构并不自动地 满足（2。3），现在可写成

$$x \wedge y \preccurlyeq z \quad \text{if and only if} \quad x \preccurlyeq [y, z]$$

对于所有x、y、$z \in \mathbf{H}$，一个*Heyting* 代数。由此可得

$$[y, z] = \bigvee \{x \in \mathbf{H} \mid x \wedge y \preccurlyeq z\}$$

引出了伪补集 $\neg x = [x, 0]$。Heyting 代数的示例很多，并且在数学基础中有着深远的影响 [107]。

Example 2.11 (Max-Plus Lattice)**.** Max-plus algebra has found plentiful applications in locomotion [106], control [109], scheduling [31], and machine learning [111]. All of max-plus algebra is based on the idea of "doing linear algebra" over a residuated lattice, instead of a field. Consider the extended reals $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty, \infty\}$ which forms a complete lattice under $\leqslant$ with the additional requirement that $-\infty \leqslant t \leqslant \infty$ for all $t \in \mathbb{R}$. Then, $(\mathbb{R}_{\max}, \wedge, \vee, -\infty, \infty, +, 0, [-,-])$ has the structure of a residuated lattice with $[s, t] = \max(t - s, 0)$. A *max-plus* or *tropical* vector space, is then the product lattice $\mathbb{R}_{\max}^d$. In some setting, this product structure is called a *weighted lattice* [110] because you can define "scalar multiplication" component-wise.

2.4 其他示例

既然关于格的大部分背景材料已经确立，我们来深入探讨一些更多的例子。

Example 2.12 (Subsets)**.** Powersets are fundamental examples of lattices to the degree that the meet and join symbols $\wedge/\vee$ are derivates of the symbols $\cap/\cup$ which normally represent intersection and union. Given a set S, the *powerset* $\wp(S)$ is a lattice under intersection and union. By definition, a union of set of subsets sets is the smallest set containing every subset. Similarly, an intersection of a set of subsets is the largest set contained in each subset. If we let $\mathbf{B} = \{0M1\}$, then powersets are equivalent to the lattice of maps $S \to \mathbf{B}$, with the inherited lattice structure

$$\begin{aligned} A \wedge B(x) &= \texttt{AND}\,(A(x), B(x))\,, \\ A \vee B(x) &= \texttt{OR}\,(A(x), B(x))\,, \end{aligned}$$

A、$B \in \mathbf{B}^S$。类似地，多重集合 是一个映射 $S \to \mathbb{N}$，通常记作 $\mathbb{N}[S]$。多重集合上的格结构如下

$$\begin{aligned} A \wedge B(x) &= \min(A(x), B(x),) \\ A \vee B(x) &= \max(A(x), B(x),) \end{aligned}$$

$A, B \in \mathbb{N}[S]$.

An interesting use-case of powerset lattices is modeling preferences. If X is a set of alternatives, for instance, restaurants), then $\mathrm{Rel}\, X := \wp(X \times X)$ is the set of all possible preference relations on X. Preference relations can be combined with the meet and join axioms, in this case, intersection and union. More interesting lattice structures are obtained by restricting $\mathrm{Rel}\, X$ to transitive-reflexive relations, as it is often argued that preferences are transitive.

Tangential to the theory of preference relations, is the theory of choice functions. A *choice function* is a map $c : \wp(X) \to \wp(X)$ such that $c(A) \subseteq A$ for all $A \in \wp(X)$. Intuitively, a choice function narrows down a subset of choices. It is established that the set of choice functions forms a lattice with pointwise union and intersection; interesting subclasses of choice functions form lattices as well [114].

例 2.13（子空间）。设 V 是一个 $\Bbbk$-向量空间。设 $\mathsf{Sub}V$ 由 V 的所有子空间组成。如果我们记 $\mathsf{Grass}(V,d)$ 为*Grassmanian*，即 d-维子空间，那么我们可以写

$$\mathrm{Sub}\, V = \coprod_{d \geqslant 0} \mathrm{Gr}(V, d).$$

$\mathsf{Sub}V$ 在子空间序下是一个格，其中 $K \leqslant W$ 如果 K 是 W 的子空间。格运算如下： $W \vee W' = \{w + w' \mid w \in W,\ w' \in W'\}$ 常记为 $W + W'$； $W \wedge W' = W \cap W'$ 这平凡地是 V 的子空间。主成分分析（PCA）及其变体可视为将数据点投影到某个特征空间 V [166] 的最优子空间 W 上。

设 $T : V \to V$ 是一个线性变换。我们称子空间 $W \subseteq V$ 是不变的，如果 $T(W) \subseteq W$。例如，1-维不变子空间由满足 $T(\mathbf{x}) = \lambda \mathbf{x}$（对某个标量 λ）的向量 $\mathbf{x}$ 张成——这些不变子空间是特征子空间。由此可知，T 的不变子空间记为 $\mathsf{Inv}T$，构成一个格，即 $\mathsf{Sub}V$ 的子格。已知不变子空间同构的条件[25]。

Example 2.14 (Partitions)**.** The set of partitions $\Pi(S)$ of a set S forms a lattice. Suppose π and π' are partitions. Then, the relation $\pi \preccurlyeq \pi'$ means that π *is finer than* π' or π *is a refinement of* π'. Put another way, for every $A \in \pi$, there is an $A' \in \pi'$ with $A \subseteq A'$. The join $\pi \vee \pi'$ is the finest common coarsening of π and π' while $\pi \wedge \pi'$ is the coarsest common refinement. Suppose $\pi = \{12|34|5\}$ and $\pi' = \{123|45\}$, then $\pi \wedge \pi' = \{12|3|4|5\}$ and $\pi \vee \pi' = \{12345\}$. Partitions are ubiquitous in engineering, from teams of autonomous agents [88] to hierarchial clustering algorithms [29]. A *subpartition* is a partition of some subset S' of the ground set S. By similar reasoning, subpartitions form a lattice denoted $\Pi_*(S)$. Maps from the time domain to the lattice of subpartitions have been used to model dynamic graph with changing node sets [92].

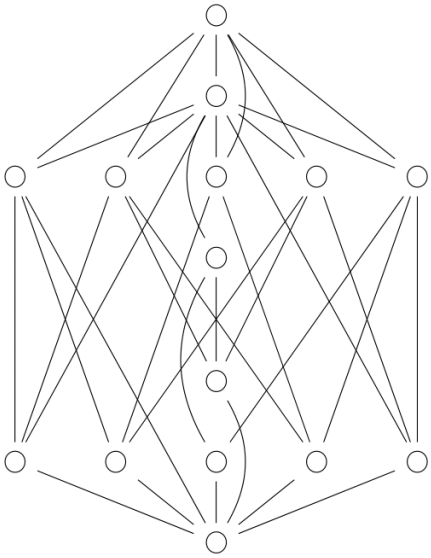


图 2.2：Π 上划分的格(X) 于$X = \{1,2,3,4\}$

Example 2.15 (Information [151])**.** The following example is due to Recall, a random variable is a measurable map $X : \Omega \to A$ to some set A which, for practical reasons, we call the *alphabet*. Depending on the context, we can think of A as a set of symbols or as a set of words. The *entropy*

$$H(X) = -\sum_{w \in A} \mathbf{Pr}\left[X^{-1}\{w\}\right] \log\left(\mathbf{Pr}\left[X^{-1}\{w\}\right]\right).$$

Entropy rougly represents the amount of information (bits) required to encode the outcome of X. A variation of entropy, relative entropy, denoted $H(Y|X)$, is the amount of information to describe the outcome of another random variable Y given that the value X generated is already known. Omitting some technical details [102], random variables over a fixed probability distribution form a poset with $X \succcurlyeq Y$ if and only if $H(Y|X) = 0$.

It is know the set of A-valued random variables under the information order forms a lattice isomoprhic to $\Pi(A)$ [102]. The *information lattice* is a sublattice $\mathbf{I}$ of $\Pi(S)$ subordinate to the σ-algebra defining the underlying probability distribution. The join of random variables $X \vee Y$ is the *joint information content* between then, exactly the joint random variable (X, Y). The meet $X \wedge Y$, on the other hand, is the *common information* between the two variables.

2.4.1 分解

代数中的一个反复出现的主题是将结构分解为尽可能小的组成部分。这引出了以下定义。

定义 2.9. 令$\mathbf{L}$ 为一个格。一个元素 $0 \neq z \in \mathbf{L}$ 若是并不可约的，即 $z = x \vee y$ 蕴含$x = z$ 或$y = z$。类似地，一个元素

$1 \neq z \in \mathbf{L}$ 是交不可约的，如果 $z = x \wedge y$ 蕴含 $z = x$ 或 $z = y$。我们将并不可约元素的集合记为 $\mathcal{J}(\mathbf{L})$。

显然，$(\mathcal{J}(\mathbf{L}), \preccurlyeq)$ 是一个偏序集，其中$x \preccurlyeq y$ 当且仅当$x \wedge y = x$。不那么显然的是，下集构成分配格。

定理2.3 （Birkhoff 对偶）。设$\mathbf{L}$ 是有限分配格，且 P 是有限偏序集。那么，

$$\mathcal{D}(\mathcal{J}(\mathbf{L}))) \cong \mathbf{L}$$

and

$$\mathcal{J}(\mathcal{D}(\mathrm{P}))) \cong \mathrm{P}.$$

证明。参见 [140, 定理10。1]。 □

Theorem 2.3 allows us to move back and forth between these, albeit restrictive, classes of posets and lattices with ease. Because $\mathcal{D}(\mathrm{Q})$ is a lattice of subsets (i.e. sublattice of the powerset $\wp(\mathrm{Q})$), Theorem 2.3 can also be interpreted as *every finite distributive lattice can be represented as a lattice of subsets*. In general, not restricted to finite posets and distributed lattices, the story is more complicated as illustrated by the following example.

Example 2.16 (Natural Numbers)**.** Consider the natural numbers $\mathbb{N}$ with the order $m \preccurlyeq n$ if and only if $n|m$, i.e. there is an $a \in \mathbb{N}$ with $n = a \cdot m$. $(\mathbb{N}, \preccurlyeq)$ is a lattice with $m \wedge n = \mathrm{lcm}(m, n)$ and $m \vee n = \gcd(m, n)$. First, notice that $\mathbb{N}$ is distributive by the classic distributive properties of gcd and lcm. Hence, $(\mathbb{N}, \preccurlyeq)$ is an (infinite) distributive lattice. The join-irreducibles are the prime number $\mathcal{J}(\mathbb{N})$ with the partial order inherited by $(\mathbb{N}, \preccurlyeq)$. It follows that $\mathcal{D}(\mathcal{J}(\mathbb{N})) = \wp(\mathfrak{p})$. However, by the Fundamental Theorem of Arithmetic, $(\mathbb{N}, \preccurlyeq) \cong \mathbb{N}[\mathfrak{p}]$—every natural number has a unique prime factorization. Hence, $\mathcal{D}(\mathcal{J}(\mathbf{L}))) \ncong \mathbf{L}$. Because $\mathbb{N}$ is not finite, we could not apply Birkhoff Duality.

3

伽罗瓦联络

Suppose P and Q are posets. A Galois connection is a pair of compatible order-preserving maps, one from P to Q, another Q to P, such that arbitrary joins and arbitrary meets are preserved in P and Q respectively. On one hand, Galois connections are share a resemblance to linear adjoints (Definition 3.1), but on the other hand, being order approximations of an inverse (Proposition 3.2), share a resemblance to Moore-Penrose pseudoinverses.

3.1 定义性质

Definition 3.1. Let $(\mathrm{P}, \preccurlyeq)$ and $(\mathrm{Q}, \preccurlyeq)$ be posets. A *Galois connection* is a pair $(f_\bullet, f^\bullet)$ of order-preserving maps $f_\bullet : \mathrm{P} \to \mathrm{Q}$ and $f^\bullet : \mathrm{Q} \to \mathrm{P}$ such that for all $x \in \mathrm{P}$, $y \in \mathrm{Q}$,

$$f_\bullet(x) \preccurlyeq y \Leftrightarrow \qquad\qquad x \preccurlyeq f^\bullet(y). \tag{3.1}$$

我们称$f_\bullet$为 左伴随 而$f^\bullet$为 右伴随。

Remark 3.1. The definition of a Galois connection is reminiscent of the definition of the adjoint of a linear transformation. If $T : V \to W$ is a map of inner-product spaces $(V, \langle\ ,\ \rangle_V)$, $(W, \langle\ ,\ \rangle_W)$, then $T^* : W \to V$ is the map defined by the following property

$$\langle T(\mathbf{x}), \mathbf{y}\rangle_W = \qquad\qquad \langle \mathbf{x}, T^*(\mathbf{y})\rangle_V. \tag{3.2}$$

Given fixed complete lattices $\mathbf{K}$ and $\mathbf{L}$, we denote the set of Galois connections between these lattices $\mathcal{L}tc(\mathbf{K}, \mathbf{L})$ (notation adapted from [69]). The set $\mathcal{L}tc(\mathbf{K}, \mathbf{L})$ is a poset with the order $f \leqslant g$ if and only if $f_\bullet(x) \preccurlyeq g_\bullet(x)$ and $g^\bullet(y) \preccurlyeq f^\bullet(y)$ for all $x, y \in \mathbf{L}$ endowing it with the structure of a complete lattice. Galois connections have an equivalent definition highlighting the property that they are the "best approximation" of an order inverse.

命题 3.1. 设 P 和 Q 是偏序集。以下各项等价。

1. $(f_\bullet, f^\bullet) : \mathrm{P} \to \mathrm{Q}$ is a Galois connection.
2. $f_\bullet : \mathrm{P} \to \mathrm{Q}$ and $f^\bullet : \mathrm{Q} \to \mathrm{P}$ are order-preserving, and $f^\bullet f_\bullet \succcurlyeq \mathrm{id}_\mathrm{P}$ and $f_\bullet f^\bullet \preccurlyeq \mathrm{id}_\mathrm{Q}$.

Proof. Suppose $f_\bullet(x) \preccurlyeq y$ if and only if $x \preccurlyeq f^\bullet(y)$ for all $x \in \mathrm{P}$, $y \in \mathrm{Q}$. Substitute $f_\bullet(x)$ for y. Then, $x \preccurlyeq f^\bullet f_\bullet(x)$. Alternatively, if we substitute $f^\bullet(y)$ for x, then we obtain $f_\bullet f^\bullet(y) \preccurlyeq y$. Conversely, suppose $f^\bullet \circ$

$f_\bullet \succcurlyeq \mathrm{id}_P$. If $f_\bullet(x) \preccurlyeq y$, then, because $f^\bullet$ is order-preserving, we have $f^\bullet(f_\bullet(x)) \preccurlyeq f^\bullet(y)$. Because $f^\bullet(f_\bullet(x)) \succcurlyeq x$, it follows by transitivity $f^\bullet(y) \succcurlyeq x$. Similarly, if $x \preccurlyeq f^\bullet(y)$, then $f_\bullet(y^\bullet) \preccurlyeq y$ implies $f_\bullet(x) \preccurlyeq y$. □

复合下伴随与上伴随会稳定化。

命题 3.2. 假设$(f_\bullet, f^\bullet)$ 是上述伽罗瓦连接。那么，

$$\begin{aligned} f_\bullet f^\bullet f_\bullet &= f_\bullet; \\ f^\bullet f_\bullet f^\bullet &= f^\bullet. \end{aligned}$$

Proof. For the first, it suffices to show that $f_\bullet f^\bullet f_\bullet(x) \succcurlyeq f_\bullet(x)$ for all $x \in P$ because $f_\bullet f^\bullet f_\bullet \preccurlyeq f_\bullet$ follows from Proposition 3.1. Also, by Propositon 3.1, $f^\bullet f_\bullet(x) \succcurlyeq x$. $f_\bullet$ is order-preserving implies $f_\bullet f^\bullet f_\bullet(x) \succcurlyeq f_\bullet(x)$. The second is dual to the first. □

命题的一个推论是 3.2 是$f^\bullet f_\bullet$ 和$f_\bullet f^\bullet$ 是幂等的。

推论 3.1. 假设 $(f_\bullet, f^\bullet)$ 如上所述是一个伽罗瓦连接。那么，

$$\begin{aligned} f^\bullet f_\bullet f^\bullet f_\bullet &= f^\bullet f_\bullet \\ f_\bullet f^\bullet f_\bullet f^\bullet &= f_\bullet f^\bullet \end{aligned}$$

命题 3.2 刻画了其中$f_\bullet$是序嵌入的伽罗瓦连接。

Proposition 3.3. Suppose $(f_\bullet, f^\bullet) : P \to Q$ is a Galois connection and $f_\bullet$ is an order-embedding. Then, $f^\bullet f_\bullet = \mathrm{id}_P$.

我们需要一个引理。

Lemma 3.1. Suppose $(f_\bullet, f^\bullet)$ is a Galois connection. Then, $f_\bullet$ is an order-embedding if and only if $f^\bullet f_\bullet$ is an order-embedding.

Proof of Lemma 3.1. Suppose $f_\bullet$ is an order-embedding. Then, by Proposition 3.2, $f_\bullet f^\bullet f_\bullet(x) \preccurlyeq f_\bullet f^\bullet f_\bullet(y)$ if and only if $x \preccurlyeq y$. This implies $x \preccurlyeq y$ if and only if

$$\begin{aligned} f^\bullet f_\bullet(x) &\preccurlyeq f^\bullet f_\bullet f^\bullet f_\bullet(y) \\ &\preccurlyeq f^\bullet f_\bullet(y) \end{aligned}$$

由推论3.1. □

Proof of Proposition 3.3. By Proposition 3.1, it suffices to show $f^\bullet f_\bullet(x) \preccurlyeq x$ for all $x \in P$. By Lemma 3.1, $f^\bullet f_\bullet(x) \preccurlyeq x$ if and only if

$$f^\bullet f_\bullet(f^\bullet f_\bullet(x)) \preccurlyeq f^\bullet f_\bullet(x).$$

根据推论 3.1，$f^\bullet f_\bullet(x) \preccurlyeq x$ 对所有$x \in X$均成立。 □

伽罗瓦连接会产生一个闭包算子和一个余闭包算子。

推论 3。2。 $f^\bullet f_\bullet$:P $\to$ P 是闭包算子。$f_\bullet f^\bullet$: Q $\to$ Q 是余闭包算子。

证明。 命题 3。1 和推论3。1 □

我们将 Fix($f^\bullet f_\bullet$) 记为 **Cl**(**K**)，并将 Fix($f_\bullet f^\bullet$) 记为 **Cl**(**L**)。

定义 3.2. 给定一个连接 $(f_\bullet, f^\bullet) : \mathbf{K} \to \mathbf{L}$，定义一个偏序集

$$\mathrm{Gal}(f) = \{(x, y) \mid f_\bullet(x) = y,\ x = f^\bullet(y)\} \subseteq \mathbf{K} \times \mathbf{L}$$

称为 *Galois* 格的f。

Gal(f) 事实上是一个完备格。

命题3.4. 设$f : \mathbf{K} \to \mathbf{L}$ 是完备格**K** 与**L**之间的伽罗瓦连接。那么，下列完**Cl**(**K**)、**Cl**(**L**)、Gal(f)是同构的完备格。

证明。 对于格结构，定理 2。1。同构由命题 3。2。

□

If $f = (f_\bullet, f^\bullet)$ is a Galois connection, $f_\bullet$ is join-preserving and $j^\bullet$ is meet-preserving. Moreover, if $f_\bullet$ is a join-preserving map, $f^\bullet$ is uniquely defined. Conversely, if $f^\bullet$ is a join-preserving map, then $f_\bullet$ is uniquely defined.

Theorem 3.1 (Adjoint Functor Theorem[1])**.** Suppose P and Q are posets and $(f_\bullet, f^\bullet) : \mathrm{P} \to \mathrm{Q}$ is a Galois connection. Then, $f_\bullet$ preserves joins and $f^\bullet$ preserves preserves meets (whenever they exist). Conversely, if $f : \mathrm{P} \to \mathrm{Q}$ is a join-preserving, there there exists a map $f^* : \mathrm{Q} \to \mathrm{P}$ given by

$$f^*(y) = \bigvee \{x \mid f(x) \preccurlyeq y\} \tag{3.3}$$

使得 (f, f^*) 是一个伽罗瓦连接。

有时，将f^* 的公式改写为以下形式会更方便:

$$f^*(y) = \bigvee f^{-1}(\downarrow y)$$

定理 3 的证明1。 在第一部分，假设$f_\bullet$是下伴随。我们证明$f_\bullet$保持并。设 $\{x_i\}_i \subseteq \mathrm{P}$ 是 P 的任意子集。那么，

$$f_\bullet\left(\bigvee_i x_i\right) \preccurlyeq y \quad \Leftrightarrow \quad \bigvee_i x_i \preccurlyeq f^\bullet(y)$$

1 该结果是 Freyd 的伴随函子定理的一个特例 [57]。

Set

$$y = \bigvee_i f(x_i).$$

然后，

$$f_\bullet\left(\bigvee_i x_i\right) \preccurlyeq \bigvee_i f(x_i) \quad \Leftrightarrow \quad \bigvee_i x_i \preccurlyeq f^\bullet(\bigvee_i f(x_i))$$

By Proposition 2.1, $\bigvee_i x_i \preccurlyeq f^\bullet(\bigvee_i f(x_i))$ holds tautologically. Hence, $f_\bullet(\bigvee_i x_i) \preccurlyeq \bigvee_i f(x_i)$ holds. By Proposition 2.1 again,

$$f_\bullet(\bigvee_i f(x_i)) \succcurlyeq \bigvee_i f(x_i)$$

，从而得到所需的等式。

In the second part, we show (3.3) is the upper adjoint of f. By Proposition 3.1, it suffices to show $f^* f(x) \succcurlyeq x$ and $f f^*(y) \preccurlyeq y$. We have

$$f^* f(x) = \bigvee f^{-1}(\downarrow f(x)).$$

但是 $\{x\} \subseteq f^{-1} f(x) \subseteq f^{-1}(\downarrow f(x))$ 意味着

$$f^* f(x) = \bigvee f^{-1}(\downarrow f(x)) \succcurlyeq \bigvee f^{-1} f(x) \succcurlyeq \bigvee \{x\} = x.$$

对于另一个不等式，我们有

$$f f^*(y) = f\left(\bigvee f^{-1}(\downarrow y)\right) \qquad = \bigvee f f^{-1}(\downarrow y)$$

由于根据第一部分，f 保持并。注意，$f f^{-1}(\downarrow y) \subseteq \downarrow y$。因此，

$$f f^*(y) = \bigvee f f^{-1}(\downarrow y) \preccurlyeq \bigvee \downarrow y = y.$$

□

对偶结果如下。

推论 3。3。 若$f : Q \to P$ 保持交，则存在一个保持并的映射$f_* : P \to Q$，使得(f_*, f) 构成一个伽罗瓦连接，定义为

$$f_*(x) = \qquad \bigwedge \{y \mid f(y) \succcurlyeq x\}.$$

3.2 来自关系的伽罗瓦连接

一些作者将伽罗瓦连接定义为伴随的反序映射，而另一些作者——如我们一样——将保序的伴随对称为伽罗瓦连接。

映射 伽罗瓦连接。我们将一对 逆变伽罗瓦连接 定义为 $(f^{\uparrow}, g^{\downarrow})$ 反序映射。

$$f^{\uparrow} : \mathbf{K} \longleftrightarrow \mathbf{L} : f^{\downarrow}$$

使得

$$f^{\downarrow}f^{\uparrow}(x) \succcurlyeq x : \qquad f^{\uparrow}f^{\downarrow}(y) \succcurlyeq y.$$

或者，等价地，

$$f^{\uparrow}(x) \succcurlyeq y \Leftrightarrow \qquad f^{\downarrow}(y) \succcurlyeq x$$

只要存在任何歧义，我们就称保序伽罗瓦连接为协变。存在一个同构。

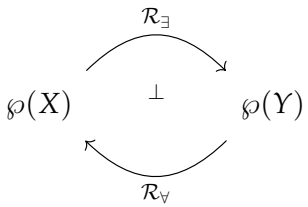
$$\mathcal{L}tc\,(\mathbf{K}^{\mathrm{op}}, \mathbf{L}) \cong \qquad \mathcal{L}tc\,(\mathbf{K}, \mathbf{L}^{\mathrm{op}})^{\mathrm{op}},$$

并且我们可以通过两种方式将 $(f^{\uparrow}, f^{\downarrow})$ 视为普通的（保序）伽罗瓦连接。

读者注意。尽管并反转映射和交反转映射的集合之间存在一一对应关系，但这种对应并不是格的同构。

There are canonical ways to construct Galois connections of each type (covariant, contravariant) from a binary relation $\mathcal{R} \subset X \times Y$. The covariant *Galois correspondence* is originally due to Schmidt [146], while the contravariant Galois correspondence is due to Ore [125].
在第9章中，我们会遇到一个以上述方式构造的伽罗瓦连接。

定理 3.2。 假设 $\mathcal{R} \in \mathrm{Rel}(X,Y)$ 是一个二元关系。那么， $\mathcal{R}$诱导出一个协变伽罗瓦连接



with

$$\mathcal{R}_{\exists}(U) = \qquad \{y \in Y \mid (\exists x) \quad x \in U \ \wedge \ x \,\mathcal{R}\, y\}, \tag{3.4}$$
$$\mathcal{R}_{\forall}(V) = \qquad \{x \in X \mid (\forall y) \quad x\mathcal{R}y \Rightarrow y \in V\}, \tag{3.5}$$

$U \subseteq X$, $V \subseteq Y$。

证明。 首先，我们证明 $\mathcal{R}_\exists$ 保并，因此我们可以应用命题 3。1以计算 $\mathcal{R}_\forall$。设 $\{U_i\}_{i\in I}$ 是 X 的子集族。然后，

$$\begin{aligned}
\mathcal{R}_\exists\left(\bigcup_{i\in I} U_i\right) = & \\
&= \{y \in Y \mid (\exists x) x \in \bigcup_{i\in I} U_i \ \wedge \ x\mathcal{R}y\} \\
&= \{y \in Y \mid (\exists x\ \exists i \in I) \quad x \in U_i \ \wedge \ x\mathcal{R}y\} \\
&= \bigcup_{i\in I}\{y \in Y \mid (\exists x) \quad x \in U_i \ \wedge \ x\mathcal{R}y\} \\
&= \bigcup_{i\in I} \mathcal{R}_\exists(U_i).
\end{aligned}$$

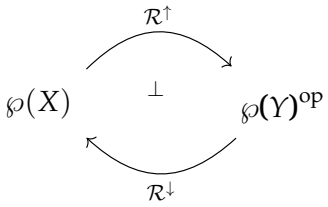
现在，我们可以应用命题 3。1 来计算 $\mathcal{R}_\forall$。令 $V \subseteq Y$。于是，

$$\begin{aligned}
\mathcal{R}_\forall(V) = & \bigcup\{U \subseteq X \mid \mathcal{R}_\exists(U) \subseteq V\} \\
= & \bigcup\{U \subseteq X \mid \{y \in Y \mid (\exists x) \quad x \in U \ \wedge \ x \ \mathcal{R} \ y\} \subseteq V\} \\
= & \{x \in X \mid (\forall y) \quad x \ \mathcal{R} \ y \Rightarrow y \in V\}
\end{aligned}$$

如所愿。 □

以下构造促成了形式概念分析（FCA）领域的诞生 [165]。

定理 3。3。 设 $\mathcal{R} \in \mathrm{Rel}(X,Y)$ 是一个二元关系。那么， $\mathcal{R}$诱导出一个逆变伽罗瓦连接



with

$$\mathcal{R}^\uparrow(U) = \{y \in Y \mid (\forall x)\,(x \in U \Rightarrow x \ \mathcal{R} \ y)\} \tag{3.6}$$

$$\mathcal{R}^\downarrow(V) = \{x \in X \mid (\forall y)\,(y \in V \Rightarrow x \ \mathcal{R} \ y)\}, \tag{3.7}$$

$U \subseteq X,\ V \subseteq Y$。

Proof. We show $\mathcal{R}^\uparrow(U) \supseteq V$ if and only if $\mathcal{R}^\downarrow(V) \supseteq U$ for all $U \subseteq X, V \subseteq Y$ since in $\wp(X)^{\mathrm{op}}$, $V \preccurlyeq V'$ if and only if $V \supseteq U'$. Observe,

$$\mathcal{R}^{\ \downarrow} = \mathcal{R}^{\mathsf{t}\uparrow}.$$

因此，只需互换X 和Y的作用即可。 □

3.3 积分变换

回顾一下，剩余格的剩余运算 $[-.-]:\ \mathbb{L}\times\mathbb{L}\to\mathbb{L}$ 具有定义性质

$$x\star y\preccurlyeq z \quad\Leftrightarrow\quad x\preccurlyeq[y,z].$$

另一种表述这一性质的方式是存在一个伽罗瓦连接

$$(-\star y,[y,-]):\mathbb{L}\to\mathbb{L}.$$

因此，我们得到以下引理。

引理 3。2。 假设$(\mathbb{L},\wedge,\vee,0,1,\star,e,\ [-,-])$是一个完备剩余格。那么，

1. $(\bigvee_i x_i)\star y=\bigvee_i(x_i\star y)$,
2. $[x,\bigwedge_i y_i]=\bigwedge_i[x,y_i]$.

证明。 应用定理3。1。 □

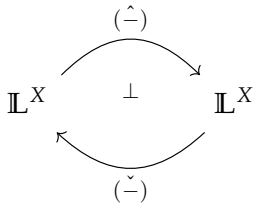
假设 $\mathbb{L}$ 是一个剩余格，并假设X 是一个集合，可能带有附加结构。一个核是一个地图

$$H:X\times X\to\mathbb{L}$$

对核上的积分可如下直接定义。[2] 设f 是一个地图$f:\ X\ \to\mathbb{L}$ 且$H:$ $X\ \times X\ \to\mathbb{L}$ 是一个核。那么，积分变换 f 是地图 $\hat{f}:\ X\ \to\mathbb{L}$ 定义为

$$\hat{f}(x)=\bigvee_{y\in X}H(x,y)\star f(y). \tag{3.8}$$

定理 3。4。 设 $\mathbb{L}$ 是一个剩余格，且 $H:X\ \times X\ \to\mathbb{L}$ 是一个核。那么，以下构成一个伽罗瓦连接



$$(f:X\to\mathbb{L})\mapsto\qquad \hat{f}(x)=\bigvee_{y\in X}H(x,y)\star f(y)$$

$$\check{g}(y)=\bigwedge_{x\in X}[H(y,x),g(y)]\hookleftarrow\qquad (g:X\to\mathbb{L})$$

2 我们给出一个比Maragos 已经引入的定义略为一般的定义 [109],但不如一个类似的范畴论定义一般 [71]。

Proof. Our strategy is to show $(\hat{-})$ is join-preserving and apply Theorem 3.1 to calculate $(\check{-})$. Suppose $f = \bigvee_{i\in I} f_i$ where $f_i \in \mathbb{L}^X$ are $\mathbb{L}$-valued functions on $\mathbb{L}$ with the order $f_i \preccurlyeq g_i$ if and only if $f_i(x) \preccurlyeq g_i(x)$ for all $x \in X$. Then,

$$\begin{aligned}\hat{f}(x) &= \bigvee_{y\in X} H(x,y) \star \left(\bigvee f_i(y)\right)\\ &= \bigvee_{y\in X}\bigvee_{i\in I} H(x,y)\star f_i(y)\\ &= \bigvee_{i\in I} \hat{f}_i(x)\end{aligned}$$

由引理3。2。

为了计算右伴随，根据定理3。1，

$$\check{g} = \bigvee\{f \mid \hat{f} \preccurlyeq g\},$$

这意味着f是其积分变换小于g的最大函数

$$\check{g}(y) = \bigvee_{f\in\mathbb{L}^X} \{f(y) \mid \hat{f}(y) \preccurlyeq g(y) \quad \forall x\}.$$

因此，对于所有 $y \in X$，f 满足

$$\bigvee_{x\in X} H(y,x) * f(x) \preccurlyeq g(y).$$

应用引理3.2，

$$H(y,x) * f(x) \preccurlyeq g(y) \quad \Leftrightarrow \quad f(x) \preccurlyeq [H(y,x), g(y)]\,.$$

因此，收紧该界，

$$\check{g}(y) = \bigwedge_{x\in X} [H(y,x), g(y)]\,.$$

□

Example 3.1 (Max-Plus Connection)**.** Suppose $\mathbb{L}$ is the residuated lattice $(\mathbb{R}_{\max}, \min, \max, -\infty, \infty, 0, +)$ with $[x,y] = y - x$ if $x, y \in \mathbb{R}_{\max} \setminus \{-\infty, \infty\}$. Then, Theorem 3.4 implies

$$(A \oslash \mathbf{x})_i = \bigvee_{i=1}^{n} a_{ij} + x_j$$

有一个上伴随，定义为 $A^\dagger \oslash \mathbf{x}$，其中 $A^\dagger$ 是$a^\dagger_{ij} = -a^{ji}$矩阵。因此，

$$(A^\dagger \oslash \mathbf{x})_j = \bigwedge_{i=1}^{n} a_{ji} - x_j.$$

因此，根据命题 3。 1，

$$A^{\dagger} \oslash (A \oslash \mathbf{x}) \geqslant \mathbf{x},$$
$$A \oslash \left(A^{\dagger} \oslash \mathbf{x}\right) \leqslant \mathbf{x},$$

and

$$A \oslash \mathbf{x} \leqslant \mathbf{y} \quad \text{if and only if} \quad \mathbf{x} \leqslant A^{\dagger} \oslash \mathbf{y}$$

对于所有 $\mathbf{x},\mathbf{y} \in \mathbb{R}_{\max}^{n}$.

3.4 范畴论

We do no assume prior knowledge of category theory. However, as is discussed elsewhere [56], there are irrefutable parallels between order theory and category theory. Reoccurring constructions pertaining to sheaves and cosheaves are further illuminated if we view them as categorical limits and colimits [133] which, here, we present in a more friendly form- as equalizers and coequalizers.

A category $\mathcal{C}$ consists of a collection of *objects* $\operatorname{Ob}\mathcal{C}$ and a collection of *morphisms* also called *arrows* $\operatorname{Mor}\mathcal{C}$ such that for any two objects $x, y \in \operatorname{Ob}\mathcal{C}$, there is a collection of morphisms $\operatorname{Hom}_{\mathcal{C}}(x,y) \subseteq \operatorname{Mor}\mathcal{C}$ between them with a map

$$\circ_{x,y,z} : \operatorname{Hom}_{\mathcal{C}}(x,y) \times \operatorname{Hom}_{\mathcal{C}}(y,z) \to \operatorname{Hom}_{\mathcal{C}}(x,z)$$

满足[3]

- $(h \circ g) \circ f = h \circ (g \circ f)$ （结合律）。
- 对于每个 $x \in \operatorname{Ob} C$，都存在一个 $\mathrm{id}_x \in \operatorname{Hom}_{\mathcal{C}}(x,x)$，使得对于每个可复合的 $f \in \operatorname{Mor}\mathcal{C}$ (恒等态射)，都有 $f \circ \mathrm{id} = f$ 且 $\mathrm{id} \circ f = f$。

通俗地说，如果你告诉我对象和箭，我就能告诉你范畴。

示例 3.1。

- Suppose $(\mathrm{P}, \preccurlyeq)$ is a preorder. Then, we may define a category $\mathcal{P}$ with $\operatorname{Ob}\mathcal{P} = \mathrm{P}$ and $\operatorname{Hom}_{\mathcal{P}}(x,y) = *$ (a one-point set) whenever $x \preccurlyeq y$ in P and $\mathcal{P}(x,y) = \emptyset$ otherwise.
- Suppose $(M, \star, e)$ is a *monoid*. Then, M is a category $\mathcal{M}$ with $\operatorname{Ob}\mathcal{M} = *$, a single object, and $\operatorname{Hom}_{\mathcal{M}}(*,*) = M$. Composition of arrows is defined according to the binary operation $\star$.

