arXiv:2304.02807v1 [math.DG] 6 Apr 2023

S^m 上自旋Yamabe型问题的解：扰动与应用

TakeshiIsobe Tian Xu

摘要

本文是旨在建立共形不变狄拉克方程存在性理论的研究计划的一部分

$$D_g\psi = f(x)|\psi|^{\frac{2}{m-1}}\psi$$

on a closed spin manifold (M, g) of dimension $m \geq 2$ with a fixed spin structure, where $f : M \to \mathbb{R}$ is a given function. The study on such nonlinear equation is motivated by its important applications in Spin Geometry: when $m = 2$, a solution corresponds to an isometric immersion of the universal covering $\widetilde{M}$ into $\mathbb{R}^3$ with prescribed mean curvature f; meanwhile, for general dimensions and $f \equiv constant$, a solution provides an upper bound estimate for the Bär-Hijazi-Lott invariant.

Comparing with the existing issues, the aim of this paper is to establish multiple existence results in a new geometric context, which have not been considered in the previous literature. Precisely, in order to examine the dependence of solutions of the aforementioned nonlinear Dirac equations on geometrical data, concrete analysis are made for two specific models on the manifold (S^m, g): the geometric potential f is a perturbation from constant with $g = g_{S^m}$ being the canonical round metric; and $f \equiv 1$ with the metric g being a perturbation of g_{S^m}, that is not conformally flat somewhere on S^m. The proof is variational: solutions of these problems are found as critical points of their corresponding energy functionals. The emphasis is that the solutions are always degenerate: they appear as critical manifolds of positive dimension. This is very different from most situations in elliptic PDEs and classical critical point theory. As corollaries of the existence results, multiple distinct embedded spheres in $\mathbb{R}^3$ with a common mean curvature are constructed, and furthermore, a strict inequality estimate for the Bär-Hijazi-Lott invariant on S^m, $m \geq 4$, is derived, which is the first result of this kind in the non-locally-conformally-flat setting.

关键词。Dirac算子；旋量Yamabe方程；摄动方法；共形类。

T. Isobe：一桥大学经济学研究生院，日本东京都国立市中 2-1，邮编 186-8601；电子邮箱：t.isobe@r.hit-u.ac.jp T. Xu：天津大学应用数学中心，中国天津市南开区，邮编 300072；电子邮箱：xutian@amss.ac.cn T. Isobe 获得日本学术振兴会（JSPS）科学研究费补助金（项目编号 20K03674）的资助，T. Xu 获得中国国家自然科学基金（NSFC 11601370）和德国亚历山大·冯·洪堡基金会的资助

数学主题分类（2010）：主分类 53C27；次分类 35R01

目录

1 引言 **2**
1.1 Existence results for the PDE problems 5
1.2 Geometric applications 8
1.3 Overview of the method 10
1.4 Organization of the paper 11

2 精确的 框架 1 **1**
2.1 化归为欧几里得空间上的问题.11 2.2 构形空间.12 2.3 具有退化条件的扰动方法.
13

3 几何势的扰动 18
3.1 拓扑度.18 3.2 主要结果的证明.24

4 对背景度量的扰动 29
4.1 Bourguignon-Gauduchon identification 30
4.2 Basic calculations 31
4.3 Proof the main result 37

A 附录 **40**
A.1 An explanation of the (k, p)-elementary matrix 40
A.2 Some basic estimates 41
A.3 Proofs of Lemma 4.6 and Proposition 4.7 43

1 引言

Let $(M, g\ \sigma)$ be an m-dimensional closed spin manifold, $m \geq 2$, with a fixed Riemannian metric g and a fixed spin structure $\sigma : P_{Spin}(M) \to P_{SO}(M)$. For the metric g, we obtain a spinor bundle $\mathbb{S}(M) \to M$, which is endowed with a natural hermitian metric $|\cdot|_g$. Sections of $\mathbb{S}(M)$ are usually called spinors. And the compatible covariant derivative on $\mathbb{S}(M)$, denoted by $\nabla^{\mathbb{S}}$, can be locally expressed as

$$\nabla^{\mathbb{S}}\varphi_\alpha = \frac{1}{4}\sum_{i,j=1}^{m} g(\nabla e_i, e_j)e_i \cdot_g e_j \cdot_g \varphi_\alpha$$

其中， $\{e_i\}$是 TM的局部正交基， $\{\varphi_\alpha\}$是 $\mathbb{S}(M)$中的局部旋量标架，右侧的符号 ∇ 表示(M, g)上的列维-奇维塔联络，而“$\cdot_g$”表示克利福德乘法。采用这些记号，我们引入Dirac算子D_g： $C^\infty(M, \mathbb{S}(M)) \to C^\infty(M, \mathbb{S}(M))$，其局部表达式为

$$D_g\psi = \sum_{i=1}^{m} e_i \cdot_g \nabla^{\mathbb{S}}_{e_i}\psi.$$

对于旋量 $\psi \in C^\infty(M, \mathbb{S}(M))$。

在共形变换下，Dirac算子的行为类似于共形拉普拉斯算子，后者出现在经典Yamabe问题的研究中，参见例如 [23,24]。因此，Dirac算子的一个旋量对应量——即 B¨ar-Hijazi-Lott 不变量 —— 在 [8,9] 中得到了研究。

从变分的观点来看，对于 $(M, g\ \sigma)$，B¨ar-Hijazi-Lott 不变量是由 $\lambda^+_{min}(M, g\ \sigma)$ 给出的正实数

$$\lambda^+_{min}(M,g,\sigma) := \inf \left\{ \frac{\left(\int_M |D_g\psi|_g^{\frac{2m}{m+1}} d\,\mathrm{vol}_g \right)^{\frac{m+1}{m}}}{\int_M (D_g\psi,\psi)_g d\,\mathrm{vol}_g} \;\middle|\; \int_M (D_g\psi,\psi)_g d\,\mathrm{vol}_g > 0 \right\}. \tag{1.1}$$

For $\tilde{g} = f^2 g$ (with $f \in C^\infty(M)$ and $f > 0$) and $\tilde{\psi} = f^{-\frac{m-1}{2}}\psi$ (via the identification of spinor bundles with respect to conformal metrics [23, 24]), we have that $\int_M (D_g\psi,\psi)_g d\,\mathrm{vol}_g = \int_M (D_{\tilde{g}}\tilde{\psi},\tilde{\psi})_{\tilde{g}} d\,\mathrm{vol}_{\tilde{g}}$ and $\int_M |D_g\psi|_g^{\frac{2m}{m+1}} d\,\mathrm{vol}_g = \int_M |D_{\tilde{g}}\tilde{\psi}|_{\tilde{g}}^{\frac{2m}{m+1}} d\,\mathrm{vol}_{\tilde{g}}$. Hence, $\lambda^+_{min}(M, g\ \sigma)$ is actually a conformal invariant. Furthermore, the Bär-Hijazi-Lott invariant is tightly connected to and behaves much like the classical Yamabe invariant. In fact, it follows from Hijazi's inequality [23] that, for $m \geq 3$, if the Yamabe invariant

$$Y(M,[g]) = \inf_{\tilde{g}\in[g]} \frac{\int_M \mathrm{Scal}_{\tilde{g}} d\,\mathrm{vol}_{\tilde{g}}}{\mathrm{Vol}(M,\tilde{g})^{\frac{m-2}{m}}}$$

是非负的，则

$$\lambda^+_{min}(M,g,\sigma) \geq \sqrt{\frac{m}{4(m-1)}} Y(M,[g]). \tag{1.2}$$

且等式 (1.2) 对标准球面成立。

与 Yamabe 不变量类似，对于任意闭自旋流形，B¨ar-Hijazi-Lott 不变量的值不会大于同维圆球面的相应值。这被称为 Aubin 不等式的自旋版本（见 [11]）：

$$\lambda^+_{min}(M,g,\sigma) \leq \lambda^+_{min}(S^m, g_{S^m}, \sigma_{S^m}) = \frac{m}{2}\omega_m^{\frac{1}{m}} \tag{1.3}$$

where $(S^m, g_{S^m}, \sigma_{S^m})$ is the m-dimensional sphere equipped with its canonical metric g_{S^m} and its unique spin structure σ_{S^m}, and ω_m stands for the volume of (S^m, g_{S^m}). In the spirit of this, the next stage would consist in showing that (1.3) is a strict inequality if (M, g) is not conformally equivalent to (S^m, g_{S^m}). However, the strict inequality in (1.3) is only verified for some special cases (for instance, if M is locally conformally flat, if D_g is invertible and if the so-called *Mass endomorphism* is not identically zero [12], and all rectangular tori [40]), and a general result is still lacking (cf. [10, 20, 22]). The methods that can be used are sometimes similar to the ones of the Yamabe problem, but since we work with Dirac operator and spinors, the reasoning is more involved as the eigenvalues of the Dirac operator tend to both $-\infty$ and $+\infty$ and there is no maximum principle.

通过考察变分问题 (1.1)，我们可以看到，相应的欧拉-拉格朗日方程是

$$D_g\psi = \lambda^+_{min}(M,g,\sigma)|\psi|_g^{\frac{2}{m-1}}\psi \quad \text{with} \int_M |\psi|_g^{\frac{2m}{m-1}} d\,\mathrm{vol}_g = 1$$

或者等价地

$$D_g\psi = |\psi|_g^{\frac{2}{m-1}}\psi \quad \text{with} \int_M |\psi|_g^{\frac{2m}{m-1}} d\,\mathrm{vol}_g = \lambda^+_{min}(M,g,\sigma)^m \tag{1.4}$$

这是一个一阶半线性偏微分方程。在闭自旋流形上，利用次临界 Sobolev嵌入的紧性，方程 (1.4) 的解的存在性已在 [9, 定理1.1] 中针对 $\lambda^+_{min}(M,g\ \sigma) < \lambda^+_{min}(S^m, g_{S^m}, \sigma_{S^m})$ 得到。

进一步地，我们可以将式（1.4）略作推广，考虑如下方程

$$D_g\psi = f(x)|\psi|_g^{\frac{2}{m-1}}\psi \quad \text{on } M, \tag{1.5}$$

where $f : M \to \mathbb{R}$ is a given function. Geometrically speaking, Eq. (1.5) is of particular interest in dimension $m = 2$ because its solution provides a strong tool for showing the existence of prescribed mean curvature surfaces (here the function f plays the role of the mean curvature). Special cases of such surfaces are constant mean curvature (CMC) surfaces (that is $f \equiv constant$) which have been studied before by completely different techniques, see for instance [21]. The correspondence between a solution of Eq. (1.5) on a Riemannian surface M and a periodic conformal immersion (possibly with branching points) of the universal covering $\widetilde{M}$ into $\mathbb{R}^3$ with mean curvature f is known as the spinorial Weierstraß representation. For details in this direction, we refer to [8, 9, 18, 29–31, 36, 42–44] and references therein.

Besides the very special case $f \equiv constant$, which reduces Eq. (1.5) to Eq. (1.4), there are only a few results about the existence of a solution to Eq. (1.5). For $f \neq constant$, Raulot [39] obtained an existence criterion for this problem which is similar to the strict inequality in (1.3). Indeed, one of his results shows that if the Dirac operator D_g is invertible, f is positive and satisfies

$$\lambda_{min,f} \cdot \big(\max_{x\in M} f\big)^{\frac{m-1}{m}} < \lambda^+_{min}(S^m, g_{S^m}, \sigma_{S^m}), \tag{1.6}$$

其中

$$\lambda_{min,f} := \inf\left\{ \frac{\Big(\int_M f^{-\frac{m-1}{m+1}} |D_g\phi|^{\frac{2m}{m+1}} d\,\mathrm{vol}_g\Big)^{\frac{m+1}{m}}}{\int_M (D_g\phi,\phi) d\,\mathrm{vol}_g} \;\Big|\; \int_M (D_g\psi,\psi)_g d\,\mathrm{vol}_g > 0 \right\},$$

then there exists a solution to Eq. (1.5). However, the validity of condition (1.6) is not very clear. Particularly, the existence results in [39] does not apply for $(M,g) = (S^m, g_{S^m})$ since the strict inequality of (1.6) is not valid on the round spheres in any circumstance. For later developments, we mention that in [27, 28] the author obtained some sufficient conditions about f under which there exists a non-trivial solution to Eq. (1.5) on (S^m, g_{S^m}), $m \geq 2$. Precisely, up to a simple scaling, [27, 28] studied the case $f(x) = 1 + \varepsilon H(x)$ for some $H \in C^2(S^m)$ and small $\varepsilon > 0$.

Much less is known about the existence of multiple solutions for Eq. (1.5). Examples can only be given on some product manifolds, see [40], or on manifolds that are invariant under some group actions, see [34]. Apart from these cases, we do not know any further multiplicity result. Hence, the purpose of this paper is to discuss new existence and multiplicity results for the problem (1.5) and, when $m = 2$, to give multiplicity results for the immersed spheres in $\mathbb{R}^3$

with prescribed mean curvature. Note that Eq. (1.5) is invariant under the canonical action of $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ on spinors. Moreover, for the cases $m \equiv 2, 3, 4 \pmod 8$, the spinor bundle has a quaternionic structure which commutes with Clifford multiplication [19, Section 1.7] or [32, Page 33, Table III]. Thus in these cases, Eq. (1.5) is invariant under the canonical action of the unit quaternions $S^3 = \{q \in \mathbb{H} : |q| = 1\}$. Thus we are interested in multiple existence of S^1 or S^3-inequivalent solutions.

我们关注两个具体问题。首先，我们研究式(1.5) 中的几何-势（即平均曲率函数）是常数的扰动这一情形，换言之，我们关注方程

$$D_{g_{S^m}}\psi = (1 + \varepsilon H(x))|\psi|^{\frac{2}{m-1}}_{g_{S^m}}\psi \quad \text{on } S^m, \tag{1.7}$$

当 $|\varepsilon| > 0$ 足够小时，并且我们旨在改进 [27,28] 中的存在性结果。其次，受 (1.3)中（严格）不等式的启发，我们引入了一族新的度量 g on S^m （接近 g_{S^m}但不共形等价），使得该方程

$$D_g\psi = |\psi|^{\frac{2}{m-1}}_g\psi \quad \text{on } S^m, \tag{1.8}$$

has multiple solutions. An important feature of Eq. (1.8) is that (as was indicated in (1.4)) the integral $\int_{S^m} |\psi|^{\frac{2m}{m-1}}_g d\,\mathrm{vol}_g$ of a solution gives a description of the Bär-Hijazi-Lott invariant. And it would be of particular interest if one can derive a conformal spectral estimate for the Dirac operator D_g so that (1.3) is a strict inequality. As we mentioned before, as far as geometric aspects are concerned, the solutions to both of the equations (1.7) and (1.8) correspond to conformal (branched) immersions of S^2 into $\mathbb{R}^3$. However, Eq. (1.8) for $m = 2$ does not reflect the geometric importance since, in view of Riemann mapping theorem, there is only one conformal structure on S^2.

1.1 PDE 问题的存在性结果

我们从方程(1.7)开始。对于函数 $H \in C^1(S^m)$，我们记 $Crit[H]$ 为H的临界集。为陈述我们的结果，我们对 H 作出以下两个始终成立的条件假设:

(H-1) $H \in C^2(S^m)$ 是一个 Morse 函数，使得 $\Delta_{g_{S^m}} H(p) \neq 0$ 对于 $p \in Crit[H]$。

(H-2) H 满足

$$\sum_{p \in Crit[H],\ \Delta_{g_{S^m}} H(p) < 0} (-1)^{\mathfrak{m}(H,p)} \neq (-1)^m$$

其中， $\mathfrak{m}(H, p)$ 是 H 在 $p \in Crit[H]$ 处的莫尔斯指数。

这里提到，条件(H-2)是众所周知的指数计数条件，最早在 [13,17] 中的标量曲率问题里被引入。

为了描述我们的一般重数结果，需要对 H 提出一些额外要求。这里我们关注函数 H 可以分解为若干个分量的情形，使得每个分量都能生成方程（1.7）的解。为此，对于一点 $p \in S^m$，记 $\pi_p : S^m \setminus \{p\} \to \mathbb{R}^m$ 为标准立体投影。接下来我们引入以下条件

(H-3) 对于某些 $l \in \mathbb{N}$、 $l \geq 2$、 H 具有以下形式

$$H(x) = \begin{cases} \displaystyle\sum_{i=1}^{l} H_i\big(\pi_{p_0}^{-1}(\pi_{p_0}(x) - z_i)\big) & x \in S^m \setminus \{p_0\} \\ \displaystyle\sum_{i=1}^{l} H_i(p_0) & x = p_0 \end{cases}$$

其中 H_1、 $\cdots H_l \in C^2(S^m)$ 有一个共同的极小点 $p_0 \in S^m$，且 z_1、 ...、 $z_l \in \mathbb{R}^m$ 是固定的。

It is clear that $H_i \circ \pi_{p_0}^{-1}$ defines a C^2-function on $\mathbb{R}^m$ and $\lim_{|y|\to\infty} H_i \circ \pi_{p_0}^{-1}(y) = H_i(p_0)$ for each i. The function $H_i\big(\pi_{p_0}^{-1}(\pi_{p_0}(\cdot) - z_i)\big) : S^m \setminus \{p_0\} \to \mathbb{R}$ can be viewed as a translation of H_i on S^m with $H_i(p_0)$ being fixed. Hence the condition (H-3) describes a function that is (approximately) concentrated on the points $\pi_{p_0}^{-1}(z_i) \in S^m \setminus \{p_0\}$, $i = 1, \ldots, l$, and is well-defined on S^m. Here, the only reason we keep p_0 being fixed is that we want to present the function H in a simple manner. And, of course, (H-3) can be replaced by other forms. Now we prove the following existence result.

定理1.1. 对于 $m \geq 2$, 令 $\mathscr{H}$ 表示 $C^2(S^m)$ 中的一个函数族，满足

$$\mathscr{H} := \big\{H \in C^2(S^m) : \textit{ } H \textit{ satisfies the conditions } (\text{H-1}) \textit{ and } (\text{H-2})\big\}.$$

Then

(1) *for a generic function $H \in \mathscr{H}$, there exists $\varepsilon_0 > 0$ such that Eq.* (1.7) *has at least $2^{[\frac{m}{2}]}$ distinct non-trivial S^1-orbits of solutions for $0 < |\varepsilon| < \varepsilon_0$. Furthermore, when $m \equiv 2, 3, 4$ (mod 8), there are at least $2^{[\frac{m}{2}]-1}$ distinct non-trivial S^3-orbits of solutions for $0 < |\varepsilon| < \varepsilon_0$;*

(2) *if $H \in C^2(S^m)$ satisfies condition* (H-3) *with generic $H_1, \ldots, H_l \in \mathscr{H}$ and let $z_1, \ldots, z_l \in \mathbb{R}^m$ be located far enough apart, then there exists $\varepsilon_0 > 0$ such that Eq.* (1.7) *has at least $2^{[\frac{m}{2}]}l$ distinct non-trivial S^1-orbits of solutions for $0 < |\varepsilon| < \varepsilon_0$. Furthermore, when $m \equiv 2, 3, 4$ (mod 8), there are at least $2^{[\frac{m}{2}]-1}l$ distinct non-trivial S^3-orbits of solutions for $0 < |\varepsilon| < \varepsilon_0$;*

(3) 若 ψ_ε 表示在 (1) 和 (2) 中得到的任一解，则

$$\int_{S^m} (1 + \varepsilon H(x)) |\psi_\varepsilon|_{g_{S^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{S^m}} = \Big(\frac{m}{2}\Big)^m \omega_m + O(\varepsilon)$$

当 $\varepsilon \to 0$，此外，只要 $|\varepsilon|$ 足够小， $|\psi_\varepsilon|_{g_{S^m}} > 0$ 在 S^m 上。

注1.2.这里，“generic”是指，存在性结果对于属于 $\mathscr{H}$的一个残余子集的 H 成立。

接下来，我们转而考察在球面上配备一族新度量时，方程 (1.8) 的存在性结果。为了刻画 S^m 上的新度量 g，我们再次使用立体投影

投影 π_p： $S^m \setminus \{p\} \to \mathbb{R}^m$ （其中 $p \in S^m$ 任意固定），从而使 $S^m \setminus \{p\}$ 上的 g 与 $\mathbb{R}^m$ 上的度量 ˜g 之间建立如下的一一对应关系：

$$\tilde{g} = f^{-2} \cdot (\pi_p^{-1})^* g, \quad f(x) = \frac{2}{1+|x|^2},\ x \in \mathbb{R}^m. \tag{1.9}$$

显然，如果 ˜g $= \mathrm{g}_{\mathbb{R}^m}$ 是典范欧几里得度量，那么 $S^m \setminus \{p\}$ 上的度量 g 可以整体延拓为标准圆度量。下文中，我们假设 ˜g 具有 ˜g $= \mathrm{g}_{\mathbb{R}^m} + \varepsilon \mathrm{h}$ 的形式，其中 h是 $\mathbb{R}^m$上的光滑对称双线性形式。特别地，让我们考虑一个具体情形

$$\tilde{g}(x) = \operatorname{diag}\big(\tilde{g}_{11}(x), \dots, \tilde{g}_{mm}(x)\big) \quad \text{with} \quad \tilde{g}_{ii}(x) = 1 + \varepsilon h_{ii}(x), \tag{1.10}$$

where $h_{ii} : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \dots, m$, are smooth functions. Let us point out here that, for a general choice of *h*, the metric *g* on $S^m \setminus \{p\}$ may be discontinuous at the point p. Hence, in order to extend *g* globally on S^m, it is natural to require the entries h_{ii}, $i = 1, \dots, m$, and their derivatives behave "nicely" at infinity. To give an idea of our multiplicity results, let us consider *h* to be compactly supported (that is to say $\cup_i \operatorname{supp} h_{ii}$ is a compact subset of $\mathbb{R}^m$). And then, we are interested in the situation where $\tilde{g}$ is not conformal to the Euclidean metric.

为清晰起见，我们为 (1.10) 中的扰动 h 构造一种简单有效的表述。下面予以精确定义。

定义 1.3。给定一个光滑的 $m \times m$ 对角矩阵函数

$$h(x) = \operatorname{diag}\big(h_{11}(x), \dots, h_{mm}(x)\big) \quad \textit{for } x \in \mathbb{R}^m,\ m \geq 2,$$

以及一个点 $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$。对于 $k \in \{1, \dots, m\}$和 $p \in [1, \infty)$，如果 $\xi \notin \operatorname{supp} \mathrm{h}_{kk}$，且对于靠近 ξ和 $i \neq k$的 $x = (x_1, \dots, x_m) \in \mathbb{R}^m$，我们称 h 在 ξ 处是 (k, p)-初等的，

$$h_{ii}(x) = h_{ii}(\xi) + c_i(x_i - \xi_i) + c_k(x_k - \xi_k) + o(|x-\xi|^p)$$

其中 $c_i \in \mathbb{R}$、 $i = 1$、...、 m是常数，且特别满足 $c_k \neq 0$。此外，如果上述h_{ii}的局部展开中 $o(|x-\xi|^p)$ 项恒等于零，那么我们称h 在 ξ 处是 (k, ∞)-初等的。由此，我们将 $p \in [1, \infty) \cup \{\infty\}$ 称为 h 在 ξ 处的余项指数。

在不打断阅读的情况下，附录A.1 中给出了此类（k, p）-初等矩阵的示例及简要说明。从定义 1.3 可以看出，"初等"矩阵是一个局部概念。在下文中，如果根据上下文可以明确我们所指的是哪个维数，我们将简单地使用"初等矩阵"这一名称来指代一个成员 h （不提及其标签编号 k, p 或位置点 ξ ）。

定理1.4。对于 $m \geq 4$，设 g 为通过(1.9)和(1.10)定义的度量，其中

$$h(x) = h^{(1)}(x) + h^{(2)}(x - x_0), \tag{1.11}$$

其中， $x_0 \in \mathbb{R}^m$ 固定，$\mathrm{h}^{(1)}$ 和 $\mathrm{h}^{(2)}$ 具有紧支集，并且是初等矩阵。另行假设，对于 $i = 1$, 2,

$$h^{(i)} \textit{ has remainder exponent } \begin{cases} p^{(i)} = \infty & m = 4, \\ p^{(i)} > 2 & m = 5, \\ p^{(i)} \geq 2 & m \geq 6. \end{cases}$$

If $|x_0|$ *is large enough, then*

(1) *Eq.* (1.8) *has at least two non-trivial solutions* ψ_1 *and* ψ_2 *provided that* $|\varepsilon| > 0$ *is small;*

(2) 对每个 $i = 1,2$, 均成立。

$$\int_{S^m} |\psi_i|_g^{\frac{2m}{m-1}} d\,\mathrm{vol}_g = \left(\frac{m}{2}\right)^m \omega_m + C_{i,m}\,\varepsilon^2 + o(\varepsilon^2),$$

其中 $C_{i,m} < 0$ 是仅依赖于 $\mathrm{h}^{(i)}$ 和维数 m的负常数$i = 1, 2$.

Remark 1.5. (1) Although we proved the case for *h* splits into two elementary matrices, Theorem 1.4 extends to the situation where *h* splits into an arbitrary finite number of elementary matrices. Here the results do not reflect the canonical action of S^1 (or S^3 when $m \equiv 2, 3, 4$ (mod 8)) as mentioned in Theorem 1.1 since each one of the solutions is generated by the components of *h* in (1.11) individually.

(2) For $p \geq 2$, let $[p] \in \mathbb{Z}$ denotes the largest integer that does not exceed p. Definition 1.3 indicates that, when *h* is (k, p)-elementary at some point ξ, all the j-th partial derivative of each h_{ii} vanish at this point for $j = 2, \ldots, [p]$. One possible extension of this condition is to consider the case where the local expansion of h_{ii} has quadratic terms. The proofs can be carried out by using similar arguments.

1.2 几何应用

如前所述，方程 (1.7) 和方程 (1.8) 的解有两项主要应用。第一项应用是，当 $m = 2$ 时，我们得到在 $\mathbb{R}^3$ 中具有指定平均曲率的浸入球面的存在性结果。第二项应用则给出了 B¨ar-Hijazi-Lott 不变量的上界估计。

Remark 1.6. (1) For $m = 2$, Theorem 1.1 implies multiple existence of conformal immersions $S^2 \to \mathbb{R}^3$ with mean curvature equals to $1 + \varepsilon H$. Precisely, if ψ_ε denotes any one of the solutions obtained in Theorem 1.1, since $|\psi_\varepsilon|_{g_{S^2}} > 0$ we can introduce a conformal metric $g_\varepsilon = |\psi_\varepsilon|^4_{g_{S^2}} g_{S^2}$ on S^2. Then, due to the conformal covariance of the Dirac operator, we see that there is a spinor field φ_ε on (S^2, g_ε) such that

$$D_{g_\varepsilon}\varphi_\varepsilon = (1 + \varepsilon H(x))\varphi_\varepsilon \quad \text{and} \quad |\varphi_\varepsilon|_{g_\varepsilon} \equiv 1.$$

因此，根据旋量Weierstraß表示，存在一个等距浸入 $\Pi : (S^2, \mathrm{g}_\varepsilon) \to (\mathbb{R}^3, \mathrm{g}_{\mathbb{R}^3})$，其平均曲率等于 $1 + \varepsilon H$。此外，由于该浸入对欧几里得体积形式的拉回为 $\Pi^*(d\,\mathrm{vol}_{g_{\mathbb{R}^3}}) = |\psi_\varepsilon|^4_{g_{S^2}} d\mathrm{vol}_{g_{S^2}}$，该浸入对应的 Willmore能量 $W(\Pi)$ 满足

$$W(\Pi) = \int_{S^2} (1 + \varepsilon H(x))^2 |\psi_\varepsilon|^4_{g_{S^m}} d\,\mathrm{vol}_{g_{S^2}} \leq 4\pi + O(\varepsilon) < 8\pi$$

为 $\varepsilon \to 0$。由于 Li-Yau 不等式 [33, 定理 6], 该浸入 Π 至多覆盖$\mathbb{R}^3$ 中的点一次。因此 Π 是一个嵌入。

(2) 定理1.4中的存在性结果直接蕴含了在(S^m,g)上关于 Dirac算子 D_g 的共形谱估计。事实上，根据 B¨ar-Hijazi-Lott 不变量 (1.1) 的定义，我们有

$$\lambda^+_{min}(S^m, g, \sigma_{S^m}) \leq \frac{\left(\int_{S^m} |D_g\psi_i|_g^{\frac{2m}{m+1}} d\,\mathrm{vol}_g\right)^{\frac{m+1}{m}}}{\int_{S^m}(D_g\psi_i, \psi_i)_g d\,\mathrm{vol}_g} = \left(\int_{S^m} |\psi_i|_g^{\frac{2m}{m-1}} d\,\mathrm{vol}_g\right)^{\frac{1}{m}} < \frac{m}{2}\omega_m^{\frac{1}{m}}$$

对于定理1.4中得到的每个 ψ_i, $i = 1,2$。因此，度量 g 满足不等式 (1.3) 中的严格不等式。类似的结果是否适用于维数 3，将会是一个很有意思的问题。

现在，通过总结注释 1.6，我们得到定理 1.1 和 1.4 的以下两个几何应用。

定理1.7. 设 H 满足定理1.1(2)的条件，则对于 $|\varepsilon| > 0$ 充分小，至少存在 l个互不相同的将(S^2, g_{S^2})嵌入到($\mathbb{R}^3$, $g_{\mathbb{R}^3}$)中的共形等距嵌入，其像的平均曲率为$1 + \varepsilon H$。

定理1.8。对于 $m \geq 4$，令g为定理1.4中的度量。则严格不等式

$$\lambda^+_{min}(S^m, g, \sigma_{S^m}) < \lambda^+_{min}(S^m, g_{S^m}, \sigma_{S^m}) \tag{1.12}$$

只要 $|\varepsilon| > 0$ 足够小，该结论就成立。

Remark 1.9. (1) Theorem 1.7 can be viewed as part of a program to understand to what extent one can use mean curvature data to determine the shape of a surface in $\mathbb{R}^3$. Usually, a generic immersion is uniquely determined up to a rigid motion by its first fundamental form and its mean curvature function, but there are some exceptions, for instance most constant mean curvature immersions. A classical result by Bonnet states that if there exists a diffeomorphism $\Psi : \Sigma_1 \to \Sigma_2$ between two closed immersed surfaces Σ_1, Σ_2 of genus zero in $\mathbb{R}^3$ such that Ψ preserves both the metric and the mean curvature function of the surfaces, then Σ_1 and Σ_2 are congruent in $\mathbb{R}^3$ (i.e., they differ by a rigid motion). In view of this, one may address the following natural questions:

问题 1. 给定函数 f： $S^2 \to \mathbb{R}$，对 f 有什么条件能够保证存在等距浸入 Π： $S^2 \to \mathbb{R}^3$，使 f 成为其平均曲率?

问题2。假设给定 f： $S^2 \to \mathbb{R}$，使其成为某个等距浸入 Π： $S^2 \to \mathbb{R}^3$的平均曲率函数。是否可能存在另一个等距浸入 $\widehat{\Pi}$： $S^2 \to \mathbb{R}^3$，它与 Π 不全等，但同样将 f 实现为其平均曲率? 如果可以，那么关于拉回度量 $\Pi^* g_{\mathbb{R}^3}$ 和 $\widehat{\Pi}^* g_{\mathbb{R}^3}$可以说些什么?

注意，在问题 2 中，我们并未假设浸入球面 $\Pi(S^2)$ 和 $\widehat{\Pi}(S^2)$ 是等距的，而这正是 Bonnet 结论的关键假设。定理 1.7 回答了问题 1 和问题 2：我们找到了关于 f 的充分条件，使其能够实现为不同等距嵌入的平均曲率，而且这些嵌入的拉回度量彼此共形相关。

(2)定理 1.8 给出了一族具体例子，使得 (1.3) 中的严格不等式成立。我们希望获得这一严格不等式，原因如下：

– (1.3) 中的严格不等式意味着不变量 $\lambda^+_{min}(M, \mathrm{g}\ \sigma)$可由一个广义度量取得，即形如 $f^2\mathrm{g}$ 的度量，其中 f 可能存在一些零点。–利用 Hijazi 不等式 [23], , 可以在 $m \geq 3$ 的情形下得到标准Yamabe问题的一个解；该问题旨在寻找 g 的共形类中具有常标量曲率的度量。

With the aim of obtaining this inequality, Ammann et al. considered in [12] the case of a locally conformally flat spin manifold $(M, g\ \sigma)$ with $\ker D_g = \{0\}$ and introduced for a fixed point $p \in M$, a further datum, the *mass endomorphism*. The main result of [12] is that the strict inequality in (1.3) holds, if g is conformally flat around p and the mass endomorphism at p has a positive eigenvalue. Meanwhile, it is also proven in [12] that the mass endomorphism always vanishes in dimension 2 and on flat tori $\mathbb{T}^m$. To describe the dependence of the mass endomorphism on the Riemannian metric, let $\mathcal{M}_{p,\mathrm{flat}}(M)$ be the set of all Riemannian metrics g on M, which are flat on a neighborhood of $p \in M$ and satisfy $\ker D_g = \{0\}$. It is proven in [10], see also [22], that for dimension $m \geq 3$, the subset of all Riemannian metrics with non-vanishing mass endomorphism at p is dense in $\mathcal{M}_{p,\mathrm{flat}}(M)$ with respect to the C^∞-topology. However, for an arbitrarily given metric g, examining the non-vanishingness of its associated mass endomorphism is still a difficult issue.

It can be seen from Theorem 1.8 that, even if the metric g is flat somewhere on M, the appearance of the non-locally-conformally-flat part is the key reason we obtain the strict inequality. It is well known that, for $m \geq 4$, the conformal flatness of (M, g) is characterized by the nullity of the Weyl tensor. However, what is the exact role of the Weyl tensor in the general inequality $\lambda^+_{min}(M, g\ \sigma) < \lambda^+_{min}(S^m, g_{S^m}, \sigma_{S^m})$ when (M, g) is non-locally conformally flat remains unexplored. To the best of our knowledge, Theorem 1.8 is the first result in the non-locally-conformally-flat setting. We consider it hence as remarkable that Theorem 1.8 might give us a good chance to uncover the very nature of the Bär-Hijazi-Lott invariant in general cases.

1.3 方法概述

定理1.1和1.4的证明依赖于在临界点理论中恰当地运用抽象扰动方法。这一思路广泛应用于许多非线性问题，例如经典的 Yamabe问题及相关课题（参见例如 [5–7, 16, 35]）。论文 [27, 28] 对如何将这一思路应用于自旋方程 (1.7) 进行了详细阐述。粗略地说，扰动方法包含两个主要步骤：首先运用 Lyapunov-Schmidt 约化，将原问题化为某个有限维变分问题；然后证明约化问题的“稳定”临界点会产生原问题的解。

正如 [27, 28], 所示，乍看之下，旋量 Yamabe 型方程的约化问题与相应的椭圆方程问题非常相似，参见例如[5, 7, 35]。然而，与椭圆情形不同，经典的抽象框架在这里无法被采用。

以直接的方式使用。这是因为，与自旋Yamabe问题相关的有限维问题是退化的，也就是说，约化泛函主项的任意临界点都不是孤立的，而这些临界点的集合表现为正维临界流形。因此，尚不清楚约化泛函的临界点是否会产生原问题的真正解。为克服这一困难，作者发展了，在 [27], 一种Morse-Bott理论的变体，并在 [28], 一种针对这类变分问题的Conley指标理论。

In this paper, our approach is different from the ones used in the aforementioned works. We have a two-step reduction procedure before finding a critical point. We start with the Lyapunov-Schmidt reduction to obtain a reduced finite dimensional variational problem, and the new perspective is the observation that the reduced functional is defined on a space endowed with a fiber bundle structure, where the fiber is non-compact and the base space is a close manifold. The key point in our argument is that the degeneracy of the reduced functional essentially occurs on the base manifold. Hence, as a second step, we get a chance to restrict the reduced functional on each fiber rather than on the total space and finally (via implicit function theorem, minimizing/maximizing methods) to show the critical points of the reduced problem give rise to solutions of the original one. An advantage of our approach is that it produces proofs that are rather simpler than the ones of [27, 28]. And this might be meaningful to provide new insight on nonperturbative problems.

1.4 论文结构安排

This paper is organized as follows: In the next section, we quickly review the transformation of the equations (1.7) and (1.8) into equivalent equations on Euclidean spaces, and then we set up our abstract perturbation framework with degenerate conditions. Section 3 is concerned with our first problem, i.e., Eq. (1.7) where the perturbation is appearing in the geometric potential. And Section 4 deals with the second problem, that is, Eq. (1.8) where the perturbation is involved in the background metric. Both in Section 3 and 4, we first collect properties of the variational problem and then apply the abstract results in Section 2 to obtain the existence results. Finally, in the Appendix, we collect some straight calculations which are needed in the arguments.

2 精确框架

2.1 化归为欧几里得空间上的问题

We first mention that the Eq. (1.7) on S^m is equivalent to an equation on $\mathbb{R}^m$ by conformal equivalence. Precisely, let $H \in C^2(S^m)$ satisfies the hypotheses of Theorem 1.1, we assume H takes its minimum at $p_0 \in S^m$. Denote by $\pi_{p_0} : S^m \setminus \{p_0\} \to \mathbb{R}^m$ the stereographic projection from p_0, we have $(\pi_{p_0}^{-1})^* g_{S^m} = f^2 g_{\mathbb{R}^m}$ where g_{S^m} is the round metric on S^m with constant sectional curvature 1 and $f(x) = \frac{2}{1+r^2}$ ($r = |x|$ and $x \in \mathbb{R}^m \cong T_{p_0} S^m$ is the stereographic coordinate with respect to π_{p_0}).

[2]另一方面，流形（M, g）上的一般方程 $D_g\psi = H(x)|\psi|^{\overline{m-1}}{}_g\psi$ 在度量的共形变换下保持不变。事实上，设 $\bar{}\mathrm{g} = e^{2u}\mathrm{g}$，其中某个函数 u 定义在 M上，则存在一个向量丛同构 F: $\mathbb{S}(M, \mathrm{g}) \to \mathbb{S}(M, \bar{}\mathrm{g})$ （此处 $\mathbb{S}(M, \mathrm{g})$ 和 $\mathbb{S}(M, \bar{}\mathrm{g})$

是相对于度量 g 和 $\bar{\mathrm{g}}$ 的 M 上的旋量丛），它是一个逐纤维等距映射，使得

$$D_{\bar{g}}\big(F(\varphi)\big) = F\big(e^{-\frac{m+1}{2}u} D_g(e^{\frac{m-1}{2}u}\varphi)\big)$$

for $\varphi \in C^\infty((M,g)\ \mathbb{S}(M,g))$ (for more detailed definitions and facts about Clifford algebras, spin structures on manifolds and Dirac operators, please consult [19, 32]). Thus, when ψ is a solution to the equation $D_g\psi = H(x)|\psi|_g^{\frac{2}{m-1}}\psi$ on (M,g), then $\varphi := F(e^{-\frac{m-1}{2}u}\psi)$ also satisfies the same equation on $(M,\bar{g})$: $D_{\bar{g}}\varphi = H(x)|\varphi|_{\bar{g}}^{\frac{2}{m-1}}\varphi$ on $(M,\bar{g})$.

将上述观察应用于带有 $M = S^m$ 的式 (1.7)，如果 $\psi \in C^1(S^m, \mathbb{S}(S^m))$ 是式 (1.7) 的一个解，那么 $\varphi = f^{\frac{m-1}{2}} F(\psi \circ \pi_{p_0}^{-1})$ 满足该方程。

$$D_{g_{\mathbb{R}^m}}\varphi = (1+\varepsilon a(x))|\varphi|_{g_{\mathbb{R}^m}}^{\frac{2}{m-1}}\varphi \quad \text{on } \mathbb{R}^m \tag{2.1}$$

其中 $a(x) = H(\pi_{p_0}^{-1}(x))$。

Conversely, by the regularity theorem for weak solutions for Eq. (1.5) (see [26, Appendix]), for any solution $\varphi \in L^{\frac{2m}{m-1}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ to Eq. (2.1) with Hölder continuous function a, there corresponds to a C^1-solution $\psi = \varphi \circ \pi_{p_0}$ to Eq. (1.7) on S^m with $H(x) = a(\pi_{p_0}(x))$. Therefore, the study of Eq. (1.7) is equivalent to the study of Eq. (2.1).

对于我们的第二个问题，即方程 (1.8)，在欧几里得空间上有一个类似的等价问题。回顾一下，根据(1.9)，我们有共形等价 $(S^m\backslash\{p\}, \mathrm{g}) \cong (\mathbb{R}^m, \tilde{\mathrm{g}})$。因此，通过重复上述论证，我们发现方程 (1.8) 等价于方程

$$D_{\tilde{g}}\varphi = |\varphi|_{\tilde{g}}^{\frac{2}{m-1}}\varphi \quad \text{on } \mathbb{R}^m \tag{2.2}$$

其中 $\tilde{\mathrm{g}} = \mathrm{g}_{\mathbb{R}^m} + \varepsilon \mathrm{h}$，正如（1.9）和（1.10）中所假设的那样。

为后续讨论起见，我们指出，方程（2.1）和（2.2）都可以看作是对未扰动方程的扰动。

$$D_{g_{\mathbb{R}^m}}\psi = |\psi|_{g_{\mathbb{R}^m}}^{\frac{2}{m-1}}\psi \quad \text{on } \mathbb{R}^m. \tag{2.3}$$

因此，在下文中，我们的扰动框架将建立在对方程（2.3）及其相关能量泛函的研究之上。

$$\mathcal{J}_0(\psi) = \frac{1}{2}\int_{\mathbb{R}^m} (\psi, D_{g_{\mathbb{R}^m}}\psi)_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} - \frac{m-1}{2m}\int_{\mathbb{R}^m} |\psi|_{g_{\mathbb{R}^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \tag{2.4}$$

其中，$(\cdot,\cdot)_{g_{\mathbb{R}^m}}$ 和 $|\cdot|_{g_{\mathbb{R}^m}}$ 分别是旋量丛 $\mathbb{S}(\mathbb{R}^m)$上的典范 Hermitian 内积及其诱导度量。

2.2 构形空间

为了从变分观点处理方程（1.5）和（2.3），有必要建立一个泛函框架。合适的函数空间是旋量场的 $H^{\frac{1}{2}}(M, \mathbb{S}(M))$和 $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$，它们在 [26,27] 中被引入。为完整起见，我们给出定义如下。

回顾一下，紧致自旋流形 (M, g) 上的 Dirac算子 D_g 在$L^2(M, \mathbb{S}(M))$ 上是自伴的，并且具有紧预解式（见 [19, 32]）。特别地，存在一个完备

由 D_g的特征旋量构成的希尔伯特空间 $L^2(M,\mathbb{S}(M))$ 的一组标准正交基 $\psi_1,\psi_2,\ldots$: $D_g\psi_k=\lambda_k\psi_k$。此外, $|\lambda_k|\to\infty$当 $k\to\infty$。

对于 $s\geq 0$, 我们定义算子 $|D_g|^s$: $L^2(M,\mathbb{S}(M))\to L^2(M,\mathbb{S}(M))$, 如下

$$|D_g|^s\psi=\sum_{k=1}^{\infty}|\lambda_k|^s\alpha_k\psi_k,$$

for $\psi=\sum_{k=1}^{\infty}\alpha_k\psi_k\in L^2(M,\mathbb{S}(M))$. Denoted by $H^s(M,\mathbb{S}(M))$ the domain of $|D_g|^s$, then it is clear that $\psi=\sum_{k=1}^{\infty}\alpha_k\psi_k\in H^s(M,\mathbb{S}(M))$ if and only if $\sum_{k=1}^{\infty}|\lambda_k|^{2s}|\alpha_k|^2<\infty$. And hence $H^s(M,\mathbb{S}(M))$ coincides with the usual Sobolev space of order s, that is $W^{s,2}(M,\mathbb{S}(M))$ (cf. [1,8]). Furthermore, we can equip $H^s(M,\mathbb{S}(M))$ with the inner product

$$\langle\psi,\varphi\rangle_{s,2}:=\mathrm{Re}(|D_g|^s\psi,|D_g|^s\varphi)_2+\mathrm{Re}(\psi,\varphi)_2$$

where $(\cdot,\cdot)_2$ is the L^2-inner product on spinors and the induced norm $\|\cdot\|_{s,2}$. In this paper, we are mainly concerned with the space $H^{\frac{1}{2}}(M,\mathbb{S}(M))$ for $M=S^m$. Notice that the spectrum of $D_{g_{S^m}}$ on S^m is bounded away from 0 and one checks easily that $\|\psi\|_{1/2}=\big||D_{g_{S^m}}|^{\frac{1}{2}}\psi\big|_2$ defines an equivalent norm on $H^{\frac{1}{2}}(S^m,\mathbb{S}(S^m))$.