3 通常省略 $\circ_{x、y、z}$ 中的下标。

3.4.1 函子

Categories are useful often because we can compare them, a perspective that has lead to connections between different branches of mathematics. Comparing two categories amounts to demonstrating a mapping between them called a functor. A *functor* F between categories $\mathcal{C}$ and $\mathcal{D}$ consists of a maping $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and a map $F_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$ such that for every $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(y, z)$,

$$\begin{aligned} F_{x,z}(g \circ f) &= F_{y,z}(g) \circ F_{x,y}(f), \\ F(\mathrm{id}_x) &= \mathrm{id}_{Fx}. \end{aligned}$$

We have already seen functors. For instance, a functor between poset categories $F : \mathcal{P} \to \mathcal{Q}$ is precisely an order-preserving map; a functor between monoid categories $F : \mathcal{M} \to \mathcal{N}$ is precisely a *monoid homomorphism*. Functors also form a category in their own right. If $\mathcal{D}$ and $\mathcal{D}$ are categories, then the $\mathcal{D}^{\mathcal{C}}$ is a category whose objects are functors $F : \mathcal{C} \to \mathcal{D}$ whose morphisms are maps between functors called natural transformations. If $F, G : \mathcal{C} \to \mathcal{D}$, then a *natural transformation* $\eta : F \Rightarrow G$ is, roughly, a set of morphisms $\{\eta_x : Fx \to Gx\}_{x \in \mathrm{Ob}\,\mathcal{C}}$ such that the following diagram commutes for every $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$

$$\begin{array}{ccc} Fx & \xrightarrow{Ff} & Fy \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_y} \\ Gx & \xrightarrow[Gf]{} & Gy \end{array}$$

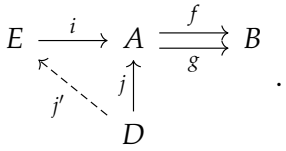
如预期，态射 $\mathrm{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$ 大致上是F 与G之间的自然变换集合范畴。

3.4.2 （余）等化子

We make use of exactly one technical construction from category theory called a *(co)equalizer*. Suppose $\mathcal{C}$ is a data category and $f, g : A \to B$ are parallel morphisms from between objects $A, B \in \mathrm{Ob}\,\mathcal{C}$ (e.g. linear transformations, join-preserving maps). By "a diagram commutes," we mean that if you compose a sequence of arrows in the diagram to form a map from a source object to a target object, you obtain the same map as if you had composed any other sequence of arrows to go from the same source object to the same target object.

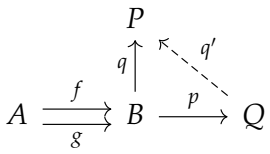
定义 3.3（等化子）。 一对平行对f、$g : A \to B$的等化子 是一个对象 E 和一个地图 $i : E \to A$， 使得对于任意地图 $j : D \to A$

存在唯一映射 $j' : D \to E$ 使下列图表交换



.

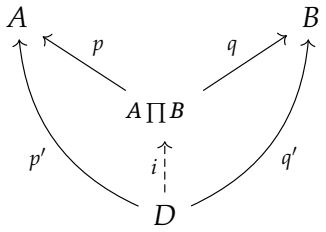
Definition 3.4 (Coequalizer)**.** The *coequalizer* of a parallel pair $f, g : A \to B$ is an object Q and a map $p : B \to Q$ such that for any map $q : B \to P$ there is a unique map $q' : Q \to P$ such that the following diagram commutes



3.4.3 （余）积

每个人都熟悉集合的积（笛卡尔积）或向量空间的积（直和）。这些构造通常可以推广到序范畴中。

Definition 3.5 (Product)**.** Suppose $A, B \in \mathrm{Ob}\,\mathcal{C}$. The *product* of A and B is an object denoted $A \prod B$ with maps $p : A \prod B \to A$ and $q : A \prod B \to B$ called *projection maps* such that for an object D and maps $p' : D \to A$ and $q' : D \to B$ there is a unique map $i : D \to A \prod B$ such that the diagram commutes



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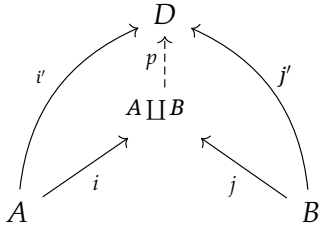
Definition 3.6 (Coproduct)**.** Suppose $A, B \in \mathrm{Ob}\,\mathcal{C}$. The *coproduct* of A and B is an object denoted $A \coprod B$ with maps $i : A \to A \coprod B$ and $j : B \to A \coprod B$ called *injection maps* such that for an object D and maps $i' : A \to D$ and $j' : B \to D$ there is a unique map $p : A \coprod B \to D$ such that the diagram commutes



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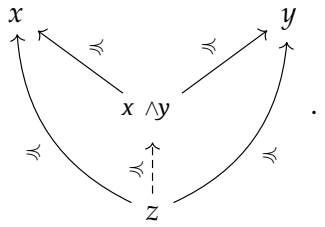
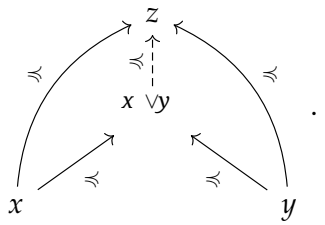
Example 3.2 (Meets & Joins)**.** Recall that a poset P is a category with an arrow $x \to y$ between two elements (objects) if and only if $x \preccurlyeq y$. The product of two elements (if it exists) is the element $x \wedge y$ defined by the universal property



.

同样，x 和y 的余积是元素$x \vee y$ ，由泛性质定义



.

任意元数的积和余积（如果存在）也以类似方式定义。对于交的情况，我们从第一原理证明以下泛性质。

引理 3。3。 设**L** 为一个完备格，I 为任意指标集，$\{x_i\}_{i\in I} \subseteq \mathbf{L}$，且 $y \in \mathbf{L}$。那么，

$$y \preccurlyeq \bigwedge_{i\in I} x_i$$

当且仅当

$$y \preccurlyeq x_i$$

对于所有 $i \in I$。

Proof. Suppose y is a lower bound of the subset $\{x_i\}_{i\in I}$. By definition, $\bigwedge_{i\in I} x_i$ is the greatest lower bound, hence, $\bigwedge_{i\in I} x_i \succcurlyeq y$. Conversely, if $y \preccurlyeq \bigwedge_{i\in I} x_i$, then y precedes the greatest lower bound of $\{x_i\}_{i\in I}$ implying y is also a lower bound by transitivity. □

We encourage the reader, as we develop the theory of lattice-valued sheaves and sheaf Laplacians, to imagine how one could extend results from posets, meets, joins and Galois connections to categories, products, coproducts, and adjunctions, respectively.

3.4.4 伴随

伽罗瓦连接是范畴论中一种称为伴随的更一般构造的特例。

Definition 3.7. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are functors. Then, L is a left adjoint (equivalently, R is a right adjoint) written $L \dashv R$ if and only if there is a bijection

$$\mathrm{Hom}_{\mathcal{D}}(Fx, y) \leftrightarrow \mathrm{Hom}_{\mathcal{D}}(x, Gy)$$

对于所有$x \in \mathrm{Ob}\,\mathcal{C}$, $y \in \mathrm{Ob}\,\mathcal{D}$。

我们已经看到，如果 $(f_\bullet, f^\bullet) : \mathbf{K} \to \mathbf{L}$ 是伽罗瓦连接，则$f_\bullet$和$f^\bullet$是函子。还需要说明的是

$$\mathrm{Hom}_{\mathbf{L}}(f_\bullet(x), y) \leftrightarrow \mathrm{Hom}_{\mathbf{K}}(x, f^\bullet(y)). \tag{3.9}$$

由于 **K** 和 **L** 都是偏序集，同态集 $\mathrm{Hom}(x,y)$ 非空当且仅当 $x \preccurlyeq y$。由此可知 (3.9) 等价于定义 3.1。

While we explictly showed some key properties of Galois connections hold, we, could have derived from more far more general properties of adjunctions. For instance, it is widely known that left adjoints preserve colimits, including coproducts, and right adjoints preserve limits, including products (Theorem 3.1), or that adjoints satisfy the triangle identities (Proposition 3.2).

3.4.5 相关示例

以下是本文中使用的全部范畴。

定义 3.8（相关范畴）。

1. Sets and functions: $\mathcal{S}et$.
2. Vector spaces over a field $\Bbbk$ and linear transformations: $\mathcal{V}ec_{\Bbbk}$
3. Real or complex Hilbert spaces spaces and linear transformations: $\mathcal{H}il_{\mathbb{R}}$ or $\mathcal{H}il_{\mathbb{C}}$.
4. Posets and order-preserving maps: $\mathcal{P}os$.
5. Monoids and monoid homomorphisms: $\mathcal{M}on$.
6. Complete lattices and join-preserving maps: $\mathcal{S}up$
7. Complete lattices and meet-meet preserving maps: $\mathcal{I}nf$.

8. 完备格与伽罗瓦连接： $\mathcal{L}tc$。对象是完备格，态射是低伴随/上伴随的对 $(f_\bullet, f^\bullet)$。复合通过

$$\circ : \mathrm{Hom}_{\mathcal{L}tc}(\mathbf{L},\mathbf{M}) \times \mathrm{Hom}_{\mathcal{L}tc}(\mathbf{K},\mathbf{L}) \to \mathrm{Hom}_{\mathcal{L}tc}(\mathbf{L},$$

$$(g_\bullet, g^\bullet) \circ (f_\bullet, f^\bullet) = (g_\bullet f_\bullet, f^\bullet g^\bullet)$$

4

网络与信号

Networks are ubiquitous, even when we do not realize it. Some examples of networks include social networks, neuronal networks, electrical networks, sensor networks, coauthorship networks, communication networks, transportion networks, and correlation networks. Often the structure of a network determines its function function [35], while other times it is the other way around [162]. Dynamical systems supported on networks, *networked dynamical systems*, exhibit behavior that is more complex than than dynamical systems on their own [163]. *Signal processing* is one view of networked dynamical systems, and it is the view we adopt.

4.1 网络

We model networks with graphs. Recall, a *multiset*, denoted $\mathbb{N}[S]$ is a set S with a map $S \to \mathbb{N}$ encoding the multiplicity of each element of S. A *directed (multi)graph* is a tuple $G = (\mathcal{V}_G, \mathcal{E}_G)$ with $\mathcal{V}_G$ a set of *nodes* and $\mathcal{E}_G \in \mathbb{N}[\mathcal{V}_G \times \mathcal{V}_G]$ is a (multi-)set of *edges*.[1] An *undirected (multi)graph* is consists of the data $G = (\mathcal{V}_G, \mathcal{E}_G)$ with $\mathcal{E}_G \in \mathbb{N}[\mathcal{V}_G \times \mathcal{V}_G / \sim]$ where we identify the $(i,j) \in \mathcal{V}_G \times \mathcal{V}_G$ with (j,i) by $(i,j) \sim (j,i)$. A graph is *simple* if $\mathcal{E}_G$ is a set and $\mathcal{E}_G$ has an empty intersection with the diagonal $\Delta(\mathcal{V}_V) = \{(i,j) \in \mathcal{V}_V \times \mathcal{V}_V \mid i = j\}$. *We hereafter assume all graphs are simple undirected graphs, unless otherwise noted. We denote nodes with lowercase letters i, j, i', j' etc. and denote edges with pairs of lowercase letters $ij, i'j'$ etc.*

In a graph, a *path* γ from $i \in \mathcal{V}_G$ to $j \in \mathcal{V}_G$ is a sequence of nodes $\{i = i_0, i_1, \ldots, i_\ell = j\}$ such that $(i_k, i_{k+1}) \in \mathcal{E}_G$ for all $0 \leqslant k \leqslant \ell$; we write the legnth of γ as $|\gamma| = \ell$. The opposite path $\gamma^{-1} = (j = i_\ell, i_{\ell-1}, \ldots, i_0 = i)$ is a path from j to i. A path is a *loop* if it has the same source and target nodes. The *diameter* of a graph is the length of the longest non-intersecting path $\mathrm{diam}(G)$. Paths form a category as follows.

Definition 4.1 (Free Category)**.** Suppose $G = (\mathcal{V}_G, \mathcal{E}_G)$ is a category. The *free category* of G is a category $\mathrm{Free}(G)$ with objects $\mathcal{V}_G$ and morphisms between $i, j \in \mathcal{V}_G$ given by paths $\gamma : i \to j$. Composition of paths is concatenation written from left to right $\gamma_1 \cdot \gamma_2$ and the identity is the trivial path denoted ε_i for each node $i \in \mathcal{V}_G$.

1 Traditionally, a directed multigraph is called a *quiver* and, equivalently, presented as the data of $Q = (\mathcal{V}_Q, \mathcal{E}_Q, h, t)$. *Head* and *tail* are maps $h, t : \mathcal{E}_Q \to \mathcal{V}_Q$ sending an edge (arrow) to its head and tail, respectively.

Suppose $i \in \mathcal{V}_G$. The *neighbors* of i are the set $\mathcal{N}_i = \{j \mid (i,j) \in \mathcal{E}_G\}$. The cardinality of $\mathcal{N}_i$ is called the *degree* of i, denoted d_i. Given an edge $(i,j) \in \mathcal{E}_G$, the *boundary* is the set $\partial(i,j) = \{i,j\}$. Given a node $i \in \mathcal{V}_G$, the *coboundary* is the (multi)set $\partial i = \{e \in \mathcal{E}_G \mid i \in \partial i\}$. For an edge $ij \in \mathcal{E}_G$, the *edge neighbors* consist of $\mathcal{N}_i = \delta i \cup \delta j - ij$. We say i is *incident* to $e \in \mathcal{E}_G$ if $i \in \partial e$. G is *d-regular* if $d_i = d$ for every $i \in \mathcal{V}_G$. G is complete *complete* if $(i,j) \in \mathcal{E}_G$ for all $i,j \in \mathcal{E}_G$, $i \neq j$. A graph is *connected* if for every $i,j \in \mathcal{V}_G$, there is a path from i to j.

If G has n nodes, then square $n \times n$ matrices encode graph structure with their sparsity patterns. A weighted graph is a graph G with weight function $\omega : \mathcal{E}_G \to \mathbb{R}_+$. A weight function is frequently encoded as a $n \times n$ matrix $[A]_{ij} = a_{ij}$ called the *adjacency matrix* of the weighted graph

$$[A]_{ij} = \begin{cases} \omega(i,j), & \text{if } ij \in \mathcal{E}_G \\ 0, & \text{otherwise} \end{cases}.$$

因此，加权图作为数据给出如下：$G = (\mathcal{V}_G, \mathcal{E}_G, A)$。度矩阵 $[D]_{ij} = d^{ij}$ 的G 是$n \times n$。

$$[D]_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}.$$

注4.1（无权图）。 如果 G 不是加权图，那么我们设定

$$\omega(i,j) = \begin{cases} 1 & ij \in \mathcal{E}_G \\ 0 & \text{otherwise} \end{cases}$$

那么，d_{ii} 是$i \in \mathcal{V}_G$的实际度d_i 。

4.2 图信号

Graph matrices are highly useful for answering questions about signals on graph, such as, is a signal smooth are there Graph signal processing [126] is concerned with $\mathbb{R}$-valued signals on graph which are functions $\mathbf{x} : \mathcal{V}_G \to \mathbb{R}$. Graph *shift operators* are matrices S that encode the graph topology via their sparsity structure and act on graph signals. Here are a few shift operators.

- 该邻接矩阵 A

$$(A\mathbf{x})_i = \sum_{j \in \mathcal{N}_i} a_{ij} x_j \quad ;$$

- 该拉普拉斯矩阵 $L = D - A$

$$(L\mathbf{x})_i = \sum_{j \in \mathcal{N}_i} a_{ij}(x_i - x_j) \tag{4.1}$$

$$= d_{ii} x_i - \sum_{j \in \mathcal{N}_i} a_{ij} x_j; \tag{4.2}$$

- 该随机游走拉普拉斯矩阵 $P = D^{-1}A$

$$(P\mathbf{x})_i = \sum_{j \in \mathcal{N}_i} \frac{1}{d_{jj}} a_{ij} x_j \quad ; \tag{4.3}$$

- 上述内容的矩阵归一化。

在所有情况下，将图移位算子应用于图信号都会产生扩散信号。信号的分量会与邻居节点的分量相结合。以初始条件 $\mathbf{x}[0]$ 反复迭代移位算子会得到各种差分方程。例如，

$$\begin{aligned} \mathbf{x}[t+1] &= \varepsilon S \mathbf{x}[t]; \\ \mathbf{x}[0] &\in \mathbb{R}^n, \end{aligned} \tag{4.4}$$

or,

$$\begin{aligned} \mathbf{x}[t+1] - \mathbf{x}[t] &= -\varepsilon S \mathbf{x}[t]; \\ \mathbf{x}[0] &\in \mathbb{R}^n \end{aligned} \tag{4.5}$$

对于步长 $\varepsilon > 0$。

Spectral properties of the graph shift operator reflect on the nature of the diffusion process. Intricate descriptions of the diffusion based on the eigenvalues and eigenvectors of S exist [36], but coarse results are based on the following facts, for the specific case $S = L$, the graph Laplacian.

事实。 方程（4.5）的与时间无关的解为核 L ，其维数等于 G 的连通分支数。

证明。方程（4.5）的解必须满足

$$0 = \mathbf{x}[t+1] - \mathbf{x}[t] = -L \cdot \mathbf{x}[t]$$

这意味着 $\mathbf{x}[t]$ 属于**L**的核。 □

事实。 假设 $\varepsilon < 2d_{\max}$，G 的最大值度。则，离散时间线性时不变系统（4。5）是全局指数稳定的。

Proof. Let $P = 1 - \varepsilon L$. The graph Laplacian is easily seen to be positive semi-definite. Hence, every eigenvalue is non-negative and has no complex part. By Gershgorin's Theorem [82], the eigenvalues of L lie in the union of closed disks

$$\bigcup_{i=1}^{n} \{z \mid |z - [L]_{ii}| \leqslant \sum_{i \neq j} |[L]_{ij}|\} \subseteq \mathbb{C}.$$

由于特征值均为实数且非负，我们可以得出 L 的特征值介于 $0 \leqslant \lambda \leqslant 2d_{\max}$ 之间。如果 λ_i 是第i^{th} 个特征值

的L，那么 $\mu_i = 1 - \varepsilon\lambda_i$ 是 i^{th} P 的特征值。因此，如果 $\varepsilon < 1/2d_{\max}$，那么

$$\begin{aligned} \mu_i &= 1 - 1/2d_{\max} \cdot \lambda_i \\ &< 1 - 1/2d_{\max} \cdot 2d_{\max} \\ &= 1 \end{aligned}$$

由于 P 的特征值的模小于一，根据线性系统的基本理论，我们得出 [34] 该系统4.5 是全局指数稳定的。

□

综上，这意味着图拉普拉斯算子的扩散过程会稳定在局部常值信号，即在图的连通分支上为常值的信号。

图信号处理 [126] 利用图扩散来滤波不同尺度的图信号。（线性）图滤子是由矩阵表示的线性变换

$$H = \sum_{k=0}^{\infty} h_k S^k$$

Filters act on signals by way of $\mathbf{z} = H\mathbf{x}$. Symmetric graph shift operators admit an orthonormal decomposition $S = V\Lambda V^\top$ where Λ is a diagonal matrix of eigenvalues and V is a column matrix of eigenvectors. Thus, maybe the best way to see what filters "do" is via the *graph Fourier transform*: $\hat{\mathbf{x}} = V^\top \mathbf{x}$. The upshot is that *filters* are componentwise determined by polynomials $\sum_{k=0}^{\infty} h_k \lambda_i^k$ in the Fourier domain for each eigenvalue λ_i of S.

4.2.1 高阶网络

尽管这项工作专注于普通网络，但我们还是不得不提及高阶网络，因为第7 章中介绍的层拉普拉斯算子同样定义在这些对象上，即使是缺乏想象力的读者也能对此有所了解。

A *simplicial complex* $\mathbb{X}$ is a collection of subsets of a node set $\mathcal{V}_G$ such that if $\sigma \in \mathbb{X}$ and $\tau \subseteq \sigma$, then $\tau \in \mathbb{X}$. As a consequence, individual nodes $i \in \mathcal{V}_G$ are elements of $\mathbb{X}$. Simply put, a simplicial complex is a *set closed under taking subsets*. An a single subset is called *simplex* (plural: *simplicies*). Simplices are graded by dimension $\dim \sigma = |\sigma| + 1$. Let $\mathbb{X}_k$ denote the set of $\sigma \in \mathbb{X}$ such that $|\sigma| = k + 1$. These subsets are called *k-simplices*. Simplices, as do graph, have boundaries and coboundaries consisting of $\partial(\sigma) = \{\tau \in \mathbb{X}_{\dim \sigma - 1} \mid \tau \subsetneq \sigma\}$ and $\delta(\sigma) = \{\beta \in \mathbb{X}_{\dim \sigma + 1} \mid \beta \supsetneq \sigma\}$.

单纯复形，如同图一样，是纯粹的组合对象，但它们可以实现为欧几里得空间的子集，就像图可以嵌入其中一样。标准 n-单纯形是空间

$\Delta[k] = \{(t_0, \ldots, t_k) \in \mathbb{R}^{k+1} \mid \sum_i t_i = 1,\ t_i \geqslant 0\ \forall i\}$. The *geometric realizaton* of $\mathbb{X}$ is the topological space $|\mathbb{X}|$ formed by gluing together standard simplices. In this way, we can interpret a simplicial complex as a collection of points, line segments, filled-in triangles, filled-in tetrehedra, and so forth. It is natural to consider face relations here instead of simple incidence relations. We write $\sigma \trianglelefteqslant \tau$ if $\sigma \subseteq \tau$ and $|\tau| = |\sigma| + 1$. The face relation poset $(\mathrm{Face}(\mathbb{X}), \trianglelefteqslant)$ is the transitive reflexive closure of the face relation. A simplicial complex $\mathbb{X}$ is *orientated* if every k-simplex is endowed with an ordering $\tau = [i_0 i_1 \ldots i_k]$. If $\sigma \trianglelefteqslant \tau$, then, neccesarily, σ consists of the data of a cyclic permutation of $\{i_0, i_1, \ldots, i_k\}$. We say the *sign* of σ is the parity of this permuation.

Hypergraphs generalize simplicial complexes, are far more practical, yet less understood. Hypergraphs model higher-order networks where relationships between agents are not required to be pair-wise (e.g. group chats), and a relationship between a number of agents does not necessarily mean that each subset of agents also share a relationship on their own. A *hypergraph* H is a tuple $(\mathcal{V}_H, \mathcal{H}_H)$ where $\mathcal{H}_H \subseteq \wp(\mathcal{V}_G)$. As such, a hypergraph inherits the partial order on $\wp(\mathcal{V}_G)$ bestowing a hypergraph with the structure of a poset $(\mathcal{H}_H, \subseteq)$. A simplicial complex, then, is synonymous with a down-closed hypergraph.

Finally, (regular) cell complexes are another generalization of simplicial complexes of a more topological flavor. Let $\mathbb{D}^k = \{x \in \mathbb{R}^{k+1} \mid \|x\| \leqslant 1\}$ be the unit disk (ball) of dimension k. A *regular cell complex* is constructed from sequence of topological spaces

$$\emptyset = \mathbb{X}_{-1} \subseteq \mathbb{X}_0 \subseteq \mathbb{X}_1 \subseteq \mathbb{X}_2 \subseteq \cdots$$

such that $\mathbb{X}_k$ is obtained from X_{k-1} by "gluing"copies of $\mathbb{D}^k$ to $\mathbb{X}_k$. For more facts about simplicial complexes and cell complexes consult a textbook on algebraic toplology [81]. Parity of faces as well as face relations port from simplicial complexes to this slightly more general setting.

4.3（超）网络的表示

我们讨论图、超图和单纯复形的各种序理论表示及其他组合表示

1. *Incidence poset*. Suppose G is a graph. Incidence forms a relation on the nodes G denoted with the symbol $\trianglelefteqslant$ according to the rule: $i \trianglelefteqslant e$ if $i \in \partial e$. For each edge ij, ther eare exactly two incidence relations $i \trianglelefteqslant ij \trianglerighteqslant j$. The transitive-reflexive closure of $\trianglelefteqslant$ completes the incidence relation to a partial order on the union $\mathcal{V}_G \cup \mathcal{E}_G$ which is denoted, in abuse of notation, P_G. We call this poset the *incidence poset* of G denoted P_G

2. 面关系偏序集。更一般地，单纯复形 $\mathbb{X}$（正则胞腔复形）的单纯形（胞腔）形成一种关系，其中

$\sigma \trianglelefteq \tau$ 当且仅当 $\sigma \in \partial\tau$。该关系的传递自反闭包算子是一个记作 $\mathsf{Face}(\mathbb{X})$的偏序集。

超图由于缺乏可扩展性和统一性而出了名地难以处理。因此，使用其他结构（如（无向）图）来表示超图很流行。

3. *Clique expansion*. The clique expansion of H is the graph G_c with $\mathcal{V}_{G_c} = \mathcal{V}_H$ and $ij \in \mathcal{E}_{G_c}$ if and only if there exists a hyperedges contianing both i and j.
4. *Line graph*. The line graph of H is the graph G_ℓ with $\mathcal{V}_{G_\ell} = \mathcal{H}_H$ and $ee' \in \mathcal{E}_{G_\ell}$ if and only if $e \cap e' \neq \emptyset$.
5. *Star espansion*. The star expansion is the bipartite graph G_s with $\mathcal{V}_{G_s} = \mathcal{V}_H \cup \mathcal{H}_H$ and $(i, e) \in \mathcal{E}_{G_s}$ if and only if $i \in \mathcal{V}_H$, $e \in \mathcal{H}_H$ and $i \in e$.

超图信号只是一个映射 $\mathbf{x}: \ \mathcal{V}_H \to \mathbb{R}$。这些图表示的一个优点是，它们为通过图信号处理来处理超图信号提供了多种方式。

We turn our attention to various order-theoretic approachs to representing hypergraphs. The lattice of down-closed sets $\mathcal{D}(\mathcal{H}_H)$ is a finite distributive lattice. If a hypergraph is a simplicial complex, then every hyperedge is down-closed, hence included in $\mathcal{D}(\mathcal{H}_H)$. The data of the star expansion is equivalent to the data of a binary *membership relation* $\mathcal{I} \subseteq \mathcal{V}_H \times \mathcal{H}_H$ with $i\mathcal{I}e$ if and only if node $i \in e$. The Galois lattice $\mathrm{Gal}(\mathcal{I})$, then, offers another lattice representaiton of a hypergraph. Of note, the Galois lattice $\mathrm{Gal}(\mathcal{I})$ is isomorphic to the lattice of subsets σ of nodes such that the intersection of the set of hyperedges containing every node in σ is again σ. Alternatively $\mathrm{Gal}(\mathcal{I})$ is isomorphic to the lattice of subsets τ of hyperedges such that the set of hyperedges containing every node in the joint intersection of nodes belonging to τ is again τ.

5

层

Sheaf theory is the mathematical study of the "local to global." This is why it is not surprising that sheaf theory has been applied in settings where local data (behavior, information) extends to global data [63, 116]. In the general setting, sheaves are indexed by a poset of open sets. Recall, a $\mathbb{X}$ be a topological space. Then, the open subsets of $\mathbb{X}$, denoted $\text{Open}(\mathbb{X})$, is a poset with inclusion $(\text{Open}(\mathbb{X}), \subseteq)$. Then, a sheaf valued in a data category $\mathcal{D}$ over a topological space $\mathbb{X}$ is a contravariant functor

$$\underline{\mathcal{F}} : \text{Open}(\mathbb{X})^{\text{op}} \to \mathcal{D}$$

In this chapter, we discuss a combinatorial version of sheaf theory called *network sheaf theory*. Instead of data being stored over open sets, it is stored over nodes and edges of an undirected graph. More details on general sheaf theory are found in Appendix A.

5.1 网络层 & 余层

Sheaves are a powerful data structure for interpreting multi-agent systems with near-arbitrary flexibility as far as the type of information/data each agent retrieves and exchanges with her its neighbors. Sheaves encode assignments of data to a geometric or topological structure. Examples of structures that one can build sheaves over are many but some common ones are graphs, simplicial complexes, cell complexes, posets, topological spaces, sites and varieties. While some of these "sheaf theories" are subtle, *network sheaf theory* is not.

Definition 5.1 (Network Sheaves & Cosheaves)**.** Suppose $G = (\mathcal{V}_G, \mathcal{E}_G)$ is a graph, P_G its incidence poset, and $\mathcal{D}$ a data category. A *network sheaf* is a functor $\underline{\mathcal{F}} : \text{P}_G \to \mathcal{D}$. A *network cosheaf* is a functor $\overline{\mathcal{F}} : {\text{P}_G}^{\text{op}} \to \mathcal{D}$.

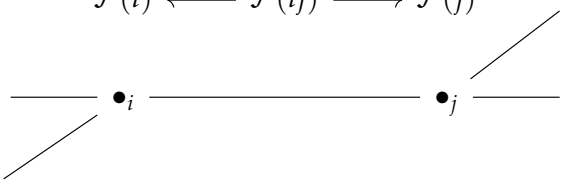
一个层 $\underline{\mathcal{F}}$ 将一个 $\underline{\mathcal{F}}(i)$ 赋给 $\mathcal{D}$ 中的每个 $i \in \mathcal{V}_G$，称为茎；将一个 $\mathcal{F}(ij)$ 赋给每条边 $ij \in \mathcal{E}_G$，以及限制映射

$$\underline{\mathcal{F}}(-) \xrightarrow{\mathcal{F}(i \trianglelefteq ij)} \underline{\mathcal{F}}(ij) \xleftarrow{\mathcal{F}(j \trianglelefteq ij)} \underline{\mathcal{F}}(j)$$



余层 $\mathcal{F}$ 将一个称为 茎的 $\mathcal{F}(i)$ 赋给每个节点 $i \in \mathcal{V}_G$（其中属于 $\mathcal{D}$），将一个 a$\mathcal{F}(ij)$ 赋给每个 $ij \in \mathcal{E}_G$，并赋予 余限制映射

$$\mathcal{F}(i) \xleftarrow{\mathcal{F}(ij \trianglerighteq i)} \mathcal{F}(ij) \xrightarrow{\mathcal{F}(ij \trianglerighteq j)} \mathcal{F}(j)$$



一个 局部（余）截面 是一个特定元素$x_i \in \mathcal{F}$ 或 $\mathcal{F}(i)$。

如果（余）层是一个类（数据结构），那么一个（余）链 是一个（余）层的实例化。

定义 5。2 （余链）。 假设 $\mathcal{F}$ 是一个取值于具有积的数据范畴、定义在 G 上的层。 $\mathcal{F}$ 的余链是积

$$C^\bullet(\underline{\mathcal{F}}; G) = \prod_{\sigma \in \mathcal{V}_G \cup \mathcal{E}_G} \underline{\mathcal{F}}(\sigma)$$

我们将$C^\bullet(G;\ \mathcal{F})$ 到节点上的投影称为0-余链

$$C^0(\underline{\mathcal{F}}; G) = \prod_{i \in \mathcal{V}_G} \underline{\mathcal{F}}(i)$$

以及到边的投影，即1-余链

$$C^1(\underline{\mathcal{F}}; G) = \prod_{ij \in \mathcal{E}_G} \underline{\mathcal{F}}(ij).$$

Informally, (co)chains are sometimes called assignments to (co)sheaves [139]. A (co)chain is like a rough vector field [99] on a manifold in that tangent vectors in a neighborhood need not vary smoothly or continuously. . One of the successes of sheaf theory is the formalization of local-to-global phenomena. Historically, sheaf theory [24] has been motivated by the problem of extending (continuous, smooth, holomorphic, rational) functions defined on open sets in a (topological, manifold, complex manifold, variety) space to larger open sets. In the specialized case of network sheaves, the *local-to-global* problem asks *if a* 0*-cochain extend to a* 1*-cochain?* In the affirmative, a 0-cochian is said to be consistent.

定义 5。3 （截面）。 假设 $\mathcal{F}$ 是一个层， $\mathcal{F}$ 是 G 上的一个余层。 $\mathcal{F}$ 的一个 截面 是一个赋值 $\mathbf{x} \in C^\bullet(\mathcal{F};\ G)$。

$$\underline{\mathcal{F}}_{i \trianglelefteq ij}(x_i) = x_{ij} = \underline{\mathcal{F}}(j \trianglelefteq ij)(x_j) \tag{5.1}$$

在每条边 ij 上都一致。 $\mathcal{F}$ 的截面集合记为$\Gamma(G;\ \mathcal{F})$。

下面的例子说明，即使在一个非常简单的例子中，限制映射以及图的拓扑结构也决定了一致赋值是否存在。

例 5.1 （常值层）。 设G 是一个任意的图。设V 是一个向量空间。记作V 的（线性）常值层是一个取值于 $\mathcal{V}ec$ 的层，具有以下数据

$$\begin{aligned}\underline{V}(i) &= V\\ \underline{V}(ij) &= V\\ \underline{V}_{i\leqslant ij} &= \mathrm{id}_V.\end{aligned}$$

当然，常值层在其他数据范畴中也有意义。它是在每个茎上都有一个特定对象且限制映射为恒等映射的网络层。

示例 5.2 （一个扭曲的层）。 令 $\mathcal{F}$ 为定义在一个 3-团上的层（图 5.1）。为了使一个 0-上链扩展为一个 1-上链，

$$x_i = -x_j$$

对于每条边ij。然而，如果在两条边上达成一致，那么在剩余的边上就不可能达成一致，因为其端点的赋值相同。因此，唯一的整体截面是零向量 $\mathbf{x} = \mathbf{0}$。

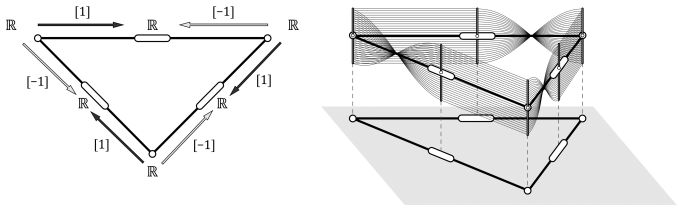


图 5.1：一个扭曲的层。

如何计算一个层的截面？G 上的一个定向是一对映射

$$(-)_{+,-} : \mathcal{E}_G \to \mathcal{V}_G,$$

将一条边ij 发送到其正端点和负端点 $ij_+ \cup ij_- = \{i, j\}$。给定一个定向的选择，截面等价于以下等化子

$$H^0(G;\underline{\mathcal{F}}) \dashrightarrow C^0(G;\underline{\mathcal{F}}) \underset{\delta_+}{\overset{\delta_-}{\rightrightarrows}} C^1(G;\underline{\mathcal{F}}) \tag{5.2}$$

其中 δ_-、δ_+ 按分量定义

$$\begin{aligned}(\delta_-\mathbf{x})_{ij} &= \underline{\mathcal{F}}(ij_- \trianglelefteq ij),\\ (\delta_+\mathbf{x})_{ij} &= \underline{\mathcal{F}}(ij_+ \trianglelefteq ij).\end{aligned}$$

映射 δ_-、$_+$ 称为 余边界映射。等化子条件规定，$H^0(G;\,\mathcal{F})$ 是$C^0(G;\,\mathcal{F})$ 的最大子对象，使得

$$(\delta_-\mathbf{x})_{ij} = (\delta_+\mathbf{x})_{ij}$$

对于每个 $\mathbf{x} \in H^0(G;\,\underline{\mathcal{F}})$。总之，

命题 5.1. $\Gamma(G;\mathcal{F}) \cong H^0(\underline{G};\ \mathcal{F})$. _

一个加法范畴是一个数据范畴，其中映射具有阿贝尔群的结构；映射可以相加、相减，并且任意两个对象之间都有一个零映射。

示例 5.3（加法范畴）。 假设数据范畴 $\mathcal{D}$ 是一个加法范畴。那么，等化子（5.2）可以重写为等化子

$$H^0(G;\underline{\mathcal{F}}) \dashrightarrow C^0(G;\underline{\mathcal{F}}) \underset{0}{\overset{\delta_+ - \delta_-}{\rightrightarrows}} C^1(G;\underline{\mathcal{F}}) \tag{5.3}$$

于是，$H^0(G;\underline{\mathcal{F}}) \cong$ 核 $(\delta_+ - \delta_-)$.

The consequence of the above example is that calculating cohomology, at least in the category of vector spaces, reduces to calculating the kernel of linear transformation. Of course, many information systems exhibit vector-valued data, but others exhibit other data types. Returning to the equalizer (5.2), if the data category is not additive, there is no clear path forward. We cannot write the equalizers as a kernel becuase each restriction map in a $\mathcal{D}$-valued network sheaf has no additive inverse. Hence, standard practices of linear and homological algebra are of no avail. We have already encountered categories that fail to be additive: the category $\mathcal{S}up$ of complete lattices and join-preserving maps, as well as the category $\mathcal{I}nf$ of complete lattices and meet-preserving maps.

5.2 同调与上同调

In the past 20 years, applied algebraic topology [59] has flourished into a mature research community. Key to this development is *persistent homology* and software computing it [127]. Suppose $\mathbb{X}$ is a simplicial complex. Then, the *homology of* $\mathbb{X}$ *in degree* k, denoted $H_k(\mathbb{X})$, is a vector space whose basis elements correspond to independent connected components ($k = 0$), cycles ($k = 1$), voids ($k = 2$), and higher dimensional versions of these ($k > 2$). The difficulty with using homology directly to understand data is that it is not robust to perturbations. This leads to the notion of persistent homology introduced in the early 2000s [49]. A filtration of simplicial complexes (spaces) is a collection of subcomplexes (subspaces) $\{\mathbb{X}\}_{t\in \mathrm{P}}$ indexed by some poset P such that

$$\mathbb{X}_s \subseteq \mathbb{X}_t \text{ whenever } \qquad\qquad s \preccurlyeq t$$

in P. Homology is functorial in the sense that continuous function between space $f : \mathbb{X} \to \mathbb{Y}$ induces a linear transformation between vector spaces $f : H_k(\mathbb{X}) \to H_k(\mathbb{X})$ [81]. Hence, the inclusion maps of a filtration induce a series of maps $H_k(\mathbb{X}_s) \to H_k(\mathbb{X}_t)$ whenever $s \preccurlyeq t$.

由链C索引的滤过可以逐点分解为称为条形码的子对象 [128] 它在总结数据的拓扑学方面取得了巨大成功 [60]。

We have not said anything about how homology or persistent homology is actually computed. This is because network sheaves have cohomology, a dualization of homology. Cohomology is defined in a general algebraic setting, but is equivalent to homology in field coeficients by the usual isomoprhism between a vector space and its dual.

定义 5.4 （上链复形）。 设 $\mathcal{D}$ 是一个带有零映射的范畴。一个（有界）*上链复形* 是对象$C^\bullet = \{C^k\}_{k\geqslant 0}$ 的一个序列，其间的映射称为*上边界映射*。

$$C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} C^3 \xrightarrow{\delta^3} \cdots$$

使得 $\delta^{k+1}\delta^k = 0$ 对于每个 $k \geqslant 0$。

如果 $\mathcal{D}$ 有核、（正规）像和商（见 [69]），则上同调 $C^\bullet$ 为

$$H^k(C^\bullet) = \qquad \text{Ker } (\delta^k)/\text{Im } (\delta^{k-1}).$$

5.3 霍奇拉普拉斯算子

Motivated by certain problems in multi-agent systems [124], decentralized algorithms for computing homology of simplicial complexes have been proposed [117]. In principle, in order to compute $H_k(\mathbb{X})$, a k-simplex (e.g. a node) can pass messages with neighboring k-simplices and eventually converge to a homology vector. These algorithms, while promising, have not been widely adopted. The concept, more broadly, of distributed computation of cohomology is pertinent to the study of sheaves, objects more conducive to distributive algorithms than unstructured simplicial complexes.

以下定义至少可以追溯到 Eckmann [48]。设$C^\bullet$ 是实（或复）希尔伯特空间范畴中的一个复形。那么，组合 *HodgeLaplacian* 是线性变换

$$\begin{aligned} \Delta_\bullet : &\qquad C^\bullet \to C^\bullet \\ \Delta_\bullet = &\qquad \delta^* \circ \delta + \delta \circ \delta^* \end{aligned} \tag{5.4}$$

霍奇拉普拉斯算子将C^k 中的余链与C^{k-1}和C^{k+1}中的余链整合起来，因此自然地推广了热方程，并引出了以下动力系统。

$$\dot{\mathbf{x}} = \qquad -\Delta_k \mathbf{x}, \tag{5.5}$$

初始条件为 $\mathbf{x}(0) \in C^k$。

此外，热方程稳定点的解释由以下定理加以证明。

定理5.1 (Hodge 定理)。 设$(C^\bullet, \delta^\bullet)$ 是一个内积空间的上链复形。那么，

$$H^k(C^\bullet) \cong \qquad \text{Ker } (\Delta^k).$$

为方便读者，我们给出证明。

Proof of the Hodge Theorem. For ease of notation, let $Z_k = \text{Ker } (\delta^k)$ and $B_k = \text{Im } (\delta^{k-1})$, called the *cocycles* and *coboundaries*, respectively. The Hodge Laplacian decomposes into a sum of symmetric, and hence positive semi-definite, linear transformations

$$\begin{aligned} \Delta_k &= \Delta_k^+ + \Delta_k^- \\ &= \delta^{k^*}\delta^k + \delta^{k-1}\delta^{k-1^*} \end{aligned}$$

称为上部-和下部- *Hodge*拉普拉斯算子。回顾一下，按照商向量空间的定义，在有限维内-积空间的范畴中

$$H^k(C^\bullet) \cong \qquad Z_k \cap B_k^\perp.$$

We have $x \in \text{Ker } (\Delta_k^+)$ if and only if $\langle x, \Delta_k^+ x\rangle_{C^k} = 0$ for all $x \in C^k$. Similarly, $x \in \text{Ker } (\Delta_k^-)$ if and only if $\langle x, \Delta_k^- x\rangle_{C^k} = 0$. Hence, it suffices to show

$$\begin{aligned} \text{Ker } (\Delta_k^+ + \Delta_k^-) &= \\ \text{Ker } (\Delta_k^+) \cap \text{Ker } (\Delta_k^-) &= \qquad Z_k \cap B_k^\perp. \end{aligned}$$

因此，x ∈核 (Δ_k) 当且仅当

$$\begin{aligned} \langle \delta^k x, \delta^k x\rangle_{C^k} &= 0 \\ \langle \delta^{k-1^*} x, \delta^{k-1^*} x\rangle_{C^k} &= 0 \end{aligned}$$

当且仅当 x ∈ 核 (δ^k) ∩核 (δ^{k-1^*})。

我们断言 核 $(\delta^{k-1^*}) = B_k^\perp$。首先，$x$ ∈核 (δ^{k-1^*}) 当且仅当

$$\langle \delta^{k-1^*} x, y\rangle_{C^{k-1}} = 0.$$

对于每个 $y \in C^{k-1}$。然后，

$$\langle x, \delta^{k-1} y\rangle_{C^k} = \qquad \langle \delta^{k-1^*} x, y\rangle_{C^{k-1}} = \qquad 0$$

这表明，x 与每个上余边界正交，当且仅当 $\delta^{k-1^*} x = 0$。

□

根据常微分方程的基本理论，热方程（5.5）可写成闭式

$$\mathbf{x}(t) = \qquad e^{-\Delta_k t}\mathbf{x}(0).$$

Because Δ_k is a positive semidefinite linear operator, the Hodge Theorem implies $x(t)$ converges to a projection of $\mathbf{x}(0)$ onto $H^k(C^\bullet)$ as $t \to \infty$. Thus, a näive algorithm for computing $H^k(C^\bullet)$ consists of the following steps.

1. 选择一个 $\mathbf{x}(0) \in C^k$.2. 求解初始条件为 $\mathbf{x}(0)$ 的热方程（5.5）。
3. 将 $\mathbf{x}(\infty)$ 加入不断扩充的 $H^k(C^\bullet)$ 基中。

A strategy to compute cohomology, such as the one above, in prinicple could be parallelized because because the Hodge Laplacian is a local operator. However, it would be our guess that practical difficulties lie in numerical stability. On the other hand, calculating the entire cohomology subspace is not always practical or necessary. Applications often call for calculating a single nearest or optimal cohomology class.

5.3.1 层拉普拉斯算子

We are primarily interested in dynamical systems on sheaves, because, after all, the final destination for this line or research is designing, learning, and controlling multi-agent systems. For sheaves valued in the data category of real or complex Hilbert spaces, the fundamental ingredients for a Hodge theory are present due to the following fact. For the following, suppose $\mathbb{X}$ is a simplicial complex or regular cell complex. Then, a *cellular sheaf* valued in a category $\mathcal{D}$ is a functor $\underline{\mathcal{F}} : \mathrm{Face}(\mathbb{X}) \to \mathcal{D}$. As a special case, if $\mathbb{X}$ is a graph, then a cellular sheaf over $\mathbb{X}$ is exactly the same data as a network sheaf.

Proposition 5.2. Suppose $\underline{\mathcal{F}} : \mathrm{Face}(\mathbb{X}) \to \mathcal{H}il$ is a sheaf of real or complex Hilbert spaces over a simplicial complex. Suppose each simplex $\sigma \in \mathbb{X}_k$ is encoded as an ordered tuple. Then, the following is a cochain complex

$$C^0(\mathbb{X};\underline{\mathcal{F}}) \xrightarrow{\delta^0} C^1(\mathbb{X};\underline{\mathcal{F}}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{k-1}} C^k(\mathbb{X};\underline{\mathcal{F}}) \xrightarrow{\delta^k} \cdots$$

with

$$C^k(\mathbb{X};\underline{\mathcal{F}}) = \bigoplus_{\sigma\in\mathbb{X}_k} \underline{\mathcal{F}}(\sigma),$$

$$\left(\delta^k \mathbf{x}\right)_\tau = \sum_{\sigma \trianglelefteq \tau} \mathrm{sgn}(\sigma)\cdot \underline{\mathcal{F}}(\sigma \trianglelefteq \tau)(x_\sigma).$$

证明。参见 [42]。 □

因此，层上同调 $H^k(\mathbb{X};\ \mathcal{F})$ 不过是复形 $C^\bullet(\underline{\mathbb{X}};\ \mathcal{F})$的上同调，而线性层拉普拉斯算子

$$L_k : C^k(\mathbb{X};\underline{\mathcal{F}}) \to C^k(\mathbb{X};\underline{\mathcal{F}})$$

只不过是$C^\bullet$的霍奇拉普拉斯算子。现在将范围限定在线性层作用于图的情形，网络层的线性层拉普拉斯算子（$k\ = 0$）具有具有以下形式

$$(L\mathbf{x})_i = \sum_{j\in\mathcal{N}_i} \underline{\mathcal{F}}^*_{i\trianglelefteq ij}\left(\underline{\mathcal{F}}_{i\trianglelefteq ij}(x_i) - \underline{\mathcal{F}}(j\trianglelefteq ij)(x_j)\right) \tag{5.6}$$

5.4 层拉普拉斯算子的例子

The examples to follow show that, indeed, network sheaf Laplacians generalize both graph Laplacians and graph connection Laplacians, each of which, in turn, approximate smooth Laplacians on manifolds. Both the graph Laplacian and the graph connection Laplacian serve as inspiration for Laplacians of lattice-valued sheaves (Chapter 7).

5.4.1 图拉普拉斯算子

我们现在以“层论”的视角重新审视图拉普拉斯算子。假设$\mathbb{M} \subseteq \mathbb{R}^N$ 是一个k维光滑（黎曼）流形。回忆一下，一个光滑流形由空间 $\mathbb{M}$ 和一个同胚-映射的数据组成

$$\varphi_U : U \to \mathbb{R}^k$$

对于每个称为坐标图的开集U，满足

$$\varphi_V \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V)$$

is a differentiable for open sets $U \cap V \neq \emptyset$. A real-valued function $f : U \to \mathbb{R}$ on an open U is said to be *smooth* if $f \circ \varphi_U^{-1} : \varphi_U(U) \to \mathbb{R}$ is differentiable. Real-valued functions on open sets define sheaf[1] $\mathcal{O}_{\mathbb{M}}$ defined by the following data

$$\begin{aligned} \mathcal{O}_{\mathbb{M}} : & \qquad \mathrm{Open}(\mathbb{M})^{\mathrm{op}} \to \mathcal{V}ec \\ \mathcal{O}_{\mathbb{M}}(U) = & \qquad \{f : U \to \mathbb{R}\} \\ \mathcal{O}_{\mathbb{M}}(V \subseteq U)(f) = & \qquad f_{|V}. \end{aligned}$$

If G is a weighted complete graph constructed from $\mathbb{M}$ by sampling points $\{x_1, x_2, \ldots, x_n\}$ uniformly from $\mathbb{M}$ and weighting edges inversely proportional to the distances of their boundary in $\mathbb{R}^N$, then the constant sheaf $\underline{\mathbb{R}}$ over G approximates the sheaf of smooth functions on $\mathbb{M}$. This weighted graph is directly weighted to the following network sheaf

$$\begin{aligned} \mathcal{F}(i) &= \mathbb{R} \\ \mathcal{F}(ij) &= \mathbb{R} \\ \mathcal{F}_{i \trianglelefteq ij}(x_i) &= e^{-\frac{\|xi - xj\|^2}{4t}} \cdot x_i \end{aligned}$$

1 参见 [24],例如

其中 $t > 0$。 $\mathcal{F}$ 的层拉普拉斯算子已计算得到

$$\begin{aligned}(L\mathbf{x})_i &= \sum_{j\in\mathcal{N}_i}\left(e^{-\frac{\|x_i-x_j\|^2}{2t}}\right)^*\left(e^{-\frac{\|x_i-x_j\|^2}{2t}}x_i - e^{-\frac{\|x_i-x_j\|^2}{2t}}x_j\right)\\ &= \left(\sum_{j\in\mathcal{N}_i}e^{-\frac{\|x_i-x_j\|^2}{t}}\right)\cdot x_i - \sum_{j\in\mathcal{N}_i}e^{-\frac{\|x_i-x_j\|^2}{t}}\cdot x_j,\\ &= d_i x_i - \sum_{j\in\mathcal{N}_i}a_{ij}x_j\end{aligned}$$

We can identify graph Laplacians of arbitrary graphs as sheaf Laplacians in the same manner by setting stalks $\mathbb{R}$ and defining both restriction maps over an edge $ij \in \mathcal{E}_G$ to be multiplication by $\sqrt{a_{ij}}$. In the above case, as the parameter t approaches zero, the graph Laplacain, and thus the sheaf Laplacian is shown to approximate the Laplace-Beltrami operator, a classical operator [100] acting on real-valued functions on $\mathbb{M}$ (i.e. the global sections of $\mathcal{O}$, i.e. the scalar fields on $\mathbb{M}$). For the following, suppose $\mathbb{M} \subseteq \mathbb{R}^N$ is a k-dimensional maniold with Laplace-Beltrami operator Δ, and suppose $\{x_1, \ldots, x_n\} \subseteq \mathbb{M}$ is collection of points sampled uniformly from $\mathbb{M}$. Set $t_n = n^{-\frac{1}{k+2+\alpha}}$ where $\alpha > 0$, and let $L_n^{t_n}$ be the graph Laplacian of the complete weighted graph on $\mathcal{V}_G = \{x_1, \ldots, x_n\}$ with weights

$$a_{ij} = e^{-\frac{\|x_i-x_j\|^2}{t_n}}.$$

定理 5.2([15])。 设 $f \in C^\infty(\mathbb{M})$。则，

$$\lim_{n\to\infty}\frac{1}{t_n(5\pi t_n)^{\frac{k}{2}}}L_n^{t_n}f(x) = \frac{1}{\mathrm{vol}(\mathbb{M})}\Delta f(x)$$

在概率论中。

5.4.2 联络拉普拉斯算子

由于图拉普拉斯算子是层拉普拉斯算子的首个关键示例，我们介绍第二个示例示例，即图联络拉普拉斯算子。

Recall, some other elements from differential geometry [86]. Suppose $\mathbb{M}$ is a k-dimensional Riemanian manifold. To each element $x \in \mathbb{M}$, the *tangent space* T_x is a vector space whose elements are equivalence classes of curves $\gamma : (-1, 1) \to \mathbb{M}$ with $\gamma(0) = x$ whose derivatives coincide. In practice, you can visualize tangent spaces as tangent lines, tangent planes etc. The *tangent bundle* is the disjoint union $T\mathbb{M} = \bigsqcup_{x\in\mathbb{M}}$ together with the projection map $T\mathbb{M} \xrightarrow{p} \mathbb{M}$. A *vector field* is a section of the tangent bundle,

$$s : \mathbb{M} \to T\mathbb{M} \quad p \circ s = \mathrm{id}_{\mathbb{M}};$$

地图 s 将每个点 $x \in \mathbb{M}$ 映射为一个切向量 $v \in T_x\mathbb{M}$。切丛的截面定义了一个层[2]

$$\begin{aligned} \Gamma : & \quad \mathrm{Open}(\mathbb{M}) \to \mathcal{V}ec \\ \Gamma(U) = & \quad \{s : U \to T\mathbb{M} \mid p \circ s = \mathrm{id}_U\} \\ \Gamma(V \subseteq U)(s) = & \quad s_{|V} \end{aligned}$$

确实， $\Gamma(U)$ 是一个称为局部截面的向量空间。 $\Gamma(\mathbb{M})$ 是定义在整个流形上的向量场空间。

Approximations of the tangent bundle lead to complications. Even if points sampled from a manifold are close, their tangent spaces will differ, in general. High-dimensional data sets are often assumed to have been sampled from low-dimensional manifold. This presents a problem because we cannot relate data in two separate local coordinates. Fortunately, this is a topic widely studied in modern differential geometry, called *parallel transport* [94]. The following is a definition or a theorem depending on how you look at it. We leave technical details to the curious reader.

Definition 5.5 ([86])**.** Suppose $\mathbb{M}$ is a k-dimensional manifold and γ is a smooth path $\gamma : [0,1] \to \mathbb{M}$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then, there is a unique invertible linear transformation

$$\mathcal{P}_\gamma : T_x\mathbb{M} \to T_y\mathbb{M}$$

称为平行移动 将一个切向量 $v \in T_x\mathbb{M}$ 发送为切向量 $\mathcal{P}_\gamma(v) \in T_y\mathbb{M}$。

平行移动会产生向量扩散，并且可以通过正交线性映射进行近似。以下内容近似表示联络拉普拉斯算子 [7]，，它与拉普拉斯-贝尔特拉米算子和霍奇拉普拉斯算子非常相近。

定义5.6 （图联络拉普拉斯算子 [11]）。 固定一个维数k，并假设$G = (\mathcal{V}_G, \mathcal{E}_G)$ 是一个带有正交线性变换数据的加权图

$$O_{ij} : \mathbb{R}^k \to \mathbb{R}^k$$

那么， 图联络拉普拉斯算子是矩阵 ∇^2

$$\begin{aligned} \nabla^2 = & \quad (\mathbb{R}^k)^n \to (\mathbb{R}^k)^n \\ (\nabla^2 \mathbf{v})_i = & \quad d_i x_i - \sum_{j \text{ß} \mathcal{N}_i} a_{ij} O_{ij} v_j \end{aligned}$$

$i \in \mathcal{V}_G$。

2 例如参见 [24]。

As a consequence of an approximation result [153] analogous to Proposition 5.2, parallel transport between tangent spaces of the manifold are approximated in the graph connection Laplacian by orthogonal linear transformations O_{ij} between the tangent spaces $O_{ij} : T_{x_i}\mathbb{M} \to T_{x_j}\mathbb{M}$. The graph connection Laplacian is a sheaf Laplacian. Suppose $\{O_i \in \mathbb{R}^{k\times K}\}_{i\in\mathcal{V}_G}$ is a family of orthogonal matrices indexed by the nodes of a weighted graph G. Construct the sheaf as follows

$$\begin{aligned}\underline{\mathcal{F}}(i) &= \mathbb{R}^k,\\ \underline{\mathcal{F}}(ij) &= \mathbb{R}^k,\\ \underline{\mathcal{F}}_{i\trianglelefteq ij}(x_i) &= \sqrt{(a_{ij})}O_i x_i\end{aligned}$$

然后，

$$\begin{aligned}(L\mathbf{x})_i &= \sum_{j\in\mathcal{N}_i}\sqrt{a_{ij}}O_i^*\left(\sqrt{a_{ij}}O_i x_i - \sqrt{a_{ij}}O_j x_j\right)\\ &= d_i x_i - \sum_{j\in\mathcal{N}_i} a_{ij}O_i^* O_j x_j\end{aligned}$$

集合范畴 $O^{ij} = O_i^* O^j$，其为正交的。

第二部分塔尔斯基拉普拉斯算子

6

格值层

我们引入一类新型的网络层，研究这些层的全局截面，以及模拟经典的层运算 [24]。

6.1 塔斯基层

我们从图上的层的定义开始，其中茎是完备的，限制映射是伽罗瓦连接。

定义 6.1 (塔斯基层)。 设 $G = (\mathcal{V}_G, \mathcal{E}_G)$ 是一个图。塔斯基层是一个函子

$$\mathcal{F} : \mathrm{P}_G \to \mathcal{L}tc$$

塔斯基层赋予一个

1. 完备格 $\mathcal{F}(i)$ 对应每个节点 $i \in \mathcal{V}_G$, 2. 完备格 $\mathcal{F}(ij)$ 对应每条边 $ij \in \mathcal{E}_G$, 3. 伽罗瓦连接

$$\mathcal{F}(i) \underset{\mathcal{F}(i \trianglelefteqslant ij)^{\bullet}}{\overset{\mathcal{F}(i \trianglelefteqslant ij)_{\bullet}}{\rightleftarrows}} \mathcal{F}(ij)$$

对于每个 $i \in \mathcal{V}_G$ 和 $j \in \mathcal{N}_i$。

Remark 6.1 (Notation)**.** Slightly simplifying cumbersome notation, we will write $(\underline{\mathcal{F}}(i \trianglelefteqslant ij), \overline{\mathcal{F}}(i \trianglelefteqslant ij)$ for the Galois connection $(\mathcal{F}(i \trianglelefteqslant ij)_{\bullet}, \mathcal{F}(i \trianglelefteqslant ij)^{\bullet})$.

Tarski 双层包含层 $\underline{\mathcal{F}}$ 和余层 $\overline{\mathcal{F}}$ 的数据，方式为

$$\begin{aligned} \underline{\mathcal{F}} &: && \mathrm{P}_G \to \mathcal{S}up, \\ \overline{\mathcal{F}} &: && \mathrm{P}_G{}^{\mathrm{op}} \to \mathcal{I}nf. \end{aligned}$$

通过选择限制映射作为下伴随，并选择余限制映射作为上伴随。

定理 6.1. 设 P 是一个偏序集。那么，以下范畴彼此等价。

$$\begin{aligned} \mathcal{L}tc^{\mathrm{P}} &\simeq && \mathcal{S}up^{\mathrm{P}} \\ \mathcal{L}tc^{\mathrm{P}} &\simeq && \mathcal{I}nf^{\mathrm{P}^{\mathrm{op}}} \end{aligned}$$

证明。 只需证明 $\mathcal{S}up \simeq \mathcal{L}tc$ 因为我们从一个自然同构 $\mathcal{L}tc^{\mathrm{P}} \simeq \mathcal{S}up^{\mathrm{P}}$ 得到一个自然同构 $\mathcal{L}tc \simeq \mathcal{S}up$。

$$\Pi : \mathcal{L}tc \to \mathcal{S}up$$

是一个等价。范畴论中的一个标准事实是，只需证明 Π 是满的、忠实的且本质上满射 [133, 定理1.5.9]。满且忠实相当于证明一个双射

$$\mathrm{Hom}_{\mathcal{L}tc}(\mathbf{K}, \mathbf{L}) \cong \mathrm{Hom}_{\mathcal{S}up}(\mathbf{K}, \mathbf{L}).$$

这直接由定理 3.1得出。论证 $\mathcal{I}nf^{\mathrm{op}} \simeq \mathcal{L}tc$，因此 $\mathcal{L}tc^{\mathrm{P}} \simeq \mathcal{I}nf^{\mathrm{Pop}}$，由对偶性得出。 □

定理 6.1 的结果是 $\mathcal{F}$ 的数据等价于 $\mathcal{F}$ 的数据或 $\mathcal{F}$的数据。

注 6.2 （双层）。 “双层”的概念是在单纯复形上阿贝尔群的层和余层的背景下引入的 [108, 120]。他们对双层的定义由同一底空间（例如图）上的一个层和一个余层以及层的茎到余层的茎之间的一个相容同态组成。在某些情况下，我们的定义与他们的一致。例如，如果 $\mathcal{F}$ 是一个单层，其伴随的满余层为 $\mathcal{F}$，则 *Tarski*双层 $(\mathcal{F}, \mathcal{F})$ 是 McPhersen-Patel 意义上的双层 [108]。

6.2 从旧层得到的新层

在以下示例中，具体对象（集合、向量空间、希尔伯特空间）的层将用正体表示；诱导的塔斯基层将用花体表示。

6.2.1 从集合层

Suppose $S : \mathrm{P}_G \to \mathcal{S}et$ is a set-valued network sheaf. Such a structure assigns the data of a set to every node and edge and (arbitrary) maps between these sets over edges. Sheaves of sets are perhaps the most basic examples of sheaves and have found their way to applications in electrical engineering, quantum theory, and topological data analysis [2, 46, 67]. The key construction is the following. Suppose $f : X \to Y$ is a function. Then, there is a Galois connection

$$\wp(X) \underset{f^{-1}(-)}{\overset{f(-)}{\rightleftarrows}} \wp(Y)$$

将X 的一个子集映射到其像，并将Y 的一个子集映射到其原像。因此，一个层 $F : \mathrm{P}_G \to \mathcal{S}et$ 通过幂集函子分解，引入了一类塔斯基层

$$\begin{array}{ccc} \mathrm{P}_G & \xrightarrow{S} & \mathcal{S}et \\ & \searrow^{\mathcal{S}} & \downarrow{\scriptstyle \wp(-)} \\ & & \mathcal{L}tc \end{array}$$

以下扩展示例可以放心跳过。

Example 6.1 (Reeb Graphs)**.** In topological data analysis, filtrations of a metric space by sublevel sets serve as a canonical example of a topological filtration. This example, roughly, subsumes Čech filtrations of point clouds [14]. The path-connected components of space is a functor $\pi_0 : \mathcal{T}op \to \mathcal{S}et$. π_0, in general, does not have a group structure. Given a cell decomposition of $\mathbb{R}$ with 0-cells

$$Z = \{t_0 < t_1 < \cdots < t_r\}$$

and a map $f : \mathbb{X} \to \mathbb{R}$, the *Reeb cosheaf* is a cosheaf valued in $\mathcal{S}et$ with stalks $\pi_0 f^{-1} I$ for open intervals $I \subseteq \mathbb{R}$ subordinate to the cell decomposition and with restriction maps induced by the functoriality of π_0 [46]. The global sections of the Reeb cosheaf correspond to path components of $\mathbb{X}$ that are born and die at particular critical values in Z. Sections of the induced Tarski sheaf consist of path components that copersist.

6.2.2 从向量值层出发

Suppose $F : \mathrm{P}_G \to \mathcal{V}ec$ is a vector-valued network sheaf. Such a structure assigns a vector space to every node and edge and linear transformations between these vector spaces over edges. Sheaves of vector spaces are key to the foundations of persistent homology [42, Section 8.2]. Suppose $T : V \to W$ is a linear transformation. Then, there is a Galois connection

$$\mathrm{Sub}\, V \underset{T^{-1}(-)}{\overset{T(-)}{\rightleftarrows}} \mathrm{Sub}\, W$$

将V 的一个子空间映射到其像，并将W 的一个子空间映射到其原像。因此，一个层 $F : \mathrm{P}_G$ 通过子空间函子[1],分解，从而引入一类塔斯基层

$$\begin{array}{ccc} \mathrm{P}_G & \xrightarrow{F} & \mathcal{V}ec \\ & \searrow^{\mathcal{F}} & \downarrow{\scriptstyle \mathrm{Sub}\,-} \\ & & \mathcal{L}tc \end{array}$$

1Grandis 将这个函子称为转移函子 [69], 我们之前称它为格拉斯曼流形 [64]。

该构造在希尔伯特空间和线性变换的（全子）范畴 $\mathcal{H}il$ 中未作改变。

示例 6.2(常值层**)**。 假设V 是向量空间，且 V 是定义在 G上的常值层。那么，诱导的塔斯基层再次是常值层 Sub$V$。

———

6.3 来自旧版的新上半格

作为定义 $\mathcal{S}$上-层的层运算的前提，我们需要在完备格上进行范畴运算。

6.3.1 积与余积

The product in $\mathcal{S}up$ is inherited from the product in $\mathcal{P}os$. Suppose P and Q are posets. Then, the *product* $\mathrm{P} \times \mathrm{Q}$ is the poset with the order $(x, y) \preccurlyeq (z, w)$ if and only if $x \preccurlyeq y$ and $y \preccurlyeq w$. The *coproduct* of P and Q, denoted $\mathrm{P} \coprod \mathrm{Q}$ is the poset on the disjoint union of P and Q (as sets) with $x \preccurlyeq y$ if $x, y \in \mathrm{P}$, $x, y \in \mathrm{Q}$, and $x \preccurlyeq y$ in either P or Q. Immediately we run into difficulty: completeness of P and Q does not imply completeness of $\mathrm{P} \coprod \mathrm{Q}$.

通常，我们会对由任意集合 I 索引的集合取笛卡尔积。

$$\prod_{i \in I} \mathbf{L}_i.$$

Elements of the product are denoted with bold letters $\mathbf{x}$. The with individual components denoted $(x_i)_{i \in I}$. The (categorical) *product* of a family of lattices $\{\mathbf{L}\}_{i \in I}$ is the cartesian product equipped with projection maps $\pi_i : \prod_{i \in I} \mathbf{L}_i \to \mathbf{L}_i$. Projection maps have lower and upper adjoints

$$\pi_i^*, \pi_{i*} : \mathbf{L}_i \to \prod_{i \in I} \mathbf{L}_i$$

$$(\pi_i^* x)_j = \begin{cases} x_i & i = j \\ 0 & i \neq j \end{cases} \qquad (\pi_{i*} x)_j = \begin{cases} x_i & i = j \\ 1 & i \neq j \end{cases}$$

范畴意义下的 余积 of $\{\mathbf{L}_i\}_{i \in I}$ is the cartesian product with maps $\pi_{i*} : \mathbf{L}_i \to \coprod_{i \in I} \mathbf{L}_i$ called inclusion maps.

$\mathcal{S}$向上 的积与余积定义了塔斯基层的链和余链。假设 $\mathcal{F}$ 是 G上的双层。那么， $\mathcal{F}$的o-余链 构成（完备）积格

$$C^0(G; \underline{\mathcal{F}}) = \prod_{i \in \mathcal{V}_G} \underline{\mathcal{F}}(i)$$

以及投影映射

$$\pi_i : C^0(G; \underline{\mathcal{F}}) \to \underline{\mathcal{F}}(i).$$

$\mathcal{F}$ 的o-链是余积格

$$C_0(G; \overline{\mathcal{F}}) = \coprod_{i \in \mathcal{V}_G} \overline{\mathcal{F}}(i)$$

以及包含映射

$$\pi_{i*} : \overline{\mathcal{F}}(i) \to C_0(G; \overline{\mathcal{F}}).$$

由于积与余积同构，因此我们将二者（在同构意义下）视为同一个对象，即双积 $\bigoplus_{i \in \mathcal{V}_G} \mathcal{F}(i)$ [133, 附录 E.5].

6.3.2 张量积与内 *Hom*

回顾一下，两个完备格之间保持并的态射构成一个完备格，称为内态射对象，下文记作 $[\mathbf{K}, \mathbf{L}]$。设 $\{f_i\}_{i \in I} \subseteq [\mathbf{K}, \mathbf{L}]$。那么，

$$\left(\bigvee f_i\right)(x) = \bigvee_{i \in I} f_i(x).$$

该 张量积 $\mathbf{K} \otimes \mathbf{L}$ 定义为[2]

$$\mathbf{K} \otimes \mathbf{L} \cong [\mathbf{K}, \mathbf{L}^{\mathrm{op}}]^{\mathrm{op}}.$$

集合范畴 $\mathbf{K} \otimes \mathbf{L}$ 由反序映射 $f \colon \mathbf{K} \to \mathbf{L}$ 组成，使得$f(\bigvee S) = \bigwedge f(S)$，或者换句话说，即逆变伽罗瓦连接。

示例 6.3 （幂集）。 如果X 和Y 是集合，那么

$$\wp(X) \otimes \wp(Y) \cong \wp(X \times Y)$$

内部同态与张量积之间有如下伴随关系。

命题 6.1. 假设$\mathbf{K}$、$\mathbf{L}$、$\mathbf{M}$ 都是完备格。那么，$[\mathbf{K} \otimes \mathbf{L}, \mathbf{M}] \cong [\mathbf{K}, [\mathbf{L}, \mathbf{M}]]$.

证明。 见 [87, 第 1]章。 □

6.3.3 等化子与余等化子

“解方程”在 $\mathcal{S}$上归根结底就是计算等化子和余等化子。

2 如果你更喜欢带有泛性质的定义，请参见 [87]。

Definition 6.2 (Equalizer & Coequalizer)**.** An *equalizer* of a pair of parrellel join-preserving maps $f, g : \mathbf{K} \to \mathbf{L}$ is a suplattice $\mathbf{E}$ and map $i : \mathbf{E} \to \mathbf{K}$ with $f \circ i = g \circ i$ such that for any map $j : \mathbf{M} \to \mathbf{K}$, there is a unique map $i' : \mathbf{M} \to \mathbf{E}$ with $i \circ i' = j$ as in the following diagram

$$\begin{array}{ccccc} \mathbf{E} & \xrightarrow{\ i\ } & \mathbf{K} & \underset{g}{\overset{f}{\rightrightarrows}} & \mathbf{L} \\ {\scriptstyle i'}\uparrow & \nearrow_{j} & & & \\ \mathbf{M} & & & & \end{array} . \tag{6.1}$$

A *coequalizer* of the pair f, g is a suplattice $\mathbf{Q}$ and a map $p : \mathbf{L} \to \mathbf{Q}$ with $p \circ f = p \circ g$ such that for any map $q : \mathbf{L} \to \mathbf{M}$, there is a unique map $p' : \mathbf{Q} \to \mathbf{M}$ with $p' \circ p = q$ as in the following diagram

$$\begin{array}{ccccc} \mathbf{K} & \underset{g}{\overset{f}{\rightrightarrows}} & \mathbf{L} & \xrightarrow{\ p\ } & \mathbf{Q} \\ & & & {\scriptstyle q}\searrow & \downarrow{\scriptstyle p'} \\ & & & & \mathbf{M} \end{array} . \tag{6.2}$$

然而，泛性质不足以进行实际计算。

命题 6.2. *Sup* 具有等化子和余等化子。

Proof. We claim $\mathbf{E} = \{x \in \mathbf{K} \mid f(x) = g(x)\}$ with the inclusion i into $\mathbf{K}$ is the equalizer of $f, g : \mathbf{K} \to \mathbf{L}$. First, $f \circ i = g \circ i$ since i is an inclusion. Second, $\mathbf{E}$ is complete, as we now show. Suppose $\{x_i\}_{i \in I} \subseteq \mathbf{E}$, then $\bigvee_{i \in I} x_i \in \mathbf{E}$ as

$$\begin{aligned} f\left(\bigvee_{i \in I} x_i\right) &= \bigvee_{i \in I} f(x_i) \\ &= \bigvee_{i \in I} g(x_i) \\ &= g\left(\bigvee_{i \in I} x_i\right). \end{aligned}$$

For coequalizers, let $\mathbf{Q} = \{y \in \mathbf{L} \mid f^*(y) = g^*(y)\}$. Then, the coequalizer of $f, g : \mathbf{L} \to \mathbf{K}$ is the lattice $\mathbf{Q}^{\mathrm{op}}$ with the projection $i^* : \mathbf{Q}^{\mathrm{op}} \to \mathbf{L}$. □

6.4 截面

For a $\mathcal{S}up$-sheaf, we define cohomology H^0 and H^1 as equalizers and coequalizers. We show H^0 is isomorphic to the complete lattice of global sections. Recall, *sections* are assignments $\mathbf{x} \in C^\bullet(G; \underline{\mathcal{F}})$ such that that satisfy

$$\underline{\mathcal{F}}(i \trianglelefteqslant ij)(x_i) = \qquad x_{ij} = \qquad \underline{\mathcal{F}}(j \trianglelefteqslant ij)(x_j) \quad i \trianglelefteqslant ij \trianglerighteqslant j \tag{6.3}$$

for every $i \trianglelefteqslant ij \trianglerighteqslant j$. Sections inherit the order on the product lattice $C^\bullet(G; \mathcal{F})$. Let $\Gamma(G; \underline{\mathcal{F}}) \subseteq C^\bullet(G; \mathcal{F})$ denote the poset of sections. In fact, $\Gamma(G; \underline{\mathcal{F}})$ is a complete lattice calculated as follows.