为便于记号, 本文中始终将 $L^q:=L^q(M,\mathbb{S}(M))$记作范数 $|\psi|_q^q=\int_M|\psi|^q d\mathrm{vol}_g$, 其中 $q\geq 1$, 并将 $2^*=\frac{2m}{m-1}$ 记作嵌入 $H^{1/2}(M,\mathbb{S}(M))\hookrightarrow L^q(M,\mathbb{S}(M))$ 的临界索博列夫指数, 其中 $1\leq q\leq 2^*$。

On $\mathbb{R}^m$, a similar function space will also be useful in our argument. Let us denote by $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ the set of spinor fields ψ on $\mathbb{R}^m$ such that $\big||D_{g_{\mathbb{R}^m}}|^{1/2}\psi\big|_2^2<\infty$ and $|\psi|_{2^*}<\infty$ with norm $\|\psi\|_{1/2,2}:=\big||D_{g_{\mathbb{R}^m}}|^{1/2}\psi\big|_2+|\psi|_{2^*}$. Here, $|D_{g_{\mathbb{R}^m}}|^{1/2}$ is defined via the Fourier transformation: $\mathscr{F}(|D_{g_{\mathbb{R}^m}}|^{1/2}\psi)(\xi)=|\xi|^{1/2}\mathscr{F}(\psi)(\xi)$ and $\big||D_{g_{\mathbb{R}^m}}|^{1/2}\psi\big|_2:=\big||\cdot|^{1/2}\mathscr{F}(\psi)\big|_2$. Notice that $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ is isomorphic to $H^{\frac{1}{2}}(S^m,\mathbb{S}(S^m))$ via the stereographic projection. The dual space of $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ will be denoted by $\mathscr{D}^{-\frac{1}{2}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ with norm $\|\cdot\|_{-1/2,2}$.

2.3 具有退化条件的微扰方法

It is clear that, on $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$, the functional $\mathcal{J}_0$ in (2.4) is well-defined and is of C^2 class. We call $\mathcal{J}_0$ the unperturbed functional. Its critical points are solutions to the unperturbed spinorial Yamabe equation (2.3). Recall that Eq. (2.1) and (2.2) are perturbations from (2.3) and it is very natural to expect that solutions of Eq. (2.1) and (2.2) can be obtained as perturbations of critical points of $\mathcal{J}_0$. In general, a well known Lyapunov-Schmidt reduction technique provides a powerful tool, see for instance [37, Chapter 10] and [15, II, 6] where the variational problem satisfies the Palais-Smale condition and [3, 4, 7] for the case that the Palais-Smale condition fails. For completeness, following [7], we sketch the idea as follows.

设 $(\mathcal{H},\langle\cdot,\cdot\rangle)$ 是一个希尔伯特空间, 其关联范数为 $\|\cdot\|:=\langle\cdot,\cdot\rangle^{1/2}$。假设给定了 $L_0\in C^2(\mathcal{H},\mathbb{R})$ 和 $\Gamma\in C^2(\mathcal{H},\mathbb{R})$。当 $\varepsilon>0$ 足够小时, 我们考虑扰动泛函

$$L_\varepsilon(z)=L_0(z)+\varepsilon\Gamma(z)+o(\varepsilon). \tag{2.5}$$

假设 L_0 具有一个非退化临界流形 $\mathcal{M}\subset\mathcal{H}$, 也就是说,

$(A1)$ $\mathcal{M}$ is a d-dimensional C^2-submanifold of $\mathcal{H}$ such that $\nabla L_0(z)=0$ for all $z\in\mathcal{M}$,

$(A2)$ $\mathcal{M}$ is 非退化，即对于所有 $z \in \mathcal{M}$，我们有 $T_z\mathcal{M} = \ker \nabla^2 L_0(z)$，

$(A3)$ $\nabla^2 L_0(z)$：对于所有 $z \in \mathcal{M}$，$\mathcal{H} \to \mathcal{H}$ 是一个指标为零的 Fredholm 算子 .

一旦条件$(A1)$-$(A3)$得到满足，在$\mathcal{H}$ 上寻找 L_ε 的临界点问题可以约化为在 $\mathcal{M}$ 上针对约化泛函 L_ε^{red}（定义如下）的同一问题。而 L_ε 的临界点将作为 L_0的临界点的小扰动获得。

令 $\mathcal{W}_z := T_z\mathcal{M}^\perp$，其中正交补是相对于 $\langle \cdot, \cdot \rangle$ 在 $\mathcal{H}$ 中定义的。我们寻找 L_ε 的临界点，其形式为 $u = z + w$，其中 $z \in \mathcal{M}$ 且 $w \in \mathcal{W}_z$。令 $P_z : \mathcal{H} \to \mathcal{W}_z$ 为到 $\mathcal{W}_z$ 上的正交投影，欧拉-拉格朗日方程 $\nabla L_\varepsilon(z + w) = 0$ 等价于

$$\begin{cases} P_z \nabla L_\varepsilon(z + w) = 0 & (\textit{auxiliary equation}) \\ (I - P_z)\nabla L_\varepsilon(z + w) = 0 & (\textit{bifurcation equation}). \end{cases} \tag{2.6}$$

然后，在条件 $(A2)$ 和 $(A3)$ 下，可以首先应用隐函数定理求解(2.6)中关于 w 的辅助方程：对于任意 $z \in \mathcal{M}$，当 ε 取足够小时，存在唯一解 $w = w_\varepsilon(z) \in \mathcal{W}_z$。此外，在任意紧子集 $\mathcal{M}_c \subset \mathcal{M}$ 上，都可以得到一致估计：

$$\mathcal{M}_c \ni z \mapsto w_\varepsilon(z) \in \mathcal{W}_z \text{ is } C^1 \text{ and } \|w_\varepsilon(z)\|,\ \|w_\varepsilon'(z)\| = O(\varepsilon) \text{ uniformly for } z \in \mathcal{M}_c. \tag{2.7}$$

关于(2.7)的证明以及对这一构造的更详细解释，请参见 [7, 第2]章。

下一步是考虑（2.6）中的分岔方程。为此，我们引入约化泛函 L_ε^{red}：$\mathcal{M} \to \mathbb{R}$ 通过

$$L_\varepsilon^{red}(z) = L_\varepsilon(z + w_\varepsilon(z))$$

于是我们有以下定理。

定理 2.1（[7]中的定理 2.12）。假设满足条件（$A1$）—（$A3$），并且对于紧子集 $\mathcal{M}_c \subset \mathcal{M}$，当 $\varepsilon > 0$ 足够小时，L_ε^{red} 存在一个临界点 $z_\varepsilon \in \mathcal{M}_c$。那么，$u_\varepsilon = z_\varepsilon + w_\varepsilon$（$z_\varepsilon$）是 $\mathcal{H}$ 上 L_ε 的一个临界点。

得益于一致估计 (2.7)，约化泛函 L_ε^{red}在以下意义下可得到良好近似：

$$L_\varepsilon^{red}(z) = L_0(z) + \varepsilon\Gamma(z) + o(\varepsilon), \quad \nabla L_\varepsilon^{red}(z) = \varepsilon\nabla\Gamma(z) + o(\varepsilon). \tag{2.8}$$

且 $L_0(z)$在 $\mathcal{M}$的任意连通分支上为常数。因此，如果 $z \in \mathcal{M}$ 以某种特定意义而言是 Γ 的非退化临界点（例如，$\nabla\Gamma$ 在 z 处的局部度数非零），那么 z在 $\mathcal{H}$ 上生成 L_ε 的一个临界点（详见 [7, 15, 37]）。

对于我们的问题，我们旨在将上述抽象框架应用于由 $L_0 = \mathcal{J}_0$ 以及定义在$\mathcal{H} = \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$上的某些扰动项所确定的泛函（见后文的（3.1）和引理4.3）。正如 [27, 第 5, 6] 节中已经证明的，$\mathcal{J}_0$ 对于定义为 $\mathcal{M}$ 的临界流形满足（$A1$）-（$A3$），其定义为

$$\mathcal{M} := \left\{ \psi_{\lambda,\xi,\gamma} : \lambda > 0,\ \xi \in \mathbb{R}^m,\ \gamma \in \mathbb{S}_m,\ |\gamma| = 1 \right\}, \tag{2.9}$$

其中

$$\psi_{\lambda,\xi,\gamma}(x)=\frac{m^{\frac{m-1}{2}}\lambda^{\frac{m-1}{2}}}{\left(\lambda^2+|x-\xi|^2\right)^{\frac{m}{2}}}(\lambda-(x-\xi))\cdot_{g_{\mathbb{R}^m}}\gamma \tag{2.10}$$

for $\lambda>0,\xi\in\mathbb{R}^m,\gamma\in\mathbb{S}_m$ with $|\gamma|=1$ ($\mathbb{S}_m$ is the spinor module, see [19,32]) and $\cdot_{g_{\mathbb{R}^m}}$ denotes the Clifford multiplication with respect to the Euclidean metric. Note that $\mathcal{M}$ is diffeomorphic to $(0,\infty)\times\mathbb{R}^m\times S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$ via the canonical map $(\lambda,\xi,\gamma)\mapsto\psi_{\lambda,\xi,\gamma}$, where $S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$ stands for the $(2^{[\frac{m}{2}]+1}-1)$-dimensional unit sphere in $\mathbb{S}_m$. And hence $\mathcal{M}$ is a non-compact manifold and the dimension of $\mathcal{M}$ is $m+2^{[\frac{m}{2}]+1}$.

By virtue of Theorem 2.1, we intend to reduce the problem to a finite dimensional variational problem on $\mathcal{M}$. But in our case, as we will see later, the reduced functional L_ε^{red} is very degenerate on $\mathcal{M}$: the perturbation term Γ in (2.8) is degenerate on $\mathcal{M}$ (in the sense that the map $\nabla\Gamma(z)|_{T_z\mathcal{M}}:T_z\mathcal{M}\to\mathbb{R}$ has a non-trivial kernel for any $z\in\mathcal{M}$). Hence, standard methods as in [6,7,15,37] do not apply.

为了使扰动框架适用，除假设($A1$)-($A3$)外，我们还对临界流形 $\mathcal{M}$ 作出以下附加条件：

($A4$) $\mathcal{M}$ 具有一个（全局）可平凡化的纤维丛结构，其基空间为紧致空间 $\mathcal{N}$，投影为 ϑ：$\mathcal{M}\to\mathcal{N}$，纤维为 $\mathcal{G}$。准确地说，存在一个保纤维微分同胚 ι： $\mathcal{G}\times\mathcal{N}\to\mathcal{M}$，使得下图交换

$$\begin{array}{ccc}\mathcal{G}\times\mathcal{N} & \xrightarrow{\ \iota\ } & \mathcal{M}\\ {\scriptstyle Proj}\downarrow & & \downarrow{\scriptstyle\vartheta}\\ \mathcal{N} & \xrightarrow{\ \mathrm{Id}\ } & \mathcal{N}\end{array}$$

($A5$) $T_\gamma\mathcal{N}\subset\ker\nabla(\Gamma\circ\iota)(g,\gamma)$ 对任意$(g,\gamma)\in\mathcal{G}\times\mathcal{N}$，其中我们已将 $T_\gamma\mathcal{N}$ 识别为总切空间 $T_{(g,\gamma)}(\mathcal{G}\times\mathcal{N})$ 的子空间。

Remark 2.2. In our application $\mathcal{N}=S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$, $\mathcal{G}=(0,+\infty)\times\mathbb{R}^m$ and $\iota(g,\gamma):=\psi_{\lambda,\xi,\gamma}$ for $g=(\lambda,\xi)\in\mathcal{G}$ and $\gamma\in\mathcal{N}$, hence we have a very natural bundle structure on $\mathcal{M}$. Particularly, we note that there is a continuous action $\mathcal{G}\times\mathcal{M}\to\mathcal{M}$ such that $\mathcal{G}$ preserves the fibers of $\mathcal{M}$ (i.e. if $(\mu,y)\in\mathcal{G}$ and $\psi_{\lambda,\xi,\gamma}\in\mathcal{M}_\gamma$ then $\psi_{\lambda,\xi,\gamma}*(\mu,y)=\psi_{\lambda\mu,\,\xi+y,\gamma}\in\mathcal{M}_\gamma$). Hence the critical manifold in (2.9) is essentially a principal $\mathcal{G}$-bundle. And since it admits a global section, we easily see that $\mathcal{M}$ is trivializable. This is the reason we introduce $(A4)$. Since $\mathcal{M}$ is parameterized via ι, condition $(A4)$ makes the variational problem even clearer: it is equivalent to consider the functional $L_\varepsilon^{red}\circ\iota:\mathcal{G}\times\mathcal{N}\to\mathbb{R}$. Comparing with the standard theory in [7,15,37], the distinct new feature $(A5)$ describes a certain degenerate situation and, particularly, it implies that $\Gamma\circ\iota(g,\gamma)$ depends only on the variables in the fiber space $\mathcal{G}$. Thus we set $\tilde{\Gamma}(g)=\Gamma\circ\iota(g,\gamma)$. For later use, we distinguish $(A5)$ into the following two cases:

$$\begin{cases}\ker\nabla(\Gamma\circ\iota)(g,\gamma)\neq T_{(g,\gamma)}(\mathcal{G}\times\mathcal{N}) & \text{for some } (g,\gamma)\in\mathcal{G}\times\mathcal{N},\\ \ker\nabla(\Gamma\circ\iota)(g,\gamma)\equiv T_{(g,\gamma)}(\mathcal{G}\times\mathcal{N}) & \text{for all } (g,\gamma)\in\mathcal{G}\times\mathcal{N},\end{cases}$$

并且我们将估 建立一些对旋量 Yamabe问题有用的抽象结果。

情形 1：对于某些 $(g,\gamma) \in \mathcal{G}\times\mathcal{N}$，$\ker \nabla(\Gamma\circ\iota)(g,\gamma) \neq T_{(g,\gamma)}(\mathcal{G}\times\mathcal{N})$

显然，在这种情 ase，$\Gamma\circ\iota(g,\gamma) \neq constant$ 在 $\mathcal{G}\times\mathcal{N}$（等价地，$\Gamma(z) \neq constant$ 在 $\mathcal{M}$）。

定理 2.3. 设 L_0、$\Gamma \in C^2$（$\mathcal{H}$、$\mathbb{R}$）如（2.5）中所示，并假设满足（$A1$）-（$A5$）。如果˜存在一个 $\bar{g}\in\mathcal{G}$，使得 $\bar{g}$ 是 Γ的严格局部极大值或极小值。那么，当 $|\varepsilon|$ 足够小时，泛函 L_ε 在 $\mathcal{M}$ 上有一个临界点。

证明。只需考虑 $\bar{g}\in\mathcal{G}$ 是严格局部极小值且 $\varepsilon>0$的情形，其他情况可类似处理。设 $\delta>0$ 和 $U\subset\mathcal{G}$ 是 $\bar{g}$ 的有界开邻域，使得

$$\tilde{\Gamma}(g) \geq \tilde{\Gamma}(\bar{g})+\delta, \quad \forall g\in\partial U.$$

由（2.8）中的第一个展开式可知

$$L_\varepsilon^{red}\circ\iota(g,\gamma) - L_\varepsilon^{red}\circ\iota(\bar{g},\gamma) = \varepsilon\big(\tilde{\Gamma}(g)-\tilde{\Gamma}(\bar{g})\big)+o(\varepsilon) > 0 \quad \forall g\in\partial U,\ \forall\gamma\in\mathcal{N}$$

当 $\varepsilon>0$ 足够小时。由于 $\overline{U}\times\mathcal{N}$ 是紧的，通过在 $\overline{U}\times\mathcal{N}$ 中极小化泛函 $L_\varepsilon^{red}\circ\iota$，并利用上述不等式，最小值在内部取得，从而得到$L_\varepsilon^{red}\circ\iota$ 的一个临界点，等价地，得到 $\iota(U\times\mathcal{N})\subset\mathcal{M}$ 中 L_ε^{red} 的一个临界点。因此，根据定理 2.1，它生成了 L_ε 的一个临界点。□

˜在定理 2.3 的假设中，Γ 的严格局部极大值或极小值˜可能是退化的。若要求 Γ 的临界点满足某种非退化条件，则可以得到更多解，如下述结果所示。

Theorem 2.4. *Let $L_0,\Gamma\in C^2(\mathcal{H},\mathbb{R})$ as in* (2.5) *and suppose that* $(A1)$-$(A5)$ *are satisfied. If $\tilde{\Gamma}$ is a Morse function on $\mathcal{G}$ and there is an open bounded subset $\Omega\subset\mathcal{G}$ such that the topological degree* $\deg(\nabla\tilde{\Gamma},\Omega,0)\neq 0$. *Then, for $|\varepsilon|$ small, the functional L_ε^{red} has at least* $\mathrm{Cat}(\mathcal{N})$ *critical points on $\mathcal{M}$.*

这里，Cat(N) 表示 $\mathcal{N}$ 的Lusternik-Schnierelman范畴，即满足 $\mathcal{N}\subset\cup_{i=1}^k A_k$的最小整数 k，其中集合 A_k 在 $\mathcal{N}$ 中是闭的且可缩的。

Proof. Fix $\gamma\in\mathcal{N}$ arbitrarily and let us consider the functional $L_\varepsilon^{red}|_{\vartheta^{-1}(\gamma)}$. By the assumption $\deg(\nabla\tilde{\Gamma},\Omega,0)\neq 0$, there exists $\bar{g}\in\Omega$ such that $\nabla\tilde{\Gamma}(\bar{g})=0$ and the local index of $\nabla\tilde{\Gamma}$ at $\bar{g}$ is non-zero. By the second expansion in (2.8) and the continuity property of the topological degree, we have

$$\deg(\nabla L_\varepsilon^{red}|_{\vartheta^{-1}(\gamma)},\iota(\Omega\times\{\gamma\}),0) = \deg(\nabla\Gamma|_{\vartheta^{-1}(\gamma)},\iota(\Omega\times\{\gamma\}),0) = \deg(\nabla\tilde{\Gamma},\Omega,0)\neq 0$$

对于较小的 ε 值。因此，对于固定的 γ，$L_\varepsilon^{red}\circ\iota(\cdot,\gamma)$ 在 $g(\gamma)\in\Omega$ 处有一个临界点。此外，我们可以选择 $g(\gamma)$，使得当 $\varepsilon\to 0$ 时，$g(\gamma)=\bar{g}+o(1)$。

If˜ 此外还是 $\mathcal{G}$上的 Morse 函数，因此根据隐函数定理，$g(\gamma)$是 $\gamma\in\mathcal{N}$的光滑函数。注意，我们可以通过条件 $g(\gamma)=\bar{g}+o(1)$，选择对所有 $\gamma\in\mathcal{N}$ 都定义的临界点全局分支$g(\gamma)$。因此，在 $\mathcal{M}$ 上寻找 L_ε^{red}的临界点，可以归结为寻找定义在 $\mathcal{N}$上的以下泛函的临界点。

$$\gamma\mapsto L_\varepsilon^{red}\circ\iota(g(\gamma),\gamma).$$

根据著名的 Lusternik-Schnierelman 理论 [41],，上述函数至少有 Cat(N) 个临界点，由此得证。□

˜注记 2.5。在定理 2.4的证明中，由于 Γ 是 Morse函数，˜ ˜
Γ 的任意临界点都是孤立的。因此，对于 Γ的每个临界点 $\bar{g}$，至少存在Cat($\mathcal{N}$) 个临界点（$g(\gamma)$ γ） $\in \mathcal{G}\times\mathcal{N}$，它们属于 L_ε^{red}，且每一个都满足 $g(\gamma) = \bar{g} + o(1)$，其中 $\varepsilon \to 0$。

情形2：ker $\nabla(\Gamma\circ\iota)(g,\gamma) \equiv T_{(g,\gamma)}(\mathcal{G}\times\mathcal{N})$ 对所有 $(g,\gamma) \in \mathcal{G}\times\mathcal{N}$

在这种情况下，我们有 $\Gamma\circ\iota(g,\gamma) \equiv constant$ 在 $\mathcal{G}\times\mathcal{N}$上。正如 [7], 中所指出的，我们需要进一步计算 L_ε^{red}的展开式中的项。为此，我们将（2.5）按 ε 的幂展开为

$$L_\varepsilon(z) = L_0(z) + \varepsilon\Gamma(z) + \varepsilon^2\Phi(z) + o(\varepsilon^2). \tag{2.11}$$

Note that $\Gamma\circ\iota(g,\gamma) \equiv constant$ on $\mathcal{G}\times\mathcal{N}$ is equivalent to $\Gamma(z) \equiv constant$ on $\mathcal{M}$. It follows that $\nabla\Gamma(z) \in \mathcal{W}_z := T_z\mathcal{M}^\perp$. Recall that $w_\varepsilon(z)$ is solution to the auxiliary equation $P_z\nabla L_\varepsilon(z+w) = 0$, hence we have $\nabla L_\varepsilon(z + w_\varepsilon(z)) \in T_z\mathcal{M}$. For a fixed $z \in \mathcal{M}$, using Taylor expansion, one sees

$$\begin{aligned}\nabla L_\varepsilon(z + w_\varepsilon(z)) &= \nabla L_0(z + w_\varepsilon(z)) + \varepsilon\nabla\Gamma(z + w_\varepsilon(z)) + o(\varepsilon)\\ &= \nabla^2 L_0(z)[w_\varepsilon(z)] + \varepsilon\nabla\Gamma(z) + \varepsilon\nabla^2\Gamma(z)[w_\varepsilon(z)] + o(\|w_\varepsilon(z)\|) + o(\varepsilon).\end{aligned}$$

然后，由公式 (2.7) 以及事实 $\nabla L_\varepsilon(z + w_\varepsilon(z)) \in T_z\mathcal{M}$，可得

$$\nabla^2 L_0(z)[w_\varepsilon(z)] + \varepsilon\nabla\Gamma(z) + o(\varepsilon) \in T_z\mathcal{M}.$$