定义 6.3. 设 $\mathcal{F}$ 是一个塔斯基层，定义在 G上。则，令$H^0(G;\ \mathcal{F})$和 $H^1(G;\ \mathcal{F})$分别为下图的等化子和余等化子[3]。

$$H^0(G;\underline{\mathcal{F}}) \dashrightarrow C^0(G;\underline{\mathcal{F}}) \underset{\delta_+}{\overset{\delta_-}{\rightrightarrows}} C^1(G;\underline{\mathcal{F}}) \dashrightarrow H^1(G;\underline{\mathcal{F}}) \quad (6.4)$$

with

$$\begin{aligned}(\delta_-)_{ij} = & \quad \underline{\mathcal{F}}(ij_- \trianglelefteqslant ij),\\ (\delta_+)_{ij} = & \quad \underline{\mathcal{F}}(ij_+ \trianglelefteqslant ij).\end{aligned}$$

根据命题 6。2,

$$\begin{aligned}H^0(G;\underline{\mathcal{F}}) \cong & \quad \{\mathbf{x} \in C^0(G;\underline{\mathcal{F}}) \mid (\delta_-\mathbf{x})_{ij} = (\delta_+\mathbf{x})_{ij} \quad \forall ij \in \mathcal{E}_G\}\\ = & \quad \{\mathbf{x} \in C^0(G;\underline{\mathcal{F}}) \mid \underline{\mathcal{F}}(i \trianglelefteqslant ij)(x_i) = \underline{\mathcal{F}}(j \trianglelefteqslant ij)(x_j) \quad \forall ij \in \mathcal{E}_G\}.\\ \cong & \quad \{\mathbf{x} \in C^0(G;\underline{\mathcal{F}}) \mid \underline{\mathcal{F}}(i \trianglelefteqslant ij)(x_i) = x_{ij} = \underline{\mathcal{F}}(j \trianglelefteqslant ij)(x_j)\}\\ = & \quad \Gamma(G;\underline{\mathcal{F}}).\end{aligned}$$

定理 6.2 (整体截面定理). 假设 $\mathcal{F}$ 是一个在 G 上的塔斯基层。那么，$H^0(G;\ \mathcal{F})$是 $C^0(G;\ \mathcal{F})$的完备拟子格。 —

Warning 6.1. Beware, $H^0(G;\underline{\mathcal{F}})$ is not in general a sublattice. $H^0(G;\underline{\mathcal{F}})$ is an equalizer. Hence, the unique map $H^0(G;\underline{\mathcal{F}}) \to C^0(G;\underline{\mathcal{F}})$ is injective (see [133, Exercise 3.1.vi]). However, joins and meets in $H^0(G;\underline{\mathcal{F}})$ do not (in general) coincide with joins and meets in $C^0(G;\underline{\mathcal{F}})$.

.

6.5 层运算

我们构造六种经典“层运算”中的四种：张量积层、层 *Hom*、拉回和推前。虽然这四种层运算会再次出现，但我们仍纳入这篇阐述，以启发未来的工作。

6.5.1 层态射 & 常值层

In order to define tensor product sheaves and sheaf hom, we first recall morphisms between sheaves. A *sheaf morphism* $\underline{\mathcal{F}} \overset{\varphi}{\Rightarrow} \underline{\mathcal{G}}$ between $\underline{\mathcal{F}}$ and $\underline{\mathcal{G}}$ consists of the data of join-preserving maps $\varphi_i : \mathcal{F}(i) \to \mathcal{G}(i)$ for every $i \in \mathcal{V}_G$ and join-preserving maps $\varphi_{ij} : \underline{\mathcal{F}}(ij) \to \underline{\mathcal{G}}(ij)$ for every $ij \in \mathcal{E}_G$ such that for every $i \in \mathcal{V}_G$ and $j \in \mathcal{N}_i$

$$\underline{\mathcal{G}}(i \trianglelefteqslant ij) \circ \varphi_i = \quad \varphi_{ij} \circ \underline{\mathcal{F}}(i \trianglelefteqslant ij). \quad (6.5)$$

所有层态射的集合记作 Hom （$\mathcal{F}, \mathcal{G}$）。 — —

3 参见 Joyal [87,第 1]章，其中证明了 $\mathcal{S}up$ 具有所有极限和余极限。

命题 6。3. Hom $(\mathcal{F}, \mathcal{G})$ 是一个完备格。

证明。 设 I 是一个任意的指标集， $\{\varphi^\alpha\}_{\alpha\in I}$ 是一族层态射。对于 $i \in \mathcal{V}_G$，记作

$$\left(\bigvee_{\alpha\in I} \varphi_i^\alpha\right)(x_i) = \bigvee_{\alpha\in I} \varphi_i^\alpha(x_i),$$

对于族 φ_{ij}^α 的并，同样如此。由此得到图表

$$\begin{array}{ccccc} \underline{\mathcal{F}}(i) & \xrightarrow{\underline{\mathcal{F}}(i\leqslant ij)} & \underline{\mathcal{F}}(ij) & \xleftarrow{\underline{\mathcal{F}}(j\leqslant ij)} & \underline{\mathcal{F}}(j) \\ \downarrow{\scriptstyle \bigvee \varphi_i^\alpha} & & \downarrow{\scriptstyle \bigvee \varphi_{ij}^\alpha} & & \downarrow{\scriptstyle \bigvee \varphi_j^\alpha} \\ \underline{\mathcal{G}}(i) & \xrightarrow{\underline{\mathcal{G}}(i\leqslant ij)} & \underline{\mathcal{G}}(ij) & \xleftarrow{\underline{\mathcal{F}}(j\leqslant ij)} & \underline{\mathcal{G}}(j) \end{array}$$

对所有 $ij \in \mathcal{E}_G$ 都交换，因为限制映射保持并性。

□

Recall, $\mathbf{B} = \{0 < 1\}$, the boolean lattice with two elements. The *boolean constant sheaf* over G is the sheaf $\underline{\mathcal{B}}$ with $\underline{\mathcal{B}}(i) = \mathbf{B}$ for all $i \in \mathcal{V}_G$, $\underline{\mathcal{B}}(ij) = \mathbf{B}$ for all $ij \in \mathcal{E}_G$, and $\underline{\mathcal{B}}(i \leqslant ij) = \mathrm{id}_{\mathbf{M}}$.

命题 6.4. 设 $\mathcal{F}$ 是图 G 上的一个 $\mathcal{S}up$-层。那么，

$$\Gamma(G;\underline{\mathcal{F}}) \cong \mathrm{Hom}(\underline{\mathcal{B}}, \underline{\mathcal{F}}).$$

Proof. A sheaf morphism φ is the data of a join-preserving map $\varphi_i : \mathbf{B} \to \underline{\mathcal{F}}(i)$ for every $i \in \mathcal{V}_G$ and a join-preserving map $\varphi_{ij} : \mathbf{B} \to \underline{\mathcal{F}}(ij)$ for every $ij \in \mathcal{E}_G$ satisfying (6.5). Therefore, φ_i "picks out" and $x_i \in \underline{\mathcal{F}}(i)$ for every node and φ_{ij} "picks out" a $x_{ij} \in \underline{\mathcal{F}}(ij)$ for every edge, such that $\underline{\mathcal{F}}(i \leqslant ij)(x_i) = x_{ij}$. This is precisely the data of a global section. □

因此，我们又得到一个证明：截面构成完备格。另一个推论是，命题 6.4中的同构蕴含一个层态射 $\mathcal{F} \overset{\varphi}{\Rightarrow} \mathcal{G}$ 在整体截格上诱导一个保并地图

$$\Gamma(G;\underline{\mathcal{F}}) \xrightarrow{\Gamma(G;\varphi)} \Gamma(G;\underline{\mathcal{G}}).$$

通过前复合。

定理 6.3. 设 P 是一个偏序集。那么， $\Gamma(G;-)$ 是一个函子

$$\Gamma(G;-) : \mathcal{S}up^{\mathrm{P}} \to \mathcal{S}up.$$

6.5.2 张量积 & 层 *Hom*

定义 6.4 (张量积层)。 设 $\mathcal{F}$ 和 $\mathcal{G}$ 是图G上的层。张量积层 $\mathcal{F} \otimes \mathcal{G}$ 是函子，满足

$$\begin{aligned} (\underline{\mathcal{F}} \otimes \underline{\mathcal{G}})(i) &= \underline{\mathcal{F}}(i) \otimes \underline{\mathcal{G}}(j) \\ (\underline{\mathcal{F}} \otimes \underline{\mathcal{G}})(ij) &= \underline{\mathcal{F}}(ij) \otimes \underline{\mathcal{G}}(ij) \end{aligned}$$

对于所有 $i \in \mathcal{V}_G$, $ij \in \mathcal{E}_G$ 以及限制映射

$$(\underline{\mathcal{F}} \otimes \underline{\mathcal{G}})(i) \xrightarrow{(\underline{\mathcal{F}} \otimes \underline{\mathcal{G}})(i \trianglelefteq ij)} (\underline{\mathcal{F}} \otimes \underline{\mathcal{G}})(ij)$$

将 $\mathcal{F}(i) \xrightarrow{f} \mathcal{G}(i)^{\mathrm{op}}$ 发送到复合

$$\overline{\mathcal{F}}(ij) \xrightarrow{\overline{\mathcal{F}}(i \trianglelefteq ij)} \overline{\mathcal{F}}(i) = \underline{\mathcal{F}}(i) \xrightarrow{f} \underline{\mathcal{G}}(i)^{\mathrm{op}} \xrightarrow{\underline{\mathcal{G}}(i \trianglelefteq ij)^{\mathrm{op}}} \underline{\mathcal{G}}(ij)^{\mathrm{op}}$$
$$\overset{(\underline{\mathcal{F}} \otimes \underline{\mathcal{G}})(i \trianglelefteq ij)(f)}{\curvearrowright}$$

6.5.3 拉回与推前

假设 $G = (\mathcal{V}_G, \mathcal{E}_G)$ 和 $H = (\mathcal{V}_H, \mathcal{E}_H)$。图同态 $p : G \to H$ 是一个严格保序映射

$$p : \mathrm{P}_G \to \mathrm{P}_H.$$

假设 p 是一个图同态 $p : G \to H$，且 $\mathcal{F}$ 是 up-层上的一个 $\mathcal{S}$。 $\mathcal{F}$ 在 p 下的拉回 层记作 $p^*\mathcal{F}$,其

$$\begin{aligned} p^*\underline{\mathcal{F}}(i) &= \underline{\mathcal{F}}(p(i)) \\ p^*\underline{\mathcal{F}}(ij) &= \underline{\mathcal{F}}(p(ij)), \\ p^*\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) &= \underline{\mathcal{F}}(p(i) \trianglelefteq p(ij))。 \end{aligned}$$

关于推前的 $\mathcal{F}$，记作$p_*\mathcal{F}$，其定义更为复杂。茎是

$$p_*\underline{\mathcal{F}}(i) = \lim_{k \trianglelefteq p(ij)} \underline{\mathcal{F}}(ij).$$

The cannonical example of a pushforward is the pushforward of the map $p : G \to *$ which is isomorphic to $\Gamma(G; \underline{\mathcal{F}})$. A canonical example of a pullback is the pullback of an embedding $H \hookrightarrow G$ called the restriction of $\underline{\mathcal{F}}$ to H.

7

塔尔斯基拉普拉斯算子

The graph Laplacian is an operator that acts on scalar fields while the connection Laplacian acts on vector fields. Their common denominator is the sheaf Laplacian [77]. In this chapter, we motivate a few candidate Laplacians for Tarski sheaves from several angles. First, from the vantage of lattice-valued consensus, then as an analogy to parallel transport, then finally as an homage to Hodge theory. Along the way, we present the Hodge-Tarski Fixed Point Theorem (Theorem 7.1) and provide algorithms to compute global sections in various settings.

7.1 共识

塔斯基拉普拉斯算子的一个实际动机是共识。

Definition 7.1 (Multi-Agent Lattice Consensus)**.** Let $\mathbf{L}$ be a lattice. Suppose $G = (\mathcal{V}_G, \mathcal{E}_G)$ is a graph. A *consensus* function is a map $\chi : \mathbf{L}^{\mathcal{V}_G} \to \mathbf{L}$. A *consensus protocol* is a map $\chi : \mathbf{L}^{\mathcal{V}_G} \to \mathbf{L}$ (consensus function) together with an algorithm whose input is an arbitrary signal $\mathbf{x} : \mathcal{V}_G \to \mathbf{L}$ and whose (desired) output is $\chi(\mathbf{x})$.

Example 7.1 (Meet-Consensus)**.** Meet-consensus takes place in a sequence of rounds. Each node, every node its state $x_i \in \mathbf{L}, i \in \mathcal{V}_G$ to each of its neighbors $j \in \mathcal{N}_i$. At the conclusion of the round, a node i computes the meet of its received messages as well as (possibly) x_i. In a simple example (Figure 7.1), two nodes (left, right) share a communication link with a common node (center). At $t = 0$ (light), each node computes meets with neighbors. At $t = 2$ (dark), a consensus is reached. The updates are defined in a recursion equation

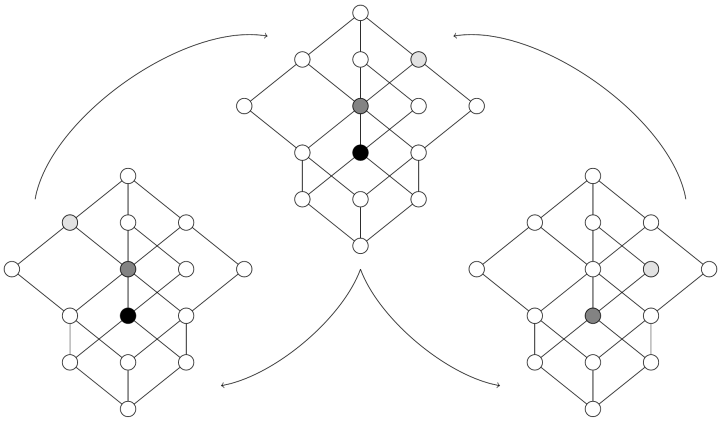


图 7.1：一种交共识算法。

$$x_i[t+1] = \qquad x_i[t] \wedge \left(\bigwedge_{j \in \mathcal{N}_i} x_j[t] \right)$$

Algorithm 1: Meet-consensus; in join-consensus, replace $\wedge$ with $\vee$.

Input: Graph $G = (\mathcal{V}_G, \mathcal{E}_G)$; $x_i \in \mathbf{L}, i \in \mathcal{V}_G$
Output: $x_i \in \mathbf{L}, i \in \mathcal{V}_G$
repeat
  for $i \in \mathcal{V}_G$ **do in parallel**
    for $j \in \mathcal{N}_i$ **do**
      $x_i \leftarrow x_i \wedge x_j$
    end
  end
until $x_i = x_j \; \forall \; ij \in \mathcal{E}_G$

Proposition 7.1. Suppose G is connected and $\mathbf{L}$ is a finite lattice. The meet-consensus algorithm (7.1) converges to $\left(\bigwedge_{j \in \mathcal{V}_G} x_j\right)_{i \in \mathcal{V}_G}$ in finite time.

证明。一轮之后，x_i 被更新为x_i 与对于每个邻居x^j 的交$j \in \mathcal{N}_i$。令 $\mathcal{N}_i^k$表示k-跳邻居i。由此可得

$$x_i[k] = \qquad \bigwedge_{j \in \mathcal{N}_i^k \cup i} x_j[0].$$

由于G 是连通的，对于任意 $i \in \mathcal{V}_G$，都保证 $\mathcal{N}_i^k \cup i \supseteq \mathcal{V}_G$。因此，

$$x_i[k] = \qquad \bigwedge_{j \in \mathcal{V}_G} x_j[0]$$

对于 $k \geqslant$直径(G)。 □

共识可以很容易地重新表述为一个整体截面问题：共识是常值层的截面的格。

命题 7.2. 假设 $\mathbf{L}$ 是G 上的常值层。则，

$$\Gamma(G; \underline{\mathbf{L}}) = \qquad \{(x_i)_{i \in \mathcal{V}_G} \mid x_i = x_j \; \forall \; ij \in \mathcal{E}_G\}.$$

证明。整体截面条件规定：$\mathrm{id}(x_i) = \mathrm{id}(x^j)$ 对于所有$ij \in \mathcal{E}_G$。

□

Γ（G；$\mathbf{L}$）的格结构与$\mathbf{L}$“相同”（在同构意义下），但并不相等。

命题 7.3. 设 G 是一个图，$\mathbf{L}$ 是一个格，并且

$$S : \qquad C^0(G;\underline{\mathbf{L}}) \to C^0(G;\underline{\mathbf{L}}) \tag{7.1}$$

$$(S\mathbf{x})_i = \qquad \bigwedge_{j\in\mathcal{N}_j} x_j \tag{7.2}$$

然后，

$$\mathrm{Post}(S) = \qquad \Gamma\left(G;\underline{\mathbf{L}}\right).$$

Proof. One direction is trivial. By Proposition 7.2, if $\mathbf{x} \in C^0(G;\underline{\mathbf{L}})$ is locally constant, then $(S\mathbf{x})_i = x_i$, hence, $S\mathbf{x} \succcurlyeq \mathbf{x}$. Now suppose $\mathbf{x} \in \mathrm{Post}(S)$. Then, for all $i \in \mathcal{V}_G$, $x_i = x_i \wedge \bigwedge_{j\in\mathcal{N}_i} x_j$ which is equivalent to $x_i \preccurlyeq \bigwedge_{j\in\mathcal{N}_i} x_j$ for all $i \in \mathcal{V}_G$. Suppose $ij \in \mathcal{E}_G$ is a particular edge. Then, both

$$x_i \preccurlyeq \qquad \bigwedge_{j\in\mathcal{N}_i} x_j \preccurlyeq \qquad x_j,$$

$$x_j \preccurlyeq \qquad \bigwedge_{i\in\mathcal{N}_j} x_i \preccurlyeq \qquad x_i.$$

因此，根据偏序集的反对称公理，对于所有 $ij \in \mathcal{E}_G$，$x_i = x^j$，且 $\mathbf{x}$ 是常值层的整体截面。 □

Beyond meet-consensus (or, dually, join-consensus), are there other consensus algorithms? To answer this question, we first describe sheaves whose sections correspond to as approximations of the constant sheaf. The following definition is due to Hansen [73, Chapter 7].

Definition 7.2 (Approximation)**.** Let G be a graph and $\underline{\mathcal{G}}$ and $\underline{\mathcal{F}}$ be Tarski sheaves over G. Then, $\underline{\mathcal{F}}$ is an approximation to $\underline{\mathcal{G}}$ if there is there is a sheaf morphism $\underline{\mathcal{F}} \overset{\varphi}{\Rightarrow} \underline{\mathcal{G}}$ such that $\underline{\mathcal{F}}(i) \cong \underline{\mathcal{F}}(j)$ for all $i \in \mathcal{V}_G$ and such that the induced join-preserving map

$$\Gamma\left(G;\underline{\mathcal{F}}\right) \to \qquad \Gamma\left(G;\underline{\mathcal{G}}\right)$$

是一个同构。

An approximation of the constant sheaf $\underline{\mathbf{L}}$ consists of a sheaf whose vertex stalks are $\mathbf{L}$ and whose lattice of sections is the same as $\Gamma\left(G;\underline{\mathbf{L}}\right)$, isomorphic to $\mathbf{L}$ if G is connected. In other words, approximations to the constant sheaf also model consensus. An approximation of $\underline{\mathbf{L}}$ is a sheaf $\underline{\mathcal{F}}$ characterized by the following diagram

$$\begin{array}{ccccc}
\underline{\mathcal{F}}(i) & \xrightarrow{\underline{\mathcal{F}}(i\trianglelefteq ij)} & \underline{\mathcal{F}}(ij) & \xleftarrow{\underline{\mathcal{F}}(j\trianglelefteq ij)} & \underline{\mathcal{F}}(j) \\
\| & & \downarrow{\scriptstyle \varphi_{ij}} & & \| \\
\mathbf{L} & \xrightarrow{\mathrm{id}} & \mathbf{L} & \xleftarrow{\mathrm{id}} & \mathbf{L}
\end{array}$$

.

There are more questions about lattice consensus than answers. If the consensus operator S is a Laplacian for the constant sheaf, what is the "sheaf Laplacian" of an approximation to the constant sheaf? or an arbitrary Tarski sheaf? Furthermore, if $\underline{\mathcal{F}}$ approximates $\underline{\mathbf{L}}$ and $\chi : C^0(\underline{\mathcal{F}}; G) \to \mathbf{L}$ is a consensus function, does there exist "sheaf dynamics" converging to the $\chi(\mathbf{x}[0])$? There is also a utility for consensus "in-between" meet- and join- consensus. In some applications, for instance, earliest rendezvous [122], min-consensus is desired. On the other hand, consider the task of forming teams amongst a set of agents. Cast as a consensus problem on the lattice of subpartitions, it would not be sensible to achieve either the finest or the coarsest organization possible.

Example 7.2 (Causal Inference)**.** Suppose a set of random variables is collected in a partial order P. For instance, random variables could indicate (transitive, reflexive, anti-symmetric) causal relationships between features. From join-consensus on the lattice of downsets $\mathcal{D}(\mathrm{P})$, a subset of P can be recovered from the poset of join-irreducibles of $\mathcal{D}(\mathrm{P})$ (isomorphic to P, Theorem 2.3), thus defining a weak consensus function $\chi : \mathrm{P}^n \to \wp(\mathrm{P})$.

Example 7.3 (Distributed Optimization)**.** A multitude of agent-based tasks are formulated or equivalent to distributed optimization [47]. Suppose $\{1, 2, \ldots, n\}$ is a group of agents with objective function f_i, $i \in \{1, 2, \ldots, n\}$. In the simplest setting, *distributed optimzation* is unconstrained optimization problem

$$\min_{\mathbf{x}} \sum_{i=1}^{n} f_i(\mathbf{x}) \tag{7.3}$$

共识和分布式优化通过以下观察相互关联，即 (7.4) 等价于

$$\begin{aligned} &\min_{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n} \sum_{i=1}^{n} f_i(\mathbf{x}_i) \\ &\qquad \text{subject to} \\ &\mathbf{x}_i = \mathbf{x}_j \quad \forall i, j \in \{0, 1, \ldots, n\} \end{aligned} \tag{7.4}$$

因此，分布式优化归结为共识，以及对局部目标函数 $f_i, i \in \{1, 2, \ldots, n\}$ 的优化。

近期研究将分布式优化推广到 G 上具有欧几里得茎的网络层 [75]:

$$\begin{aligned} &\min_{\mathbf{x} \in C^0(G; \underline{\mathcal{F}})} \sum_{i \in \mathcal{V}_G} f_i(\mathbf{x}_i) \\ &\qquad \text{subject to} \\ &\mathbf{x} \in H^0(G; \underline{\mathcal{F}}). \end{aligned} \tag{7.5}$$

其中，每个 f_i 都是凸函数 $f_i : \mathcal{F}(i) \to \underline{\mathbb{R}}_\circ$

Optimization algorithms maximizing [119] or minimizing [159] real-valued functions on lattices are restricted to certain classes of functions (e.g. submodular, supermodular, increasing difference). An algorithm solving (7.5) for $\underline{\mathcal{F}}$, a Tarski sheaf, is an open problem whose solution, we speculate, will involve clever modulations between the tasks of running lattice consensus and performing "gradient" updates of the local objective functions $f_i, i \in \mathcal{V}_V$.

7.2 平行移动

SchreiberandWaldorf [147]以及其他人 [95]论证了流形上的平行移动 $\mathbb{M}$ 取值于任意范畴 $\mathcal{D}$ 由一个函子建模

$$P : \mathrm{Path}(\mathbb{M}) \to \mathcal{D}.$$

from the path groupoid of $\mathbb{M}$. $\mathrm{Path}(\mathbb{M})$ is a category whose objects are points from $\mathbb{M}$ and whose morphisms are, roughly, homotopy classes of paths between points [80]. In the sequel, given a graph G and a Tarski sheaf $\mathcal{F}$ on G, we construct a functor

$$P^{\mathcal{F}} : \mathrm{Free}(G) \to \mathcal{P}os$$

发送路径 $\gamma : i \to j$ 到一个保序映射 $P^{\mathcal{F}}_{\gamma} : \mathcal{F}(i) \to \mathcal{F}(j)$，模拟平行移动。在我们看来，平行移动为通过通信网络中一条路径连接的智能体的局部截面之间的消息传递建模。

定义 7.3（平行移动）**.** 设G 是一个图，且 $\mathcal{F}$ 是G上的一个塔斯基层。设$j \in \mathcal{N}_i$。那么，沿路径$i \to j$ 的平行移动 是地图

$$\begin{aligned} P^{\mathcal{F}}_{i\to j} : & \quad \mathcal{F}(i) \to \mathcal{F}(j) \\ P^{\mathcal{F}}_{i\to j}(x_i) = & \quad \overline{\mathcal{F}}(j \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i). \end{aligned}$$

If γ is an arbitrary path from i to j, then $P^{\mathcal{F}}_{\gamma}$ is defined as the (unique) composition of parallel transport maps along paths of length one. The parallel transport of the trivial path ε_i based at $i \in \mathcal{V}_G$ is defined to be the identity map on $\mathcal{F}(i)$.

Parallel transport is functorial by construction (i.e. $P^{\mathcal{F}}_{\gamma_1 \cdot \gamma_2} = P^{\mathcal{F}}_{\gamma_2} \circ P^{\mathcal{F}}_{\gamma_1}$). Furthermore, parallel transport is a functor into the catgory of posets and order-preserving maps because $P^{\mathcal{F}}_{i\to j}$, as a composition of an join- and meet- preserving map, is not (in general) join- or meet- preserving. If γ and γ' are two distinct paths from i to j, in is not the case (in general) that

$$P^{\mathcal{F}}_{\gamma}(x_i) = P^{\mathcal{F}}_{\gamma}(x_j) \quad \forall i, j \in \mathcal{V}_G.$$

在上述条件成立时， $\mathcal{F}$ 被称为平坦。

7.2.1 整体单值化

Having defined parallel transport of Tarski sheaves, an abundance of potential geometric construction wait at the door including connections, curvature, characteristic classes, and more. Immediately, we are in a position to define holonomy, a construction which, roughly speaking, measures the degree to which parallel transport around a closed loop fails to preserve orientations of tangent vectors.

定义 7.4 （整体单值化）。 设 G 是一个图。令 $\Omega_i(G)$ 表示以特定的 $i \in \mathcal{V}_G$ 为基点的 G 中的环。那么， $\mathcal{F}$ 的整体单值化是有序幺半群

$$\mathrm{Hol}_i(G;\underline{\mathcal{F}}) = \{P_\gamma^{\mathcal{F}} \mid \gamma \in \Omega_i(G)\}$$

Holonomy is a monoid because the composition of loops yields a composition of transport maps and the trivial path ε_i yields an identity. It is a poset because $\mathrm{Hol}_i(G;\underline{\mathcal{F}})$ inherits the order of the poset of order-preserving maps $\mathrm{End}_{\mathcal{P}os}(\underline{\mathcal{F}}(i))$.

7.2.2 整体截面

平行移动刻画了截面。我们首先说明，截面有助于沿任意（网络）路径将“切向量”（格元素）传输为相容的“切向量”（格元素）。

引理 7.1 （截面路径引理）. 假设 $\mathcal{F}$ 是G 上的塔斯基层，且 $\mathbf{x}\in \Gamma(G;\mathcal{F})$ 是一个截面。那么，对于从 i 到 j的每条路径 γ，

$$P_\gamma^{\mathcal{F}}(x_i) \succcurlyeq x_j.$$

证明。 我们按路径长度进行归纳：

基础情形。 假设 γ 是路径 $i \to j$ （长度为一）。那么，由于 $\mathbf{x}$ 是一个截面，

$$\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) \succcurlyeq \underline{\mathcal{F}}(j \trianglelefteq ij)(x_j)$$

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$$\overline{\mathcal{F}}(j \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) \succcurlyeq x_j.$$

Inductive Step. Suppose γ is a path of length ℓ. Suppose, without loss of generality, that γ is a path from i to j, and γ' is a path from the same i to j'. By the inductive hypothesis, $P_\gamma^{\mathcal{F}}(x_i) \succcurlyeq x_j$. Then,

$$\begin{aligned} P_{\gamma'}^{\mathcal{F}}(x_i) &\succcurlyeq \bigwedge_{j'\in\mathcal{N}_j} P_{j\to j'}^{\mathcal{F}} P_\gamma^{\mathcal{F}}(x_i) \\ &\succcurlyeq \bigwedge_{j'\in\mathcal{N}_j} P_{j\to j'}^{\mathcal{F}}(x_j) \\ &\succcurlyeq x_{j'}. \end{aligned}$$

□

对于长度为一的路径，邻域路径，引理7。1 有一个部分逆命题。

引理 7。2 （邻域路径引理）。 假设 $\mathcal{F}$ 是 G 上的塔斯基层，并且假设

$$P^{\mathcal{F}}_{i\to j}(x_i) \succcurlyeq x_j, \quad i \in \mathcal{V}_G, j \in \mathcal{N}_i$$

对于所有$\mathbf{x} \in C^0(G; \mathcal{F})$。那么，$\mathbf{x} \in \Gamma(G; \mathcal{F})$。

证明。 根据假设，以下两者对于所有 $\mathbf{x} \in C^0(G; \mathcal{F})$上的 $ij \in \mathcal{E}_G$均成立。

$$\begin{aligned}\overline{\mathcal{F}}(j \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) &\succcurlyeq x_j,\\ \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(j \trianglelefteq ij)(x_j) &\succcurlyeq x_i.\end{aligned}$$

因此，由于伽罗瓦连接的性质，

$$\begin{aligned}\underline{\mathcal{F}}(j \trianglelefteq ij)(x_j) &\succcurlyeq \underline{\mathcal{F}}(i \trianglelefteq ij)(x_i),\\ \underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) &\succcurlyeq \underline{\mathcal{F}}(j \trianglelefteq ij)(x_j),\end{aligned}$$

这意味着二者相等。 □

Remark 7.1 (Proof of Lemma 7.2)**.** The argument in the proof of the Neighborhood Lemma breaks downs when you try to generalize it to arbitrary paths. For instance, if $\gamma = (i, j, k)$ is a path between distinct nodes, then

$$P^{\mathcal{F}}_{\gamma}(x_i) = \overline{\mathcal{F}}(k \trianglelefteq jk)\underline{\mathcal{F}}(j \trianglelefteq jk)\overline{\mathcal{F}}(j \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) \succcurlyeq x_k$$

并不意味着

$$P^{\mathcal{F}}_{\gamma}(x_k) = \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(j \trianglelefteq ij)\overline{\mathcal{F}}(j \trianglelefteq jk)\underline{\mathcal{F}}(k \trianglelefteq jk)(x_k) \succcurlyeq x_i,$$

仅此而已

$$\underline{\mathcal{F}}(j \trianglelefteq jk)\overline{\mathcal{F}}(j \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) \succcurlyeq \underline{\mathcal{F}}(k \trianglelefteq jk)(x_k).$$

7.3 霍奇-塔斯基定理

最后，在本节中，我们为塔斯基层定义层拉普拉斯算子，最终得到一个以霍奇风格的不动点定理来计算整体截面的结果。

7.3.1 塔斯基拉普拉斯算子

定义 7。5。 设 $\mathcal{F}$ 是G上的塔斯基层。塔斯基拉普拉斯是一个算子，定义

$$\begin{aligned}L :\ & C^0(G; \underline{\mathcal{F}}) \longrightarrow C^0(G; \underline{\mathcal{F}})\\ (L\mathbf{x})_i =\ & \bigwedge_{j \in \mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(j \trianglelefteq ij)(x_j).\end{aligned}$$

回顾，格 交投影$\mathbf{L}$ 到子集 自然数集 $\subseteq \{1, 2, \ldots, n\}$是地图

$$\begin{aligned} \chi : \quad & \mathbf{L}^n \to \mathbf{L} \\ \chi(\mathbf{x}) = \quad & \bigwedge_{i \in N} x_j \end{aligned}$$

然后，我们立即注意到，TarskiLaplacian 是两个操作的复合：(1) 平行移动，以及 (2) 交投影。因此，

$$(L\mathbf{x})_i = \bigwedge_{j \in \mathcal{N}_i} P^{\mathcal{F}}_{j \to i}(x_j). \tag{7.6}$$

此外，请回顾塔斯基不动点定理的条件2.2.

引理 7。3. 塔斯基拉普拉斯算子是完备格上的保序映射。

证明。 通过类似于格多项式是保序的论证[18], L 是保序函数的复合。 $C^0(G; \mathcal{F})$ 是完备格的乘积，因此是完备的。 □

7.3.2 扩散

对于以下命题，令 $\mathcal{N}_i^k$ 表示k跳邻域 of $i \in \mathcal{V}_G$.[1] 假设$W \subseteq \mathcal{V}_G$ 是节点的子集。那么， π_W 是$C^0(G; \mathcal{F})$ 投影到W上的结果。

命题 7.4 (局部性)。 假设 $\mathcal{F}$ 是一个定义在G 上的塔斯基层，其塔斯基拉普拉斯算子为L。假设$i \in \mathcal{V}_G$。那么，塔斯基拉普拉斯算子被限制在k跳邻居的补集上

$$L^k_{\mathcal{V}_G \setminus \mathcal{N}_i^k} : \quad \prod_{j \in \mathcal{V}_G \setminus \mathcal{N}_i^k} \underline{\mathcal{F}}(j) \to \prod_{j \in \mathcal{V}_G \setminus \mathcal{N}_i^k} \underline{\mathcal{F}}(j)$$

是恒等映射。

As an operator, the Tarski Laplacian is said to be *local* in the sense that $(L^k\mathbf{x})_i$ only depends only on the assignment to the nodes $j \in \mathcal{N}_i^k$. Familiar examples of other local operators include the graph Laplacian, graph connection Laplacian, and the linear sheaf Laplacian.

7.3.3 与联络拉普拉斯算子的相似性

回顾一下，在联络拉普拉斯算子中，

$$(L\mathbf{x})_i = \quad \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} w_{ij} O_{ij} x_j,$$

1 令 $\mathcal{N}_i^0 = \{i\}$。

邻居的切向量 $\{\mathbf{x}_j\}_{j\in\mathcal{N}_i}$ 由近似于$T_{x_i}\mathbb{M}$ 与$T_{x_i}\mathbb{M}$之间平行移动的正交映射 O^{ij} 进行变换。接下来，接收节点消息 $\{O^{ij}\mathbf{x}_j\}_{j\in\mathcal{N}_i}$，并通过加权求和进行聚合。

In the Tarski Laplacian (7.6), lattice elements of neighbors $\{x_j\}_{j\in\mathcal{N}_i}$ are transformed by order-preserving maps $P^{\mathcal{F}}_{j\to i}$ subordinate to $\mathcal{F}$ that model parallel transport . Next, node messages $\{P^{\mathcal{F}}_{j\to i}(x_j)\}_{j\in\mathcal{N}_i}$ are received and aggregated with a meet-projection onto $\mathcal{N}_j$.

7.3.4 一个不动点定理

Recall, the Hodge Theorem (Theorem 5.1) characterizes the fixed points of heat equation (5.5). We now state a culminating result characterising the suffix points of the Tarski Laplacian with the sections $\Gamma(G;\underline{\mathcal{F}})$ of a Tarski sheaf $\underline{\mathcal{F}}$ over G.

定理 7.1 （塔斯基-Hodge 定理）。 设 $\mathcal{F}$ 是定义在图 G 上的塔斯基层，其塔斯基拉普拉斯算子 L 为：$C^0(G;\mathcal{F})\to C^0(G;\mathcal{F})$。那么，

$$\mathrm{Post}(L) = H^0(G;\underline{\mathcal{F}}). \tag{7.7}$$

证明依赖于第7节2中的平行移动理论。

*Tarski-Hodge*定理的证明。根据截面路径引理（引理 7。1）和邻域引理（引理 7。2），

$$\mathbf{x}\in\Gamma(G;\underline{\mathcal{F}})$$

当且仅当

$$P^{\mathcal{F}}_{j\to i}(x_j) \succcurlyeq x_i$$

对所有 $i\in\mathcal{V}_G$，$j\in\mathcal{N}_i$。根据交的泛性质（引理3。3），

$$P^{\mathcal{F}}_{j\to i}(x_j) \succcurlyeq x_i$$

对所有 $j\in\mathcal{N}_i$ 当且仅当

$$\bigwedge_{j\in\mathcal{N}_i} P^{\mathcal{F}}_{j\to i}(x_j) \succcurlyeq x_i.$$

因此，$(L\mathbf{x})_i\succcurlyeq x_i$对每个$i\in\mathcal{V}_V$当且仅当$\mathbf{x}\in H^0(G;\mathcal{F})$。 _ □

等价地，我们可以将上述内容改写为算子 $L\wedge\mathrm{id}$ 的不动点，因为 $L(\mathbf{x})\succcurlyeq\mathbf{x}$ 当且仅当 $L(\mathbf{x})\wedge\mathbf{x}$。

推论 7.1. 固定$(L\wedge\mathrm{id}) = H^0(G;\mathcal{F})$. _

Remark 7.2. Indeed, the Hodge-Tarski Theorem (Theorem 7.1) is a "Hodge-style fied point theorem." The Euler discretization of the heat equation (5.5) (step-size $\alpha > 0$) is the following

$$\mathbf{x}[t+1] - \mathbf{x}[t] = -\alpha\Delta_k\mathbf{x} \tag{7.8}$$

由此可得

$$\text{Fix}(\text{id} - \alpha\Delta) \cong H^k(C^\bullet)$$

通过 Hodge 定理（定理 (Δ_K) 中的核$H^0(C^\bullet)$ 之间的同构5.1）。然后，将 (7.8) 与

$$\text{不动点(恒等} \wedge L) = H^k(G;\underline{\mathcal{F}}). \tag{7.9}$$

第二个（同样显然的）推论是由整体截面与 H^0 之间的同构得出的（命题 5.1）。

推论 7.2. 固定$(L \wedge \text{id}) \cong \Gamma(G;\mathcal{F})$. __

最后一个推论为定理 6。2 提供了第三种替代证明：塔斯基层的整体截面是 Γ （G; $\mathcal{F}$） 的一个完备拟子格。

—

推论 7。3。 $H^0(G;\mathcal{F})$ 是完备格。此外，存在从 $\Gamma(G;\mathcal{F})$ 到 0-上链的完备格 $C^0(G;\mathcal{F})$ 的序嵌入。 — —

Proof. The Tarski Laplacian is an order-preserving map on a complete lattice $C^0(G;\underline{\mathcal{F}})$ (Proposition 7.3). Therefore, by Corollary 7.3 and the Tarski Fixed Point (Theorem 2.2), the isomorphism between $\Gamma(G;\underline{\mathcal{F}})$ and $H^0(G;\underline{\mathcal{F}})$ followed by the inclusion of $H^0(G;\underline{\mathcal{F}})$ into $C^0(G;\underline{\mathcal{F}})$ is an order-embedding. □

7.4 迈向算子理论

By Corollary 7.3, global sections of Tarski sheaves have arbitrary meets and joins, forming a complete lattice. In the context of multi-agent systems, this implies the set of consistent assignments of lattice-valued data over a network of agents forms a lattice and results in two merging operations (meet/join) between consistent assignments, sections. In this section, we introduce two new operators: the closure and the Helmholtzian. We discuss a strategy for computing the explicit lattice structure of $H^0(G;\underline{\mathcal{F}})$ of a Tarski sheaf.

7.4.1 一个闭包算子

我们在 $C^0(G;\ \mathcal{F})$ 上介绍另一个不是拉普拉斯算子的局部算子。

定义 7.6 (闭包)。 该闭包算子 是一个自同态。

$$E: \qquad C^0(G;\underline{\mathcal{F}}) \to C^0(G;\underline{\mathcal{F}}) \tag{7.10}$$

$$(E\mathbf{x})_i = \bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i). \tag{7.11}$$

Recall, if $(f_\bullet, f^\bullet)$ is a Galois connection, then $f^\bullet f_\bullet \succcurlyeq \mathrm{id}$ (Proposition 3.1). The closure, then, "intensifies" local data. To be precise, E is a closure operator on the complete lattice $C^0(G;\underline{\mathcal{F}})$.

定理 7.2 （闭包定理）。 算子 E 是 $C^0(G;\ \mathcal{F})$上的闭包算子。

—

Proof. It is trivial to check E is order-preserving. By Proposition 3.1 and the universal property of meets (Lemma 3.3), $(E\mathbf{x})_i \succcurlyeq x_i$ for all $i \in \mathcal{V}_G$. Hence, E is inflationary. To show E is idempotent, we show $E^2(\mathbf{x}) \preccurlyeq \mathbf{x}$ and $E^2(\mathbf{x}) \succcurlyeq \mathbf{x}$. For the first inequality,

$$\begin{aligned}
(E^2\mathbf{x})_i = &\bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)\Big(\bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i)\Big)\\
\preccurlyeq &\bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)\overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i)\\
= &\bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i)\\
= &\ (E\mathbf{x})_i,
\end{aligned}$$

其中倒数第二个等式由引理3。2得出。对于第二个不等式,

$$\begin{aligned}
(E^2\mathbf{x})_i = &\bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij) \bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i)\\
\succcurlyeq &\bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i)
\end{aligned}$$

由 $E\mathbf{x} \succcurlyeq \mathbf{x}$ 和单调性。因此， $E^2 = E$。 □

闭包算子 E 的不动点是$C^0(G;\ \mathcal{F})$的一个准子格。

—

推论 7.4. $\mathrm{Fix}(E)$ 是一个完备格。

证明。 应用定理2。 1。 □

此外， 我们可以显式计算交和并

$$\begin{aligned}
\mathbf{x} \wedge_{\mathrm{Fix}(E)} \mathbf{y} = &\ \mathbf{x} \wedge \mathbf{y},\\
\mathbf{x} \vee_{\mathrm{Fix}(E)} \mathbf{y} = &\ E(\mathbf{x} \vee \mathbf{y}).
\end{aligned}$$

以下内容建立了闭包算子与塔斯基拉普拉斯算子之间的联系。

引理 7.4. 设 $\mathbf{x} \in H^0(G;\ \mathcal{F})$。则，$\underline{L}(\mathbf{x}) = E(\mathbf{x})$。

证明。 设 $\mathbf{x} \in H^0(G;\ \underline{\mathcal{F}})$。那么，对所有 $i \in \mathcal{V}_G$,

$$\begin{aligned}(L\mathbf{x})_i = & \bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(j \trianglelefteq ij)(x_j)\\ = & \bigwedge_{j\in\mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i)\end{aligned}$$

由于 $\mathcal{F}(j \trianglelefteq ij)(x_j) = \mathcal{F}(i \trianglelefteq ij)(x_i)$ 对于每个 $ij \in \mathcal{E}_G$。 □

7.4.2 截面的交与并

Lemma 7.4 suggests that one may attempt to restrict L to $H^0(G;\underline{\mathcal{F}})$ to obtain a closure operator. The following shows that $H^0(G;\underline{\mathcal{F}})$ is invariant under L. A similar result was shown by Cunninghame-Green & Butkovic [41] in a highly-specialized case (Figure 1.4) of max-plus restriction maps over the network with two nodes and one edge.

引理 7.5. 假设 $\mathbf{x} \in H^0(G;\mathcal{F})$。那么，$L(\mathbf{x}) \in H^0(G;\ \mathcal{F})$。

证明。 根据霍奇-塔斯基定理（定理 7.1），只需证明 $L(\mathbf{x}) \succcurlyeq \mathbf{x}$，这由 L 的单调性和传递性得出。 □

Warning 7.1. It would be tempting to conclude that $L|_{H^0(G;\underline{\mathcal{F}})} : H^0(G;\underline{\mathcal{F}}) \to H^0(G;\underline{\mathcal{F}})$ is a closure operator, and, then, one may compute meets and joints in $H^0(G;\underline{\mathcal{F}})$ explicitly by Theorem 2.1. However, this argument is invalid because, in the definition of the Tarski Laplacian, the product lattice structure on $C^0(G;\underline{\mathcal{F}})$ does not, in general, coincide with the lattice structure on $H^0(G;\underline{\mathcal{F}})$. Hence, this argument is circular.

7.4.3 塔斯基-亥姆霍兹算子

塔斯基拉普拉斯算子作用于 $C^0(G;\ \underline{\mathcal{F}})$。我们定义另一个算子，类似于图亥姆霍兹算子 [104]，它作用于 $C^1(G;\ \mathcal{F})$。

定义 7.7 (亥姆霍兹算子)。 设 $\mathcal{F}$ 是 G 上的塔斯基层。塔斯基-亥姆霍兹算子 是算子 $\mathfrak{L} : C^1(G;\mathcal{F}) \to C^1(G;\ \mathcal{F})$ 定义

$$(\mathfrak{L}\mathbf{x})_{ij} = \left(\bigvee_{j'\in\mathcal{N}_i - j} \underline{\mathcal{F}}(i \trianglelefteq ij)\overline{\mathcal{F}}(i \trianglelefteq ij')(x_{ij'})\right) \vee \left(\bigvee_{i'\in\mathcal{N}_j - i} \underline{\mathcal{F}}(j \trianglelefteq ij)\overline{\mathcal{F}}(j \trianglelefteq i'j)(x_{i'j})\right)$$

我们猜测，格 $H^1(G;\ \mathcal{F})$ 作为余等化子（5.2）构造而成，与塔斯基-亥姆霍兹算子的前缀点一致。

猜想 7.1. 假设 $\mathcal{F}$ 是 G 上的塔斯基层，其塔斯基-亥姆霍兹算子为 $\mathfrak{L}$: $C_1(G;\mathcal{F}) \to C_1(G;\ \mathcal{F})$。那么，

$$\mathrm{Pre}(\mathfrak{L}) = H^1(G;\underline{\mathcal{F}}). \tag{7.12}$$

7.5 算法

本章最后讨论两种受 TarskiLaplacian 启发的不同算法。

7.5.1 热流

塔斯基拉普拉斯算子启发我们在 $C^0(G;\mathcal{F})$ 上定义离散时间热方程：

$$\mathbf{x}[t+1] = (L \wedge \mathrm{id})\mathbf{x}[t] \tag{7.13}$$

初始条件为 $\mathbf{x}[0] \in C^0(G;\mathcal{F})$。局部而言，这些更新可以表示为一个分布式算法（算法 2）。为保证算法终止，我们需要一个假设。

假设 7.1. (下降链条件) 积格 $C^0(\mathcal{F};G)$ 满足下降链条件（定义 2.3）。

算法 2： 热流

输入： 双层 $\mathcal{F}$ 在图 $G = (\mathcal{V}_G, \mathcal{E}_G)$上的层；初始 $x_i \in \underline{\mathcal{F}}(i), i \in \mathcal{V}_G$

输出： $x_i \in \mathcal{F}(\underline{i})$，$i \in \mathcal{V}_G$

重复

  对于 $i \in \mathcal{V}_G$，**并行执行**

    对于 $j \in \mathcal{N}_i$ **执行**

      $x_i \leftarrow x_i \wedge \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(j \trianglelefteq ij)(x^j)$

    end

  end

直到 $\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) = \underline{\mathcal{F}}(j \trianglelefteq ij)(x^j)\ \forall ij \in \mathcal{E}_G$

命题 7.5 (完备性)。 假设 $\mathcal{F}$ 是一个定义在 G上的塔斯基层，满足假设 7。1。则，算法 2 在有限次迭代内收敛。

Proof. $L \wedge \mathrm{id} \preccurlyeq \mathrm{id}$ (trivially) and the recursion $\mathbf{x}[t] = (L \wedge \mathrm{id})^t\mathbf{x}[0]$ imply $\mathbf{x}[s] \preccurlyeq \mathbf{x}[t]$ whenever $s \leqslant t$. Hence, $\{\mathbf{x}[t]\}_{t\geqslant 0}$ forms a descending chain in $C^0(G;\underline{\mathcal{F}})$. □

推论 7。5。 假设 $\mathcal{F}$ 是 G 上的一个塔斯基层，满足假设 7。1，则算法 2 收敛到某个 $\mathbf{x} \in H^0(G;\ \mathcal{F})$。

Proof. By Proposition 7.5, the Algorithm 2 converges. Hence, there is a t_0 such that $(L \wedge \mathrm{id})^t\mathbf{x}[t] = (L \wedge \mathrm{id})^t\mathbf{x}[t_0]$ for all $t \geqslant t_0$. Therefore, by the Tarski-Hodge Theorem, $(L \wedge \mathrm{id})^t\mathbf{x}[t_0] \in H^0(G;\underline{\mathcal{F}})$. □

如果偏序集是分级的，那么它满足下降链条件。

推论 7。6。 假设 $C^0(G;\ \mathcal{F})$ 是分级的。那么，算法 2 也会在有限时间内终止，并收敛到一个截面。

热流算法（算法 2）以视觉形式呈现（图 7。2）。

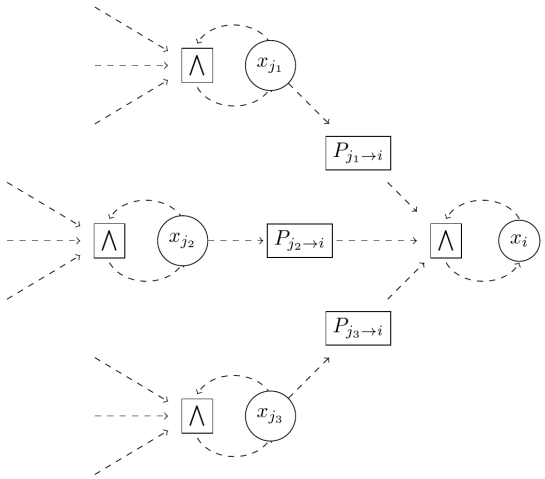


图 7.2：塔斯基层上的热流。

7.5.2 *Gossip*算法

Gosip algorithms are a general class of decentralized algorithms that are broadly characterized as having each exchange information with one or a few neighboring nodes each round. The advantages of gossip over Laplacian-based methods include fault tolerance and a relaxation of assumption that nodes update their states synchronously [90]. In order to describe an asynchronous version of the Heat Flow algorithm, we will need some notation. Let $\tau : \mathbb{N} \to \wp(\mathcal{V}_G)$ be a *broadcast sequence*. At time t, the nodes $\tau(t) \subseteq \mathcal{V}_G$ broadcast to all neighbors $j \in \mathcal{N}_i$. This is quite general. For instance, if $\tau(t) = \mathcal{V}_G$, then we recover Algorithm 2. However, if $\tau(t)$ is a single node, then we have a truly asynchronous algorithm. Additionally, we require nodes to continually exchange information.

假设 7.2. (活性) 对于每个 $i \in \mathcal{V}_G$ 和 $t_0 \in \mathbb{N}$，都存在$t \geqslant t_0$，使得 $i \in \tau(t)$。

HeatFlow 算法（算法 2）通过仅聚合给定时刻正在广播的节点所发送的经平行移动的格值信息而得到修改（算法 3）。

或者，我们可以将该算法改写为递归方程。

$$\mathbf{x}[t+1] = L_t\big(\mathbf{x}[t]\big) \wedge \mathbf{x}[t] \tag{7.14}$$

称为*Gossip* 方程，其时变 Tarski拉普拉斯算子L_t定义如下。

算法3： Gossip算法

输入： 图 G = 上的双层 $\mathcal{F}(\mathcal{V}_G, \mathcal{E}_G)$；初始 $x_i \in \mathcal{F}(i), i \in \mathcal{V}_G$；广播序列 τ：$\mathbb{N} \to \wp(\mathcal{V}_G)$.

对于 $t \in \mathbb{N}$ **执行**

  for $i \in \tau(t)$ **do**

    对于 $j \in \mathcal{N}_i$ **执行**

      发送至$(j) \leftarrow \underline{\mathcal{F}}(i \trianglelefteq ij)(x_i)$

      receiveFrom$(i) \leftarrow$sendTo(j)

      $x_j \leftarrow x_j \wedge \overline{\mathcal{F}}(j \trianglelefteq ij)$ (`receiveFrom`(i))

    end

  end

end

Definition 7.8 (Time-Varying Tarski Laplacian)**.** Suppose G is a graph and $\mathcal{F}$ is a Tarski sheaf over G and $\tau : \mathbb{N} \to \wp(\mathcal{V}_G)$ satisfies Assumption 7.2. The *asynchronous Tarski Laplacian*:[2] is the time-varying operator

$$\begin{aligned} \tilde{L}_t : \quad & \mathbb{N} \times C^0(G; \underline{\mathcal{F}}) \to C^0(G; \underline{\mathcal{F}}) \\ (\tilde{L}_t)_i = \quad & \bigwedge_{j \in \tau(t) \cap \mathcal{N}_i} \overline{\mathcal{F}}(i \trianglelefteq ij) \underline{\mathcal{F}}(j \trianglelefteq ij)(x_j). \end{aligned} \tag{7.15}$$

现在，我们陈述一个关于 Gossip算法的整体截面的均衡结果。

Theorem 7.3 (Theorem 1 [135])**.** Suppose G is a graph and $\mathcal{F}$ is a Tarski sheaf over G. Suppose $\tau : \mathbb{N} \to \wp(\mathcal{V}_G)$ is a firing sequence satisfying Assumption 7.1. Suppose $\tilde{L}_t$ is the asynchronous Tarski Laplacian. Then, if τ is a broadcast sequence satisfying Assumption 7.2, the sections $H^0(G; \underline{\mathcal{F}})$ coincide with the time-independent solutions of the recursion equations

$$\mathbf{x}[t+1] = (L_t \wedge \mathrm{id})\mathbf{x}[t] \tag{7.16}$$

初始条件为 $\mathbf{x}[0] \in C^0(G; \mathcal{F})$.

推论 7.7。假设 G，$\mathcal{F}$ 且 τ 满足假设 7。1 和 7。2。则，算法3 在有限次迭代内终止，并收敛到一个 $\mathbf{x} \in H^0(G; \mathcal{F})$。

在附录B中，我们回顾了有关流言算法（算法3）收敛速度的一些实验结果。

2 如果 ~~$\tau(t) \cap \mathcal{N}_i = \emptyset$~~，则 ~~$(\tilde{L}_t)_i = 1$~~ 因为 $\bigwedge \emptyset = 1$。

第三部分应 用

8

信号

Signal processing is a broad area of engineering devoted to the study of various sorts of signals. Some examples of signals include audio, video, speech, image, communication, GPS, sonar, lidar, and radar, among others. In our view, signal processing is synonymous with information processing.

At a high level of abstraction, a signal is a map $x : \Omega \to \mathcal{C}$ where Ω is a domain and $\mathcal{C}$ is an object in a data category. Classically, $\mathcal{C}$ is a finite-dimensional vector space and each dimension of $\mathcal{C}$ is called a *channel* or *feature*, however, the same terminology may be used if $\mathcal{C}$ is an arbitrary object in a data category. The set of signals over Ω valued in $\mathcal{C}$ is denoted $\mathcal{X}(\Omega, \mathcal{C})$.

示例8.1. 一些传统示例：

> 波形。假设 T 是波形的周期。那么，Ω 是区间 $[0,T]$ 具有且 $\mathcal{C} = \mathbb{R}$ （电势、位移等）。
>
> 图像. Ω 是一个网格 $\mathbb{Z} \times \mathbb{Z}$ （像素）和 $\mathcal{C} = \mathbb{R}^d$ （强度），其中通常 $d = 1$ （灰度）或 3（RBG）。
>
> 网络. Ω 是一个图 $G = (\mathcal{V}_G, \mathcal{E}_G)$ 和 $\mathcal{C} = \mathbb{R}^d$；每个通道都是一个特征。在实践中，通道通常是从数据中生成的特征，而图则根据节点/智能体之间的相似性度量（例如皮尔逊相关系数）构建。

In Chapter 6 we introduced lattice-valued sheaves as a way to generalize sheaves valued in sets and vector spaces. If every stalk in a lattice-valued sheaf over a graph G is the same lattice, say $\mathbf{L}$, then it is perfectly reasonable to identify 0-cochains $C^0(G; \underline{\mathcal{F}})$ with signals $\mathcal{X}(\mathcal{V}_G, \mathbf{L})$. We might, then, hope for some of the themes of signal processing to port to lattice-valued sheaves.

8.1 代数信号处理

Algebraic signal processing (ASP) is a unified signal processing framework based on the representation theory of associative algebras [51]. One limitation of ASP, however, is that $\mathcal{X}(\Omega, \mathcal{C})$ must be a vector space in this perspective. We first basics of representation theory.

定义 8.1 (代数)。 令 $\Bbbk$ 为一个域。一个结合代数 $\Bbbk$，或简称为$\Bbbk$-代数，是一个向量空间A ，配备一个结合双线性映射[1]

$$\cdot : A \times A \to A$$

A $\Bbbk$-algebra is *unital* if there exists an element $1 \in A$ with $a \cdot 1 = 1 \cdot a = a$. An *algebra homomorphism* between algebras $(A, \cdot)$ and $(A', \cdot')$ is a linear transformation $\rho : A \to A'$ such that $\rho(a \cdot b) = \rho(a) \cdot' \rho(b)$.

我们提供了几个代数的示例，其表示a在工程学中颇受关注。

Example 8.1 (Polynomials)**.** Polynomials with coeficients in $\Bbbk$ form an algebra denoted $\Bbbk[t]$; multiplication is given by multiplication of polynomials. Multinomials, in the same way, form an algebra $\Bbbk[t_1, t_2, \ldots, t_n]$. The unit is the constant polynomial $1 \in \Bbbk[t]$ or $\Bbbk[t_1, t_2, \ldots, t_n]$.

Example 8.2 (Endomorphism Algebra)**.** Suppose V is a vector space. Then, End V is a $\Bbbk$-algebra with composition. If V is finite-dimensional, End V is isomorphic to the algebra $\mathrm{Mat}_{n\times n}(\Bbbk)$ of $n \times n$ matrices with coefficients in $\Bbbk$. The unit is the identity linear transformation.

Example 8.3 (Monoid Algebra)**.** Let $(M, \star, e)$ be a monoid. The *monoid algebra* denoted $\Bbbk[M]$ (sometimes called the *monoid ring*) is the free vector space generated by formal linear combinations of elements of M

$$\sum_{m\in M} h_m[m], \quad |\{h_m \in \Bbbk \mid h_m \neq 0\}| < \infty.$$

$\Bbbk[M]$ 中的乘法定义在基元素上

$$[m][n] = \qquad\qquad [m \cdot n],$$

然后通过线性延拓。代数 $\Bbbk[M]$ 是有单位元的，其单位元为 $[e]$。等价地，幺半群代数由带乘法的函数向量空间 $\Bbbk^M$ 构成

$$(f \circledast g)(m) = \qquad \sum_{p\cdot q=m} f(p)g(q) \quad f, g \in \Bbbk^M, \quad m, p, q \in M.$$

幺半群代数在有限幺半群表示的研究中是整的 [155],有限群表示理论的推广 [150]。

例 8.4(幺半群代数)。 假设M 是半格M ={x、y、z,1} 其乘法表如下

1 等价地，一个线性映射 $A \otimes A \to A$

$\wedge$	x	y	z	1
x	x	z	z	x
y	z	y	z	y
z	z	z	z	z
1	x	y	z	1

那么，假设 $h = [x] - [y]$ 且 $h' = [1] - 2[y] - [z],$，那么

$$\begin{aligned} h \cdot h' = & \\ = & \quad ([x] - [y]) \cdot ([1] - 2[y] - [z]) \\ = & \quad [x \wedge 1] - 2[x \wedge y] - [x \wedge z] - [y \wedge 1] + 2[y \wedge y] + [y \wedge z] \\ = & \quad [x] - 2[z] - [z] - [y] + 2[y] + [z] \\ = & \quad [x] - 2[z] + [y] \end{aligned}$$

例子 8.5 (关联代数)。设 P 是一个局部有限[2] 偏序集。那么，P 的关联代数记作 $\mathbb{k}\mathrm{P}$，它是函数向量空间上的一个 $\mathbb{k}$-代数。

$$f : \mathbb{I}\mathrm{P} \to \mathbb{k}$$

与乘法搭配

$$(f \circledast g)([x, y]) = \sum_{x \preccurlyeq y \preccurlyeq z} f([x, z]) g([z, y]).$$

等价地，$\mathbb{k}\mathrm{P}$ 是在具有适当幺半群结构的 $\mathbb{I}\mathrm{P}$ 上的幺半群代数。

Example 8.6 (Path Algebra)**.** A *quiver* is a tuple $Q = (\mathcal{V}_Q, \mathcal{E}_Q, h, t)$ consisting of sets $\mathcal{V}_Q, \mathcal{E}_Q$ (vertices, arrows) and maps $h, t : \mathcal{E}_Q \to \mathcal{V}_Q$ (head, tail). A path in a quiver is a sequence $\gamma = a_0 a_1 \dots a_\ell$ of elements of $\mathcal{E}_Q$ such that $h(a_i) = t(a_{i+1})$. Trivial paths e_i are indexed by $\mathcal{V}_Q$. Let $\mathrm{Path}(Q)$ denote the set of paths in Q. Then, the *path algebra* $\mathbb{k}Q$ is the free vector space generated by $\mathrm{Path}(Q)$ with a product of basis elements

$$\gamma_1 \cdot \gamma_2 = \begin{cases} \gamma_1 \gamma_2 & h(\gamma_1) = t(\gamma_2) \\ 0 & h(\gamma_1) \neq t(\gamma_2) \end{cases}$$

假设A 是一个 $\mathbb{k}$-代数。A 的表示是一个向量空间V，以及一个代数同态 $\rho : A \to \mathrm{End}(V)$。$A$ 的表示构成一个范畴 $\mathcal{R}ep_A$，其态射如下。

定义 8.2 (交织映射)。设(V, ρ) 和(V, ρ') 是A的表示。那么，一个交织映射 是线性变换 $\theta : V \to V'$，满足

$$\rho'_a \circ \theta = \theta \circ \rho_a$$

对于所有 $a \in A$。

2 如果每个闭区间 $[x,y]$ 都是有限偏序集，则称偏序集是局部有限的。当然，有限偏序集都是局部有限的

在上述示例中，信号空间与各个带有表示的代数自然配对，从而引出以下定义。

定义 8.3（代数信号模型）。一个代数信号模型 是一个三元组$(A, \mathcal{X}, \rho)$。A 是一个 $\Bbbk$-代数，其元素称为滤子；$\mathcal{X}$是一个向量空间，其元素称为信号；ρ 是一个表示

$$\rho : \qquad A \to \mathrm{End}(\mathcal{X}).$$

给定一个滤波器 $h \in A$，线性变换[3] $\rho_h : \mathcal{X} \to \mathcal{X}$ 是实例化h的。对信号$x \in \mathcal{X}$ 施加 ρ_h 称为卷积。

Remark 8.1. The original formulation of algebraic signal processing regards an algebraic signal model as an algebra A of filters together with an A-module. However, the data of an A module is the same as the data of a representation $\rho : A \to \mathrm{End}(\mathcal{X})$ by standard facts of representation theory [51]. We prefer to think of representations as maps to emphasize relationships between models.

现在我们将展示，熟悉的和不熟悉的信号处理框架都归入代数信号处理的范畴。

Example 8.7 (Graph Signal Processing [129])**.** Let $G = (\mathcal{V}_G, \mathcal{E}_G)$ be a weighted graph with (graph) shift operator $S : \mathbb{R}^{\mathcal{V}_G} \to \mathbb{R}^{\mathcal{V}_G}$. There are several common choices for S, including the (normalized) adjacency or Laplacian matrix. A *graph filter* is an element of the algebra $\Bbbk[t]$. The signal space $\mathcal{X}(\mathcal{V}_G, \mathbb{R})$ consists is the vector space of functions $\mathbb{R}^{\mathcal{V}_G}$. Convolution, then, is a representation

$$\rho : \mathbb{R}[t] \to \mathrm{End}\left(\mathbb{R}^{\mathcal{V}_G}\right).$$

由于 $\mathbb{R}[t]$ 由单项式t生成，一个表示由 $\rho(t)$ 唯一确定，我们选择其为图移位算子。因此，图滤波器由

$$\begin{aligned} \mathbf{z} &= \mathbf{h} \circledast \mathbf{x} \\ &= \rho_{\mathbf{h}} \cdot \mathbf{x} \\ &= \left(\sum_{k=0}^{\infty} h_k S^k\right) \cdot \mathbf{x} \end{aligned}$$

让我们考虑一个图滤波器的具体示例，即热核 [126]。热核是一种参数化线性图滤波器，记作H_t，其作用于图信号时

$$H_t \cdot \mathbf{x} = \left(\sum_{k=0}^{\infty} \frac{t^k e^{-t}}{k!} S^k\right) \cdot \mathbf{x}, \quad t \geqslant 0.$$

3 我们通常将 $\rho_h : \mathcal{X} \to \mathcal{X}$ 用来表示元素 $h \in A$ 在 ρ 下的像，即线性变换。

A graph filter, in practice, will typically have $h_k = 0$ for all $k \geqslant K$ where $K \ll n$. While there is a computational justification for this, there is also a theoretical one. Suppose $n = |\mathcal{V}_G|$. Then, by the Cayley-Hamilton Theorem, we can always write S^n as a linear combination of $\{I, S, S^2, \ldots, S^{n-1}\}$. Hence, we have an alternate description of graph filters as a representation of a smaller algebra

$$\rho : \Bbbk[t]/(t^n - 1) \to \mathrm{End}(\Bbbk^{\mathcal{V}_G})$$

其中 $\Bbbk[t]/(t^n - 1)$ 是次数至多为$n - 1$的多项式代数。

Example 8.8 (Quiver Signal Processing [130])**.** Suppose $Q = (\mathcal{V}_Q, \mathcal{E}_Q, h, t)$ is a quiver. A *representation* $\boldsymbol{\pi}_\bullet$ of Q is an assignment of a vector space $\boldsymbol{\pi}(v)$ for every $e \in \mathcal{V}_Q$ and a linear transformation $\boldsymbol{\pi}(e) : \boldsymbol{\pi}(t(e)) \to \boldsymbol{\pi}(h(e))$ for every $e \in \mathcal{E}_Q$. The *total space* of a representation is the direct sum

$$\mathrm{Tot}(\boldsymbol{\pi}) = \bigoplus_{i \in \mathcal{V}_Q} \boldsymbol{\pi}(i).$$

事实是，Q 的表示与 $\Bbbk Q$ [128]的表示等价。对于其中一个方向，假设 $\boldsymbol{\pi}$ 是Q的一个表示。那么，我们定义

$$\rho^{\pi} : \Bbbk Q \to \mathrm{End}\left(\mathrm{Tot}(\boldsymbol{\pi})\right)$$

通过将路径 $\gamma = e_1 e_2 \ldots e_\ell$ 映射为线性映射的复合

$$\boldsymbol{\pi}(\gamma) = \boldsymbol{\pi}(e_\ell) \cdots \boldsymbol{\pi}(e_2)\boldsymbol{\pi}(e_1),$$

并将其 ρ^{π} 线性延拓到整个定义域 $\Bbbk Q$。作用于 $\mathrm{Tot}(\boldsymbol{\pi})$,

$$(\rho^{\pi}([\gamma]) \cdot \mathbf{x})_j = \begin{cases} \boldsymbol{\pi}(\gamma) \cdot x_i & h(\gamma) = j,\ t(\gamma) = i \\ 0 & \text{otherwise} \end{cases}$$

因此，Q 的表示与代数信号模型 $(A、\ \mathcal{X}、\ \rho) = (\Bbbk Q、\ \mathrm{Tot}(\boldsymbol{\pi})、\ \rho^{\pi})$ 相对应。

示例 8.9 (幺半群代数**)**。 假设 M 是一个幺半群。那么，在幺半群信号处理中，$A = \Bbbk[G]$ 和 $\mathcal{X} = \Bbbk^M$。幺半群卷积是一种表示

$$\rho : \Bbbk[M] \to \mathrm{End}(\Bbbk^M). \tag{8.1}$$

移位算子对应于 $\Bbbk[M]$ 的生成元，而后者又对应于 M 的生成元（如果存在）。

8.2 格信号处理

将重点缩小到格，我们首先研究定义域为（半）格的信号，然后研究通道为格的信号，最后研究定义域和通道均为格的信号。

8.2.1 半格上的信号

Püschel and Wendell proposed [131] a signal processing framework on meet- or join- semilattices. In this case, Ω is a semilattice and $\mathcal{C}$ is the real or complex number. In our view, their key definition is convolution of semilattice signals, although, in a series of paper, they build up a complete theory including Fourier transforms [132], filters for denoising [148], and an algorithm for sampling [164].