因此我们得出

$$w_\varepsilon(z) = -\varepsilon K_z(\nabla\Gamma(z)) + o(\varepsilon),$$

其中， K_z 表示将 $\nabla^2 L_0(z)$ 限制在 $\mathcal{W}_z$上所得映射的逆。现在，我们可以将 $L_\varepsilon^{red}(z)$ 展开为：
$=L_\varepsilon(z + w_\varepsilon(z))$

$$\begin{aligned}L_\varepsilon^{red}(z) &= L_0(z) + \frac{1}{2}\nabla^2 L_0(z)[w_\varepsilon(z), w_\varepsilon(z)] + \varepsilon\Gamma(z) + \varepsilon\nabla\Gamma(z)[w_\varepsilon(z)] + \varepsilon^2\Phi(z) + o(\varepsilon^2)\\ &= L_0(z) + \varepsilon\Gamma(z) + \varepsilon^2\Big(\Phi(z) - \frac{1}{2}\langle K_z(\nabla\Gamma(z)), \nabla\Gamma(z)\rangle\Big) + o(\varepsilon^2).\end{aligned}$$

这里我们强调， $L_0(z)$ 和 $\Gamma(z)$ 都是 $\mathcal{M}$ 上的常数。因此，通过稍微改变记号，对于每个固定的 ε，上述展开都可以视为 (2.8) 中所考虑情形的一个类似结果。因此，类似于定理 2.3，我们有以下结果。

定理 2.6.设 L_0、 Γ、 $\Phi \in C^2(\mathcal{H}$、 $\mathbb{R})$ 如 (2.11)中所述，并假设($A1$)-($A5$)得到满足。如果 $\Gamma\circ\iota \equiv constant$ 在 $\mathcal{G}\times\mathcal{N}$ 上，且存在一个开有界子集 $U \subset \mathcal{G}$ 使得

$$\inf_{\gamma\in\mathcal{N}}\Big(\min_{\partial U}\hat{\Phi}\big|_{\vartheta^{-1}(\gamma)} - \min_{\overline{U}}\hat{\Phi}\big|_{\vartheta^{-1}(\gamma)}\Big) > 0 \quad or \quad \sup_{\gamma\in\mathcal{N}}\Big(\max_{\partial U}\hat{\Phi}\big|_{\vartheta^{-1}(\gamma)} - \max_{\overline{U}}\hat{\Phi}\big|_{\vartheta^{-1}(\gamma)}\Big) < 0,$$

其中 $\hat{}^{\Phi}\big|_{\vartheta^{-1}(\gamma)} = \hat{}\Phi\circ\iota(\cdot,\gamma)$，并且

$$\hat{\Phi}(z) := \Phi(z) - \frac{1}{2}\langle K_z(\nabla\Gamma(z)), \nabla\Gamma(z)\rangle \quad for\ z \in \mathcal{M}.$$

那么，当 $|\varepsilon|$ 很小时，泛函 L_ε 在 $\mathcal{M}$上有一个临界点。

最后但同样重要的是，类似于定理 2.4，我们有

定理 2.7. 设 $L_0, \Gamma, \Phi \in C^2(\mathcal{H}, \mathbb{R})$ 如 (2.11) 中所示，并假设满足 $(A1)$-$(A5)$。进一步假设 $\mathcal{N}$ 是单连通的。若 $\Gamma \circ \iota \equiv constant$ 在 $\mathcal{G} \times \mathcal{N}$ 上，并且对每个 $\gamma \in \mathcal{N}$，$\hat{}\,\hat{}$ 若 $\Phi|_{\vartheta^{-1}(\gamma)} = \Phi \circ \iota(\cdot, \gamma)$ 是 $\mathcal{G}$ 上的 Morse 函数，且存在一个有界开子集 $\Omega(\gamma) \subset \mathcal{G}\hat{}$，使得拓扑度 $\deg(\nabla \Phi|_{\vartheta^{-1}(\gamma)}\ \iota(\Omega(\gamma) \times \{\gamma\}), 0) \neq 0$。那么，当 $|\varepsilon|$ 足够小时，泛函 L_ε^{red} 在 $\mathcal{M}$ 上至少有 $\mathrm{Cat}(\mathcal{N})$ 个临界点。

证明。根据假设，对于每个 $\gamma \in \mathcal{N}$，存在一个临界点 $g(\gamma) \in \Omega(\gamma) \subset \mathcal{G}$。对于任意 $\gamma_0 \in \mathcal{N}$，根据莫尔斯条件，我们可以选取 γ_0 的一个小邻域，使得$g(\gamma)$在该邻域内关于 γ 光滑地变化。这些临界点的局部分支由于 $\mathcal{N}$ 的单连通性而可以全局延拓，从而得到一个在 $\mathcal{N}$ 上全局定义的光滑 $g(\gamma)$。然后证明过程与定理 2.4 中相同。 □

3 几何势的扰动

在本节中，我们处理问题（2.1），即，

$$D_{g_{\mathbb{R}^m}}\psi = (1 + \varepsilon a(x))|\psi|^{2^*-2}_{g_{\mathbb{R}^m}}\psi \quad \text{on } \mathbb{R}^m$$

其中 $2^* = \frac{2m}{m-1}$。相应的泛函可写成如下形式

$$\mathcal{J}(\psi) = \mathcal{J}_0(\psi) + \varepsilon\Gamma(\psi) \tag{3.1}$$

其中

$$\mathcal{J}_0(\psi) = \frac{1}{2}\int_{\mathbb{R}^m}(\psi, D_{g_{\mathbb{R}^m}}\psi)_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}} - \frac{1}{2^*}\int_{\mathbb{R}^m}|\psi|^{2^*}_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

and

$$\Gamma(\psi) = -\frac{1}{2^*}\int_{\mathbb{R}^m} a(x)|\psi|^{2^*}_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\ .$$

我们打算利用定理2.4来得到定理1.1。为此，我们需要详细了解(2.9)-(2.10)中提到的临界流形 $\mathcal{M}$ 上的泛函 Γ 的行为。

3.1 拓扑次数

回顾一下，对于函数 $H \in C^2(S^m)$，我们假设 H 在 $p_0 \in S^m$ 处取得其最小值，并记π_{p_0}：$S^m \setminus \{p_0\} \to \mathbb{R}^m$ 立体投影，然后函数 a 定义为 $a(x) := H(\pi^{-1}_{p_0}(x))$，其中 $x \in \mathbb{R}^m$。

在下文中，我们始终假设 $a = H \circ \pi^{-1}_{p_0}$对于某个函数 $H \in C^2(S^m)$，此外，条件 (H-1) 和 (H-2) 也得到满足。

引理 3.1。函数 a 具有以下性质：

(1) $a \in L^\infty(\mathbb{R}^m) \cap C^2(\mathbb{R}^m)$ *and there is a constant* $C_0 > 0$ *such that*

$$|\partial_i a(x)| \le C_0(1 + |x|^2)^{-1} \quad \textit{and} \quad |\partial_i\partial_j a(x)| \le C_0(1 + |x|^2)^{-\frac{3}{2}}.$$

where $\partial_i = \frac{\partial}{\partial x_i}$ *and* $\partial_i\partial_j = \frac{\partial^2}{\partial x_i \partial x_j}$.

(2) *a is a Morse function. The number of critical points of a on $\mathbb{R}^m$ is finite and $\Delta_{g_{\mathbb{R}^m}} a(x) \neq 0$ for $x \in Crit[a]$. Moreover*

$$\sum_{x\in Crit[a],\ \Delta_{g_{\mathbb{R}^m}} a(x)<0} (-1)^{\mathfrak{M}(a,x)} \neq (-1)^m$$

其中， $Crit[a]$ 表示 a 在 $\mathbb{R}^m$ 上的临界集，而 $\mathfrak{M}(a,x)$是 a在 x处的莫尔斯指标。

(3)存在 $R_0 > 0$ 使得 $\nabla a(x) \cdot x < 0$ 对于 $|x| \geq R_0$ 且

$$\int_{B_R} \nabla a(x) \cdot x \, d\,\mathrm{vol}_{g_{\mathbb{R}^m}} < 0$$

对于任意 $R > R_0$。

证明。关于（1）和（2），详细证明请参见 [27, 引理 4。1] 注意，（3）的类似陈述已经在 [28, 引理 3。2]中得到证明。尽管 [28,引理 3。2] 中的维数被固定为 $m = 3$，但同样的论证仍然适用，因此略去证明。 □

注意，对于 $\psi_{\lambda,\ \xi,\ \gamma} \in \mathcal{M}$ （见（2.10））,

$$\Gamma(\psi_{\lambda,\xi,\gamma}) = -\frac{m-1}{2m} \int_{\mathbb{R}^m} a(x) |\psi_{\lambda,\xi,\gamma}|_{g_{\mathbb{R}^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

and

$$|\psi_{\lambda,\xi,\gamma}|_{g_{\mathbb{R}^m}} = \frac{m^{\frac{m-1}{2}} \lambda^{\frac{m-1}{2}}}{\left(\lambda^2 + |x-\xi|^2\right)^{\frac{m-1}{2}}}.$$

于是显然， $\Gamma(\psi_{\lambda,\ \xi,\ \gamma})$ 与因子 $\gamma \in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$ 无关。因此，为了研究 $\Gamma(\psi_{\lambda,\ \xi,\ \gamma})$, 只需考虑（相差一个常数）

$$\Psi(\lambda,\xi) := \int_{\mathbb{R}^m} a(x) |\psi_{\lambda,\xi,\gamma}|_{g_{\mathbb{R}^m}}^{\frac{2m}{m-1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} = m^m \int_{\mathbb{R}^m} \frac{a(\lambda x + \xi)}{(1+|x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

其中最后一个等式由变量代换得到。现在，由引理 3.1 (1) 可容易验证，对于任意 $\xi \in \mathbb{R}^m$,

$$\lim_{\lambda\to 0^+} \Psi(\lambda,\xi) = C_1 a(\xi), \quad \text{where } C_1 = m^m \int_{\mathbb{R}^m} \frac{1}{(1+|x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}, \tag{3.2}$$

$$\partial_\lambda \Psi(\lambda,\xi) = m^m \int_{\mathbb{R}^m} \frac{\nabla a(\lambda x + \xi) \cdot x}{(1+|x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

and

$$\lim_{\lambda\to 0^+} \partial_\lambda \Psi(\lambda,\xi) = m^m \int_{\mathbb{R}^m} \frac{\nabla a(\xi) \cdot x}{(1+|x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} = 0 \tag{3.3}$$

基于上述观察，并借鉴 Ambrosetti 等人在 [5], 中的思路，我们旨在将 Ψ 在 $(-\infty,\infty)$ 上推广为 $\times \mathbb{R}^m$

$$\widetilde{\Psi}(0,\xi) = C_1 a(\xi), \quad \widetilde{\Psi}(\lambda,\xi) = \Psi(-\lambda,\xi) \text{ for } \lambda < 0.$$

~由 (3.2) 和 (3.3) 可知， Ψ 在 $(-\infty,\infty)$ 上具有 $C^1\times\mathbb{R}^m$，并且

$$\partial_\lambda\widetilde{\Psi}(0,\xi)=0\quad \text{for } \xi\in\mathbb{R}^m.$$

此外，我们有

$$\partial_i\widetilde{\Psi}(0,\xi)=\lim_{\lambda\to 0}m^m\int_{\mathbb{R}^m}\frac{\partial_i a(\lambda x+\xi)}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}=C_1\partial_i a(\xi)\quad \text{for } \xi\in\mathbb{R}^m.$$

因此，我们得到以下一一对应关系：

$$\xi\in Crit[a]\Longleftrightarrow(0,\xi)\in Crit[\widetilde{\Psi}] \tag{3.4}$$

Next, we will characterize the local index of $\widetilde{\Psi}$ at critical points of the form $(0,\xi)$ and evaluate the topological degree of $\widetilde{\Psi}$ in a bounded subset of $(0,\infty)\times\mathbb{R}^m$. In what follows (for the proofs of Lemma 3.2, Lemma 3.3 and Corollary 3.4), since the calculation is essentially the same as in [28, Section 3], we shall sketch most of the details for general case $m\geq 3$. Here, for the sake of completeness, we mainly specialize to the proofs for the case $m=2$, where some additional arguments are required.

~引理3.2。 Φ 形如$(0,\xi)$ $\in$~$(-\infty,\infty)\times\mathbb{R}^m$的临界点至多只有有限个。此外，这些临界点都是孤立的，并且 Ψ 在临界点 $(0,\xi)$ 处的局部指标为

$$\mathrm{ind}\left(\nabla\widetilde{\Phi},(0,\xi)\right)=\begin{cases}(-1)^{\mathfrak{M}(a,\xi)} & \textit{if } \Delta_{g_{\mathbb{R}^m}}a(\xi)>0,\\ (-1)^{\mathfrak{M}(a,\xi)+1} & \textit{if } \Delta_{g_{\mathbb{R}^m}}a(\xi)<0.\end{cases}$$

*Pro*的。对于 $m\geq 3$，由 [27, 第 7] 节中的计算可直接得出

$$\lim_{\lambda\to 0}\partial_\lambda^2\widetilde{\Psi}(\lambda,\xi)=C_2\Delta_{g_{\mathbb{R}^m}}a(\xi)\quad \text{where } C_2=m^{m-1}\int_{\mathbb{R}^m}\frac{|x|^2}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

$$\lim_{\lambda\to 0}\partial_i\partial_j\widetilde{\Psi}(\lambda,\xi)=C_1\partial_1\partial_j a(\xi)$$

and

$$\lim_{\lambda\to 0}\partial_i\partial_\lambda\widetilde{\Psi}(\lambda,\xi)=0$$

~因此， Φ 在 $(0,\xi)$ 处的 Hessian 矩阵为

$$\nabla^2\widetilde{\Phi}(0,\xi)=\begin{pmatrix}C_2\Delta_{g_{\mathbb{R}^m}}a(\xi) & O\\ O & C_1\nabla^2 a(\xi)\end{pmatrix}.$$

~若 $(0,\xi_0)\in Crit[\Phi]$，则可知

$$\widetilde{\Phi}(\lambda,\xi)\sim\widetilde{\Phi}(0,\xi_0)+C_2\Delta_{g_{\mathbb{R}^m}}a(\xi_0)\lambda^2+C_1\nabla^2 a(\xi_0)[\xi-\xi_0,\xi-\xi_0]$$

只要(λ,ξ)接近$(0,\xi_0)$即可。因此，$(0,\xi_0)$ 是孤立的，结论由指标的乘法性质得出。

~ ~对于 $m = 2$, $\partial_\lambda^2\Phi(0,\xi) = \lim_{\lambda\to 0}\partial_\lambda^2\Phi(\lambda,\xi)$ 不存在。因此，我们需要了解 $\partial_\lambda\Phi(\lambda,\xi)$ 在 $\lambda \to 0$ 时的详细~行为。

为此，对于 $\lambda > 0$，记作

$$\begin{aligned}\partial_\lambda\Phi(\lambda,\xi) &= 4\sum_{i=1}^{2}\int_{\mathbb{R}^2}\frac{\partial_i a(\lambda x+\xi)x_i}{(1+|x|^2)^2}d\,\mathrm{vol}_{g_{\mathbb{R}^2}}\\ &= 4\sum_{i=1}^{2}\int_{|x|\le\frac{1}{\lambda}}\frac{\partial_i a(\lambda x+\xi)x_i}{(1+|x|^2)^2}d\,\mathrm{vol}_{g_{\mathbb{R}^2}} + 4\sum_{i=1}^{2}\int_{|x|\ge\frac{1}{\lambda}}\frac{\partial_i a(\lambda x+\xi)x_i}{(1+|x|^2)^2}d\,\mathrm{vol}_{g_{\mathbb{R}^2}}\\ &= 4I_1(\lambda,\xi) + 4I_2(\lambda,\xi).\end{aligned}$$

注意，根据泰勒定理，$a \in C^2(\mathbb{R}^2)$，我们有

$$\partial_i a(\lambda x+\xi) = \partial_i a(\xi) + \lambda\nabla\partial_i a(\xi)\cdot x + o(|\lambda x|)$$

其中，$o(|\lambda x|) \to 0$ 作为 $|\lambda x| \to 0$。因此，我们推得

$$\begin{aligned}I_1(\lambda,\xi) &= \sum_{i=1}^{2}\int_{|x|\le\frac{1}{\lambda}}\frac{\partial_i a(\xi)x_i}{(1+|x|^2)^2}d\,\mathrm{vol}_{g_{\mathbb{R}^2}} + \sum_{i=j}^{2}\sum_{i=1}^{2}\int_{|x|\le\frac{1}{\lambda}}\frac{\lambda\partial_j\partial_i a(\xi)x_j x_i}{(1+|x|^2)^2}d\,\mathrm{vol}_{g_{\mathbb{R}^2}}\\ &\quad + o\Big(\int_{|x|\le\frac{1}{\lambda}}\frac{\lambda|x|^2}{(1+|x|^2)^2}d\,\mathrm{vol}_{g_{\mathbb{R}^2}}\Big)\\ &= \frac{\lambda}{2}\Delta_{g_{\mathbb{R}^2}}a(\xi)\int_{|x|\le\frac{1}{\lambda}}\frac{|x|^2}{(1+|x|^2)^2}d\,\mathrm{vol}_{g_{\mathbb{R}^2}} + o(\lambda\ln\lambda)\\ &= -\pi\lambda\ln\lambda\big(\Delta_{g_{\mathbb{R}^2}}a(\xi) + o(1)\big) + o(\lambda)\end{aligned}$$

并且，根据引理 3.1 (1),

$$I_2(\lambda,\xi) = O\Big(\int_{|x|\ge\frac{1}{\lambda}}\frac{|x|}{(1+|x|^2)^2}d\,\mathrm{vol}_{g_{\mathbb{R}^2}}\Big) = O(\lambda)$$

作为 $\lambda \to 0$。因此，我们有

$$\partial_\lambda\Phi(\lambda,\xi) = -4\pi\lambda\ln\lambda\big(\Delta_{g_{\mathbb{R}^2}}a(\xi) + o(1)\big)$$

~作为 $\lambda \to 0$。如果$(0,\xi_0) \in Crit[\Psi]$，由于 $\Delta_{g_{\mathbb{R}^2}}a(\xi_0) \neq 0$，显然该函数

$$\lambda \mapsto -4\pi\Delta_{g_{\mathbb{R}^2}}a(\xi_0)\lambda\ln\lambda$$

与函数 $\lambda \mapsto \Delta_{g_{\mathbb{R}^2}}a(\xi_0)\lambda$ 同伦，且当 λ 接近 0 时，该同伦过程中不会取值 0。因此

$$\mathrm{ind}\left(\partial_\lambda\widetilde{\Psi}(\cdot,\xi_0),0\right) = \begin{cases}1 & \text{if } \Delta_{g_{\mathbb{R}^2}}a(\xi_0) > 0,\\ -1 & \text{if } \Delta_{g_{\mathbb{R}^2}}a(\xi_0) < 0.\end{cases}$$

因此，由于指标的乘法性质，该断言立即成立。

□

引理 3.3。对于 $m \geq 2$，存在 $R > 0$ 使得 $Crit[\widetilde{\Psi}] \subset B_R^{m+1} := \{(\lambda,\xi) \in (-\infty,\infty) \times \mathbb{R}^m : \lambda^2 + |\xi|^2 \leq R^2\}$ 并且

$$\deg(\nabla\widetilde{\Psi}, B_R^{m+1}, 0) = (-1)^{m+1}.$$

证明。尽管 [28] 中的论证毫无困难地适用于一般情形 $m \geq 2$，为求完整起见，我们将证明过程概述如下。

我们首先证明存在 $R > 0$，使得

$$\nabla\widetilde{\Psi}(\lambda,\xi)\cdot(\lambda,\xi) < 0 \tag{3.5}$$

对于 $(\lambda,\xi) \in (-\infty,\infty) \times \mathbb{R}^m$，其中 $\lambda^2 + |\xi|^2 \geq R^2$。

为此，我们来计算

$$\begin{aligned}\nabla\widetilde{\Psi}(\lambda,\xi)\cdot(\lambda,\xi) &= m^m \int_{\mathbb{R}^m} \frac{\nabla a(\lambda x+\xi)\cdot(\lambda x+\xi)}{(1+|x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \\ &= m^m \lambda^{-m} \int_{\mathbb{R}^m} \frac{\lambda^{2m}\nabla a(x)\cdot x}{(\lambda^2+|x-\xi|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\,. \end{aligned} \tag{3.6}$$

注意，对于任意 $r > 0$，我们可以将上述积分分解为

$$\begin{aligned}\int_{\mathbb{R}^m} \frac{\lambda^{2m}\nabla a(x)\cdot x}{(\lambda^2+|x-\xi|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} &= \int_{|x|\leq r} \frac{\lambda^{2m}\nabla a(x)\cdot x}{(\lambda^2+|x-\xi|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \\ &\quad + \int_{|x|\geq r} \frac{\lambda^{2m}\nabla a(x)\cdot x}{(\lambda^2+|x-\xi|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \\ &= I_1(r,\lambda,\xi) + I_2(r,\lambda,\xi)\end{aligned}$$

容易验证，根据引理 3.1 (3) 的前半部分，$I_2(r,\lambda,\xi) < 0$ 当 $r > R_0$ 时成立。另一方面，我们可以将 $I_1(r,\lambda,\xi)$ 分解为

$$\begin{aligned} I_1(r,\lambda,\xi) &= \int_{|x|\leq r} \frac{\lambda^{2m}(\nabla a(x)\cdot x)_+}{(\lambda^2+|x-\xi|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} - \int_{|x|\leq r} \frac{\lambda^{2m}(\nabla a(x)\cdot x)_-}{(\lambda^2+|x-\xi|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \\ &\leq \max_{|x|\leq r} \frac{\lambda^{2m}}{(\lambda^2+|x-\xi|^2)^m} \int_{|x|\leq r} (\nabla a(x)\cdot x)_+ d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \\ &\quad - \min_{|x|\leq r} \frac{\lambda^{2m}}{(\lambda^2+|x-\xi|^2)^m} \int_{|x|\leq r} (\nabla a(x)\cdot x)_- d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \end{aligned}$$

其中，$(\nabla a(x)\cdot x)_+$ 和 $(\nabla a(x)\cdot x)_-$ 分别是 $\nabla a(x)\cdot x$ 的正部和负部。接下来，根据引理 3.1 (3) 的后半部分，我们指出

$$\delta \int_{|x|\leq r} (\nabla a(x)\cdot x)_+ d\,\mathrm{vol}_{g_{\mathbb{R}^m}} + \int_{|x|\leq r} \nabla a(x)\cdot x\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}} < 0 \tag{3.7}$$

条件是 $r \geq R_0$ 和 δ 足够小且固定。令

$$M(r,\lambda,\xi) = \max_{|x|\leq r} \frac{\lambda^{2m}}{(\lambda^2+|x-\xi|^2)^m} = \begin{cases} \dfrac{\lambda^{2m}}{(\lambda^2+(|\xi|-r)^2)^m} & \text{if } |\xi| > r \\ 1 & \text{if } |\xi| \leq r \end{cases}$$

and

$$m(r,\lambda,\xi) = \min_{|x|\leq r} \frac{\lambda^{2m}}{(\lambda^2+|x-\xi|^2)^m} = \frac{\lambda^{2m}}{(\lambda^2+(r+|\xi|)^2)^m},$$

我们发现 $\lim_{\lambda^2+|\xi|^2\to\infty} \frac{M}{m}(r,\lambda,\xi)\ (r,\lambda,\xi) = 1$。因此，存在 $R_1 > 0$ 使得

$$1 \leq \frac{M(r,\lambda,\xi)}{m(r,\lambda,\xi)} \leq 1+\delta$$

对于 $\lambda^2+|\xi|^2 \geq R_1$。现在，由 (3.7) 可得以下估计。

$$\begin{aligned} I_1(r,\lambda,\xi) &\leq M(r,\lambda,\xi)\int_{|x|\leq r}(\nabla a(x)\cdot x)_+ d\,\mathrm{vol}_{g_{\mathbb{R}^m}} - m(r,\lambda,\xi)\int_{|x|\leq r}(\nabla a(x)\cdot x)_- d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \\ &\leq m(r,\lambda,\xi)\Big(\delta\int_{|x|\leq r}(\nabla a(x)\cdot x)_+ d\,\mathrm{vol}_{g_{\mathbb{R}^m}} + \int_{|x|\leq r}\nabla a(x)\cdot x d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\Big) < 0, \end{aligned}$$

这表明（3.5）。

Since (3.5) suggests $Crit[\widetilde{\Psi}] \subset B_R^{m+1}$ for some $R > 0$ large, it remains to evaluate the topological degree of $\widetilde{\Psi}$. Note that (3.5) forces $-t(\lambda,\xi)+(1-t)\nabla\widetilde{\Psi}(\lambda,\xi) \neq 0$ for any $(\lambda,\xi) \in \partial B_R^{m+1}$ and $t \in [0,1]$. It follows that $\nabla\widetilde{\Psi}(\cdot,\cdot)$ is homotopic to $-id$ on B_R^{m+1} through homotopy which does not take value 0 on ∂B_R^{m+1} and, particularly,

$$\deg(\nabla\widetilde{\Psi}, B_R^{m+1}, 0) = \deg(-id, B_R^{m+1}, 0) = (-1)^{m+1}.$$

这就完成了证明。 □

~为了求得 Ψ 在 $(0,\infty)\times\mathbb{R}^m$上的拓扑度，我们将 $Crit[\Psi]$ 分解为

$$Crit[\widetilde{\Psi}] = Crit_+[\widetilde{\Psi}] \cup Crit_0[\widetilde{\Psi}] \cup Crit_-[\widetilde{\Psi}]$$

其中

$$Crit_\pm[\widetilde{\Psi}] = \big\{(\lambda,\xi) \in Crit[\widetilde{\Psi}] : \pm\lambda > 0\big\} \quad \text{and} \quad Crit_0[\widetilde{\Psi}] = \big\{(\lambda,\xi) \in Crit[\widetilde{\Psi}] : \lambda = 0\big\}.$$

~由 Ψ的对称性，我们有

$$Crit_-[\widetilde{\Psi}] = \big\{(\lambda,\xi) :\ (-\lambda,\xi) \in Crit_+[\widetilde{\Psi}]\big\}.$$

感谢 Le　　根据引理3.2 和3.3，我们可以找到一个有界域 $\Omega_+ \subset (0,\infty)\times\mathbb{R}^m$ 使得

$$\Omega_+ \subset B_R^{m+1} \quad \text{and} \quad Crit_+[\widetilde{\Psi}] \subset \Omega_+.$$

作为引理 3.3 的推论，我们有

推论 3.4. deg(∇Ψ, Ω+, 0) $\neq 0$。

~ ~证明。由于 $\Psi = \Psi|_{(0,\infty)\times\mathbb{R}^m}$，只需计算 deg(∇Ψ, Ω+, 0)。

反设 deg(∇ $\widetilde{\Psi}$, Ω+, 0) $= 0$。令

$$\Omega_- = \big\{(\lambda,\xi) : (-\lambda,\xi) \in \Omega_+\big\}.$$

~ ~那么，显然 $Crit_-[\Psi] \subset \Omega_-$，并且根据假设，deg(∇Ψ,Ω−,0) $= 0$。由度的可加性，根据引理 3.2 和 3.3，我们可以看出

$$\begin{aligned}
(-1)^{m+1} &= \deg(\nabla\widetilde{\Psi}, B_R^{m+1}, 0) \\
&= \deg(\nabla\widetilde{\Psi}, \Omega_+, 0) + \deg(\nabla\widetilde{\Psi}, \Omega_-, 0) + \deg(\nabla\widetilde{\Psi}, B_R^{m+1} \setminus (\Omega_+ \cup \Omega_-), 0) \\
&= 0 + 0 + \sum_{(0,\xi)\in Crit_0[\widetilde{\Psi}]} \operatorname{ind}\big(\nabla\widetilde{\Psi}, (0,\xi)\big) \\
&= \sum_{\xi\in Crit[a],\, \Delta_{g_{\mathbb{R}^m}} a(\xi)>0} \operatorname{ind}(\nabla a, \xi) - \sum_{\xi\in Crit[a],\, \Delta_{g_{\mathbb{R}^m}} a(\xi)<0} \operatorname{ind}(\nabla a, \xi). \qquad (3.8)
\end{aligned}$$

另一方面，由引理 3.1 (3)，我们可以仿照引理 3.3 的证明进行推导，得到

$$\deg(\nabla a, B_R, 0) = (-1)^m$$

其中 $B_R = \{x \in \mathbb{R}^m : \ |x| \le R\}$。由于

$$\deg(\nabla a, B_R, 0) = \sum_{\xi\in Crit[a],\, \Delta_{g_{\mathbb{R}^m}} a(\xi)>0} \operatorname{ind}(\nabla a, \xi) + \sum_{\xi\in Crit[a],\, \Delta_{g_{\mathbb{R}^m}} a(\xi)<0} \operatorname{ind}(\nabla a, \xi),$$

由 (3.8) 可得

$$\sum_{\xi\in Crit[a],\, \Delta_{g_{\mathbb{R}^m}} a(\xi)>0} \operatorname{ind}(\nabla a, \xi) = 0 \quad \text{and} \quad \sum_{\xi\in Crit[a],\, \Delta_{g_{\mathbb{R}^m}} a(\xi)<0} \operatorname{ind}(\nabla a, \xi) = (-1)^m.$$

因此，我们有

$$\sum_{x\in Crit[a],\ \Delta_{g_{\mathbb{R}^m}} a(x)<0} (-1)^{\mathfrak{M}(a,x)} = \sum_{\xi\in Crit[a],\, \Delta_{g_{\mathbb{R}^m}} a(\xi)<0} \operatorname{ind}(\nabla a, \xi) = (-1)^m$$

这与引理 3.1 (2) 矛盾。 □

3.2 主要结果的证明

对于 $l \in \mathbb{N}$，令 a_1、...、$a_l \in L^\infty(\mathbb{R}^m) \cap C^2(\mathbb{R}^m)$为满足引理 3.1 中所述性质的函数。对于 z_1, $\ldots z_l \in \mathbb{R}^m$，我们考虑

$$a(x) = \sum_{i=1}^{l} a_i(x - z_i).$$

那么对于 $(\lambda,\xi) \in (0,\infty) \times \mathbb{R}^m$ 函数 Ψ 取以下形式

$$\Psi(\lambda,\xi) = \sum_{i=1}^{l} \Psi_i(\lambda,\xi)$$

其中

$$\Psi_i(\lambda,\xi) = m^m \int_{\mathbb{R}^m} \frac{a_i(\lambda x + \xi - z_i)}{(1+|x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

注意， Ψ_i 与变量 $\gamma \in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$ 无关。

上一节中的证明表明，存在有界域 $\Omega_i \subset (0,\infty)\times\mathbb{R}^m$, $i=1,\ldots,l$，使得 $Crit_+[\Psi_i] \subset \Omega_i(z_i)$ 且

$$\deg(\nabla\Psi_i, \Omega_i(z_i), 0) \neq 0$$

其中 $Crit_+[\Psi_i] = \{(\lambda,\xi) : \nabla\Psi_i(\lambda,\xi) = 0 \text{ 和 } \lambda > 0\}$ 和 $\Omega_i(z_i) = \{(\lambda,x) : (\lambda, x - z_i) \in \Omega_i\}$。因此，为了应用定理 2.4，我们还需完成以下任务，

$(T1)$ $\nabla\Psi_i(\lambda,\xi) \to 0$ 作为 $\lambda + |\xi - z_i| \to \infty$;

$(T2)$ 对于 $a_i (i = 1, \ldots, l)$ 的一般选择， Ψ_i 是 (λ,ξ) 的 Morse 函数。

注意到，如果 z_1、……、 z_l 彼此相距很远，则 $\Omega_i(z_i) \cap \Omega_j(z_j) = \emptyset$ 只要 $i \neq j$。因此，如果上述两个陈述得到证明，则 Ψ 在每个 $\Omega_i(z_i)$ 中至少有一个临界点，只要 $\min_{1\le i\neq j\le l} |z_i - z_j|$ 足够大。然后，回忆一下

$$\Gamma(\psi_{\lambda,\xi,\gamma}) = -\frac{m-1}{2m}\Psi(\lambda,\xi) \quad \text{for } \psi_{\lambda,\xi,\gamma} \in \mathcal{M}$$

and

$$\mathcal{J}(\psi) = \mathcal{J}_0(\psi) + \varepsilon\Gamma(\psi) \quad \text{for } \psi \in \mathscr{D}^{1/2}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m)).$$

By applying Theorem 2.4 and Remark 2.5, the functional $\mathcal{J}$ has at least $l \cdot \mathrm{Cat}\left(S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)\right)$ critical points. However, these critical points may appear as an S^1-orbit (or S^3-orbit when $m = 2,3,4 \pmod 8$) of a single solution. To find non-S^1-equivalent (or S^3-equivalent) solutions, we observe that, due to the S^1-equivariance (or S^3-equivariance) property of Eq. (2.6), our construction of $w_\varepsilon = w_\varepsilon(z)$ can be done equivariantly with respect to these actions. Namely, $w_\varepsilon(z)$ satisfies $w_\varepsilon(\tau \star z) = \tau \star w_\varepsilon(z)$ for $z \in \mathcal{M}$ and $\tau \in S^1$ (or $\tau \in S^3$), where "$\star$" stands for the corresponding group action. Then the reduced functional L_ε^{red} is S^1 (or S^3)-invariant, $L_\varepsilon^{red} \circ \iota(g, \tau\star\gamma) = L_\varepsilon^{red} \circ \iota(g,\gamma)$ for all $(g,\gamma) \in \mathcal{G}\times\mathcal{N}$ and $\tau \in S^1$ (or $\tau \in S^3$ when $m = 2,3,4$ (mod 8)). Therefore, L_ε^{red} can be reduced to be defined on $\mathcal{G} \times \mathcal{N}/S^1$ (or $\mathcal{G}\times\mathcal{N}/S^3$ when $m = 2,3,4$ (mod 8)). We define $\hat{L}_\varepsilon^{red}(g,[\gamma]) = L_\varepsilon^{red}(g,\gamma)$, where $[\gamma] \in \mathcal{N}/S^1$ (or $\mathcal{N}/S^3$) is the equivalence class represented by $\gamma \in \mathcal{N}$. Since $\mathcal{N} = S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$ is homeomorphic to the unit sphere in $\mathbb{C}^{2^{[\frac{m}{2}]}}$ (and in $\mathbb{H}^{2^{[\frac{m}{2}]-1}}$ when $m = 2,3,4$ (mod 8)), $\mathcal{N}/S^1 \cong \mathbb{C}P^{2^{[\frac{m}{2}]}-1}$ ($\mathcal{N}/S^3 \cong \mathbb{H}P^{2^{[\frac{m}{2}]-1}-1}$ when $m = 2,3,4$ (mod 8)). To estimate the Lusternik-Schnirelmann category of these quotients, recall that the category of a topological space X is estimated from below by the cuplength, $\mathrm{Cat}(X) \ge \mathrm{CL}(X) + 1$. Here the cuplength of X is defined as the greatest integer $k \ge 0$ such that there exist $\omega_1,\ldots,\omega_k \in \mathrm{H}^*(X;\mathbb{Z})$ so that $\deg\omega_j > 0$ and $\omega_1 \cup \cdots \cup \omega_k \neq 0$.

The cohomology rings of $\mathbb{C}P^{2^{[\frac{m}{2}]}-1}$ and $\mathbb{H}P^{2^{[\frac{m}{2}]-1}-1}$ are respectively isomorphic to truncated polynomial rings $\mathrm{H}^*(\mathbb{C}P^{2^{[\frac{m}{2}]}-1};\mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{2^{[\frac{m}{2}]}})$ and $\mathrm{H}^*(\mathbb{H}P^{2^{[\frac{m}{2}]-1}-1};Z) = \mathbb{Z}[\beta]/(\beta^{2^{[\frac{m}{2}]-1}})$, where $\alpha \in H^2(\mathbb{C}P^{2^{[\frac{m}{2}]}-1};\mathbb{Z})$ and $\beta \in H^4(\mathbb{H}P^{2^{[\frac{m}{2}]-1}-1};Z)$. Thus the cuplength of the quotient $\mathcal{N}/S^1$ and $\mathcal{N}/S^3$ are $\mathrm{CL}(\mathcal{N}/S^1) = 2^{[\frac{m}{2}]} - 1$ and $\mathrm{CL}(\mathcal{N}/S^3) = 2^{[\frac{m}{2}]-1} - 1$ when $m = 2,3,4$ (mod 8). Therefore, we have $\mathrm{Cat}(\mathcal{N}/S^1) \geq 2^{[\frac{m}{2}]}$ and $\mathrm{Cat}(\mathcal{N}/S^3) \geq 2^{[\frac{m}{2}]-1}$ when $m = 2,3,4$ (mod 8). From this, we conclude that the functional $\mathcal{J}$ has at least $2^{[\frac{m}{2}]}l$ critical points whose S^1-orbits are different in the general case $m \geq 2$ and when $m = 2,3,4$ (mod 8) there are at least $2^{[\frac{m}{2}]-1}l$ critical points whose S^3-orbits are different.

为了研究这些解的零点集，我们首先根据备注 2.5注意到：设 ψ_ε 为上述得到的解之一，则 ψ_ε 收敛于某个 $\psi_{\lambda,\xi,\gamma} \in \mathcal{M}$，当 $\varepsilon \to 0$ 时。注意，

$$\lim_{|x|\to\infty} f^{-\frac{m-1}{2}}(x)|\psi_{\lambda,\xi,\gamma}(x)|_{g_{\mathbb{R}^m}} = \left(\frac{m}{2}\right)^{\frac{m-1}{2}} \lambda^{\frac{m-1}{2}} > 0, \quad \text{where } f(x) = \frac{2}{1+|x|^2}$$

由此可知 $\psi_{\lambda,\xi,\gamma}$ 对应于方程的一个解 φ_0

$$D_{g_{S^m}}\varphi = |\varphi|^{2^*-2}_{g_{S^m}}\varphi \quad \text{on } S^m$$

通过立体投影（见第 2节），将$|\varphi_0|_{g_{S^m}} > 0$ 在 S^m上。因此，由于 $\{\psi_\varepsilon\}$ 的收敛性，如果我们将 ψ_ε 拉回到 S^m 上，我们有

$$\left|(f^{-\frac{m-1}{2}}\psi_\varepsilon)\circ\pi_p\right|_{g_{S^m}} = |\varphi_0|_{g_{S^m}} + o_\varepsilon(1) > 0$$

作为 $\varepsilon \to 0$。因此，方程 (1.7) 的相应解在 S^m 上没有零点。

在下文中，上述陈述（$T1$）和（$T2$）的证明将在引理 3.5 和命题 3.6 中给出。

引理3.5。对于每个 i、$\nabla\Psi_i$（λ、ξ）$\to 0$ 作为 $\lambda + |\xi - z_i| \to \infty$。

证明。此 由引理 3.1（1）和控制收敛定理可轻易推出。

□

命题 3.6. 对于 $H \in C^2(S^m)$和 $(\lambda,\xi) \in (0,\infty) \times \mathbb{R}^m$，我们定义

$$\Psi(\lambda,\xi;H) = m^m \int_{\mathbb{R}^m} \frac{a(\lambda x+\xi)}{(1+|x|^2)^m} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

其中 $a = H\circ\pi_p^{-1}$(π_p： $S^m \setminus \{p\} \to \mathbb{R}^m$ 是相对于固定的 $p \in S^m$ 的立体投影）。那么存在一个残差子集 $\mathcal{A} \subset C^2(S^m)$，使得对于所有 $H \in \mathcal{A}$， Ψ 是 $(0,\infty) \times \mathbb{R}^m$上的 Morse 函数。

回顾一下，巴拿赫空间中的子集 $\mathcal{S}$ $\mathcal{B}$ 是剩余集，如果 $\mathcal{S}$ 包含 $\mathcal{B}$的开稠子集的可数交。根据贝尔纲定理，剩余集在 $\mathcal{B}$中稠密。

为了证明上述命题，我们需要做一些准备。首先，我们定义

$$\mathcal{F}(H,\lambda,\xi) = \nabla\Psi(\lambda,\xi;H) \in \mathbb{R}^{m+1}.$$

让我们考虑以下普适临界集

$$\mathcal{C} = \big\{(H,\lambda,\xi) \in C^2(S^m)\times(0,\infty)\times\mathbb{R}^m : \mathcal{F}(H,\lambda,\xi) = 0\big\},$$

where $C^2(S^m)\times(0,\infty)\times\mathbb{R}^m$ is a separable Banach manifold modeled by $C^2(S^m)\times\mathbb{R}\times\mathbb{R}^m$ with a norm $\|(K,\mu,\zeta)\| = \|K\|_{C^2(S^m)}+|\mu|+|\zeta|$ for $(K,\mu,\zeta)\in T_{(H,\lambda,\xi)}(C^2(S^m)\times(0,\infty)\times\mathbb{R}^m) = C^2(S^m)\times\mathbb{R}\times\mathbb{R}^m$. Note that $(H,\lambda,\xi)\in\mathcal{C}$ if and only if $(\lambda,\xi)\in(o,\infty)\times\mathbb{R}^m$ is a critical point of $\Psi(\cdot,\cdot;H)$.

引理 3.7。 $0\in\mathbb{R}^{m+1}$ 是 $\mathcal{F}$的正则值，也就是说，导数

$$\nabla\mathcal{F}(H,\lambda,\xi) : C^2(S^m)\times\mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^{m+1}$$

在任意 $(H,\lambda,\xi)\in\mathcal{C}$ 处是满射。

证明。令 $(H,\lambda,\xi)\in\mathcal{C}$，我们有

$$\mathcal{F}(H,\lambda,\xi) = \Big(m^m\int_{\mathbb{R}^m}\frac{\nabla a(\lambda x+\xi)\cdot x}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}},\ m^m\int_{\mathbb{R}^m}\frac{\nabla a(\lambda x+\xi)}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\Big)\in\mathbb{R}\times\mathbb{R}^m.$$

我们将证明， $\nabla\mathcal{F}$ 在子空间 $C^2(S^m)\times\{0\}\times\{0\}$ 上的限制是满射。注意，

$$\begin{aligned}\nabla\mathcal{F}(H,\lambda,\xi)[K,0,0] &= \Big(m^m\int_{\mathbb{R}^m}\frac{\nabla b(\lambda x+\xi)\cdot x}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}},\ m^m\int_{\mathbb{R}^m}\frac{\nabla b(\lambda x+\xi)}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\Big)\\ &= \mathcal{F}(K,\lambda,\xi)\end{aligned}$$

其中 $b = K\circ\pi_p^{-1}$对于 $K\in C^2(S^m)$。

反设 $C^2(S^m)\times\{0\}\times\{0\}$ 经由 $\nabla\mathcal{F}(H,\lambda,\xi)$ 映射后的像是 $\mathbb{R}^{m+1}=\mathbb{R}\times\mathbb{R}^m$ 的真子空间。那么存在一个非零向量 $(\lambda_0,\xi_0)\in\mathbb{R}\times\mathbb{R}^m$，满足$\xi_0=(\xi_1,\dots,\xi_m)\in\mathbb{R}^m$，它消去该像。也就是说，

$$0 = \nabla\mathcal{F}(H,\lambda,\xi)[K,0,0]\cdot(\lambda_0,\xi_0) = m^m\int_{\mathbb{R}^m}\frac{\nabla b(\lambda x+\xi)\cdot(\lambda_0 x+\xi_0)}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}} \tag{3.9}$$

对于任意的 $K\in C^2(S^m)$ 和 $b = K\circ\pi_p^{-1}$。

For $1\le i\le m+1$, let $\beta_i\in C^\infty(S^m)$ be a smooth function obtained by restricting the i-th coordinate function of $\mathbb{R}^{m+1}$ on S^m, i.e., $\beta_i(x) = x_i|_{S^m}$ where $x=(x_1,\dots,x_{m+1})$. We define $K_i\in C^\infty(S^m)$ by $K_i(x) = \beta_i\big(\pi_p^{-1}(\frac{\pi_p(x)-\xi}{\lambda})\big)$. In other words, K_i is the i-th component of the conformal diffeomorphism of S^m which is $\mathbb{R}^m\ni x\mapsto\frac{x-\xi}{\lambda}$ in term of the identification $S^m\cong\mathbb{R}^m\cup\{\infty\}$ by the stereographic projection π_p. Then we have $b_i(x) = K_i\circ\pi_p^{-1}(x) = \beta_i\big(\pi_p^{-1}(\frac{x-\xi}{\lambda})\big)$ for $x\in\mathbb{R}^m$.