定义 8.4. 假设$(S, \wedge, 1)$是一个（交）半格。[4] 假设$\mathbf{f} \in \mathbb{k}^S$是一个信号，且$\mathbf{h} \in \mathbb{k}^S$是一个滤波器。那么，卷积 由$\mathbf{f}$和$h$定义如下：

$$(\mathbf{h} \circledast \mathbf{f})_y = \sum_{x \in S} h_x f_{x \wedge y}. \tag{8.2}$$

如有任何歧义，根据 S是交半格还是并半格，卷积分别写作 $\mathbf{h} \circledast_\wedge \mathbf{f}$ 或 $\mathbf{h} \circledast_\vee \mathbf{f}$。

卷积可以借助移位算子以更紧凑的形式表示。

$$\begin{aligned} T_x : & \quad \mathbb{k}^S \to \mathbb{k}^S \\ (T_x \mathbf{f})_y = & \quad f_{x \wedge y}. \end{aligned}$$

因此，

$$\mathbf{h} \circledast - = \sum_{x \in S} h_x T_x. \tag{8.3}$$

是一个映射，将滤子$\mathbf{j}$映射为 $\mathbb{k}^S$上的自同态。让我们考察一下移位算子的某些性质。

引理 8。1。 假设 $x, y \in \mathbf{L}$。那么，

1. T_x 是线性的。
2. $T_{x \wedge y} = T_x T_y$。 3. T_x 是幂等的。 4. $T_x T_y = T_y T_x$。

证明。 线性显然。对于（2），

$$\begin{aligned} (T_x T_y \mathbf{f})_z = & \quad (T_y \mathbf{f})_{z \wedge x} \\ = & \quad f_{(z \wedge x) \wedge y} \\ = & \quad f_{(x \wedge y) \wedge z} \\ = & \quad (T_{x \wedge y} \mathbf{f})_z. \end{aligned}$$

由于半格中的交运算具有幂等性和交换性，因此 (3) 和 (4) 可由 (2) 推出。 □

4 不失一般性，由于 S^{op} 是有界并半格。

以下定理确立了过滤的结合律。

引理 8.2. 假设 $\mathbf{h}$,$\mathbf{h}' \in \Bbbk^S$ 是滤子，且$f \in \Bbbk^S$ 是信号。那么，

$$\mathbf{h} \circledast (\mathbf{h}' \circledast \mathbf{f}) = \mathbf{h}' \circledast (\mathbf{h} \circledast \mathbf{f}).$$

证明。设 x、$y \in S$。

$$\mathbf{h} \circledast \mathbf{f} = \sum_{x \in S} h_x T_x \mathbf{f}$$

使得

$$\begin{aligned} \mathbf{h} \circledast (\mathbf{h}' \circledast \mathbf{f}) &= \sum_{y \in S} h_y T_y \left(\sum_{x \in S} h'_x T_x \mathbf{f} \right) \\ &= \sum_{y \in S} \sum_{x \in S} h_y h'_x T_x T_y \mathbf{f} \end{aligned}$$

因为 T^y 是线性的。然后，根据引理8。2，

$$\sum_{y \in S} \sum_{x \in S} h_y h'_x T_x T_y = \sum_{y \in S} \sum_{x \in S} h'_x h_y T_y T_x.$$

反转该过程意味着 $\mathbf{h} \circledast (\mathbf{h}' \circledast \mathbf{f}) = \mathbf{h}' \star (\mathbf{h} \circledast \mathbf{f})$。 □

由此可知，晶格卷积是一个具有$A = \Bbbk[S]$, $\mathcal{X} = \Bbbk^S$ 的代数信号模型，并且

$$\begin{aligned} \rho : \Bbbk[S] &\to \operatorname{End}\left(\Bbbk^S\right) \\ \rho\left(\sum_{x \in S} h_x [x]\right) &= \sum_{x \in S} h_x T_x. \end{aligned} \tag{8.4}$$

Proposition 8.1. Convolution acting on the left (or right by commutativity) $\mathbf{h} \circledast - : \Bbbk^S \to \Bbbk^S$ (8.2) coincides with the algebra homomorphism ρ_h (8.4) applied to an algebraic filter $h \in A$.

证明。 将滤子$\mathbf{h} \in \Bbbk^S$ 与多项式的系数对应起来

$$h = \sum_{x \in S} h_x [x].$$

Then, for $h \in \Bbbk[S]$, $\mathbf{f} \in \Bbbk^S$ define $\rho_h \cdot \mathbf{f}$ by $\mathbf{h} \circledast \mathbf{f}$. By Lemma 8.1, $\rho_h \in \operatorname{End}(\Bbbk^S)$. However, we still need to show that ρ is an algebra homomorphism. Suppose $h, h' \in \Bbbk[S]$. Then,

$$\begin{aligned} h \cdot h' &= \left(\sum_{x \in S} h_x [x] \right) \cdot \left(\sum_{y \in S} h'_y [y] \right) \\ &= \sum_{z = x \wedge y} h_x h'_y T_z \end{aligned}$$

引理 8.2并应用 ρ 得到

$$\begin{aligned}
\rho_{h\cdot h'} &= \sum_{z=x\wedge y} h_x h'_y \rho([z]) \\
&= \sum_{x,y\in S} h_x h'_y T_{x\wedge y} \\
&= \sum_{x\in S}\sum_{y\in S} h_x h'_y T_x T_y \\
&= \sum_{x\in S} h_x T_x \left(\sum_{x\in S} h'_y T_y\right) \\
&= \rho_h \rho_{h'}.
\end{aligned}$$

在h 中，线性是显然的。 □

8.2.2 格上的信号

We argue signal processing on lattices is not equivalent to signal processing on semilattices. Suppose $(\mathbf{L}, \wedge, \vee, 0, 1)$ is a bounded lattice. We view $\mathbf{L}$ as a bounded semilattice, and thus a monoid, in two seperate ways: a meet-semilattice denoted $(L_\wedge, \wedge, 1)$ and a join-semilattice $(L_\vee, \vee, 0)$. However, the absorption identities (Definition 2.5) further restrict the structure of a lattice than merely a poset that is simulatneously a meet- and join- semilattice. For one, there are two families of shift operators, one for each element $x \in \mathbf{L}$

$$\begin{aligned}
(T^\vee_x \mathbf{f})_y &= f_{x\vee y} \\
(T^\wedge_x \mathbf{f})_y &= f_{x\wedge y}
\end{aligned}$$

每个族都实例化滤子

$$\begin{aligned}
\rho^\wedge &: \Bbbk[L_\wedge] \to \mathrm{End}(\Bbbk^{\mathbf{L}}) \\
\rho^\vee &: \Bbbk[L_\vee] \to \mathrm{End}(\Bbbk^{\mathbf{L}})
\end{aligned}$$

给定一个有界格 $(\mathbf{L}, \wedge, \vee, \mathbf{0}, \mathbf{1})$，我们研究代数信号模型之间的关系

$$\begin{aligned}
\mathcal{M}_\wedge &= (\Bbbk[L_\wedge], \Bbbk^{\mathbf{L}}, \rho^\wedge), \\
\mathcal{M}_\vee &= (\Bbbk[L_\vee], \Bbbk^{\mathbf{L}}, \rho^\vee).
\end{aligned}$$

命题 8.2. 设 $\mathbf{L}$ 是一个带有移位算子 $\{T^\wedge_a、T^\vee_a\}_{a\in\mathbf{L}}$ 的格，定义为

$$\begin{aligned}
(T^\wedge_a \mathbf{f})_x &= f_{x\wedge a}, \\
(T^\vee_a \mathbf{f})_x &= f_{x\vee a}.
\end{aligned}$$

那么，

$$T^\vee_x T^\wedge_x = T^\wedge_x T^\vee_x.$$

此外，

$$T_x^{\vee} T_x^{\wedge} \mathbf{f} = f_x \mathbf{1}$$

其中 $\mathbf{1} \in \Bbbk^{\mathbf{L}}$ 是常数单位信号。

证明。 假设 $\mathbf{f} \in \Bbbk^{\mathbf{L}}$。吸收恒等式（Ch 蕴含

$$\begin{aligned}(T_x^{\vee} T_x^{\wedge} \mathbf{f})_y &= (T_x^{\wedge} \mathbf{f})_{x \vee y} \\ &= f_{(x \vee y) \wedge x} \\ &= f_x.\end{aligned}$$

显然，$(T_x^{\wedge} T_x^{\vee} \mathbf{f})^y = f_x$ 也成立。 □

推论 8。1。 $T_x^{\wedge} T_x^{\vee}$和 $T_x^{\vee} T_x^{\wedge}$是幂等的。

证明。 注意 $T_x^{\wedge} \mathbf{1} = \mathbf{1}$ 和 $T_x^{\vee} \mathbf{1} = \mathbf{1}$。然后，根据命题8。2，

$$\begin{aligned}T_x^{\vee} T_x^{\wedge} T_x^{\vee} T_x^{\wedge} \mathbf{f} &= T_x^{\vee} T_x^{\wedge} f_x \mathbf{1} \\ &= f_x \mathbf{1} \\ &= T_x^{\vee} T_x^{\wedge} \mathbf{f}\end{aligned}$$

□

现在我们将介绍原文中呈现的一些傅里叶分析内容 [132]。设$\mathbf{L}$ 是一个有界格。令

$$1_{\{y \preccurlyeq x\}} = \begin{cases} 1 & y \preccurlyeq x \\ 0 & y \not\preccurlyeq x \end{cases}$$

be the indicator function on the boolean variable $\{y \preccurlyeq x\}$. Define a signals $\mathbf{f}^{\preccurlyeq y} = (1_{\{y \preccurlyeq x\}})_{x \in \mathbf{L}}$ and $\mathbf{f}^{\succcurlyeq y} = (1_{\{y \succcurlyeq x\}})_{x \in \mathbf{L}}$.

Lemma 8.3 (Theorem 1 [132])**.** Suppose $\mathbf{L}$ is a lattice with shifts $\{T_x^{\wedge}, T_x^{\vee}\}_{x \in \mathbf{L}}$. Then, $T_x^{\wedge} \mathbf{f}^{\preccurlyeq y} = 1_{y \preccurlyeq x} \mathbf{f}^{\preccurlyeq y}$ and $T_x^{\vee} \mathbf{f}^{\succcurlyeq y} = 1_{y \succcurlyeq x} \mathbf{f}^{\succcurlyeq y}$. Furthermore, $\{\mathbf{f}^{\preccurlyeq y}\}_{y \in \mathbf{L}}$ is an eigenbasis of $\Bbbk^{\mathbf{L}}$ with respect to $T_x^{\wedge}$. Dually, $\{\mathbf{f}^{\succcurlyeq y}\}_{y \in \mathbf{L}}$ is an eigenbasis of $\Bbbk^{\mathbf{L}}$ with respect to $T_x^{\vee}$.

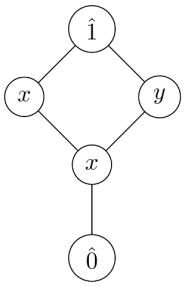
证明。 有关 $T_x^{\wedge}$ 的情形，请参见 [132] 。$T_x^{\vee}$ 的情形可由对偶原理推出。 □

Every basis vector $\mathbf{f}^{\preccurlyeq y}$ (dually, $\mathbf{f}^{\succcurlyeq y}$) is an eigenvector of $T_x^{\wedge}$ with eigenvalue $1_{\{x \preccurlyeq y\}}$. Similarly, for $T_x^{\vee}$. Hence, the only eigenvalues of $T_x^{\wedge}$ and $T_x^{\vee}$ are 0 and 1. Define a change-of-basis transformation $\theta : \Bbbk^{\mathbf{L}} \to \Bbbk^{\mathbf{L}}$ by sending each $\mathbf{f}^{\preccurlyeq y} \in \{\mathbf{f}^{\preccurlyeq y}_{y \in \mathbf{L}}$ to $\mathbf{f}^{\succcurlyeq y} \in \{\mathbf{f}^{\succcurlyeq y}_{y \in \mathbf{L}}$. For the remainder of this section, suppose $\mathbf{L}$ is finite. Then, we can place a total order on the elements $\mathbf{L}$ (e.g. topological sort), labeling elements $\mathbf{L} = \{y_1, y_2, \ldots, y_m\}$. Now, in the finite case, the linear transformation θ is represented by an invertible matrix $\Theta = [\theta_{ij}]$ uniquely determined

通过将基 $\{\mathbf{f}^{\succcurlyeq y}\}_{y\in\mathbf{L}}$ 中的每个元素写成基元素 $\{\mathbf{f}\}^{\preccurlyeq y}\in {}_{y\mathbf{L}}$的线性组合。

$$\mathbf{f}^{\succcurlyeq y_i} = \sum_{i=1}^{m}\theta_{i,j}\mathbf{f}^{\preccurlyeq y_j}.$$

示例8.10. 假设$\mathbf{L}$ 是格



我们将$\mathbf{L} = \{\hat{}0、z、x、y、\hat{}1\}$中的元素排序。我们用$B_\wedge$表示其列为交基向量的矩阵

$$B_\wedge = \begin{bmatrix}\mathbf{f}^{\preccurlyeq\hat{0}} & \mathbf{f}^{\preccurlyeq z} & \mathbf{f}^{\preccurlyeq x} & \mathbf{f}^{\preccurlyeq y} & \mathbf{f}^{\preccurlyeq\hat{1}}\end{bmatrix} = \begin{bmatrix}1&1&1&1&1\\0&1&1&1&1\\0&0&1&0&1\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix},$$

Clearly, $B_\vee = B_\wedge^\top$ because $[B_\wedge]_{i,j} = 1_{\{x_i\preccurlyeq x_j\}}$ Then, $\Theta B_\wedge = B_\vee$ implies $\Theta = B_\vee B_\wedge^{-1}$. In our case, θ sends the signal

$$\mathbf{f} = \left(f_{\hat{0}}, f_z, f_x, f_y, f_{\hat{1}}\right)$$

到信号

$$\theta(\mathbf{f}) = \left(f_{\hat{0}} - f_z, f_{\hat{0}} - f_x - f_y + f_{\hat{1}}, f_{\hat{0}} - f_y, f_{\hat{0}} - f_x, f_{\hat{0}}\right).$$

我们现在给出一个“不可能性结果”，表明表示 $\rho^\wedge$ 与 $\rho^\vee$ 之间的同构意味着底层格是平凡格 $\mathbf{L} = \{0 = 1\}$。

定理8.1. 设 $\mathbf{L}$ 是一个格。那么，

$$\theta\left(T_x^\wedge\mathbf{f}\right) = T_x^\vee\theta(\mathbf{f}) \tag{8.5}$$

当且仅当 $\mathbf{L} = \{0 = 1\}$.

证明。一个方向是显然的。设（8。5）成立。那么，根据引理8。3，我们可以写成 $\mathbf{f} = \sum_{y \in \mathbf{L}} h_y \mathbf{f}^{\preccurlyeq y}$ 其中 $h_y \in \Bbbk$。一方面，

$$\begin{aligned}
\theta\left(T_x^{\wedge}\mathbf{f}\right) &= \theta T_x^{\wedge}\left(\sum_{y\in\mathbf{L}} h_y \mathbf{f}^{\preccurlyeq y}\right) \\
&= \sum_{y\in\mathbf{L}} h_y \theta T_x^{\wedge y}\mathbf{f}^{\preccurlyeq y} \\
&= \sum_{y\in\mathbf{L}} h_y 1_{y\preccurlyeq x}\theta(\mathbf{f}^{\preccurlyeq y}) \\
&= \sum_{y\in\mathbf{L}} h_y 1_{y\preccurlyeq x}\mathbf{f}^{\succcurlyeq y}
\end{aligned}$$

另一方面，

$$\begin{aligned}
T_x^{\vee}\theta(\mathbf{f}) &= T_x^{\vee}\theta\left(\sum_{y\in\mathbf{L}} h_y \mathbf{f}^{\preccurlyeq y}\right) \\
&= \sum_{y\in\mathbf{L}} h_y T_x^{\vee}\theta(\mathbf{f}^{\preccurlyeq y}) \\
&= \sum_{y\in\mathbf{L}} h_y T_x^{\vee}\mathbf{f}^{\succcurlyeq y} \\
&= \sum_{y\in\mathbf{L}} h_y 1_{y\succcurlyeq x}\mathbf{f}^{\succcurlyeq y}
\end{aligned}$$

因此，对于给定的固定$x \in \mathbf{L}$，只有当$y \preccurlyeq x$ 当且仅当$y \succcurlyeq x$时，这一条件才必须成立。这只有在$\mathbf{L} = \{\hat{0} = \hat{1}\}$ 是一个单元素格时才可能。□

我们给出格上信号及其卷积的两个示例。

例 8.11 （编码）。 给定格中的一个元素 $x \in \mathbf{L}$，我们可以将其编码为信号 $\mathbf{1}_x : \mathbf{L} \to \mathbb{R}$，其中

$$\mathbf{1}_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

In the signal-processing literature, this is called a *one-hot* encoding since exactly one component of the signal is "hot," while the rest are "cold." In practice, noise is sometimes added to one-hot encodings for numerical stability. The one-hot encoding is useful for applying continuous information processing techniques to lattices (algebraic structures). For instance, if $\mathbf{L}$ is finite, $\mathbf{L} = \{\hat{0} = y_1, y_2, \ldots, y_m = \hat{1}\}$ is an total ordering of the elements of $\mathbf{L}$, and $\mathcal{F}$ is an approximation of the constant sheaf $\underline{\mathbf{L}}$ over G, then the Tarski Laplacian can be identifies as a map $\mathbf{L} : \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$. In general, the Tarski Laplacian of a lattice-valued sheaf whose stalks are finite lattices can be expressed as a map

$$L : \mathbb{R}^{\sum_{i\in\mathcal{V}_G} |\mathcal{F}(i)|} \to \mathbb{R}^{\sum_{i\in\mathcal{V}_G} |\mathcal{F}(i)|}$$

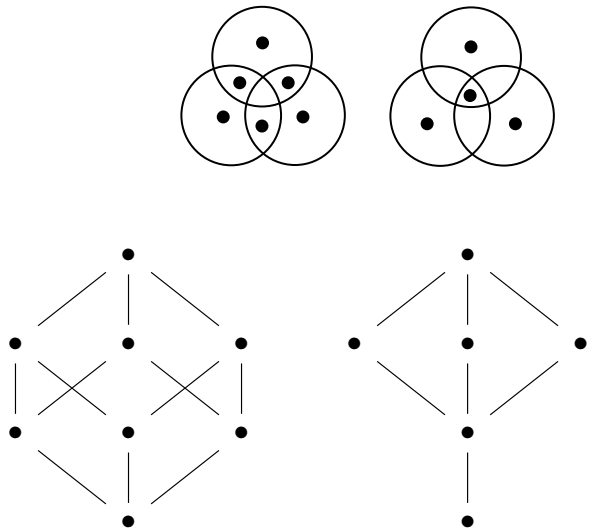


图 8.1：超图及其伽罗瓦格。

More efficient encodings of lattice elements is a subject of interest. For instance, if a lattice is finite, then every element is a join of join irreducible elements. Hence, you could represent an element of a finite lattice with a few-hot encoding, however, not uniquely.

Example 8.12 (Hypergraph Convolution)**.** With mimicking graph signal processing [168] as one approach to filtering signals on hypergraphs, lattice convolution, as was suggested in the introductory paper [132], is another approach. Suppose $H = (\mathcal{V}_H, \mathcal{H}_H)$ is a hypergraph. Recall, a hypergraph signal is a map $\mathbf{f} : \mathcal{V}_H \to \mathbb{R}$. Suppose $\mathcal{I}$ is the incidence relation of H. Then, the Galois lattice is the following

$$\mathrm{Gal}(\mathcal{I}) = \quad \{(\sigma, \tau) \subseteq \mathcal{V}_H \times \mathcal{H}_H \mid \cap_{i \in \sigma} i^{\uparrow} = \tau, \sigma = \cap_{e \in \tau} e^{\downarrow}\},$$

我们在此上下文中将其记为 $\mathsf{Gal}(H)$ 。回想一下，$\mathsf{Gal}(H)$ 与以下不动点格同构

$$\begin{aligned} \mathrm{Cl}(\wp(\mathcal{V}_H)) &= \quad \{\sigma \subseteq \mathcal{V}_H \mid \sigma^{\uparrow\downarrow} = \sigma\} \subseteq \quad \wp(\mathcal{V}_H), \\ \mathrm{Cl}(\wp(\mathcal{H}_H)) &= \quad \{\tau \subseteq \mathcal{H}_H \mid \tau^{\downarrow\uparrow} = \tau\} \subseteq \quad \wp(\mathcal{H}_H). \end{aligned}$$

关于$H = (\mathcal{V}_H, \mathcal{H}_H)$ 上的信号通过聚合被迁移为 $\mathsf{Gal}(H)$ 上的信号。我们假设最大值是一个不错的选择，因为超图扩散能量泛函 [33]具有以下形式

$$Q(\mathbf{f}) = \quad \sum_{e \in \mathcal{H}_H} \max_{i,j \in e} (f_i - f_j)^2.$$

8.2.3 偏序集上的信号

如果 Ω 是有限偏序集 P，且 $\mathcal{C}$ 再次取实数或复数，那么就可以将格上的信号处理框架移植到一个新的

Birkhoff 对偶性下有限偏序集的框架（定理2.3）。在一个方向上，给定一个有限偏序集 P，存在一个同构

$$\begin{aligned} \eta : & \quad \mathrm{P} \xrightarrow{\cong} \mathcal{J}\left(\mathcal{D}\left(\mathrm{P}\right)\right) \\ \eta(p) = & \quad \downarrow p \end{aligned} \tag{8.6}$$

which implies the join-irreducibles of the distributive lattice $\mathcal{D}\left(\mathrm{P}\right)$ are in one-to-one correspondence with the elements of P. Hence, each $p \in \mathrm{P}$ corresponds to a unique join-irreducible $\downarrow p$. This implies the monoid algebra $\Bbbk[\mathcal{D}\left(\mathrm{P}\right)_{\vee}]$ has a generating set $\mathcal{G} = \{[\downarrow p]\}_{p \in \mathrm{P}}$. Hence, the shift operators in the representation framework are given by the images of $g \in \mathcal{G}$ under the representation

$$\rho_{\vee} : \quad \Bbbk\left[\mathcal{G}\right] \to \mathrm{End}\left(\Bbbk^{\mathcal{D}(\mathrm{P})}\right).$$

Recent work, takes a similar approach and generalizes (semi)lattice signal processing to signal processing on directed acyclic graphs (DAGs) [149]. The authors of this work do not, however, construct convolutions via Birkhoff Duality as we suggest.

8.2.4 格值信号

On the other hand, suppose Ω is an abelian group (e.g. $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$) and $\mathcal{C}$ is a lattice. A lattice-valued signal, then, is a map $\mathbf{f} : A \to \mathbf{L}$. Lattice-valued signals are a natural taxum of signal to study. For instance, if $\mathcal{C}$ is the lattice of partitions of the nodes of a dynamic graph, a signal $f \in \mathcal{X}(\mathbb{Z}, \Pi\left(\mathcal{V}_G\right))$ would model the merging and splitting of connected components of a network as connections between nodes are established/disconnected [92].

Definition 8.5 (Lattice-Valued Convolution [110])**.** Suppose $(\mathbb{L}, \wedge, \vee, 0, 1, \star, e, \to)$ is a residuated lattice and $(A, +, 0)$ an abelian group. Define the following *sup-convolution*, $f, h \in \mathcal{X}(A, \mathbf{L})$,

$$(h \circledast f)(x) = \quad \bigvee_{s \in A} h(s) \star f(t - s). \tag{8.7}$$

下面给出一个在时间域中导出非线性滤子的示例。

示例 8.13 （一维 Max-Plus 卷积）。假设 Ω 是 $\mathbb{Z}$，且 $\mathcal{C}$ 是格 $\mathbb{R}_{\max}$。如果h、f： $\mathbb{Z} \to \mathbb{R}_{\max}$ 是信号，那么

$$(h \circledast f)(t) = \quad \max_{s \in \mathbb{Z}} h(s) + f(t - s).$$

当然， $\mathcal{C}$ 可以包含多个通道（即 $\mathcal{C} = \mathbb{R}_{\max}^{d}$），在这种情况下，每个通道都可以并行处理。

设 $\alpha \in \mathbb{L}$ 是一个剩余格的元素。如果$f \in \mathcal{X}(\Omega, \mathbb{L})$ 是定义在某个域 Ω上的剩余格值信号，我们将信号f 关于$a \in \mathbf{L}$的标量乘法定义为

$$(a \star f)(t) = \quad a \star f(t) \quad \forall t \in \Omega.$$

If $f : \mathbb{Z} \to \mathcal{C}$ is a signal, then its domain translation is $T_s f : \mathbb{Z} \to \mathcal{C}$ given by $T_s f(t) = f(t - s)$. We say an operator $D : \mathbf{L}^{\mathbb{Z}} \to \mathbf{L}^{\mathbb{Z}}$ is *dilation-translation invariat* if the following conditions are satisfied

$$\begin{aligned} D\left(\bigvee_{i\in I} f_i\right) &= \bigvee_{i\in I} D(f_i) \\ D(a \star f) &= a \star D(f) \\ T_s D(f) &= D(T_s f) \end{aligned}$$

对于所有$f \in \mathbb{L}^{\mathbb{Z}}$、$a \in \mathbb{L}$、$s \in \mathbb{Z}$。第一项要求指出，$D$ 是关于 $\mathbb{L}^{\mathbb{Z}}$ 上格结构的保并映射。第二项和第三项规定了等变性。

以下是这种信号处理方式最接近“代数结果”的内容。

定理 8.2 (定理 3 [109]). 一个算子 $D : \mathbf{L}^{\mathbb{Z}} \to \mathbf{L}^{\mathbb{Z}}$ 是膨胀平移不变的，当且仅当每个信号 $f \in \mathbb{L}^{\mathbb{Z}}$ 都可表示为

$$f(t) = \bigvee_{s\in\mathbb{Z}} f(s) \star D(q)(t - s)$$

其中$q : \mathbb{Z} \to \mathbf{L}$ 是称为脉冲的信号，其定义为

$$q(t) = \begin{cases} e & t = 0 \\ 0 & t \neq 0 \end{cases}$$

我们在此提出如下定义。

定义 8.6 （Püschel-Maragos 卷积）. 设 P 是一个有限分布格，且 $\mathbb{L}$ 是一个剩余格。那么，h、$f : \mathrm{P} \to \mathbb{L}$ 的 Püschel-Maragos 卷积是地图

$$\begin{aligned} h \circledast f : & \quad \mathrm{P} \to \mathbb{L} \\ (h \circledast f)(x) = & \bigvee_{y\in\mathbf{K}} h(y) \star f(x \wedge y) \end{aligned}$$

定理 8.3. 假设$f : \mathrm{P} \to \mathbb{L}$ 保持并。那么，$h \circledast f : \mathrm{P} \to \mathbb{L}$也保持并。此外，地图

$$\begin{aligned} - \circledast f : & \quad \mathbb{L}^{\mathrm{P}} \to \mathbb{L}^{\mathrm{P}}, \\ (h \circledast f)(x) = & \bigvee_{y\in\mathrm{P}} f(x \wedge y) * h(y) \end{aligned}$$

具有右伴随

$$\begin{aligned} - \circledast' f : & \quad \mathbb{L}^{\mathrm{P}} \to \mathbb{L}^{\mathrm{P}}, \\ (h \circledast' f)(y) = & \bigwedge_{x\in\mathrm{P}} f(x \wedge y) \to h(y). \end{aligned}$$

证明。由定理3。4。 □

8.3 卷积神经网络

Convolution neural networks (neural networks in general) consist of a series of layers. Each layer ℓ has an input signal $\mathbf{f}^{(\ell-1)}$ and an output signal $\mathbf{f}^{(\ell)}$ so that the output to layer ℓ is the input of layer $\ell+1$. The input layer is $\mathbf{x} = \mathbf{f}^{(0)}$; the output layer is denoted $\mathbf{y}$. In each layer, the following operations may be performed.

1.对输入$\mathbf{h}^{\ell}$ 信号使用滤子组进行滤波。

2. 聚合滤过信号。

3. 应用非线性 σ_ℓ （例如聚合、ReLU 或 sigmoid）。

4.（可选）池化。

在我们看来，卷积神经网络属于信号处理的范畴。

8.3.1 格神经网络

Convolutional layers in a lattice convolutional neural network (L-CNN) have two flavors. Meet- and join- convolutional layers. Fix a finite lattice $\mathbf{L}$, and choose an (arbitrary) total order on the elements $\mathbf{L} = \{\hat{0} = y_1, x_2, \ldots, x_m = \hat{1}\}$. Hence, we can identify a lattice signal $f : \mathbf{L} \to \mathbb{R}$ as a vector $\mathbf{f} \in \mathbb{R}^m$. For layer ℓ, let F_ℓ denote the number of features in that layer. Each feature has an associated lattice signal, stored in a matrix

$$\mathbf{F}^{(\ell)} = \{\mathbf{f}_i\}_{i=1}^{F_{\ell-1}} \in \mathbb{R}^{m \times F_\ell}.$$

类似地，滤子存储在张量中

$$\mathbf{H}^{(\ell)} = \{\mathbf{h}_{i,j}^{(\ell)}\}_{\substack{i=1,\ldots,F_{\ell-1} \\ j=1,\ldots,F_\ell}} \in \mathbb{R}^{m \times F_{\ell-1} \times F_\ell}.$$

然后，交-卷积层是

$$\begin{aligned} \mathbf{F}_j^{\ell} = & \quad \sigma_\ell \left(\sum_{i=1}^{F_{\ell-1}} \mathbf{h}_{i,j}^{(\ell)} \circledast_\wedge \mathbf{f}_j \right) \\ j \in & \quad \{1, 2, \ldots, F_\ell\}, \end{aligned}$$

而一个并卷积层是

$$\begin{aligned} \mathbf{F}_j^{\ell} = & \quad \sigma_\ell \left(\sum_{i=1}^{F_{\ell-1}} \mathbf{h}_{i,j}^{(\ell)} \circledast_\vee \mathbf{f}_j \right) \\ j \in & \quad \{1, 2, \ldots, F_\ell\}. \end{aligned}$$

任一选择都包括使用滤子进行格卷积 $\mathbf{h}^{j}{}_{i}$、聚合（求和），随后应用非线性函数 σ_ℓ。关于实现的说明，因为

signals in every layer are supported on the same lattice, a Fourier basis $\{f^{\leqslant y_i}\}_{i=1,\dots,m}$ suffices to write every convolution operation as a matrix multiplication. In practice, you may implement a L-CNN with a few fully-connected (MLP [68]) layers, for example, in a classifier.

Example 8.14 (Point Cloud Classification)**.** A point cloud $\mathbb{Y}$ is a finite subset of $\mathbb{R}^d$. For instance, point clouds can be collected from LiDAR (Light Detection and Ranging) systems. There is a well-known pipeline to "approximate" $\mathbb{Y}$ as a simplicial complex. Given resolution $\varepsilon > 0$, define the *Vietoris-Rips complex*

$$\mathrm{VR}_\varepsilon(\mathbb{Y}) = \quad \{\sigma \in \wp(\mathbb{Y}) \mid B_\varepsilon(x) \cap B_\varepsilon(y) \neq \emptyset \quad \forall x, y \in \sigma\}.$$

This leads to a filtration $\mathbb{X}_\varepsilon = \mathrm{VR}_\varepsilon$. It has been argued that, while this filtration is robust to some kinds of noise, it is not robust to outliers. It was originally suggest by Carlsson and Zomorodian [30] to filter $\mathbb{Y}$ also by density ρ (e.g. estimated with k-nearest neighbors). The intuition is that certain topological features can be captured at different sampling resolutions. Let

$$\mathrm{VR}_\varepsilon(\mathbb{Y}^{\geqslant \rho})$$

denote the Vietoris-Rips complex of the sub- point cloud of point with density greater than or equal to ρ. Then, $\mathbb{X}_{(\varepsilon,\rho)} = \mathrm{VR}_\varepsilon(\mathbb{Y}^{\geqslant \rho})$ clearly forms a bifiltration of simplicial complexes. Bi-filtrations have several known topological invariants: the Hilbert function and bi-graded Betti numbers [101]. Upon discretizing and rescaling the domain of ε and ρ, each invariant is seen as a signal on the grid lattice $\mathbf{L} = [m] \times [n]$ where $[n] = \{0 < 1 < \cdots < n\}$. To this end, we trained a L-CNN [134] in order to classify point clouds in the ModelNet10 dataset [167] strictly using topological invariants produced with the RIVET software package [101] as features. The results are summarized in Appendix B.

8.3.2 迈向图神经网络（图神经网络）

Graph Neural Networks (GNNs) are a powerful models offering more efficient and generalizable solutions to node classification, network classification, and representation learning tasks. It is widely argued that graph diffusion is the driving force behind graph neural networks, the impetus for our study of foundational models of graph neural networks based upon the Tarski Laplacian (as well as other linear sheaf Laplacians [12, 20, 74]).

We will review three types of GNNs, each of which brings to light different aspects of neural information processing on graphs. First we establish some common notation. Suppose $G = (\mathcal{V}_G, \mathcal{E}_G, \omega)$ is a weighted graph with $n = |\mathcal{V}_G|$ nodes, (normalized) adjacency matrix $A \in \mathbb{R}^{n\times n}$, degree matrix $D \in \mathbb{R}^{n\times n}$, and (normalized) graph Laplacian $L \in \mathbb{R}^{n\times n}$. Suppose each layer ℓ of the GNN has F_ℓ features so that

$\mathbf{X}^{(\ell)} \in \mathbb{R}^{n \times \ell}$ is a feature matrix of layer ℓ. Each layer is associated another matrix $\mathbf{H}^{(\ell)} \in \mathbb{R}^{F_\ell \times F_{\ell+1}}$ called the *weight matrix* or *channel-mixing* matrix. The entries of each $\mathbf{H}^{(\ell)}$ are learnable parameters. Finally, each layer has a nonlinear function σ_ℓ that is either pointwise (e.g. ReLU, sigmoid) or a localized nonlinear function such as local maximum or median [143].

1.图卷积网络（GCNs） [93] 是融合了图扩散和多层感知器元素的深度神经网络。层的形式为

$$\mathbf{X}^{(\ell+1)} = \sigma_\ell\left(L\mathbf{X}^{(\ell)}\mathbf{H}^{(\ell)}\right), \tag{8.8}$$

或者，在局部，

$$x_{i,g}^{(\ell+1)} = \sigma_\ell\left(\sum_{f=1}^{F_\ell} h_{f,g}^{(\ell)}\left(L\mathbf{x}_f^{(\ell)}\right)_i\right), \quad \forall i \in \mathcal{V}_G,\ g \in \{1,2,\ldots,F_{\ell+1}\}.$$

每层只进行一次扩散步骤。图卷积会发生，但每个图滤波器只有一个非零参数。

2. 消息传递神经网络（MPNNs） [65] 采取相当一般的形式。

a) Each node $i \in \mathcal{V}_G$ receives messages from neighboring nodes consisting via a function $\psi_{i,j}^{(\ell)}\left(\mathbf{f}_i^{(\ell)}, \mathbf{f}_j^{(\ell)}, a_{ij}\right)$ (learned parameters possible).

b) Node $i \in G$ aggregates the messages received by neighbors with a map

$$\square : \mathbb{R}^{\mathcal{N}_i} \to \mathbb{R},$$

例如，求和、最大值或平均值。

c) 然后，节点通过更新函数 $\varphi^{(\ell)}$（可能包含学习得到的参数）更新其状态，具体取决于先前的状态 $\mathbf{f}^{(\ell)}{}_i$ 和聚合后的消息 $\mathbf{m}^{(\ell+1)}{}_i$。

总之，

$$\begin{aligned}\mathbf{m}_i^{(\ell+1)} &= \square_{w \in \mathcal{N}_i} \psi_{i,j}^{(\ell)}\left(\mathbf{x}_i^{(\ell)}, \mathbf{x}_j^{(\ell)}, a_{ij}\right), \\ \mathbf{x}_i^{(\ell+1)} &= \varphi_i^{(\ell)}\left(\mathbf{x}_i^{(\ell)}, \mathbf{m}_i^{(\ell+1)}\right).\end{aligned} \tag{8.9}$$

3. 图卷积神经网络（GCNNs） [58] 体现了一种观点，即图神经网络由滤子构成，并强调谱域。每一层可以包含多个扩散。

发生在一个k跳邻居中聚合信息，而不是在一个1跳邻居中。

$$x_{i,g}^{(\ell+1)} = \sigma_\ell \left(\sum_{f=1}^{F_\ell} \sum_{k=0}^{K_\ell} h_{f,g,k}^{(\ell)} \left(L^k \mathbf{x}_f^{(\ell)} \right)_i \right), \quad \forall i \in \mathcal{V}_G,\ g \in \{1, 2, \ldots, F_{\ell+1}\}.$$

热流的一次迭代 7.13 可以视为消息传递神经网络中的一层，如下所示。假设 $\mathcal{F}$ 是定义在G上的塔斯基层。然后，给定以下内容，

$$\begin{aligned} \psi_{i,j}^{(\ell)}(\mathbf{x}_i^{(\ell)}, \mathbf{x}_j^{(\ell)}, a_{ij}) &= & P_{j \to i}^{\mathcal{F}}(\mathbf{x}_j^{(\ell)}) \\ \bigsqcap_{j \in \mathcal{N}_i} \mathbf{x}_j &= & \bigwedge_{j \in \mathcal{N}_i} \mathbf{x}_j \\ \varphi_i^{(\ell)}(\mathbf{x}_i^{(\ell)}, \mathbf{m}_i^{(\ell+1)}) &= & \mathbf{x}_i^{(\ell)} \wedge \mathbf{m}_i^{(\ell+1)}, \end{aligned}$$

热流是一个消息传递神经网络。

An open question remains whether the Tarski Laplacian inspires convolutional neural networks. With respect to an underlying lattice-valued sheaf over G, we hypothesize that signals on lattice $\mathcal{F}(i)$ are correlated related to signals on neighboring lattices $\{\mathcal{F}(j)\}_{j \in \mathcal{N}_i}$. This leads to a possible notion of signal convolution relative to sheaf.