现在，我们将 K_i b_i 3 9 $\nabla b_i\ x\ =\frac{1}{\lambda}\nabla\ \beta_i\circ\pi_p^{-1}\ \ \frac{x-\xi}{\lambda}$ $\nabla b_i\ \lambda x+$ 和 代入 (.) 。由于 () () () 且 () () ()，我们得到 $\xi\ =\frac{1}{\lambda}\nabla\ \beta_i\circ\pi_p^{-1}\ \ x$

$$\int_{\mathbb{R}^m}\frac{\nabla(\beta_i\circ\pi_p^{-1})(x)\cdot(\lambda_0 x+\xi_0)}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}} = 0 \tag{3.10}$$

对于 $i=1,\ .....,\ m+1$。

Note that $\beta_i\circ\pi_p^{-1}(x)=\frac{2x_i}{1+|x|^2}$ for $i=1,\dots,m$ and $\beta_{m+1}\circ\pi_p^{-1}(x)=\frac{|x|^2-1}{1+|x|^2}$ (here, without loss of generality, we assume p is the north pole). We first take $i=m+1$. Since $\nabla(\beta_{m+1}\circ\pi_p^{-1})(x)=\frac{4x}{(1+|x|^2)^2}$, (3.10) gives

$$0=\int_{\mathbb{R}^m}\frac{4x\cdot(\lambda_0 x+\xi_0)}{(1+|x|^2)^{m+2}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}=\lambda_0\int_{\mathbb{R}^m}\frac{4|x|^2}{(1+|x|^2)^{m+2}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}},$$

因此 $\lambda_0=0$。

对于 $1\le i\le m$，我们有 $\partial_j(\beta_i\circ\pi_p^{-1})(x)=\frac{2\delta_{ij}}{1+|x|^2}-\frac{4x_ix_j}{1+|x|^2()^2}$。因此，由 (3.10) 和 $\lambda_0=0$ 可得

$$\begin{aligned}0&=\int_{\mathbb{R}^m}\frac{1}{(1+|x|^2)^m}\Big[\Big(\frac{2}{1+|x|^2}-\frac{4x_i^2}{(1+|x|^2)^2}\Big)\xi_i-\sum_{j\neq i}\frac{4x_ix_j\xi_j}{(1+|x|^2)^2}\Big]d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\&=\xi_i\int_{\mathbb{R}^m}\frac{2(1+|x|^2-2x_i^2)}{(1+|x|^2)^{m+2}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.\end{aligned}$$

由于

$$\int_{\mathbb{R}^m}\frac{x_i^2}{(1+|x|^2)^{m+2}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}=\frac{1}{m}\int_{\mathbb{R}^m}\frac{|x|^2}{(1+|x|^2)^{m+2}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}<\frac{1}{2}\int_{\mathbb{R}^m}\frac{1}{(1+|x|^2)^{m+1}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}},$$

我们有 $\xi_i=0$ 对于 $i=1,\dots,m$。因此我们有 $(\lambda_0,\xi_0)=0$，这是一个矛盾。因此 $\nabla\mathcal{F}$ 到子空间 $C^2(S^m)\times\{0\}\times\{0\}$ 的限制映射是满射。 □

由于 $\nabla\mathcal{F}(H,\lambda,\xi)$的核对于 $(H,\lambda,\xi)\in\mathcal{C}$ 具有一个有补子空间（因为其余维数是有限的），根据逆函数定理，$\mathcal{C}\subset C^2(S^m)\times(0,\infty)\times\mathbb{R}^m$ 是一个C^1-子流形。

令 $\mathcal{P}\colon\mathcal{C}\to C^2(S^m)$ 为投影，即 $\mathcal{P}(H,\lambda,\xi)=H$。接下来我们证明：

引理 3.8. $\mathcal{P}\colon\ \mathcal{C}\to C^2(S^m)$ 是一个指标为 0 的 Fredholm 映射。

证明。我们首先注意到，$\mathcal{C}$ 在（H，λ，ξ）处的切空间 $\in\mathcal{C}$ 是

$$T_{(H,\lambda,\xi)}\mathcal{C}=\big\{(K,\mu,\zeta)\in C^2(S^m)\times(0,\infty)\times\mathbb{R}^m:\ \nabla\mathcal{F}(H,\lambda,\xi)[K,\mu,\zeta]=0\big\}.$$

由于 $\nabla\mathcal{P}(H,\lambda,\xi)[K,\mu,\zeta]=K$对于 $(K,\mu,\zeta)\in T_{(H,\lambda,\xi)}\mathcal{C}$，我们看到 $(K,\mu,\zeta)\in\ker\nabla P(H,\lambda,\xi)$ 当且仅当 $K=0$，因此

$$\ker\nabla P(H,\lambda,\xi)\cong\ker\nabla\mathcal{F}(H,\lambda,\xi)[0,\cdot,\cdot]\big|_{\{0\}\times\mathbb{R}\times\mathbb{R}^m}.$$

另一方面，定义一个映射

$$\begin{aligned}\mathcal{Z}:\ C^2(S^m)&\longrightarrow\mathbb{R}^{m+1}/\operatorname{im}\nabla\mathcal{F}(H,\lambda,\xi)[0,\cdot,\cdot]\big|_{\{0\}\times\mathbb{R}\times\mathbb{R}^m}\\K&\longmapsto[\mathcal{F}(K,\lambda,\xi)]\end{aligned}$$

其中，$[v]$ 表示商空间中向量 $v\in\mathbb{R}^{m+1}$ 的等价类。然后，根据事实

$$\nabla\mathcal{F}(H,\lambda,\xi)[K,\mu,\zeta]=\mathcal{F}(K,\lambda,\xi)+\nabla\mathcal{F}(H,\lambda,\xi)[0,\mu,\zeta]$$

for all $(K,\mu,\zeta)\in C^2(S^m)\times\mathbb{R}\times\mathbb{R}^m$, it follows that $\ker\mathcal{Z}=\operatorname{im}\nabla\mathcal{P}(H,\lambda,\xi)$. To proceed, let us claim that $\mathcal{Z}$ is surjective. In fact, since $\nabla\mathcal{F}(H,\lambda,\xi)$ is surjective by Lemma 3.7, we see that for any vector $v\in\mathbb{R}^{m+1}$ there exists $(K,\mu,\zeta)\in C^2(S^m)\times\mathbb{R}\times\mathbb{R}^m$ such that

$$v=\nabla\mathcal{F}(H,\lambda,\xi)[K,\mu,\zeta]=\mathcal{F}(K,\lambda,\xi)+\nabla\mathcal{F}(H,\lambda,\xi)[0,\mu,\zeta].$$

这意味着 $[v]=[\mathcal{F}(K,\lambda,\xi)]$ 在商空间 $\mathbb{R}^{m+1}/\operatorname{im}\nabla\mathcal{F}(H,\lambda,\xi)[0,\cdot,\cdot]\big|_{\{0\}\times\mathbb{R}\times\mathbb{R}^m}$ 中。因此，映射 $\mathcal{Z}$ 诱导出一个同构

$$\operatorname{coker}\nabla\mathcal{P}(H,\lambda,\xi)=\frac{C^2(S^m)}{\operatorname{im}\nabla\mathcal{P}(H,\lambda,\xi)}=\frac{C^2(S^m)}{\ker\mathcal{Z}}\cong\frac{\mathbb{R}^{m+1}}{\operatorname{im}\nabla\mathcal{F}(H,\lambda,\xi)[0,\cdot,\cdot]\big|_{\{0\}\times\mathbb{R}\times\mathbb{R}^m}}.$$

因此，映射 $\mathcal{P}$ 是 Fredholm的，且 $\operatorname{ind}\nabla\mathcal{P}(H,\lambda,\xi)=\operatorname{ind}\nabla\mathcal{F}(H,\lambda,\xi)\big|_{\{0\}\times\mathbb{R}\times\mathbb{R}^m}=0$。 □

命题 3.6 的证明。根据引理 3.8， $\mathcal{P}:\mathcal{C}\to C^2(S^m)$是一个 C^1-Fredholm 映射，其定义域和值域为可分的 C^1-Banach流形，指标为 0。由 Sard-Smale定理， $\mathcal{P}$ 的正则值集合在 $C^2(S^m)$中是稠密剩余集。对于任意正则值 $H\in C^2(S^m)$，引理 3.8 的证明表明

$$0=\dim\operatorname{coker}\nabla\mathcal{P}(H,\lambda,\xi)=\dim\frac{\mathbb{R}^{m+1}}{\operatorname{im}\nabla\mathcal{F}(H,\lambda,\xi)[0,\cdot,\cdot]\big|_{\{0\}\times\mathbb{R}\times\mathbb{R}^m}}.$$

注意到

$$\nabla\mathcal{F}(H,\lambda,\xi)[0,\mu,\zeta]=\nabla^2\Psi(\lambda,\xi;H)[(\mu,\zeta)],\quad\text{for }(\mu,\zeta)\in\mathbb{R}\times\mathbb{R}^m=\mathbb{R}^{m+1}$$

我们发现 Hessian 矩阵 $\nabla^2\Psi(\lambda,\xi;H)$ 是非退化的。这意味着 $\Psi(\cdot,\cdot;H)$是 $(0,\infty)\times\mathbb{R}^m$ 上的 Morse 函数。证明至此完成。 □

Remark 3.9. Let $H\in C^2(S^m)$ be an arbitrary function satisfying conditions (H-1) and (H-2), then there exists $\delta>0$ such that $\tilde{H}\in C^2$ with $\|H-\tilde{H}\|_{C^2(S^m)}<\delta$ satisfies the same conditions. Proposition 3.6 implies that for any given $H\in C^2(S^m)$ satisfying conditions (H-1) and (H-2), there exists $\tilde{H}\in C^(S^m)$, close to H in C^2-norm, which satisfies the same conditions and the function

$$(\lambda,\xi)\longmapsto m^m\int_{\mathbb{R}^m}\frac{\tilde{a}(\lambda x+\xi)}{(1+|x|^2)^m}d\operatorname{vol}_{g_{\mathbb{R}^m}}$$

是一个 Morse 函数，其中 $\tilde{a}(x):=\tilde{H}\circ\pi_p^{-1}(x)$。因此，我们的断言“$\Psi$ 对于 a 的一般选择是 Morse 函数”得到验证。

4 背景度量的扰动

在本节中，我们考虑方程

$$D_g\psi=|\psi|_g^{2^*-2}\psi\quad\text{on }S^m\tag{4.1}$$

涉及一个扰动度量，其中 $2^*=\frac{2m}{m-1}$。具体而言，通过使用立体投影π_p： $S^m\setminus\{p\}\to\mathbb{R}^m$（对于固定的 $p\in S^m$），度量 g 由 (1.9) 和 (1.10) 定义。在此情形下，(4.1) 等价于方程。

$$D_{\tilde{g}}\psi=|\psi|_{\tilde{g}}^{2^*-2}\psi\quad\text{on }\mathbb{R}^m.\tag{4.2}$$

4.1 Bourguignon-Gauduchon 识别方式

在本部分，我们将汇集一些基本结果，这些结果将使我们能够将(4.2)的能量泛函$\mathcal{J}$写成适合使用前面提到的微扰方法的形式。为此，我们需要确定在 $\mathbb{R}^m$ 上相对于典范欧几里得度量$g_{\mathbb{R}^m}$ 和新度量 ˜g的Dirac算子和旋量场。Bourguignon和Gauduchon的构造 [14] 为我们提供了这种必要的确定对应关系。

首先，我们用 $\odot^2_+(\mathbb{R}^m)$ 表示正定对称双线性型的空间，即 $\mathbb{R}^m$ 上的度量。接着，对于足够小的 $\varepsilon > 0$，我们考虑恒等映射。

$$\begin{aligned} \mathrm{Id} : (\mathbb{R}^m, g_{\mathbb{R}^m}) &\to (\mathbb{R}^m, \tilde{g}) \\ x &\mapsto x \end{aligned}$$

以及映射

$$\begin{aligned} \tilde{G} : \mathbb{R}^m &\to \odot^2_+(\mathbb{R}^m) \\ x &\mapsto \tilde{G}_x := (\tilde{g}_{ij}(x))_{ij} \end{aligned}$$

which associates to a point $x \in \mathbb{R}^m$ the matrix of the coefficients of the metric $\tilde{g}$ at this point, expressed in the basis $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \dots, m$. Notice that $\tilde{G}_x \in \odot^2_+(\mathbb{R}^m)$, hence G_x is invertible and there is a unique matrix $B_x \in \odot^2_+(\mathbb{R}^m)$ such that $B_x^2 = \tilde{G}_x^{-1}$. Let $b_{ij}(x)$, $i, j = 1, \dots, m$, be the entries of B_x, we have

$$\begin{aligned} B_x : (T_x\mathbb{R}^m \cong \mathbb{R}^m, g_{\mathbb{R}^m}) &\to (T_x\mathbb{R}^m, \tilde{g}_x) \\ v = \sum_k v_k \partial_k &\mapsto B_x(v) := \sum_j \big(\sum_k b_{jk}(x) v_k \big) \partial_j \end{aligned}$$

为每个 $x \in \mathbb{R}^m$ 定义一个等距映射。由于矩阵 B_x 随 x 光滑变化，我们得到 $SO(m)$-主丛之间的一个同构：

$$\begin{array}{ccc} P_{SO}(\mathbb{R}^m, g_{\mathbb{R}^m}) & \xrightarrow{\eta} & P_{SO}(\mathbb{R}^m, \tilde{g}) \\ \downarrow & & \downarrow \\ \mathbb{R}^m & \xrightarrow{\mathrm{Id}} & \mathbb{R}^m \end{array}$$

其中 $\eta\{v_1, \dots, v_m\} = \{B(v_1) \dots, B(v_m)\}$ 是 $(\mathbb{R}^m, g_{\mathbb{R}^m})$ 上的定向标架 $\{v_1, \dots, v_m\}$。注意到映射 η 与 $SO(m)$ 的右作用可交换，因此它可以提升到自旋结构：

$$\begin{array}{ccc} P_{Spin}(\mathbb{R}^m, g_{\mathbb{R}^m}) & \xrightarrow{\tilde{\eta}} & P_{Spin}(\mathbb{R}^m, \tilde{g}) \\ \downarrow & & \downarrow \\ \mathbb{R}^m & \xrightarrow{\mathrm{Id}} & \mathbb{R}^m \end{array}$$

因此，我们得到旋量丛 $\mathbb{S}(\mathbb{R}^m, g_{\mathbb{R}^m})$ 与 $\mathbb{S}(\mathbb{R}^m, \tilde{g})$ 之间的一个同构：

$$\begin{aligned} \mathbb{S}(\mathbb{R}^m, g_{\mathbb{R}^m}) := P_{Spin}(\mathbb{R}^m, g_{\mathbb{R}^m}) \times_\rho \mathbb{S}_m &\longrightarrow \mathbb{S}(\mathbb{R}^m, \tilde{g}) := P_{Spin}(\mathbb{R}^m, g) \times_\rho \mathbb{S}_m \\ \psi = [s, \varphi] &\longmapsto \tilde{\psi} = [\tilde{\eta}(s), \varphi] \end{aligned} \tag{4.3}$$

其中，ρ 是复旋量表示，$[s, \varphi]$ 表示(s, φ) 在 $Spin(m)$ 作用下的等价类。这确定了旋量场。

对于 Dirac 算子，正如 [11, 命题 3.2], 所示，该识别可表示为以下公式

$$D_{\tilde{g}}\tilde{\psi} = \widetilde{D_{g_{\mathbb{R}^m}}\psi} + W \cdot_{\tilde{g}} \tilde{\psi} + X \cdot_{\tilde{g}} \tilde{\psi} + \sum_{i,j}(b_{ij} - \delta_{ij})\tilde{\partial}_i \cdot_{\tilde{g}} \widetilde{\nabla_{\partial_j}\psi} \tag{4.4}$$

其中 $\cdot_{\tilde{g}}$ 表示相对于度量 ˜g 的克利福德乘法,

$$W = \frac{1}{4} \sum_{\substack{i,j,k \\ i\neq j\neq k\neq i}} \sum_{\alpha,\beta} b_{i\alpha}(\partial_\alpha b_{j\beta}) b^{-1}_{\beta k}\, \tilde{\partial}_i \cdot_{\tilde{g}} \tilde{\partial}_j \cdot_{\tilde{g}} \tilde{\partial}_k,$$

其中， b^{-1}_{ij} 是 B的逆矩阵的元素， $\tilde{\partial}_i = B(\partial_i)$以及

$$X = \frac{1}{2}\sum_{i,k} \tilde{\Gamma}^i_{ik}\tilde{\partial}_k,$$

其中 $\tilde{\Gamma}^k_{ij} = $ ˜g($\tilde{\nabla}_{\tilde{\partial}_i}\tilde{\partial}_j, \tilde{\partial}_k$) 为第二类克里斯托费尔符号。

Remark 4.1. On spin manifolds, since the tangent bundle is embedded in the bundle of Clifford algebra, vector fields have two different actions on spinors, i.e. the Clifford multiplications and the covariant derivatives. Here, to distinguish the two actions on a spinor ψ, we denote $\partial_i \cdot_{g_{\mathbb{R}^m}} \psi$ the Clifford multiplication of ∂_i and $\nabla_{\partial_i}\psi$ the covariant derivative with respect to the metric $g_{\mathbb{R}^m}$ (respectively, $\tilde{\partial}_i \cdot_{\tilde{g}} \tilde{\psi}$ the Clifford multiplication of $\tilde{\partial}_i$ and $\tilde{\nabla}_{\tilde{\partial}_i}\tilde{\psi}$ the covariant derivative with respect to the metric $\tilde{g}$). For functions, we shall simply denote $\partial_i u$ for its partial derivative.

4.2 基本计算

以下展开式源于初等计算。

引理 4.2. 设由 (1.10) 给出 ˜g，则有

$$\sqrt{\det \tilde{G}} = 1 + \frac{\varepsilon}{2}\operatorname{tr} h + \varepsilon^2\Big(\frac{1}{8}(\operatorname{tr} h)^2 - \frac{1}{4}\operatorname{tr}(h^2)\Big) + o(\varepsilon^2), \tag{4.5}$$

$$B = I - \frac{\varepsilon}{2}h + \frac{3\,\varepsilon^2}{8}h^2 + o(\varepsilon^2) \tag{4.6}$$

and

$$B^{-1} = I + \frac{\varepsilon}{2}h - \frac{\varepsilon^2}{8}h^2 + o(\varepsilon^2). \tag{4.7}$$

˜采用 Bourguignon-Gauduchon 对应的记号，对于 ψ in $\mathbb{S}(\mathbb{R}^m,$ ˜g) 中的旋量，与 (4.2) 相关的能量泛函定义为

$$\mathcal{J}(\tilde{\psi}) = \frac{1}{2}\int_{\mathbb{R}^m} (\tilde{\psi}, D_{\tilde{g}}\tilde{\psi})_{\tilde{g}}\, d\operatorname{vol}_{\tilde{g}} - \frac{1}{2^*}\int_{\mathbb{R}^m} |\tilde{\psi}|^{2^*}_{\tilde{g}}\, d\operatorname{vol}_{\tilde{g}}\,. \tag{4.8}$$

事实证明， $\mathcal{J}$是一个涉及 ε的泛函，因为 ˜g由(1.10)给出。现在，通过 Bourguignon-Gauduchon同一化，我们得到如下展开:

引理 4.3. 设 $\tilde{g}$ 由 (1.10) 给出，则

$$\mathcal{J}(\tilde{\psi}) = \mathcal{J}_0(\psi) + \varepsilon\Gamma(\psi) + \varepsilon^2\Phi(\psi) + o(\varepsilon^2),$$

where

$$\mathcal{J}_0(\psi) = \frac{1}{2}\int_{\mathbb{R}^m} (\psi, D_{g_{\mathbb{R}^m}}\psi)_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}} - \frac{1}{2^*}\int_{\mathbb{R}^m} |\psi|^{2^*}_{g_{\mathbb{R}^m}}\, d\,\mathrm{vol}_{g_{\mathbb{R}^m}},$$

$$\Gamma(\psi) = \int_{\mathbb{R}^m} \frac{\mathrm{tr}\, h}{2}\Big[\frac{1}{2}\big(\psi, D_{g_{\mathbb{R}^m}}\psi\big)_{g_{\mathbb{R}^m}} - \frac{1}{2^*}|\psi|^{2^*}_{g_{\mathbb{R}^m}}\Big] - \frac{1}{4}\sum_i h_{ii}\,\mathrm{Re}\,\big(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi, \psi\big)_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

and

$$\begin{aligned}\Phi(\psi) = \int_{\mathbb{R}^m} \Big(\frac{1}{8}(\mathrm{tr}\, h)^2 - \frac{1}{4}\,\mathrm{tr}(h^2)\Big)\Big[\frac{1}{2}\big(\psi, D_{g_{\mathbb{R}^m}}\psi\big)_{g_{\mathbb{R}^m}} - \frac{1}{2^*}|\psi|^{2^*}_{g_{\mathbb{R}^m}}\Big] \\ + \frac{1}{16}\sum_i \Big(3h_{ii}^2 - 2(\mathrm{tr}\, h)h_{ii}\Big)\,\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi, \psi)_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\end{aligned}$$

对于 $\psi \in \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$。

证明。我们首先将 $\mathcal{J}$ 的二次部分按 ε 展开。利用 (4.4) 以及事实 $X \in T\mathbb{R}^m$，可得

$$\mathrm{Re}(\tilde{\psi}, D_{\tilde{g}}\tilde{\psi})_{\tilde{g}} = \mathrm{Re}(\tilde{\psi}, \widetilde{D_{g_{\mathbb{R}^m}}\psi})_{\tilde{g}} + \mathrm{Re}(W \cdot_{\tilde{g}} \tilde{\psi}, \tilde{\psi})_{\tilde{g}} + \sum_{i,j}(b_{ij} - \delta_{ij})\,\mathrm{Re}(\tilde{\partial}_i \cdot_{\tilde{g}} \widetilde{\nabla_{\partial_j}\psi}, \tilde{\psi})_{\tilde{g}}.$$

由于 (4.3) 中定义的映射 $\psi \mapsto \psi$ 在每个纤维上都是等距的，我们得到

$$\mathrm{Re}(\tilde{\psi}, \widetilde{D_{g_{\mathbb{R}^m}}\psi})_{\tilde{g}} = \mathrm{Re}\,\big(\psi, D_{g_{\mathbb{R}^m}}\psi\big)_{g_{\mathbb{R}^m}}, \quad \mathrm{Re}(\tilde{\partial}_i \cdot_{\tilde{g}} \widetilde{\nabla_{\partial_j}\psi}, \tilde{\psi})_{\tilde{g}} = \mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_j}\psi, \psi)_{g_{\mathbb{R}^m}}$$

and

$$\mathrm{Re}(W \cdot_{\tilde{g}} \tilde{\psi}, \tilde{\psi})_{\tilde{g}} = \frac{1}{4}\sum_{\substack{i,j,k\\ i\neq j\neq k\neq i}} \Big(\sum_{\alpha,\beta} b_{i\alpha}(\partial_\alpha b_{j\beta})b^{-1}_{\beta k}\Big)\,\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \partial_j \cdot_{g_{\mathbb{R}^m}} \partial_k \cdot_{g_{\mathbb{R}^m}} \psi, \psi)_{g_{\mathbb{R}^m}}.$$

Pla $\tilde{}$ $\tilde{}$inly, $\mathrm{Re}(W \cdot_{\tilde{g}} \psi, \psi)_{\tilde{g}} \equiv 0$ 对于 $m = 2$。对于 $m \geq 3$，由 (4.6) 和 (4.7) 可知

$$\begin{cases} b_{i\alpha} = \delta_{i\alpha} - \dfrac{\varepsilon}{2}h_{i\alpha} + \dfrac{3\,\varepsilon^2}{8}\displaystyle\sum_l h_{il}h_{l\alpha} + o(\varepsilon^2), \\ \partial_\alpha b_{j\beta} = -\dfrac{\varepsilon}{2}\partial_\alpha h_{j\beta} + \dfrac{3\,\varepsilon^2}{8}\displaystyle\sum_l \Big(\partial_\alpha h_{jl}h_{l\beta} + h_{jl}\partial_\alpha h_{l\beta}\Big) + o(\varepsilon^2), \\ b^{-1}_{\beta k} = \delta_{\beta k} + \dfrac{\varepsilon}{2}h_{\beta k} - \dfrac{\varepsilon^2}{8}\displaystyle\sum_l h_{\beta l}h_{lk} + o(\varepsilon^2). \end{cases}$$

Hence

$$\begin{aligned} b_{i\alpha}(\partial_\alpha b_{j\beta})b^{-1}_{\beta k} = -\frac{\varepsilon}{2}\delta_{\beta k}\delta_{i\alpha}\partial_\alpha h_{j\beta} + \frac{\varepsilon^2}{4}\partial_\alpha h_{j\beta}\big(\delta_{\beta k}h_{i\alpha} - \delta_{i\alpha}h_{\beta k}\big) \\ + \frac{3\delta_{\beta k}\delta_{i\alpha}\varepsilon^2}{8}\sum_l \Big(\partial_\alpha h_{jl}h_{l\beta} + h_{jl}\partial_\alpha h_{l\beta}\Big) + o(\varepsilon^2).\end{aligned}$$

Note that we have assumed $h_{ij}=0$ for $i\neq j$, we soon get

$$b_{i\alpha}(\partial_\alpha b_{j\beta})b^{-1}_{\beta k}=o(\varepsilon^2).$$

因此

$$\begin{aligned}\mathrm{Re}(\tilde{\psi},D_{\tilde{g}}\tilde{\psi})_{\tilde{g}}=\mathrm{Re}\left(\psi,D_{g_{\mathbb{R}^m}}\psi\right)_{g_{\mathbb{R}^m}}&-\frac{\varepsilon}{2}\sum_i h_{ii}\ \mathrm{Re}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi,\psi)_{g_{\mathbb{R}^m}}\\&+\frac{3\,\varepsilon^2}{8}\sum_i h_{ii}^2\ \mathrm{Re}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi,\psi)_{g_{\mathbb{R}^m}}+o(\varepsilon^2)\end{aligned}\tag{4.9}$$

˜由于 $|\psi|_{\tilde{g}}=|\psi|_{g_{\mathbb{R}^m}}$，由（4.5）和（4.9）可得

$$\begin{aligned}\mathcal{J}(\tilde{\psi})=\mathcal{J}_0(\psi)+\varepsilon\int_{\mathbb{R}^m}&\frac{\mathrm{tr}\,h}{2}\Big[\frac{1}{2}\big(\psi,D_{g_{\mathbb{R}^m}}\psi\big)_{g_{\mathbb{R}^m}}-\frac{1}{2^*}|\psi|^{2^*}_{g_{\mathbb{R}^m}}\Big]\\&-\frac{1}{4}\sum_i h_{ii}\,\mathrm{Re}\left(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi,\psi\right)d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\+\varepsilon^2\int_{\mathbb{R}^m}&\Big(\frac{1}{8}(\mathrm{tr}\,h)^2-\frac{1}{4}\,\mathrm{tr}(h^2)\Big)\Big[\frac{1}{2}\big(\psi,D_{g_{\mathbb{R}^m}}\psi\big)_{g_{\mathbb{R}^m}}-\frac{1}{2^*}|\psi|^{2^*}_{g_{\mathbb{R}^m}}\Big]\\&-\frac{\mathrm{tr}\,h}{8}\sum_i h_{ii}\,\mathrm{Re}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi,\psi)_{g_{\mathbb{R}^m}}\\&+\frac{3}{16}\sum_i h_{ii}^2\ \mathrm{Re}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi,\psi)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}+o(\varepsilon^2)\end{aligned}$$

由此完成证明。 □

回忆一下在（2.9）中定义的临界流形 $\mathcal{M}\subset\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$，上述引理的一个直接推论是

推论 4.4. 对于 $\psi_{\lambda,\ \xi,\ \gamma}\in\mathcal{M}$，其中 $\lambda>0$、$\xi\in\mathbb{R}^m$ 和 $\gamma\in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$，我们有

$$\Gamma(\psi_{\lambda,\xi,\gamma})=0$$

and

$$\Phi(\psi_{\lambda,\xi,\gamma})=\frac{m^{m-1}\lambda^m}{16}\int_{\mathbb{R}^m}\frac{\mathrm{tr}(h^2)-(\mathrm{tr}\,h)^2}{\left(\lambda^2+|x-\xi|^2\right)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.$$

证明。对于 $\psi_{\lambda,\xi,\gamma}\in\mathcal{M}$，很容易验证它满足方程

$$D_{g_{\mathbb{R}^m}}\psi_{\lambda,\xi,\gamma}=|\psi_{\lambda,\xi,\gamma}|^{2^*-2}_{g_{\mathbb{R}^m}}\psi_{\lambda,\xi,\gamma}$$

在 $\mathbb{R}^m$ 上。此外，根据 $\psi_{\lambda,\ \xi,\ \gamma}$ 的显式表达式，我们得到

$$|\psi_{\lambda,\xi,\gamma}|_{g_{\mathbb{R}^m}}=\frac{m^{\frac{m-1}{2}}\lambda^{\frac{m-1}{2}}}{\left(\lambda^2+|x-\xi|^2\right)^{\frac{m-1}{2}}}$$

and

$$\begin{aligned}\nabla_{\partial_i}\psi_{\lambda,\xi,\gamma}(x) = &-\frac{m^{\frac{m+1}{2}}\lambda^{\frac{m-1}{2}}}{\left(\lambda^2+|x-\xi|^2\right)^{\frac{m}{2}+1}}(x_i-\xi_i)(\lambda-(x-\xi))\cdot_{g_{\mathbb{R}^m}}\gamma\\ &+\frac{m^{\frac{m-1}{2}}\lambda^{\frac{m-1}{2}}}{\left(\lambda^2+|x-\xi|^2\right)^{\frac{m}{2}}}(-\partial_i)\cdot_{g_{\mathbb{R}^m}}\gamma.\end{aligned} \tag{4.10}$$

因此，根据克利福德乘法的规则，我们得到 $\partial_i \cdot_{g_{\mathbb{R}^m}} \partial_i = -1$ 和

$$\begin{aligned}\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{\lambda,\xi,\gamma}(x) = &-\frac{m^{\frac{m+1}{2}}\lambda^{\frac{m-1}{2}}}{\left(\lambda^2+|x-\xi|^2\right)^{\frac{m}{2}+1}}(x_i-\xi_i)\,\partial_i\cdot_{g_{\mathbb{R}^m}}(\lambda-(x-\xi))\cdot_{g_{\mathbb{R}^m}}\gamma\\ &+\frac{m^{\frac{m-1}{2}}\lambda^{\frac{m-1}{2}}}{\left(\lambda^2+|x-\xi|^2\right)^{\frac{m}{2}}}\gamma\end{aligned} \tag{4.11}$$

and

$$\mathrm{Re}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{\lambda,\xi,\gamma},\psi_{\lambda,\xi,\gamma})_{g_{\mathbb{R}^m}} = \frac{m^{m-1}\lambda^m}{\left(\lambda^2+|x-\xi|^2\right)^m}\quad \text{for } i=1,\dots,m.$$

因此，

$$\Gamma(\psi_{\lambda,\xi,\gamma}) = \int_{\mathbb{R}^m}\left(\frac{\mathrm{tr}\,\boldsymbol{h}}{2}\frac{1}{2m}\frac{m^m\lambda^m}{\left(\lambda^2+|x-\xi|^2\right)^m}-\frac{\mathrm{tr}\,\boldsymbol{h}}{4}\frac{m^{m-1}\lambda^m}{\left(\lambda^2+|x-\xi|^2\right)^m}\right)d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\equiv 0$$

and

$$\Phi(\psi_{\lambda,\xi,\gamma}) = \frac{m^{m-1}\lambda^m}{16}\int_{\mathbb{R}^m}\frac{\mathrm{tr}(\boldsymbol{h}^2)-(\mathrm{tr}\,\boldsymbol{h})^2}{\left(\lambda^2+|x-\xi|^2\right)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.$$

□

由引理 4.3 和推论 4.4 可知，定理 2.6 足以证明此处的存在性问题。因此，接下来我们将研究其行为。

$$\hat{\Phi}(z) := \Phi(z) - \frac{1}{2}\langle K_z(\nabla\Gamma(z)),\nabla\Gamma(z)\rangle$$

on a fiber $\vartheta^{-1}(\gamma)$ for a fixed $\gamma\in\mathcal{N} = S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$, where K_z stands for the inverse of $\nabla^2\mathcal{J}_0(z)$ restricted to $\mathcal{W}_z := T_z\mathcal{M}^\perp\subset\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$. Note that $\vartheta^{-1}(\gamma)$ is parameterized by $\mathcal{G} = (0,\infty)\times\mathbb{R}^m$, it is very natural to study the values of $\hat{\Phi}(\psi_{\lambda,\xi,\gamma})$ when $\lambda\to 0$ and $\lambda+|\xi|\to\infty$.

引理 4.5. 假设我们满足引理 4.3 的假设条件。则有

$$\lim_{\lambda\to 0}\Phi(\psi_{\lambda,\xi,\gamma}) = C_0\big(\mathrm{tr}(\boldsymbol{h}^2)-(\mathrm{tr}\,\boldsymbol{h})^2\big)(\xi)\quad \textit{for any } \xi\in\mathbb{R}^m,$$

其中

$$C_0 = \frac{m^{m-1}}{16}\int_{\mathbb{R}^m}\frac{1}{\left(1+|x|^2\right)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.$$

证明。通过变量代换，可以容易地推出

$$\Phi(\psi_{\lambda,\xi,\gamma}) = \frac{m^{m-1}}{16}\int_{\mathbb{R}^m} \frac{\big(\operatorname{tr}(h^2) - (\operatorname{tr} h)^2\big)(\lambda x+\xi)}{\big(1+|x|^2\big)^m} d\operatorname{vol}_{g_{\mathbb{R}^m}}.$$

因此我们有了

$$\lim_{\lambda\to 0}\Phi(\psi_{\lambda,\xi,\gamma}) = \frac{m^{m-1}}{16}\int_{\mathbb{R}^m} \frac{1}{\big(1+|x|^2\big)^m} d\operatorname{vol}_{g_{\mathbb{R}^m}} \big(\operatorname{tr}(h^2) - (\operatorname{tr} h)^2\big)(\xi).$$

□

接下来的两个结果，即引理4.6和命题 4.7，是我们论证所需的主要组成部分。为清晰起见，由于这两个结果的证明包含了本文的主要难点，现将其推迟至附录 A.3。

引理 4.6. 假设满足引理 4.3 的假设，则有以下结论。

$$\lim_{\lambda\to 0}\big\langle K_{\psi_{\lambda,\xi,\gamma}}(\nabla\Gamma(\psi_{\lambda,\xi,\gamma})), \nabla\Gamma(\psi_{\lambda,\xi,\gamma})\big\rangle = C_1\big(\operatorname{tr}(h^2) - (\operatorname{tr} h)^2\big)(\xi) \quad \textit{for any } \xi\in\mathbb{R}^m,$$

其中

$$C_1 = \frac{m^{m-1}}{4}\int_{\mathbb{R}^m} \frac{|x|^2}{\big(1+|x|^2\big)^{m+1}} d\operatorname{vol}_{g_{\mathbb{R}^m}}.$$

由引理 4.5 和 4.6 可得

$$\lim_{\lambda\to 0}\hat{\Phi}(\psi_{\lambda,\xi,\gamma}) = \Big(C_0 - \frac{1}{2}C_1\Big)\big(\operatorname{tr}(h^2) - (\operatorname{tr} h)^2\big)(\xi) = 0.$$

since $C_0 = \frac{1}{2}C_1$. With this observation in mind, to see whether $\hat{\Phi}$ has local maximum or minimum on the fiber $\vartheta^{-1}(\gamma)$, we have to go further to study the asymptotic behavior of $\hat{\Phi}(\psi_{\lambda,\xi,\gamma})$ as $\lambda\to 0$. By using the concept of elementary matrix in Definition 1.3, the following proposition characterize the exact expansion of $\hat{\Phi}(\psi_{\lambda,\xi,\gamma})$ with respect to λ small.

Proposition 4.7. *For $m\geq 4$, assume that we are in the hypotheses of Lemma 4.3. Let $k\in\{1,\dots,m\}$, $p\in[2,\infty]$ and $h=\operatorname{diag}(h_{11},\dots,h_{mm})$ be (k,p)-elementary at a point $\xi\in\mathbb{R}^m$ with $\partial_k h_{ii}(\xi)\equiv c_k\neq 0$, for $i\neq k$. If*

$$\begin{cases} p=\infty & m=4,\\ p>2 & m=5,\\ p\geq 2 & m\geq 6,\end{cases}$$

then

$$\hat{\Phi}(\psi_{\lambda,\xi,\gamma}) = -\frac{3m^{m-2}(m-1)(m-2)c_k^2}{128}\lambda^2\int_{\mathbb{R}^m}\frac{|x|^2}{(1+|x|^2)^m} d\operatorname{vol}_{g_{\mathbb{R}^m}} + o(\lambda^2) \quad \textit{as } \lambda\to 0.$$

ˆ 特别地，对于较小的 λ 值， $\Phi(\psi_{\lambda,\xi,\gamma}) < 0$ 。

ˆ下一个命题表明， $\Phi(\psi_{\lambda,\xi,\gamma})$ 随着 $\lambda+|\xi|\to\infty$ 的增大而衰减。

命题 4.8. 假设满足引理 4.3 的条件。那么

$$\hat{\Phi}(\psi_{\lambda,\xi,\gamma})\to 0 \quad as\ \lambda+|\xi|\to\infty.$$

Proof. Let $r:\mathbb{R}^m\setminus\{0\}\to\mathbb{R}^m\setminus\{0\}$ be defined as $r(x)=-\frac{x}{|x|^2}$. Then, for each $(\lambda,\xi,\gamma)\in(0,\infty)\times\mathbb{R}^m\times S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$, we can define $\psi^\sharp_{\lambda,\xi,\gamma}(x)=\psi_{\lambda^\sharp,\xi^\sharp,\gamma}(x)$, where $\lambda^\sharp=\frac{\lambda}{\lambda^2+|\xi|^2}$ and $\xi^\sharp=\frac{-\xi}{\lambda^2+|\xi|^2}$. Then $\psi^\sharp_{\lambda,\xi,\gamma}\in\mathcal{M}$ and it follows from a direct computation that that

$$\big|\psi^\sharp_{\lambda,\xi,\gamma}(x)\big|_{g_{\mathbb{R}^m}}=\big|\psi_{\lambda^\sharp,\xi^\sharp,\gamma}(x)\big|_{g_{\mathbb{R}^m}}=\frac{1}{|x|^{m-1}}|\psi_{\lambda,\xi,\gamma}(r(x))|_{g_{\mathbb{R}^m}}.$$

粗略地说，该映射 $\psi_{\lambda,\xi,\gamma}\mapsto\psi^\sharp_{\lambda,\xi,\gamma}$ 可以被解释为 $\mathcal{M}$ 中无穷远处的元素与接近 $\psi_{t,0,\gamma}$ 且 $t>0$ 很小时的元素之间的一一对应。

对于任意函数 $P\colon\mathbb{R}^m\to\mathbb{R}$，令我们定义 $P^\sharp(x)=P(r(x))$。注意到 r是共形的，共形因子为 $\frac{1}{|x|^4}$，即 $r^*\mathrm{g}_{\mathbb{R}^m}=\frac{1}{|x|^4}\mathrm{g}_{\mathbb{R}^m}$，我们得到

$$\begin{aligned}\int_{\mathbb{R}^m}P(x)|\psi_{\lambda,\xi,\gamma}(x)|^{2^*}_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}&=\int_{\mathbb{R}^m}P(r(x))|\psi_{\lambda,\xi,\gamma}(r(x))|^{2^*}_{g_{\mathbb{R}^m}}\frac{1}{|x|^{2m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\&=\int_{\mathbb{R}^m}P^\sharp(x)\big|\psi^\sharp_{\lambda,\xi,\gamma}(x)\big|^{2^*}_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}.\end{aligned}\tag{4.12}$$

此外，注意到

$$\mathrm{Re}\left(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi^\sharp_{\lambda,\xi,\gamma}(x),\psi^\sharp_{\lambda,\xi,\gamma}(x)\right)_{g_{\mathbb{R}^m}}=\frac{1}{m}\big|\psi^\sharp_{\lambda,\xi,\gamma}(x)\big|^{2^*}_{g_{\mathbb{R}^m}}$$

我们有

$$\begin{aligned}&\mathrm{Re}\int_{\mathbb{R}^m}P(x)(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{\lambda,\xi,\gamma},\psi_{\lambda,\xi,\gamma})_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\&\quad=\mathrm{Re}\int_{\mathbb{R}^m}P^\sharp(x)\left(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi^\sharp_{\lambda,\xi,\gamma},\psi^\sharp_{\lambda,\xi,\gamma}\right)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\end{aligned}\tag{4.13}$$

对于 $i=1,\dots,m$。

现在令 $\tilde{\mathrm{g}}^\sharp(x):=\tilde{\mathrm{g}}(r(x))$，并考虑将 (4.8) 中的 $\tilde{\mathrm{g}}$ 替换为 $\tilde{\mathrm{g}}^\sharp$ 后得到的相应新扰动泛函 $\mathcal{J}^\sharp$。与引理 4.3 类似，我们有如下展开式

$$\mathcal{J}^\sharp(\tilde{\psi})=\mathcal{J}_0(\psi)+\varepsilon\Gamma^\sharp(\psi)+\varepsilon^2\Phi^\sharp(\psi)+o(\varepsilon^2).$$

并且由 (4.12) 和 (4.13)，我们推得大发快

$$\Gamma^\sharp(\psi^\sharp_{\lambda,\xi,\gamma})=\Gamma(\psi_{\lambda,\xi,\gamma})\quad\text{and}\quad\Phi^\sharp(\psi^\sharp_{\lambda,\xi,\gamma})=\Phi(\psi_{\lambda,\xi,\gamma}).$$

此外，我们还有

$$\hat{\Phi}^\sharp(\psi^\sharp_{\lambda,\xi,\gamma})=\hat{\Phi}(\psi_{\lambda,\xi,\gamma}).$$

因此，通过应用引理 4.5 和 4.6，我们得到

$$\lim_{\lambda+|\xi|\to\infty}\hat{\Phi}(\psi_{\lambda,\xi,\gamma})=\lim_{\lambda+|\xi|\to\infty}\hat{\Phi}^\sharp(\psi^\sharp_{\lambda,\xi,\gamma})=\lim_{\substack{\lambda^\sharp\to0\\\xi^\sharp\to0}}\hat{\Phi}^\sharp(\psi_{\lambda^\sharp,\xi^\sharp,\gamma})=0,$$

which completes the proof. □

Remark 4.9. From Proposition 4.7, 4.8 and Theorem 2.6 we can immediately conclude that: there exists a bounded open set $U \subset (0,\infty)\times\mathbb{R}^m$, independent of $\gamma \in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$, such that $\hat{\Phi}(\psi_{\lambda,\xi,\gamma})$ achieves a strict minimum in dimension $m \geq 4$ in U, and hence the perturbed spinorial Yamabe equation (4.2) has a solution.

4.3 主要结果的证明

证明将分几个步骤进行，我们先从一些基本估计开始。

引理 4.10。当 $z \in \mathcal{M}$ 时，算子 $K_z = \nabla^2\mathcal{J}_0(z)^{-1}$ 一致有界。

证明。对于固定的 $\gamma \in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$，我们记 $z_0 = \psi_{1,0,\gamma}$。对于任意 $g=(\lambda,\xi) \in (0,\infty)\times\mathbb{R}^m$，我们在 $\mathscr{D}^{1/2}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ 中的元素上定义右作用如下

$$(\psi\cdot g)(x) = \lambda^{-\frac{m-1}{2}}\psi\Big(\frac{x-\xi}{\lambda}\Big) \quad \text{for } \psi\in\mathscr{D}^{1/2}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m)).$$

Then it is easy to check that $z_0\cdot g = \psi_{\lambda,\xi,\gamma}$ and the inverse action is defined by $g^{-1} = (\lambda^{-1},-\lambda^{-1}\xi)$. Moreover, by standard Fourier transformation, we can see that the action of g is an isometry, i.e. $\|\psi\cdot g\| = \|\psi\|$ for any $\psi\in\mathscr{D}^{1/2}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$ and $g\in(0,\infty)\times\mathbb{R}^m$.