Definition 8.7 (Convolution Relative to a Sheaf)**.** Suppose $G = (\mathcal{V}_G, \mathcal{E}_G)$ is a graph and $\mathcal{F}$ is a Tarski sheaf over G. Suppose $\mathbf{F} = \{\mathbf{f}_i\}_{i \in \mathcal{V}_G}$ is a collection of signals over lattices indexed by $\mathcal{V}_G$, suppose $\mathbf{H} = \{\mathbf{h}^{i,j}\}_{i,j \in \mathcal{V}_H}$ in a filter bank, and suppose $L : C^0(G; \underline{\mathcal{F}}) \to C^0(G; \underline{\mathcal{F}})$ is the Traski Laplacian. Then, *convolution of* $\mathbf{F}$ *by a filterbank* $\mathbf{H}$ *relative to* $\mathcal{F}$ is the operation

$$(\mathbf{H} \star_{\mathcal{F}} \mathbf{F})_i = \sum_{j \in \mathcal{V}_G} \sum_{\mathbf{y} \in C^0(G; \underline{\mathcal{F}})} h_{y_i}^{i,j} T^{\wedge}_{(L\mathbf{y})_i} \mathbf{f}_i \tag{8.10}$$

9

语义学

我们的第二个应用领域可以宽泛地描述为语义。在本章中，我们在两种不同的语境下探讨语义：时间逻辑和多智能体逻辑。

In logic, syntax consists of strung-together symbols, while semantics is found "under the hood." Semantics also has a linguistic connotation as something like "subsurface layers of meaning," and, of course, meaning is highly dependent on context. Thus, semantics is synonymous with a formal context which we call a *model* and others call a *structure*. A model consists of a set of *states* and rules for verifying whether a given state satisfies a logic formula. Model checking [9] is a subfield of formal verification in computer science dedicated to solving the following problems.

问题 9.1 (模型检验)**.** 给定一个模型、一个特定状态s以及一个公式 φ（使用适当的语言），判断s是否满足 φ，记作 $(M,s) \models \varphi$。

问题 9.2 (有效性)**.** 给定一个模型 M 和一个公式 φ，判断每个状态是否满足给定公式，或者等价地，检查每个s 是否满足 $\neg\varphi$。如果 φ 有效，通常写作$M \models \varphi$。

问题 9.3 (规划)**.** 给定一个 M 和一个公式 φ，生成一个状态s，使得 $(M,s) \models \varphi$（如果存在这样的状态）。

For models whose rules are specified by satisfying a given relation, we observe that Galois connections induced by relations (Theorem 9.3) transform semantic content. Thus, we argue, as others have done [96], that order theory is practical in semantics. To date, we see formal theories of semantics having been applied to single- and multi-agent autonomous systems as well as artificial intelligence. It is in the area of autonomy that we hope our work can make an impact.

9.1 时序逻辑

Temporal logic is a modal logic with much success in interpreting abstract systems by specifying logical constraints a system must satisfy as well as the time or duration. Two critical properties that temporal logic model checking address are *saftey*, something "bad" will never happen, and *liveness*, something "good" will always happen, at least eventually. More recently, temporal logic control systems have been utilized in path-planning [53, 154] and multi-agent autonomous systems [88].

9.1.1 迁移系统

The logic of time, temporal logic, is powerful for modeling systems. We utilize a popular method of abstracting away a system. We begin with a set of *atomic propositions*, denoted Φ. Elements of Φ are labeled p_1, p_2 etc. In the most elementary setting, these propositions have two possible truth values $\{\texttt{false}, \texttt{true}\}$ and typically designate facts about a system, for instance, whether an agent is located in particular region of a map, or whether there is a communication link between a pair of agents. In temporal logic, the truth values of atomic propositions are time-variant.

定义 9.1. （带标记的非确定性）迁移系统是一个元组$\mathbf{TS} = (Q, Q0, \Sigma, \delta, \Phi, \pi)$，其中

1.Q，一个状态集合范畴，

2. $Q_0 \subseteq Q$，一个初始状态的集合范畴，

3. Σ，一个控制输入的集合范畴，

4. δ，一个迁移映射，将状态 $q \in Q$ 和控制输入$\sigma \in \Sigma$ 映射到可能后继状态的一个子集

$$\delta : Q \times \Sigma \to \wp(Q), \tag{9.1}$$

5. Φ，一个原子命题集合范畴（真或假），

6. π，一个将状态 $q \in Q$ 映射到在 q 处为真的原子命题集合的地图

$$\pi : Q \to \wp(\Phi).$$

注意，我们本可以将迁移映射写成带标记关系。

$$\to \subseteq Q \times \Sigma \times Q$$

通过柯里化 δ。因此，迁移系统可以方便地绘制成有向图，其中箭由 Σ 标记为其子集。在一个确定性迁移系统中，（9。1）被替换为

$$\delta : Q \times \Sigma \to Q.$$

并且我们假设 $Q_0 = \{Q_0\}$。在一个无标签迁移系统中，我们忽略Σ 并将（9。1）替换为

$$\delta : Q \to \wp(Q), \tag{9.2}$$

等价地，关系 $\to \subseteq Q \times Q$。如果Q、 Σ 和 Φ 都是有限的，我们称该迁移系统是有限的。

Remark 9.1. In a transition system, uncertainty is not (ordinarily) modeled with probability, but agnostic to the likelihood of a given state q transitioning to any number or states $q' \in \delta(q, \sigma)$. We can form an unlabeled transition system from a labeled one via

$$\tilde{\delta}(q) = \bigcup_{\sigma \in \Sigma} \delta(q, \sigma).$$

无标记迁移系统等价于一个二元关系$\to \subseteq Q \times Q$。

示例 9.1（控制系统）. 考虑离散时间不变线性系统

$$\mathbf{x}[t+1] = A\mathbf{x}[t] + B\mathbf{u}[t] \tag{9.3}$$
$$\mathbf{y}[t+1] = C\mathbf{x}[t+1] \tag{9.4}$$

with $A \in \mathbb{R}^{n \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times p}$, $\mathbf{u} \in \mathbb{R}^p$, $C \in \mathbb{R}^{m \times n}$, $\mathbf{y} \in \mathbb{R}^m$. Linear systems are ubiquitous in control theory, for instance, model-predictive control [28]. The system (9.3) is rewritten as the following transition system:

$$\begin{aligned} Q &= \mathbb{R}^n \\ \Sigma &= \mathbb{R}^p \\ \Phi &= \mathbb{R}^m \\ \delta(\mathbf{x},\mathbf{u}) &= \{A\mathbf{x} + B\mathbf{u}\} \\ \pi(x) &= \{C\mathbf{y}\}. \end{aligned}$$

令 Q^ω 表示由Q中的元素组成的无限序列集合。

$$Q^\omega = \{\alpha : \mathbb{N} \to Q\}$$

which we call traces. If $\tau \in Q^\omega$, the *suffix* of τ is the trace $\tau[t_0]$ starting at $t_0 \geqslant 0$. Suppose $\sigma_{in} \in \Sigma^\omega$ is a sequence of control inputs, for instance, driving directions: *turn left at the light, go straight at the stop sign*. Then,

$$\begin{aligned} \tau(t) &= \delta(\sigma(t), \tau(t-1)) \\ \tau(0) &\in Q_0 \end{aligned}$$

定义了一个轨迹 $\tau \in \wp(Q)^\omega$。最后，轨迹通过指定 Φ 的一系列子集

$$\sigma_{out}(t) = \pi\left(\tau(t)\right).$$

在我们描述如何使输出 σ_{out} 服从时序逻辑约束之前，请考虑以下关键示例。

示例 9.2 (网络连通性 [136])。 假设 $G = (\mathcal{V}_G, \mathcal{E}_G)$ 是一个无向图。[1] 一个动态图是地图

$$G_t : \mathbb{N} \to \text{Subgraph}\,(G) \tag{9.5}$$

is a dynamic graph with edges $\mathcal{E}_{Gt} \subseteq \mathcal{E}_G$ collecting edges of the subgraph G_t at time t. At each discrete time instant, nodes in the edge $ij \in \mathcal{E}_G$ toggle their status from on to off or from off to on or are on standby. These actions are represented by the following control inputs

$$\Sigma = \prod_{ij \in \mathcal{E}_G} \{\sigma_{switch}, \quad \sigma_{sleep}\}.$$

状态被表示为

$$\mathbf{Q} = \{-1, +1\}^{|\mathcal{E}_G|}.$$

状态转换由函数控制。

$$\delta(\mathbf{s}, \boldsymbol{\sigma})_{ij} = \begin{cases} -q_{ij} & \sigma^{ij} = \sigma_{switch} \\ q_{ij} & \sigma^{ij} = \sigma_{sleep} \end{cases}$$

给定 $\mathbf{q} \in Q, \quad \sigma \in \Sigma$。令

$$\Phi = \{p^{ij}\}_{ij \in \mathcal{E}_G}$$

其中 p^{ij} 是命题

节点 i 与节点 j 之间存在连接。

那么，当然，

$$\pi(\mathbf{s}) = \bigcup_{q_{ij} > 0} \pi^{ij}.$$

Hence, a control input $\sigma_{in} \in \Sigma^\omega$ determines an output $\sigma_{out} \in \wp(\Phi)^\omega$ specifying a subset of edges $\mathcal{E}_{Gt}$ that are connected at time t. So we see the defined transitions system $\mathbf{TS} = (\mathbf{Q}, \mathbf{q}_0, \Sigma, \Phi, \pi)$ is equivalent to a dynamic graph (9.5). Shortly, we will find it helpful to be able to reason about dynamic graphs with transition systems.

9.1.2 线性时序逻辑

线性时序逻辑（LTL）具有以下语法。

$$\texttt{true} \quad | \quad p \quad | \quad \varphi \wedge \psi \quad | \quad \neg\varphi \quad | \quad \bigcirc \varphi \quad | \quad \varphi \mathcal{U} \psi \quad | \quad \Diamond \varphi \quad | \quad \Box \varphi.$$

There needs explanation. If p is an atomic proposition in Φ, then p is a formula. If φ and ψ are formulas, then all of the operations of propositional calculus also constitute formulas. The other symbols are interpreted: $\bigcirc\varphi$ *next time*, $\varphi\mathcal{U}\psi$ *until*, $\Diamond\varphi$ *eventually*, $\Box\varphi$ *always*. Derived

1 不失一般性，我们可以将G 视为顶点 $\mathcal{V}_G$上的完全图

算子包括 $\square\Diamond\varphi$ 无限多次 和 $\Diamond\square\varphi$ 从某个时刻起永远。由此语法构成的所有公式的集合是 $\mathcal{L}^{LTL}(\Phi)$。

LTL 的语义如下。设 $\sigma \in \wp(\Phi)^{\omega}$ 是一个输入单词，$p \in \Phi$，以及 $\varphi, \psi \in \mathcal{L}^{LTL}(\Phi)$。那么，

$$
\begin{aligned}
\sigma \models\ & p & \Leftrightarrow\ & & & p \in \sigma(0),\\
\sigma \models\ & \varphi \wedge \psi & \Leftrightarrow\ & & & \sigma \models \varphi,\ \sigma \models \psi,\\
\sigma \models\ & \neg\varphi & \Leftrightarrow\ & & & \sigma \not\models \varphi,\\
\sigma \models\ & \bigcirc\varphi & \Leftrightarrow\ & & & \sigma[1] \models \varphi,\\
\sigma \models\ & \varphi\mathcal{U}\psi & \Leftrightarrow\ & \exists\, t_0 \geqslant 0 \quad \text{such that} \quad \sigma[t_0] \models \psi \quad \text{and} \quad \sigma[t'] \models \varphi \quad \forall 0 \leqslant t' < t_0, & &\\
\sigma \models\ & \Diamond\varphi & \Leftrightarrow\ & & \exists t_0 \geqslant 0 \quad \text{such that} \quad & \sigma[t_0] \models \varphi,\\
\sigma \models\ & \square\varphi & \Leftrightarrow\ & & \sigma[t'] \models \varphi \quad \forall t' \geqslant 0. &
\end{aligned}
$$

对于一个轨迹 $\tau \in \wp(Q)^{\omega}$，如果对应的输出工作为 $\sigma_{in} \models \varphi$，我们记作 $\tau \models \varphi$。

回到语法，了解哪些词满足给定命题这一概念很有用。给定 φ，φ 的内涵 是

$$[\![\varphi]\!] = \{\tau \in \wp(Q)^{\omega} \mid \tau \models \varphi\}$$

满足 φ 的单词收集在集合范畴 $\mathbf{Words}(\varphi) = \pi([\![\varphi]\!])$中。

给定一个迁移系统 **TS** 和一个命题 φ 存在一个计算处理流程，用于验证 φ 在 **TS** [9]中有效。该过程包括将公式“转换” φ 为 Büchi 自动机。

定义 9.2. A 非确定性 *Büchi* 自动机（*NBA*） 是一个元组$\mathbf{B} = (S, S0, \Sigma, \delta, F)$ 其中

1. S, 一个集合范畴的状态,
2. S_0, 个初始状态的集合,
3. Σ, 一个控制输入集合范畴,
4. δ, 一个迁移映射
$$\delta : S \times \Sigma \to \wp(S),$$
5. $F \subseteq S$, 一个接受状态的集合范畴。

An *input word* $\sigma \in \Sigma^{\omega}$ is *accepted* if there is at least one corresponding sequence $\mathbf{s} = s(0)s(1)s(2)\ldots$ such $s(0) \in S_0$ and the set $\{t \mid s(t) \in F\}$ is countably infinite. The *language* of $\mathbf{B}$, denoted $\mathcal{L}(\mathbf{B})$ consists of the set of input words σ that are accepted by $\mathbf{B}$.

定理 9。1（定理 5.37 [9]）。 对于任意 *LTL* 公式 $\varphi \in \mathcal{L}^{LTL}(\Phi)$，存在一个（广义[2]）非确定性 Büchi 自动机 $\mathbf{G}\varphi$，满足$\Sigma = \wp(\Phi)$，使得

$$\mathbf{Words}(\varphi) = \mathcal{L}(\mathbf{G}_{\varphi}).$$

2 广义 Büchi 非确定自动机（GNBA）比普通 NBA 稍微更一般。不过，GNBA 可以转换为 NBA [9, 定理 4.56]。

Once we have φ translated into $\mathbf{B}_\varphi$, we construct the *product Büchi automaton* (PBA) $\mathbf{P} = \mathbf{TS} \otimes \mathbf{B}_\varphi$ [9, p. 200] which can be viewed as a directed graph via (9.2). After taking a quotient of the digraph of $\mathbf{P}$ by strongly connected components, use Djistra's algorithm [38] to search for paths beginning at an initial state $(q_0, s_0) \in Q \times S$ that visit an accepting state in $Q \times F$ infinitely often. A projection of this path in the product automaton $\mathbf{P}$ onto the original transition system $\mathbf{TS}$ yields a trace $\tau \models \varphi$. We say $\mathbf{TS} \models \varphi$ if there does not exist a trace τ such that $\tau \models \neg\varphi$.

命题9.1 (复杂度 [9]). 模型检验问题 $\mathbf{TS} \models \varphi$ 的复杂度是

$$\mathcal{O}\left(|\mathbf{TS}|2^{|\varphi|}\right)$$

其中 $|\mathbf{TS}|$ 是 $\mathbf{TS}$ 中状态和迁移的数量，$|\varphi|$ 大致是 φ 中符号的数量。

9.1.3 分布式模型检验

In the following, we consider a network $G = (\mathcal{V}_G, \mathcal{E}_G)$ of agents and a number of atomic propositions indexed by the agents $i \in \mathcal{V}_G$. In stead of model-checking a formula φ for the entire system, a strategy is to model-check *local formulas* φ_i.

Problem 9.4 (Distributed LTL Model-Checking). Suppose $G = (\mathcal{V}_G, \mathcal{E}_G)$ is a graph and $\mathbf{TS} = (Q, Q_0, \Sigma, \delta, \Phi)$ is a transition system such that Φ is indexed by $\mathcal{V}_G$ and a (possibly empty) set $\mathcal{A}$ not dependent on the graph G.

$$\Phi = \{p^i_A\}_{i\in\mathcal{V}_G, A\in\mathcal{A}}.$$

令 $\Phi_i = \{p^j{}_A\}_{j\in\mathcal{N}_i\cup i}$，$A\in\mathcal{A}$ 为支持于$i \in \mathcal{V}_G$邻域的原子命题集合范畴。令

$$\varphi = \bigwedge_{i\in\mathcal{V}_G} \varphi_i$$

其中 $\varphi_i \in \mathcal{L}(\Phi_i)$。给定 $\sigma \in \Sigma^\omega$，判断 $\sigma \models \varphi$。

我们列举一些分布式模型检测问题的示例。

Example 9.3 (Intermittent Connectivity). The following example is taken from [88]. Suppose robots with limited communication (e.g. submarine robots) each travel in between two rendezvous locations $\mathbf{v}_i$ and $\mathbf{v}_j$ in order to commuicate with other agents. Then, let $G = (\mathcal{V}_G, \mathcal{E}_G)$ be the graph with nodes indexing rendevous locations $\mathbf{v}_{ij}$ and edges indexing robots r_{ij}. This intermittent connectivity problem is a distributed model-checking problem with

$$\varphi_i = \Box\Diamond \bigwedge_{j\in\mathcal{N}_i} p^{ij}_{\mathbf{v}_i}$$

其中 $p^{ij}_{\mathbf{v}_i}$ 是命题：

> *Robot* r_{ij} *is located in the region* $\mathcal{R}_i = \{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_{ij}\| \leqslant \varepsilon\}$.

局部公式 φ_i 指定：

> 所有分配到会合点 $\mathbf{v}_i$ 的机器人都同时位于区域 $\mathcal{R}_i$，且这种情况无限次发生。

Example 9.4 (Epidemics)**.** Let $\mathcal{A} = \{S, I, R\}$ represent the status *susceptible*, *infected*, and *recovered*. Suppose $\Phi = \{p^i_{A \in \mathcal{A}}\}$. For instance, p^i_I is the proposition *agent* i *is infected*. The popular susceptible-infected-recovered (SIR) model can be viewed as a distributed model-checking problem with

$$\varphi_i = \qquad \Box\left(\bigvee_{j \in \mathcal{N}_i} p^i_S \Rightarrow \Diamond p^j_I\right) \wedge \Box\left(p^i_I \Rightarrow \Diamond p^i_R\right).$$

我们将 φ_i 解释为：

> 如果智能体 i的至少一个邻居被感染，那么智能体 i 最终也会被感染；如果智能体 i 被感染，她最终会康复。

当然，还有类似的模型，例如易感-感染-易感（SIS）、易感-感染-康复-死亡（SIRD）。请注意，流行病模型既可用于模拟谣言，也可用于模拟生理疾病。

Example 9.5 (Non-Interfering Connectivity [136])**.** Consider an arbitrary undirected graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ and the transition system **TS** defined in Example 9.2 modeling network connectivity. Recall, $p^{ij}, ij \in \mathcal{E}_G$ is the proposition

> 通信链路 ij 处于活动状态。

考虑局部公式

$$\varphi_i = \qquad \bigwedge_{j \in \mathcal{N}_i} \varphi_{ij}$$

其中 φ_{ij} 是公式

$$\varphi_{ij} = \qquad \Box\left(p^{ij} \Rightarrow \bigvee_{i'j' \in \mathcal{N}_{ij}} \neg p^{i'j'}\right) \wedge \Box\Diamond p^{ij}. \tag{9.6}$$

φ_{ij} 可以解释为：

> 如果通信链路 ij 处于激活状态，那么所有相邻的通信链路都必须关闭。此外，每条通信链路都必须无限次地变为激活状态。

我们提出了 [136] 一种分布式模型检测问题的部分解决方案，该方案适用于上述示例。

选择一个节点子集 $\mathcal{V}_C \subseteq \mathcal{V}_G$ 的节点，我们称其为命令节点。

2. After selecting a global proximity radius $R \in \mathbb{N}$ and a local proximity radius $r \leqslant \lfloor R/2 \rfloor$, form the *command graph* with nodes $\mathcal{V}_C$ and $ij \in \mathcal{E}_C$ if and only if i and j are connected by a shortest path of length no greater than r in G. Construct a family of subgraphs $G_i, i \in \mathcal{V}_C$ induced by the node sets

$$\mathcal{V}_{G_i} = \{j \in \mathcal{V}_G \mid d(i,j) \leqslant r\};$$

G_i 是包含 $\mathcal{V}_{G_i}$ 的G 的最小子图。

For $i \in \mathcal{V}_C$, let $\mathbf{TS}_i$ be the subtransaction system of $\mathbf{TS}$ (Problem 9.4) restricted to G_i. Model checking each $\mathbf{TS}_i \models \bigwedge_{j \in \mathcal{V}_{G_i}} \varphi_j$, $i \in \mathcal{V}_C$ is now substantially less expensive than model-checking $\mathbf{TS} \models \varphi$.

9.2 多智能体逻辑

For more details, we suggest the reader consult the standard reference [52]. For the following, suppose a system has agents labeled $\mathcal{V}_G = \{1, 2, \ldots, n\}$. In multi-agent logic, we develop a mathematical language to reason about information held by individuals and groups of individuals in the system. Information, in our setting, consists of formulas and their truth values, contrasting the perspective of Chapter 8 which views information as signals, boxes of numbers. In temporal logic, there are two modalities $\Box$ and $\Diamond$. In multi-agent logic, we have a modality for every agent $K_1, K_2, \ldots, K_n$.

定义 9.3 (克里普克模型)**.** 一个克里普克框架 由一个S 状态集合 组成，并带有二元关系

$$\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n.$$

一个 克里普克模型，记作 M，是一个框架 $(S, \mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n)$ 以及一个 赋值。

$$\pi : S \to \wp(\Phi).$$

显然，通过柯里化，赋值与关系 $\models$ 在 S 和 Φ 之间构成双射，我们将其称为 解释；我们在赋值和解释之间自由地来回转换

$$(M, s) \models p \quad \Leftrightarrow \quad p \in \pi(s).$$

多智能体逻辑区别于普通命题逻辑，其关系集合 $\mathcal{K}_i$ 编码了语法的语义。

模态算子写作K_i。连同命题逻辑的联结词，多智能体逻辑的完整语法如下

$$\texttt{true} \quad | \quad p \quad | \quad \varphi \wedge \psi \quad | \quad \neg\varphi \quad | \quad i \quad | \quad K_i\varphi.$$

The set of all sentences formed via the above syntax over the atomic propositions Φ is called the language $\mathcal{L}_n(\Phi)$. The set of all n-agent Kripke models over Φ is denoted $\mathcal{M}_n(\Phi)$. We can *interpret* a $\varphi \in \mathcal{L}_n(\Phi)$ with the following inductive rules:

$$(M,s) \models p \quad \Leftrightarrow \quad p \in \pi(s) \tag{9.7}$$

$$(M,s) \models \varphi \wedge \psi \quad \Leftrightarrow \quad (M,s) \models \varphi, (M,s) \models \psi \tag{9.8}$$

$$(M,s) \models \neg\varphi \quad \Leftrightarrow \quad (M,s) \not\models \varphi \tag{9.9}$$

$$(M,s) \models K_i\varphi \quad \Leftrightarrow \quad (M,t) \models \varphi \quad \forall t \in S \quad \text{such that} \quad s\mathcal{K}_i t \tag{9.10}$$

It is the last rule that gives a Kripke model the nickname, a "possible worlds" model because $K_i\varphi$ is satisfied at a state if every neighboring state t with $s\ \mathcal{K}_i t$ satisfies φ. As suggested by notation, the recommended mental model for the formula $K_i\varphi$ is *agent i knows* φ. We will see shortly why this is just a "mental model." The modal logic $\mathcal{L}_n(\Phi)$ could then be coined a *epistemic logic*. We write $M \models \varphi$ if $(M,s) \models \varphi$ for all $s \in S$. In this case, we would say φ is *valid* in M.

由基本语法和语义还可推导出其他逻辑算子。假设$A \subseteq \mathcal{V}_G$ 是智能体的子集。那么，

$$E_A\varphi = \bigwedge_{i\in A} K_i\varphi$$

这在我们的心智模型中表示A中的每个人都知道 φ。 定义

$$C_A\varphi = \bigwedge_{k\geqslant 1} E_A^k\varphi$$

which in our mental model specifies that φ is *common knowledge*, or, in other words, everyone knows that everyone knows that everyone knows φ *ad infinitum*. Note that E_G is not in the language $\mathcal{L}_n(\Phi)$. To include common knowledge, we will need to augment our language to include the symbols C_G. Finally, let D_A be the operator with semantics

$$(M,s) \models D_A\varphi \quad \Leftrightarrow \quad (M,t) \models \varphi \quad (\forall t \in S) \quad (s,t) \in \bigcap_{i\in A} \mathcal{K}_i.$$

在我们的思维模式中，D_A 提供了分布式知识 这一概念，其直观含义是群 $A \subseteq \mathcal{V}_G$ 中所有智能体的"合并"知识蕴含 φ。我们将 $\mathcal{L}_n^C$, $\mathcal{L}_n^D$, $\mathcal{L}_n^{CD}$记作增强的

分别是具有共同知识、分配知识以及共同知识和分配知识的语言。

For the time being, we will narrow our focus on $\mathcal{L}_n$. Let us briefly describe this logic axiomatically. An axiom system consists of axioms and inference rules. An axiom is a valid formula while an inference rule is a reduction of a set of valid formulas in a context to another formula. The axiom system $\mathsf{K}_n(\Phi)$ consists of the following rules and axioms:[3]

(A1) 命题演算的重言式。

(A2) $(K_i\varphi \wedge K_i(\varphi \wedge \psi)) \Rightarrow K_i\psi \quad \forall i \in \mathcal{V}_G$ (Distribution Axiom)

(R1) 由 φ 和 $\varphi \Rightarrow \psi$ 推出 ψ (肯定前件律)

(R2) 由 φ 推得 $K_i\varphi$ (知识泛化)

以下定理保证：给定 Φ，在 $\mathbf{K}_n$ 中可证明的每个公式在 $\mathcal{M}_n$ 中都有效；反之，在 $\mathcal{M}_n$ 中有效的每个公式都可在 $\mathbf{K}_n$ 中证明。

定理 9.2 (定理 3.1.3 [52])。 公理系统 $\mathbf{K}_n(\Phi)$ 是语言 $\mathcal{L}_n(\Phi)$ 相对于$\mathcal{M}_n(\Phi)$的可靠且完备的公理化。

区分知识与信念的额外公理，产生了其他公理系统，在语义上对应于 $\mathcal{M}_n$ [52]的子类。

(A3)$K_i\varphi \Rightarrow \varphi$ (知识公理)

(A4) $K_i\varphi \Rightarrow K_iK_i\varphi$ (正内省)

(A5) $\neg K_i\varphi \Rightarrow K_i\neg K_i\varphi$ (负内省)

(A6) $\neg K_i$(假) (一致性)

(A7) $\varphi \Rightarrow K_i\neg K_i\neg\varphi$ (单调性)

We will not say much more about the resulting correspondences between classes of Kripke models and axiom systems except that axiom systems with more axioms than K_n restrict the class of allowable Kripke relation $\mathcal{K}_i$. For instance, the axiom system called $\mathsf{S5}_n$ (K_5 plus A3, A4, A5) restricts $\mathcal{M}_n$ to the class of models $\mathcal{M}_n^{rst}$ whose Kripke relations $\mathcal{K}_i$ are equivalence relations. A case can be made, thus, that $\mathcal{M}_n^{rst}$ models knowledge. Hence, we interpret the formula $K_i\varphi$ as *agent i knows* φ.

In another instance, the axiom system called $\mathsf{KD45}_n$ (K_5 plus A4, A5, A6) restricts $\mathcal{M}_n$ to the class of models $\mathcal{M}_n^{elt}$ whose Kripke relations $\mathcal{K}_i$ are Euclidean, serial and transitive. $\mathsf{KD45}_n$ is reasonably interpreted as

3 为方便起见，我们按照 Fagin 等人的方式标记公理和推理规则 [52]。

modeling belief with K_i interpreted as *agent i believes* φ and, of special note, $\neg K_i \neg \varphi$ as *agent i doesn't disbelieve* φ. With this interpretation in mind, notice the absence of A3 in $\mathsf{KD45}_n$; believing something does not make it true.

9.2.1 克里普克-伽罗瓦连接

Recall from our discussion of temporal logic, intension allows you to pass from syntax and semantics. In Kripke semantics, the *intension* of a formula describes the semantic content of that formula, the subset of states for which a given formula is true. In general, we refer to a subset $e \subseteq \wp(S)$ as an *event*.

定义 9.4 (内涵**).** 假设M 是一个克里普克模型且 $\varphi \in \mathcal{L}_n(\Phi)$ 一个公式。然后，

$$[\![\varphi]\!] = \{s \in S \mid (M, s) \models \varphi\}$$

is the *intension* of φ. We say φ has more *semantic content* than ψ if $[\![\varphi]\!] \supseteq [\![\psi]\!]$. We say two formulae $\varphi, \psi \in \mathcal{L}_n(\Phi)$ are *semantically equivalent* if $[\![\varphi]\!] = [\![\psi]\!]$ and write $\varphi \equiv \psi$.

命题 9.2. 给定模型 M，语义等价是 $\mathcal{L}_n$（Φ）上的等价关系。

Let $\mathcal{L}_n(\Phi)/\simeq$ denote the set of equivalence classes of formulas where individual classes are denoted $[\varphi]$. Semantic equivalence is not to say that two formulas φ and ψ are the same, but that they are the same in the context M. We now begin an investigation of how syntax and semantics interact in multi-agent logic. Let $(-)^c$ denote complement in the lattice $\wp(S)$.

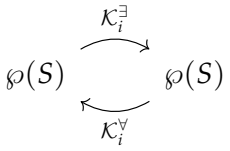
引理 9.1. 设 $\varphi, \psi \in \mathcal{L}_n(\Phi)$ 且 $M = (S, K_1, \ldots, K_n, \Phi, \pi)$。那么，

$$\begin{aligned} [\![\varphi \wedge \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\neg\varphi]\!] &= [\![\varphi]\!]^c \\ [\![\mathtt{true}]\!] &= S \end{aligned}$$

证明。 练习。 □

Of course, other identities relating the syntax of propositional logic and the structure of the lattice $\wp(S)$ are easily shown. However, lattice-theoretic interpretations of Kripke modalities are much less trivial. In the following theorem, we establish a Galois connection on $\wp(S)$ which "respects" semantic equivalence.

Theorem 9.3 (Kripke-Galois Connection)**.** Suppose M is a Kripke model $M = S, \mathcal{K}_1, \ldots, \mathcal{K}_n, \Phi, \pi)$, Then, there is a family of Galois connections indexed by $i \in \{1, 2, \ldots, n\}$



由.....给出

$$\begin{aligned} \mathcal{K}_i^{\exists}(e) = & \quad \{t \in S \mid (\exists s)\,(s \in e \wedge s\mathcal{K}_i t)\} \\ \mathcal{K}_i^{\forall}(e) = & \quad \{s \in S \mid (\forall t)\,(s\mathcal{K}_i t \Rightarrow t \in e)\} \end{aligned} \tag{9.11}$$

使得

$$\begin{aligned} \mathcal{K}_i^{\forall}([\![\varphi]\!]) = & \quad [\![K_i\varphi]\!] \\ \mathcal{K}_i^{\exists}([\![\varphi]\!]) = & \quad [\![\neg K_i\neg\varphi]\!]. \end{aligned} \tag{9.12}$$

对于所有 $\varphi \in \mathcal{L}_n(\Phi)$。

证明。 根据定理 3。 2，有序对 $(\mathcal{K}_i^{\exists}, \mathcal{K}_i^{\forall})$ 是一个伽罗瓦连接。还需证明（9。 12）。首先，

$$\begin{aligned} \mathcal{K}_i^{\forall}([\![\varphi]\!]) = & \\ = & \quad \{s \in S \mid (\forall t) \quad s\mathcal{K}_i t \Rightarrow t \models \varphi\} \\ = & \quad [\![K_i\varphi]\!]. \end{aligned}$$

其次，

$$\begin{aligned} \mathcal{K}_i^{\exists}([\![\varphi]\!]) = & \\ = & \quad \{t \in S \mid (\exists s) \quad s \models \varphi \ \wedge \ s\mathcal{K}_i t\} \\ = & \quad \{t \in S \mid \neg\,((\forall s) \quad s \not\models \varphi \vee \neg(s\mathcal{K}_i t))\} \\ = & \quad \{t \in S \mid \neg\,((\forall s) \quad s\mathcal{K}_i t \Rightarrow s \not\models \varphi)\} \\ = & \quad \{t \in S \mid \neg\,((\forall s) \quad s\mathcal{K}_i t \Rightarrow s \in \neg\varphi)\} \\ = & \quad S - \{t \in S \mid (\forall s) \quad s\mathcal{K}_i t \Rightarrow s \in \neg\varphi\} \\ = & \quad S - [\![K_i\neg\varphi]\!] \\ = & \quad [\![\neg K_i\neg\varphi]\!] \end{aligned}$$

其中最后一个等式由引理9。 1。 □

推论 9.1. $\mathbf{K}_i\neg K_i\neg\varphi$ 的语义内容大于 φ，而 $\neg K_i\neg K_i\varphi$ 的语义内容小于 φ。

证明。根据命题3。 1，

$$\begin{aligned} \mathcal{K}_i^{\forall}\mathcal{K}_i^{\exists}([\![\varphi]\!]) \supseteq & \quad [\![\varphi]\!] \\ \mathcal{K}_i^{\exists}\mathcal{K}_i^{\forall}([\![\varphi]\!]) \subseteq & \quad [\![\varphi]\!]. \end{aligned}$$

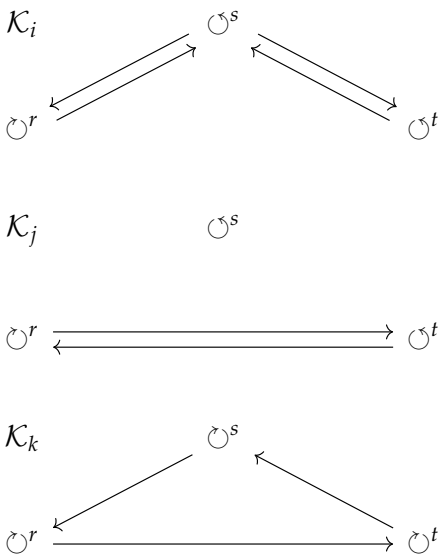
此外，

$$\mathcal{K}_i^{\forall}\mathcal{K}_i^{\exists}([\![\varphi]\!]) = [\![K_i\neg K_i\neg\varphi]\!]$$
$$\mathcal{K}_i^{\exists}\mathcal{K}_i^{\forall}([\![\varphi]\!]) = [\![\neg K_i\neg K_i\varphi]\!]$$

由引理9.1. □

Thus, applying $\mathcal{K}_i^{\forall}\mathcal{K}_i^{\exists}$ or $\mathcal{K}_i^{\exists}\mathcal{K}_i^{\forall}$ increases or decreases the information content, respectively. It may not be obvious, but the reader may check for herself than $\mathcal{K}_i^{\forall}\mathcal{K}_j^{\exists}$ in general neither increases or decreases semantic content.

Example 9.6 (Alice, Bob & Eve)**.** Consider a network G with three agents (Alice, Bob and Eve) represented by nodes i, j, k respectively. Suppose Alice and Bob can communicate $ij \in \mathcal{E}_G$, Bob and Eve can communicate $jk \in \mathcal{E}_G$, but not Alice and Eve $ik \notin \mathcal{E}_G$. There are three states in our system $S = \{r, s, t\}$. The Kripke relations $\mathcal{K}_i, \mathcal{K}_j, \mathcal{K}_k$ for Alice, Bob and Eve are depicted by the following directed graphs.



每个关系都会导出一个伽罗瓦连接，如下表所示。

e	$\mathcal{K}_i^{\exists}(e)$	$\mathcal{K}_i^{\forall}(e)$	$\mathcal{K}_j^{\exists}(e)$	$\mathcal{K}_j^{\forall}(e)$	$\mathcal{K}_k^{\exists}(e)$	$\mathcal{K}_k^{\forall}(e)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{r\}$	$\{r,\ s\}$	$\emptyset$	$\{r,\ t\}$	$\emptyset$	$\{r,\ t\}$	$\emptyset$
$\{s\}$	$\{r,s,t\}$	$\emptyset$	$\{s\}$	$\{s\}$	$\{r,\ s\}$	$\emptyset$
$\{t\}$	$\{s,\ t\}$	$\emptyset$	$\{r,\ t\}$	$\emptyset$	$\{s,\ t\}$	$\emptyset$
$\{r,s\}$	$\{r,s,t\}$	$\{r\}$	$\{r,\ s,\ t\}$	$\{s\}$	$\{r、\ s、\ t\}$	$\{s\}$
$\{r,\ t\}$	$\{r,s,t\}$	$\{r,\ s,\ t\}$	$\{r,t\}$	$\{r,\ t\}$	$\{r,s,t\}$	$\{r\}$
$\{s,t\}$	$\{r,\ s,\ t\}$	$\{r,s,t\}$	$\{r,s,t\}$	$\{r,s,t\}$	$\{r、\ s、\ t\}$	$\{t\}$
$\{r,s,t\}$	$\{r,\ s,\ t\}$	$\{r,s,t\}$	$\{r,s,t\}$	$\{r,s,t\}$	$\{r,s,t\}$	$\{r,s,t\}$

9.2.2 克里普克层

现在我们引入一类格值网络层和余层，其茎是表示逻辑命题语义内-容的幂集，而其限制映射和余限制映射是克里普克-伽罗瓦连接。

Definition 9.5 (Kripke Sheaves)**.** Suppose G is a graph with nodes labeled $\mathcal{V}_G = \{1, 2, \ldots, n\}$. Suppose $M = (S, \mathcal{K}_1, \ldots, \mathcal{K}_n, \Phi, \pi)$ is a Kripke model. A *Kripke bisheaf* is a Tarski sheaf $\mathcal{F}_M$ characterized by the following.

1. Each stalk $\mathcal{F}_M(i)$ is the powerset $\wp(S)$.
2. Each stalk $\mathcal{F}_M(ij)$ is the powerset $\wp(S)$.
3. $\underline{\mathcal{F}}_M(i \trianglelefteq ij) = \mathcal{K}_i^{\exists}$ for all $i \in \mathcal{V}_G, j \in \mathcal{N}_i$.
4. $\overline{\mathcal{F}}_M(i \trianglelefteq ij) = \mathcal{K}_i^{\forall}$ for all $i \in \mathcal{V}_G, j \in \mathcal{N}_i$.

$\mathcal{F}_M$ 的 0-上链是积 $\underline{C}^0(G; \mathcal{F}_M) = \wp(S)^n$。对于余链，由此从语法转向语义，令

$$[\![\boldsymbol{\varphi}]\!] = \qquad\qquad ([\![\varphi_i]\!])_{i \in \mathcal{V}_G} \in \wp(S)^n$$

对于公式元组 $\boldsymbol{\varphi} \in \mathcal{L}_n(\Phi)^n$。我们来考虑一些特殊情况。

- Suppose $\mathcal{K}_i$ is the identity relation on S; $s\mathcal{K}_i s$ for all $s \in S$, no other elements are in $\mathcal{K}_i$. Then, $(\mathcal{K}_i^{\exists}, \mathcal{K}_i^{\forall}) = (\mathrm{id}, \mathrm{id})$ and $\underline{\mathcal{F}}_M$ is the constant sheaf $\underline{\wp(S)}$. Note, in this case, $[\![\varphi]\!] = [\![K_i\varphi]\!]$ so that the constant sheaf reduces a network sheaf model of the semantics for ordinary propositional logic.
- Suppose $\mathcal{K}_i$ is a (possibly different) equivalence relation for each i, so that $\mathcal{M}_n^{rst}$ is a class of models for $\mathsf{S5}_n$. The restriction map $\mathcal{K}_i^{\exists}$ sends an event e to the union of equivalence classes of elements of e. In particular, $\mathcal{K}_{\exists}$ sends equivalence classes to equivalence classes. The corestriction map $\mathcal{K}_i^{\forall}$ sends an event e to the set of elements of S whose equivalence class is contained in e.

正如我们的记号所示，0-上链通过公式的内涵来模拟公式元组。我们将研究这一类层的截面。

Proposition 9.3. Suppose $\mathcal{F}_M$ is a Kripke sheaf with the above data. Suppose $\boldsymbol{\varphi} \in \mathcal{L}_n(\Phi)^n$ is a tuple of formulas indexed by the nodes of G. Then, $[\![\boldsymbol{\varphi}]\!] \in H^0(G; \underline{\mathcal{F}}_M)$ if and only if

$$K_i \neg \varphi_i \equiv K_j \neg \varphi_j \quad \forall ij \in \mathcal{E}_G.$$

证明。 回想一下，根据整体截面判据，$\mathbf{e} \in \Gamma(G; \mathcal{F}_M)$ 当且仅当

$$\underline{\mathcal{F}}_M(i \trianglelefteq ij)(e_i) = \underline{\mathcal{F}}_M(j \trianglelefteq ij) \quad \forall ij \in \mathcal{E}_G.$$

这里，令 e_i 等于 $[\![\varphi_i]\!]$，其中 $\varphi_i \in \mathcal{L}_n(\Phi)$ 是某个公式，因此

$$\mathcal{K}_i^{\exists}([\![\varphi_i]\!]) = \mathcal{K}_j^{\exists}([\![\varphi_j]\!]) \quad \forall ij \in \mathcal{E}_G.$$

根据引理 9.1 和定理9.3，

$$\begin{aligned} \mathcal{K}_i^{\exists}([\![\varphi_i]\!]) &= [\![\neg K_i \neg \varphi_i]\!] \\ &= [\![K_i \neg \varphi_i]\!]^c \\ &= \mathcal{K}_i^{\forall}([\![\neg \varphi_i]\!])^c \\ &= \mathcal{K}_i^{\forall}([\![\varphi_i]\!]^c)^c \end{aligned}$$

这意味着，

$$\mathcal{K}_i^{\forall}([\![\varphi_i]\!]^c)^c = \mathcal{K}_j^{\forall}([\![\varphi_j]\!]^c)^c,$$

对两边取补集并再次应用 $[\![\varphi]\!]^c = \neg[\![\varphi]\!]$，可得

$$\mathcal{K}_i^{\forall}([\![\neg \varphi_i]\!]) = \mathcal{K}_j^{\forall}([\![\neg \varphi_j]\!]).$$

因此，

$$[\![K_i \neg \varphi_i]\!] = [\![K_j \neg \varphi_j]\!].$$

□

命题的证明9。 3 表明，在某些条件下，$K_i \varphi_i$在语义上也等价于 $K^j \varphi_j$。

命题 9.4. 假设除上述条件外，$\mathcal{K}_i$ 对所有 $i \in \mathcal{V}_G$ 都是串行的。那么，$[\![\varphi]\!] \in H^0(G; \mathcal{F}_M)$ 当且仅当

$$K_i \varphi_i \equiv K_j \varphi_j.$$

我们将需要一个引理。

引理 9.2. 假设 $\mathcal{K}_i$ 是串行的。那么，$\mathcal{K}_i^{\forall}(\emptyset) = \emptyset$。

引理的证明 9.2. 根据命题 3.1 和定理 9.3，

$$\mathcal{K}_{\forall}(\emptyset) = \bigcup \{e \subseteq S \mid \mathcal{K}_i^{\exists}(e) = \emptyset\}.$$

Because $\mathcal{K}_i$ is serial, for every $s \in S$, there is at least one $t \in S$ with $s \mathcal{K}_i t$. Hence, $\mathcal{K}_i^{\exists}(e) = \emptyset$ if and only if $e = \emptyset$. Thus, $\mathcal{K}_{\forall}(\emptyset) = \emptyset$. □

Proof of Proposition 9.4. By Proposition 9.3, $[\![\boldsymbol{\varphi}]\!]$ is a section if and only if $[\![K_i\neg\varphi_i]\!] = [\![K_j\neg\varphi_j]\!]$ for all $ij \in \mathcal{E}_G$, which holds if and only if $[\![K_i\varphi_i]\!] = [\![K_j\varphi_j]\!]$ for all $ij \in \mathcal{E}_G$ by the following.

Claim. $\mathcal{K}_i^{\forall}([\![\varphi_i]\!]^c) = \mathcal{K}_i^{\forall}([\![\varphi]\!])^c$.

根据命题3.1 （再次）和定理9.3,

$$\mathcal{K}_i^{\forall}\left([\![\varphi_i]\!] \cap [\![\varphi_i]\!]^c\right) = \qquad \mathcal{K}_i^{\forall}\left([\![\varphi_i]\!]\right) \cap \mathcal{K}_i^{\forall}\left([\![\varphi_i]\!]^c\right).$$

另一方面,

$$\begin{aligned}\mathcal{K}_i^{\forall}\left([\![\varphi_i]\!] \cap [\![\varphi_i]\!]^c\right) &= \qquad \mathcal{K}_i^{\forall}(\varnothing)\\ &= \qquad \varnothing\end{aligned}$$

因此,

$$\mathcal{K}_i^{\forall}\left([\![\varphi_i]\!]^c\right) = \qquad \mathcal{K}_i^{\forall}\left([\![\varphi_i]\!]\right)^c,$$

or,

$$[\![K_i\neg\varphi_i]\!] = \qquad [\![K_i\varphi_i]\!]^c. \tag{9.13}$$

Finally, $[\![K_i\neg\varphi_i]\!] = [\![K_j\neg\varphi_j]\!]$ is equivalent to $[\![K_i\varphi_i]\!]^c = [\![K_j\varphi_j]\!]^c$ which is of course equivalent to $[\![K_i\varphi_i]\!] = [\![K_j\varphi_j]\!]$ by taking set complements of both sides. □

Proposition 9.3 and Proposiotin 9.4 suggest we can interpret global sections of Kripke sheaves as local *knowledge consensus*. We say $\mathcal{F}$ is *star-shaped* if for all $i \in \mathcal{V}_G$ and for all $j, j' \in \mathcal{N}_i$, $\underline{\mathcal{F}}(i \trianglelefteq ij) = \underline{\mathcal{F}}(i \trianglelefteq ij')$. However, by the following Lemma, local knowledge consensus necessarily extends to global knowledge consensus.

Lemma 9.3. Suppose $\mathcal{F}$ is a star-shaped Tarski sheaf over a G, and suppose $\mathbf{x} \in \Gamma(G; \underline{\mathcal{F}})$. Then, $x_{ij} = x_{i'j'}$ for every pair of edges $ij, i'j' \in \mathcal{E}_G$ in the same connected component of G.

证明。 根据星形性质,

$$\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) = \qquad \underline{\mathcal{F}}(i \trianglelefteq ij')(x_i)$$

对于所有 $i \in \mathcal{V}_G$、j、$j' \in \mathcal{N}_i$。然而，因为 $\mathbf{x} \in \Gamma(G; \underline{\mathcal{F}})$,

$$\begin{aligned}\underline{\mathcal{F}}(j \trianglelefteq ij)(x_j) &= \qquad \underline{\mathcal{F}}(i \trianglelefteq ij)(x_i)\\ \underline{\mathcal{F}}(j' \trianglelefteq ij')(x_{j'}) &= \qquad \underline{\mathcal{F}}(i \trianglelefteq ij')(x_i)\end{aligned}$$

这意味着 $x^{ij} = x^{ij'}$。由连通性可知，将 $\mathcal{F}$ 的截面投影到 $C^1(G; \underline{\mathcal{F}})$ 上时，截面是常数。 □

前面的所有结果最终导出了以下定理。

Theorem 9.4. Suppose $M = (S, \mathcal{K}_1, \ldots, \mathcal{K}_n, \Phi, \pi)$ is a Kripke model supported on a graph $G = (\mathcal{V}_V, \mathcal{E}_E)$ such that every $\mathcal{K}_i$ is serial and $\mathcal{F}_M$ its corresponding Kripke bisheaf. Then, $[\![\boldsymbol{\varphi}]\!] \in H^0(G; \underline{\mathcal{F}}_M)$ if and only if for every $i, j \in \mathcal{V}_G$ in the same connected component

$$K_i\varphi_i \equiv K_j\varphi_j.$$

推论 9。2。 特别地，若 $\boldsymbol{\varphi}$ 为值为 φ 的常数，则

$$[\![K_i\varphi]\!] = [\![K_j\varphi]\!]$$

对于所有 $i,j \in \mathcal{V}_G$ 位于同一连通分支中。

9.2.3 克里普克拉普拉斯算子

现在，我们将重点转向网络上的语义动态。0-上链 $\wp(S)^n$ 上的扩散动态通过 $\mathcal{F}_M$ 的塔斯基拉普拉斯算子定义。

Definition 9.6 (Kripke Laplacian)**.** Suppose $M = (S, \mathcal{K}_1, \ldots, \mathcal{K}_n, \Phi, \pi)$ is Kripke model and G is a graph with nodes $\mathcal{V}_G = \{1, 2, \ldots, n\}$ as before. Then, the *Kripke Laplacian* is the operator

$$\begin{aligned} L : \quad & \wp(S)^n \to \wp(S)^n \\ (L\mathbf{e})_i = \quad & \bigcap_{j \in \mathcal{N}_i} \mathcal{K}_i^{\forall}\mathcal{K}_i^{\exists}(e_i) \end{aligned} \tag{9.14}$$

Example 9.7 (Alice, Bob & Eve)**.** Let us revisit the network G, frame $(S, \mathcal{K}_i, \mathcal{K}_j, \mathcal{K}_j)$, and Galois connections $(\mathcal{K}_i^{\exists}, \mathcal{K}_i^{\forall}), (\mathcal{K}_j^{\exists}, \mathcal{K}_j^{\forall}), (\mathcal{K}_k^{\exists}, \mathcal{K}_k^{\forall})$ from Example 9.6. Then, the semantic sheaf $\mathcal{F}_M$ is depicted by the following diagram

$$\wp(S) \underset{\mathcal{K}_i^{\forall}}{\overset{\mathcal{K}_i^{\exists}}{\rightleftarrows}} \wp(S) \underset{\mathcal{K}_j^{\forall}}{\overset{\mathcal{K}_j^{\exists}}{\rightleftarrows}} \wp(S) \underset{\mathcal{K}_j^{\forall}}{\overset{\mathcal{K}_j^{\exists}}{\rightleftarrows}} \wp(S) \underset{\mathcal{K}_k^{\forall}}{\overset{\mathcal{K}_k^{\exists}}{\rightleftarrows}} \wp(S)$$

$$\bullet_i \text{ ——————— } \bullet_j \text{ ——————— } \bullet_k$$

克里普克拉普拉斯是该算子

$$L(e_i, e_j, e_k) = \left(\mathcal{K}_i^{\forall}\mathcal{K}_j^{\exists}(e_j), \mathcal{K}_j^{\forall}\mathcal{K}_i^{\exists}(e_i) \cap \mathcal{K}_j^{\forall}\mathcal{K}_k^{\exists}(e_k), \mathcal{K}_k^{\forall}\mathcal{K}_j^{\exists}(e_j)\right)$$

我们计算几个初始 0-余链的 Kripke 拉普拉斯算子的值。

e_i	e_j	e_k	$(L\mathbf{e})_i$	$(L\mathbf{e})_j$	$(L\mathbf{e})_k$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{r\}$	$\{s\}$	$\{t\}$	$\emptyset$	$\{s\}$	$\emptyset$
$\{s\}$	$\{r\}$	$\{t\}$	$\{r, s, t\}$	$\{r,s,t\}$	$\{r\}$
$\{t\}$	$\{t\}$	$\{t\}$	$\{r, s, t\}$	$\{r,s,t\}$	$\{r\}$
$\{s\}$	$\{s\}$	$\{s\}$	$\emptyset$	$\{s\}$	$\emptyset$
$\{r, s, t\}$	$\{r,s,t\}$	$\{r, s, t\}$	$\{r, s, t\}$	$\{r,s,t\}$	$\{r,s,t\}$

现在我们可以应用第6 章和7中建立的理论。

定理 9.5. 给定数据 M 和 G，整体截面 $\Gamma(G; \mathcal{F}_M)$构成 $\wp(S)^n$ 的完备拟子格。

证明。 应用定理6。 2。 □

We argue that the lattice $\Gamma(G; \underline{\mathcal{F}}_M)$ is an algebraic model for the pair (M, G). Suppose $\mathcal{K}_i$ is the identity relation for every $i \in \mathcal{V}_G$. Then, $\Gamma(G; \underline{\mathcal{F}}_M)$ is isomorphic to $\wp(S)$. This makes sense because in this case, modalities $\mathcal{K}_i$ act as the identity, which reduces the semantics to propositional calculus, a Boolean algebra.

定理 9。 6。 设L 是$(M,)$的 Kripke 拉普拉斯算子。则，

$$\mathrm{Post}(L) = \qquad \Gamma(G; \underline{\mathcal{F}}_M).$$

证明。 应用定理7。 1。 □

定理 9.7. 整体截面与全局时间不变解重合。

$$e_i[t+1] = \qquad (L\mathbf{e})_i \cap e_i \quad \forall i \in \mathcal{V}_G.$$

证明。幂集满足下降链条件。应用命-题 7。 5。

□

在附录B中，我们通过随机生成一个网络和局部 Ker 关系来进行实验。我们使用流言算法（算法3）计算热流。

9.2.4 扩散知识

和之前一样，假设 G 是一个图，且 Φ 是一个原子命题集合。

定义 9.7. 假设 $\varphi \in \mathcal{L}_n(\Phi)$。那么，对于 $i \in \mathcal{V}_V$，句法拉普拉斯是 $\mathcal{L}_n(\Phi)$ 中定义的公式

$$\left(\tilde{L}\boldsymbol{\varphi}\right)_i = \qquad \bigwedge_{j \in \mathcal{N}_i} K_j \neg K_i \neg \varphi.$$

引理 9。 4。 假设 $\boldsymbol{\varphi} \in \mathcal{L}_n(\Phi)^n$ 且 $M \in \mathcal{M}_n(\Phi)$。则，

$$[\![(L\varphi)_i]\!] = \qquad (L[\![\boldsymbol{\varphi}]\!])_i.$$

证明。根据引理9。 1 和定理9。 3，

$$\begin{aligned}
[\![L_i\varphi_i]\!] &= \\
&= [\![\bigwedge_{j \in \mathcal{N}_i} K_i \neg K_j \neg \varphi_j]\!] \\
&= \bigwedge_{j \in \mathcal{N}_i} [\![K_i \neg K_j \neg \varphi_j]\!] \\
&= \bigwedge_{j \in \mathcal{N}_i} \mathcal{K}_i^{\forall} \left([\![\neg K_j \neg \varphi_j]\!]\right) \\
&= \bigwedge_{j \in \mathcal{N}_i} \mathcal{K}_i^{\forall} \mathcal{K}_j^{\forall} [\![\varphi_j]\!].
\end{aligned}$$

□

假设 $\boldsymbol{\varphi}$ 是常数，值为 φ。那么，将$L_i\varphi$ 定义为以下公式是合理的：

$$\bigwedge_{j\in\mathcal{N}_j} \neg K_j \neg \varphi.$$

在 $\mathsf{KD45}_n$ 中，对信念进行建模时，$L_i\varphi$ 可以被解释为

> *If Bob is a neighbor of Alice, then Alice believes that Bob doesn't disbelieve φ.*

同样，在 $\mathsf{S5}_n$ 中，建模知识 $L_i\varphi$ 可以解释为

> 如果 *Bob* 是 *Alice* 的邻居，那么 *Alice* 知道 *Bob* 并非不知道 φ。

在这两种情况下，以及通常在我们的心智模型中，我们可以将 φ 的$L_i\varphi$扩散性知识 称为。

通过模仿共同知识算子 $E_A\varphi$ 和 $C_A\varphi$，我们既可以将扩散知识的概念扩展到节点的任意子集，又可以通过迭代L_i 算子来整合高阶信息。假设 $A \subseteq \mathcal{V}_G$ 是一个智能体群。然后，定义

$$\begin{aligned} L_A\varphi &= \bigwedge_{i\in A} L_i\varphi \\ L_A^\omega\varphi &= \bigwedge_{k\geqslant 1} L_A^k\varphi. \end{aligned}$$

还有许多问题有待解答。例如，能否从$L^{\mathcal{V}_G}$或$L^\omega\varphi$推导出$E_{\mathcal{V}_G}\varphi$、$C_{\mathcal{V}_G}\varphi$ 或 $D_{\mathcal{V}_G}\varphi$？在 K_n 或其他公理系统中会出现什么情况？这是否取决于G的结构？

第四部分

附录

A

层论

In Chapter 5 and 6 we were introduced to network sheaves and, more specifically, lattice valued network sheaves. However, those with some degree of familiarity with sheaves, may not even recognize that a functor $\underline{\mathcal{F}} : \mathrm{Ind}(G) \to \mathcal{S}up$ is properly a sheaf. We may also wonder what lattice-valued sheaf theory looks like in the general case.

回顾一下，拓扑空间是由数据 $(\mathbb{X}, \mathrm{Open}(\mathbb{X}))$ 构成，其中包含一个称为开集[1] 的子集偏序集，并满足以下条件

1. $\emptyset \in$打开$(\mathbb{X})$.
2. $\mathbb{X} \in$打开$(\mathbb{X})$.
3. 如果 $\{U\}_{i \in I} \subseteq$ 开集$(\mathbb{X})$ 是任意的开集族，那么它们的并集 $\bigcup_{i \in I} U_i \in$ 开集$(\mathbb{X})$ 也是开集。

定义 A。1（预层）。设 $\mathbb{X}$ 是一个拓扑空间，$\mathcal{D}$ 是一个范畴。那么，一个关于 $\mathbb{X}$ 取值于 $\mathcal{D}$ 的预层是一个逆变函子

$$\underline{\mathcal{F}} : \mathrm{Open}(\mathbb{X})^{\mathrm{op}} \to \mathcal{D}$$

具体而言，对于每个开集U，预层是如下数据

$$\begin{aligned} \underline{\mathcal{F}}(U) &\in && \mathrm{Ob}(\mathcal{D}) \\ \underline{\mathcal{F}}(V \subseteq U) &\in && \mathrm{Hom}_{\mathcal{D}}(\underline{\mathcal{F}}(V), \underline{\mathcal{F}}(U)) \end{aligned}$$

for $V \subseteq U$ open. The architypical example of a presheaf with $\mathcal{D} = \mathcal{S}et$ over a space $\mathbb{X}$ is the following. For $U \in \mathrm{Open}(\mathbb{X})$, assign the set of $\mathbb{R}$-valued continuous functions to U and assign the restriction of a particular map on U to the open subset $V \subseteq U$

$$\begin{aligned} \underline{\mathcal{F}}(U) &= && \{f : U \to \mathbb{R}\} \\ \underline{\mathcal{F}}(V \subseteq U) &= && f_{|V} \end{aligned}$$

Suppose $\mathcal{A}$ and $\mathcal{B}$ are categories. A functor $\mathcal{A} \to \mathcal{B}$ is *faithful* if for every pair of objects $x, y \in \mathrm{Ob}(\mathcal{A})$, the map $\hom_{\mathcal{A}}(x, y) \hookrightarrow \hom_{\mathcal{B}}(T(x), T(y))$ is injective. Then, a *concrete category* $\mathcal{D}$ is a category that admits a faithful functor $\mathcal{D} \to \mathcal{S}et$.

1 The notion of an open set is quite general, but it coincides with familiar intuition. For instance, if $(\mathbb{M}, d)$ is a metric space, then a subset $U \subseteq \mathbb{M}$ is open in the *metric space topoloy* if and only for every $x \in U$ there is an $\varepsilon > 0$ such that the (open) ball $B_\varepsilon(x) = \{y \in \mathbb{M} \mid d(x, y) < \varepsilon\}$ is contained in U. The topology formed by arbitrary unions and finite intersections of balls in$\mathbb{M}$ is called the *metric space topology*.

Example A.1 (Suplattices)**.** The category $\mathcal{S}up$ is concrete. The functor $\mathcal{S}up \to \mathcal{S}et$ sends a complete lattice $\mathbf{L}$ to its underlying set L and a join-preserving map $f : \mathbf{K} \to \mathbf{L}$ to the ordinary function $f : K \to L$. Faithfulness is trivial because morphisms in $\mathcal{S}up$ are maps between sets that preserve structure; if two functions are equal after forgetting the join-preserving structure, then, of course, the structure-preserving maps were equal.

We define sheaves with arbitrary algebraic strucutre by first defining sheaves valued in sets. An *open cover* of $U \subseteq \mathbb{X}$ consists of a collection of open sets $\{U_i\}_{i\in I}$ such that $\bigcup_{i\in I} U_i \supseteq U$. Let U_{ij} denote the open set $U_i \cap U_j$. The following definition is taken from [107], while not standard, is one of a handful of equivalent definitions of a sheaf.

Definition A.2 (Sheaf)**.** Suppose $\underline{\mathcal{F}}$ is a presheaf of sets over X. Then, $\underline{\mathcal{F}}$ is a *sheaf* if for every open set $U \in \mathrm{Open}(\mathbb{X})$ and open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of U, the following is an equalizer

$$\underline{\mathcal{F}}(U) \dashrightarrow \prod_{i\in I} \underline{\mathcal{F}}(U_i) \underset{\prod_{i\in I} \underline{\mathcal{F}}(U_{ij}\subseteq U_j)}{\overset{\prod_{j\in I} \underline{\mathcal{F}}(U_{ij}\subseteq U_i)}{\rightrightarrows}} \prod_{i,j\in I} \underline{\mathcal{F}}(U_{ij})$$

那么，取值于具体范畴 $\mathcal{D}$ 的预层 $\mathcal{F}$ 若在以下图表中复合后构成预层，即为层 。

$$\begin{array}{ccc} \mathrm{Open}(\mathbb{X})^{\mathrm{op}} & \xrightarrow{\underline{\mathcal{F}}} & \mathcal{D} \\ & \searrow & \downarrow \\ & & \mathcal{S}et \end{array}$$

是一个层。

Global sections of a sheaf over a space are simply the object $\underline{\mathcal{F}}(\mathbb{X})$. However, this definition is not very useful because it is not constructable. In certain situations, $\mathbb{X}$ is presented with a cell decomposition or, in real-life examples, is approximated with as a graph or simplicial complex.

Definition A.3. Suppose $(\mathrm{P}, \trianglelefteq)$ is a poset. Then, the *Alexandrov topology* on P, denoted $\mathrm{Alex}(\mathrm{P})$, consists of up-sets: subsets $U \subseteq \mathrm{P}$ with $x \in U$, $y \trianglerighteq u$ implies $y \in U$.

回忆一下，如果一个范畴具有所有（小）极限和余极限，那么分别称其为 完备 或 余完备 。[2]

定理 A。1。 范畴 $\mathcal{S}$向上 的完备格和保并映射是完备且余完备的。

2 我们在此不回顾极限和余极限。关于极限和余极限的良好引言，请参见 [133, Chpater3]。

Proof. See Joyal [87, Chapter 1] for a detailed explanation. Alternatively, $\mathcal{S}up$ has products, equalizers, coproducts, and coequalizers, making $\mathcal{S}up$ complete and cocomplete [133, Theorem 3.4.12]. □

We now state a result by Curry which adresses the common (invalid) critique that functors out of posets (e.g. Tarski sheaves) are not really sheaves. The follwing theorem requires the notion of a Kan extension [133] and is stated in a form dual to the original result.

Theorem A.2 (Theorem 4.8 [44])**.** Suppose P is a poset and $\mathcal{C}$ is a complete category. Suppose $\underline{\mathcal{F}} : \mathrm{P} \to \mathcal{C}$ is a functor (e.g. network sheaf), and suppose $\iota : \mathrm{P} \to \mathrm{Alex}(\mathrm{P})^{\mathrm{op}}$ is the functor associating an element $x \in \mathrm{P}$ to the open set $\uparrow x$. Then, the right Kan extension $\mathrm{Ran}_{\iota}(\underline{\mathcal{F}})$ of $\underline{\mathcal{F}}$ along ι is a sheaf over the Alexandrov topology $\mathrm{Alex}(\mathrm{P})$.

推论 A。1. 一个塔斯基层 $\mathcal{F} : \mathrm{P}_G \to \mathcal{L}tc$ 是 $\mathbf{Alex}(\mathrm{P}_G)$空间上的一个层（自然同构于该层）。

证明。 定理6.1，定理 A.1 和定理 A.2。 □

B

实 验

b.1 Gossip算法

In the following example, we demonstrate the validity of the gossip algorithm introduced in Chapter 7 for an example sheaf introduced in Chapter 9. Suppose $\mathcal{F}$ is a Tarski sheaf, such that for every $ij \in \mathcal{E}_G$, $\underline{\mathcal{F}}(ij)$ is not only a complete lattice but also a metric space $(\underline{\mathcal{F}}(ij), d_{ij})$. We measure "how close" an $\mathbf{x} \in C^0(G; \underline{\mathcal{F}})$ is to being a section with the following notion of *Dirichlet energy* or *total variation*.

$$V(\mathbf{x}) \;=\; \sum_{ij \in \mathcal{E}_G} d_{ij}\left(\underline{\mathcal{F}}(i \trianglelefteq ij)(x_i), \underline{\mathcal{F}}(j \trianglelefteq ij)(x_j)\right).$$

以下内容显而易见

事实。 设 $\mathcal{F}$是 G 上的塔斯基层，且 $\mathbf{x} \in C^0(G;\ \mathcal{F})$。则，$V(\underline{\mathbf{x}}) = 0$ 当且仅当 $\mathbf{x} \in H^0(G;\ \mathcal{F})$。 —

Recall the definition of the time-varying Tarki Laplacian (Definition 7.8). In the following experiment, we generate random geometric graphs $G = (\mathcal{V}_G, \mathcal{E}_G)$ with parameters $|\mathcal{V}_G| = 40$ and $r = 0.08$. Specifically, locations $y_i \in \mathbb{R}^2$ of each $i \in \mathcal{V}_G$ are sampled uniformly at random from $[0,1]^2$; $ij \in \mathcal{E}_G$ if and only if $\|x_i - x_j\| \leqslant r$.

We construct a Kripke sheaf $\mathcal{F}_M$ (Definition 9.5) by generating a Kripke model M (really, just a frame) with randomized Kripke relations $\mathcal{K}_i$, $i \in \mathcal{V}_G$. In our experiment, we choose a fixed number of states $S = \{1, 2, \ldots, 10\}$ and wire each $(a, b) \in \mathcal{K}_i$ randomly (with probability $p = 0.9$ if $a = b$, $p = 0.1$ otherwise). The relations $\mathcal{K}_i$ are generated independently. We select a random firing sequence $\tau : \mathbb{N} \to \wp(\mathcal{V}_G)$ which serves the purpose of determining which nodes broadcast to their neighbors at time t. In our experiment, we randomly select a single node to broadcast each round. For a series of random initial assignments $\mathbf{x}[0]$, we run the heat flow dynamics using the asynchronous Kripke Laplacian (Definition 9.6). The results are summarized below (Figure B.1).

b.2 一个格点卷积神经网络（l-cnn）

We use a portion of the ModelNet10 dataset [167] as a source of point clouds. This dataset consists of 100s of 3D CAD models which represent objects from 1 of 10 classes. We sample points from the 3D models to produce finite metric spaces embedded in $\mathbb{R}^3$, then compute the corresponding multidimensional persistence modules [101], from

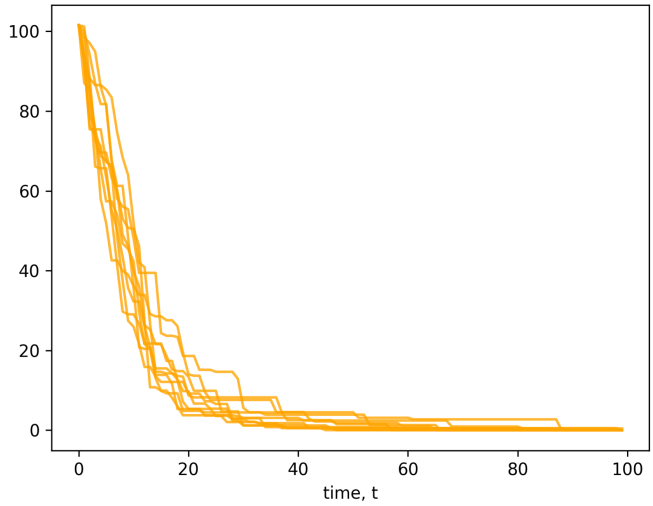


图 B.1：对于带有随机化克里普克层的几何随机图上的流言算法，10 组随机初始条件下热流的狄利克雷能量。

其中我们生成的特征被用作卷积神经网络分类器的输入。

The pipline thus begins with a 3D polyhedral model, of which 3000 vertices are sampled to produce a point cloud in $\mathbb{R}^3$. This point cloud then produces a bifiltered simplicial complex, whose degree-0 persistent homology we calculate using RIVET [101], sampled at a discrete grid of 40×40 points, producing lattice-indexed signals given by the Hilbert function and the multi-graded Betti numbers ξ_0, ξ_1, ξ_2; four features in total. These are then passed to the classifier, which produces a class prediction, in this case one of 10 possible objects. As the filter function on these data sets, we use a codensity function that is the inverse sum distance from a point x to the k-nearest neighbors ($k = 100$). The name codensity is appropriate because the points in the densest regions of $\mathcal{M}$ appear earlier in the filtration.

We compare the performance of two convolutional networks on this classification task. One uses is a L-CNN and the other is a standard CNN. Each has three convolutional layers followed by two fully connected layers. The lattice-based convolution layers are of the form $\alpha \cdot \mathrm{MeetConv}(x) + (1-\alpha) \cdot \mathrm{JoinConv}(x)$ for a hyperparameter $\alpha \in [0,1]$ ($\alpha = 0.5$) Thus, *lattice layers* are a hybrid of join- and meet- convolutions. All convolution kernels have dimension 4×4, hidden convolution layers have 16 features, and the final convolution layer has 8 features. The first two convolution layers are followed by max-pooling layers with a 2×2 kernel. For the lattice convolutional layers, the support of the kernel lay in an evenly spaced 4×4 grid of points in $[m] \times [n]$, while the standard convolutional layers had a standard kernel. The inner fully connected layer has 32 features. We use a cross-entropy loss function with a softmax in the final layer. The

lattice-based convolutional architecture is summarized in Figure B.2. The networks are trained with Adam gradient algorithm with learning rate 2×10^{-4} for a total of 300 epochs. We hold out 10% of the data for testing. Results are shown in Figure B.2.

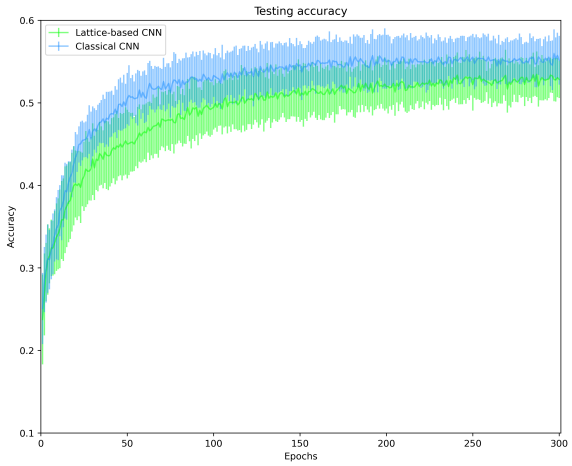


图 B.2：L-CC（格点卷积神经网络）的测试/训练准确率与经典 CNN 的比较。

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