注意到 $\nabla^2\mathcal{J}_0(z_0)$ 在 $\mathcal{W}_{z_0} = T_{z_0}\mathcal{M}^\perp$ 上是自伴且可逆的，因此存在常数 $c_\gamma>0$，使得对任意 $w\in\mathcal{W}_{z_0}$，存在 $v(w)\in\mathcal{W}_{z_0}$ 满足 $\|v(w)\|=1$ 并且

$$\nabla^2\mathcal{J}_0(z_0)[w,v(w)] = \big\|\nabla^2\mathcal{J}_0(z_0)[w]\big\| \geq c_\gamma\|w\|.$$

这里常数 c_γ 的下标源于这样一个事实：z_0（隐含地）依赖于γ。下文中，对任意 $w\in\mathcal{W}_{z_0}$，我们简单地将 $w_g = w\cdot g$ 记作 $g=(\lambda,\xi)\in(0,\infty)\times\mathbb{R}^m$ 和 $z_g := z_0\cdot g$。我们指出，通过变量代换,

$$\mathrm{Re}\int_{\mathbb{R}^m}(D_{g_{\mathbb{R}^m}}w_g, v(w)_g)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}} = \mathrm{Re}\int_{\mathbb{R}^m}(D_{g_{\mathbb{R}^m}}w, v(w))_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}},$$

$$\mathrm{Re}\int_{\mathbb{R}^m}|z_g|^{\frac{2}{m-1}}(w_g, v(w)_g)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}} = \mathrm{Re}\int_{\mathbb{R}^m}|z_0|^{\frac{2}{m-1}}(w, v(w))_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^m}|z_g|^{\frac{4-2m}{m-1}}\,\mathrm{Re}(z_g,w_g)_{g_{\mathbb{R}^m}}\,\mathrm{Re}(z_g,v(w)_g)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &\quad= \mathrm{Re}\int_{\mathbb{R}^m}|z_0|^{\frac{4-2m}{m-1}}\,\mathrm{Re}(z_0,w)_{g_{\mathbb{R}^m}}\,\mathrm{Re}(z_0,v(w))_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\end{aligned}$$

因此可推出

$$\nabla^2\mathcal{J}_0(z_g)[w,v(w_{g^{-1}})_g] = \nabla^2\mathcal{J}_0(z_0)[w_{g^{-1}},v(w_{g^{-1}})] \geq c_\gamma\|w_{g^{-1}}\| = c_\gamma\|w\|$$

因此，我们看到，$\|K_{z_\rho}\|$ 的算子范数对于所有 $g\in(0,\infty)\times\mathbb{R}^m$ 均由 $\frac{1}{c_\gamma}$ 一致有界，因为 $K_{z_g} = \nabla^2\mathcal{J}_0(z_g)^{-1}$。注意到 $\gamma\in S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$，其中 $S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$是一个紧流形，我们有 c_γ 在 γ 中一致地远离 0。并且由于 K_{z_0} 关于 γ的连续性，我们得出 K_z 对所有 $z\in\mathcal{M}$ 均一致有界，证明完毕。

□

引理 4.11。设 ˜g 由 (1.10) 给出。假设矩阵函数 h 具有紧支撑。那么

$$\|\nabla\Gamma(\psi_{\lambda,\xi,\gamma})\| \to 0 \quad \textit{as } \lambda \to \infty \textit{ unifromly in } (\xi,\gamma).$$

证明。取足够大的 $R>0$，使得 $\operatorname{supp} \mathrm{h} \subset B_R$，其中 B_R 是 $\mathbb{R}^m$ 中以原点为中心、半径为 R 的开球。对于任意 $v \in \mathscr{D}^{1/2}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$，由 (A.12) 可得

$$\begin{aligned}
|\langle \nabla\Gamma(\psi_{\lambda,\xi,\gamma}), v\rangle| &\le C\Big(\int_{B_R} |\nabla\psi_{\lambda,\xi,\gamma}|\cdot|v| + |\psi_{\lambda,\xi,\gamma}|\cdot|v| d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big)\\
&\le C\Big(\int_{B_R} \frac{\lambda^{\frac{m-1}{2}}}{\big(\lambda^2+|x-\xi|^2\big)^{\frac{m}{2}}}\cdot|v| + \frac{\lambda^{\frac{m-1}{2}}}{\big(\lambda^2+|x-\xi|^2\big)^{\frac{m-1}{2}}}\cdot|v| d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big)\\
&\le C\big(\lambda^{-\frac{m+1}{2}} + \lambda^{-\frac{m-1}{2}}\big)\|v\|.
\end{aligned}$$

因此，$\|\nabla\Gamma(\psi_{\lambda,\xi,\gamma})\| \to 0$ 当 $\lambda\to\infty$ 在 (ξ,γ) 中一致地。 □

Next, let us consider a further situation of the perturbed metric in (1.10). Precisely, we consider the matrix h in the form of (1.11). In this case, the matrix h can be decomposed as $h(x) = h^{(1)}(x) + h^{(2)}(x-x_0)$, and it is clear that $h^{(1)}(\cdot)$ and $h^{(2)}(\cdot - x_0)$ have disjoint supports provided that $|x_0|$ is sufficiently large. In this setting, by substituting in (4.8) with the metrics $\tilde{g}^{(1)} = g_{\mathbb{R}^m} + \varepsilon h^{(1)}$ and $\tilde{g}^{(2)}(\cdot - x_0) = g_{\mathbb{R}^m} + \varepsilon h^{(2)}(\cdot - x_0)$, we obtain $\Gamma^{(1)}$, $\Gamma^{(2),x_0}$, $\Phi^{(1)}$, $\Phi^{(2),x_0}$, etc., the corresponding functionals. It is easy to check that

$$\Gamma^{(2),x_0}(\psi_{\lambda,\xi,\gamma}) = \Gamma^{(2)}(\psi_{\lambda,\xi-x_0,\gamma}) \quad \text{and} \quad \Phi^{(2),x_0}(\psi_{\lambda,\xi,\gamma}) = \Phi^{(2)}(\psi_{\lambda,\xi-x_0,\gamma}).$$

因此，我们得到

$$\hat{\Phi}^{(2),x_0}(\psi_{\lambda,\xi,\gamma}) = \hat{\Phi}^{(2)}(\psi_{\lambda,\xi-x_0,\gamma}).$$

如果 $|x_0|$ 足够大，使得 $\mathrm{h}^{(1)}$ 和 $\mathrm{h}^{(2)}(\cdot - x_0)$ 的支撑集不相交，我们发现

$$\Gamma(\psi_{\lambda,\xi,\gamma}) = \Gamma^{(1)}(\psi_{\lambda,\xi,\gamma}) + \Gamma^{(2),x_0}(\psi_{\lambda,\xi,\gamma}) = \Gamma^{(1)}(\psi_{\lambda,\xi,\gamma}) + \Gamma^{(2)}(\psi_{\lambda,\xi-x_0,\gamma}),$$

$$\Phi(\psi_{\lambda,\xi,\gamma}) = \Phi^{(1)}(\psi_{\lambda,\xi,\gamma}) + \Phi^{(2)}(\psi_{\lambda,\xi-x_0,\gamma}),$$

and

$$\nabla\Gamma(\psi_{\lambda,\xi,\gamma}) = \nabla\Gamma^{(1)}(\psi_{\lambda,\xi,\gamma}) + \nabla\Gamma^{(2),x_0}(\psi_{\lambda,\xi,\gamma}).$$

ˆ 为了获得关于 Φ 的类似分解，我们需要以下引理。

引理 4.12. 给定 $\Lambda>0$ 任意地，则

$$\|\nabla\Gamma^{(1)}(\psi_{\lambda,\xi,\gamma})\|\cdot\|\nabla\Gamma^{(2),x_0}(\psi_{\lambda,\xi,\gamma})\| \to 0 \quad \textit{as } |x_0|\to\infty$$

uniformly in $(\lambda,\xi,\gamma) \in (0,\Lambda]\times\mathbb{R}^m \times S^{2^{[\frac{m}{2}]+1}-1}(\mathbb{S}_m)$.

证明。与引理 4.10类似，我们首先固定 $R>0$ large，使得$\operatorname{supph}^{(1)} \cup \operatorname{supph}^{(2)} \subset B_R$。然后，对于任意 $v \in \mathscr{D}^{1/2}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$,

$$|\big\langle \nabla\Gamma^{(1)}(\psi_{\lambda,\xi,\gamma}), v\big\rangle| \le C^{(1)}\Big(\int_{B_R} \frac{\lambda^{\frac{m-1}{2}}}{\big(\lambda^2+|x-\xi|^2\big)^{\frac{m}{2}}}\cdot|v| + \frac{\lambda^{\frac{m-1}{2}}}{\big(\lambda^2+|x-\xi|^2\big)^{\frac{m-1}{2}}}\cdot|v| d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big)$$

其中 $C^{(1)}>0$ 取决于 $\|\mathfrak{h}^{(1)}\|_{L^\infty}$ 和 $\|\nabla\mathfrak{h}^{(1)}\|_{L^\infty}$。因此，如果 $|\xi|>R$，我们可以看出，

$$\|\nabla\Gamma^{(1)}(\psi_{\lambda,\xi,\gamma})\|\le C^{(1)}\Big(\frac{\Lambda^{\frac{m-1}{2}}}{\operatorname{dist}(\xi,B_R)^m}+\frac{\Lambda^{\frac{m-1}{2}}}{\operatorname{dist}(\xi,B_R)^{m-1}}\Big)$$

对于所有 $\lambda\in(0,\Lambda]$ 以及 $\gamma\in S^{2^{[\frac{m}{2}]+1}-1}$({v4)。类似地，对于某些 $C^{(2)}>0$，如果 $|\xi-x_0|>R$ 我们有

$$\|\nabla\Gamma^{(2),\,x_0}(\psi_{\lambda,\xi,\gamma})\|=\|\nabla\Gamma^{(2)}(\psi_{\lambda,\xi-\xi_0,\gamma})\|\le C^{(2)}\Big(\frac{\Lambda^{\frac{m-1}{2}}}{\operatorname{dist}(\xi-x_0,B_R)^m}+\frac{\Lambda^{\frac{m-1}{2}}}{\operatorname{dist}(\xi-x_0,B_R)^{m-1}}\Big)$$

对于所有 $\lambda\in(0,\Lambda]$ 以及 $\gamma\in S^{2^{[\frac{m}{2}]+1}-1}$({v4)。

注意到

$$\operatorname{dist}(\xi,B_R)+\min\{|\xi|,\,R\}=|\xi|\quad\text{and}\quad\operatorname{dist}(\xi-x_0,B_R)+\min\{|\xi-x_0|,\,R\}=|\xi-x_0|.$$

由三角不等式可直接得出，若 $|x_0|$ 较大（例如 $|x_0|>4R$），则必有 dist(ξ,B_R) $\ge\frac14|x_0|$ 或 dist$(\xi-x_0,B_R)\ge\frac14|x_0|$，因此

$$\|\nabla\Gamma^{(1)}(\psi_{\lambda,\xi,\gamma})\|\cdot\|\nabla\Gamma^{(2),\,x_0}(\psi_{\lambda,\xi,\gamma})\|\le\frac{C\Lambda^{\frac{m-1}{2}}}{|x_0|^{m-1}}\to0$$

当 $|x_0|\to\infty$ 在(λ,ξ,γ) 中一致地 $\in(0,\Lambda]\times\mathbb{R}^m\times S^{2^{[\frac{m}{2}]+1}-1}$(𝕊$_m$。 □

由引理 4.10–4.12 可立即得到 $\hat\Phi(\psi_{\lambda,\xi,\gamma})$ 的如下分解。

推论 4.13. 以下结论成立。

$$\big\langle K_{\psi_{\lambda,\xi,\gamma}}(\nabla\Gamma^{(1)}(\psi_{\lambda,\xi,\gamma})),\nabla\Gamma^{(2),\,x_0}(\psi_{\lambda,\xi,\gamma})\big\rangle\to0$$

当 $|x_0|\to\infty$ 时，在$(\lambda,\xi,\gamma)\in(0,\infty)\times\mathbb{R}^m\times S^{2^{[\frac{m}{2}]+1}-1}$(𝕊$_m$) 中一致地。特别地，

$$\hat\Phi(\psi_{\lambda,\xi,\gamma})=\hat\Phi^{(1)}(\psi_{\lambda,\xi,\gamma})+\hat\Phi^{(2)}(\psi_{\lambda,\xi-x_0,\gamma})+o(1),$$

其中 $o(1)\to0$ 当 $|x_0|\to\infty$时，在(λ,ξ,γ) 中一致地 $\in(0,\infty)\times\mathbb{R}^m\times S^{2^{[\frac{m}{2}]+1}-1}$(𝕊$_m$)。

现在我们可以证明我们的主要结果了。

定理1.4的证明。对于 $m\ge4$，根据命题4.7和4.8，我们可以选择

$$U_R^{(1)}:=\{(\lambda,\xi)\in(0,\infty)\times\mathbb{R}^2:\,\lambda+|\xi|<R\}$$

and

$$U_R^{(2)}:=\{(\lambda,\xi)\in(0,\infty)\times\mathbb{R}^2:\,\lambda+|\xi-x_0|<R\}$$

其中 $R>0$ 取足够大，使得对于所有 $\gamma\in S^{2^{[\frac{m}{2}]+1}-1}$(𝕊$_m$),

$$\min_{(\lambda,\xi)\in\partial U_R^{(1)}}\hat\Phi^{(1)}(\psi_{\lambda,\xi,\gamma})-\min_{(\lambda,\xi)\in U_R^{(1)}}\hat\Phi^{(1)}(\psi_{\lambda,\xi,\gamma})\ge\frac1R$$

and

$$\min_{(\lambda,\xi)\in\partial U_R^{(2)}} \hat{\Phi}^{(2)}(\psi_{\lambda,\xi-x_0,\gamma}) - \min_{(\lambda,\xi)\in U_R^{(2)}} \hat{\Phi}^{(2)}(\psi_{\lambda,\xi-x_0,\gamma}) \geq \frac{1}{R}$$

因此，由推论 4.13 可知，对于 $|x_0| \gg R$，存在 $\delta > 0$，使得 $\{(\lambda,\xi) : \hat{\Phi}(\psi_{\lambda,\xi,\gamma}) < -\delta\}$ 在 $U_{(1)R}$和 $U_R^{(2)}$中分别有两个不连通的分支。因此，应用定理 2.6，$\Phi(\psi_{\lambda,\xi,\gamma})$ 的两个不同局部极小值会导出方程 (4.2) 的两个不同解（从而通过立体投影的拉回得到方程 (4.1) 的两个不同解），当 ε 取较小值时亦然。 □

A 附录

A.1 (k,p)-初等矩阵的解释

这里我们指出，(k,p)-初等矩阵 $k = 1$、.....、m 的线性组合可以生成一个非常大的对角矩阵族。为了更清楚地考察这一点，我们将范围限定在三维空间中（即使处于其他维数，也不会造成混淆）。首先，令 $\mathbf{h}^{(k)}$, $k = 1, 2, 3$ 为在同一点 $\xi = (\xi_1, \xi_2, \xi_3)$ 处的 (k, p)-初等矩阵，$\in \mathbb{R}^3$其中相应的 $c_k \neq 0$ 为非零系数，即我们有局部表达式。

$$h^{(1)}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h_{22}^{(1)}(x) & 0 \\ 0 & 0 & h_{33}^{(1)}(x) \end{pmatrix}, \quad h^{(2)}(x) = \begin{pmatrix} h_{11}^{(2)}(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33}^{(2)}(x) \end{pmatrix}$$

and

$$h^{(3)}(x) = \begin{pmatrix} h_{11}^{(3)}(x) & 0 & 0 \\ 0 & h_{22}^{(3)}(x) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$\left.\begin{aligned} h_{22}^{(1)}(x) &= a_{22}^{(1)} + c_1(x_1-\xi_1) + c_2^{(1)}(x_2-\xi_2) + o(|x-\xi|^p) \\ h_{33}^{(1)}(x) &= a_{33}^{(1)} + c_1(x_1-\xi_1) + c_3^{(1)}(x_3-\xi_3) + o(|x-\xi|^p) \end{aligned}\right\} \text{ for } h^{(1)},$$

$$\left.\begin{aligned} h_{11}^{(2)}(x) &= a_{11}^{(2)} + c_1^{(2)}(x_1-\xi_1) + c_2(x_2-\xi_2) + o(|x-\xi|^p) \\ h_{33}^{(2)}(x) &= a_{33}^{(2)} + c_2(x_2-\xi_2) + c_3^{(2)}(x_3-\xi_3) + o(|x-\xi|^p) \end{aligned}\right\} \text{ for } h^{(2)},$$

$$\left.\begin{aligned} h_{11}^{(3)}(x) &= a_{11}^{(3)} + c_1^{(3)}(x_1-\xi_1) + c_3(x_3-\xi_3) + o(|x-\xi|^p) \\ h_{22}^{(3)}(x) &= a_{22}^{(3)} + c_2^{(3)}(x_2-\xi_2) + c_3(x_3-\xi_3) + o(|x-\xi|^p) \end{aligned}\right\} \text{ for } h^{(3)},$$

where $a_{ii}^{(k)}$ and $c_i^{(k)}$, $i, k = 1, 2, 3$, are some given real constants. Then, a linear combination of $h^{(k)}$, say for instance $\alpha h^{(1)} + \beta h^{(2)} + \gamma h^{(3)}$ for $\alpha, \beta, \gamma \in \mathbb{R}$, gives rise to a matrix of the form (in local expansion around the point ξ)

$$\begin{pmatrix} a_{11} + \boldsymbol{b}_1 \cdot (x-\xi) & 0 & 0 \\ 0 & a_{22} + \boldsymbol{b}_2 \cdot (x-\xi) & 0 \\ 0 & 0 & a_{33} + \boldsymbol{b}_3 \cdot (x-\xi) \end{pmatrix} + o(|x-\xi|^p) \tag{A.1}$$

with $a_{11}=\beta a_{11}^{(2)}+\gamma a_{11}^{(3)}$, $a_{22}=\alpha a_{22}^{(1)}+\gamma a_{22}^{(3)}$, $a_{33}=\alpha a_{33}^{(1)}+\beta a_{33}^{(2)}$ characterizing the zero-order terms and the vector-valued coefficients

$$\boldsymbol{b}_1=(\beta c_1^{(2)}+\gamma c_1^{(3)},\beta c_2,\gamma c_3),\quad \boldsymbol{b}_2=(\alpha c_1,\alpha c_2^{(1)}+\gamma c_2^{(3)},\gamma c_3),\quad \boldsymbol{b}_3=(\alpha c_1,\beta c_2,\alpha c_3^{(1)}+\beta c_3^{(2)})$$

characterizing the first-order terms. Such specific local expansion is satisfied by a large class of diagonal matrix functions. Moreover, notice that (A.1) is defined only around one spatial point, we can consider such kind of expressions around an arbitrary finite number of spatial points. And even more, there is no restriction on the global behavior of $h^{(k)}$, $k=1,2,3$. This is why we use the word "*elementary*" in Definition 1.3.

A.2 一些基本估计

下文中，当存在常数 $C>0$ 使得 $f\le Cg$ 时，我们用记号 $f\lesssim g$ 表示两个函数 f 和 g。于是，根据 (2.10) 中 $\psi_{1,\ 0,\ \gamma}$ 的显式表达式，我们得到

$$|\psi_{1,0,\gamma}(x)|\lesssim\frac{1}{(1+|x|^2)^{\frac{m-1}{2}}}\quad\text{and}\quad|\nabla\psi_{1,0,\gamma}(x)|\lesssim\frac{1}{(1+|x|^2)^{\frac{m}{2}}}.$$

给定 $R>\delta>0$，令 $\lambda\searrow 0$ 为一个参数，对于任意 $\varphi\in\mathscr{D}^{\frac12}(\mathbb{R}^m,\ \mathbb{S}(\mathbb{R}^m))$ 且有 $\|\varphi\|=1$，我们估计：

$$\begin{aligned}\Big|\operatorname{Re}\int_{|x|\ge\frac{\delta}{\lambda}}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{1,0,\gamma},\varphi)_{g_{\mathbb{R}^m}}d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big|&\le\Big(\int_{|x|>\frac{\delta}{\lambda}}|\nabla_{\partial_i}\psi_{1,0,\gamma}|^{\frac{2m}{m+1}}d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big)^{\frac{m+1}{2m}}|\varphi|_{\frac{2m}{m-1}}\\&\lesssim\Big(\int_{\frac{\delta}{\lambda}}^{\infty}\frac{r^{m-1}}{(1+r^2)^{\frac{m^2}{m+1}}}dr\Big)^{\frac{m+1}{2m}}\\&\lesssim\lambda^{\frac{m-1}{2}}\qquad\text{for } m\ge2,\end{aligned}\tag{A.2}$$

$$\begin{aligned}\Big|\operatorname{Re}\int_{|x|\le\frac{\delta}{\lambda}}x_j(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{1,0,\gamma},\varphi)_{g_{\mathbb{R}^m}}d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big|&\lesssim\Big(\int_0^{\frac{\delta}{\lambda}}\frac{r^{\frac{2m}{m+1}+m-1}}{(1+r^2)^{\frac{m^2}{m+1}}}dr\Big)^{\frac{m+1}{2m}}\\&=\Big(\int_0^1+\int_1^{\frac{\delta}{\lambda}}\frac{r^{\frac{2m}{m+1}+m-1}}{(1+r^2)^{\frac{m^2}{m+1}}}dr\Big)^{\frac{m+1}{2m}}\\&\lesssim\begin{cases}\lambda^{-\frac12}&m=2,\\|\ln\lambda|^{\frac23}&m=3,\\1&m\ge4,\end{cases}\end{aligned}\tag{A.3}$$

$$\Big|\operatorname{Re}\int_{\frac{\delta}{\lambda}\le|x|\le\frac{R}{\lambda}}x_j(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{1,0,\gamma},\varphi)d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big|\lesssim\Big(\int_{\frac{\delta}{\lambda}}^{\frac{R}{\lambda}}\frac{r^{\frac{2m}{m+1}+m-1}}{(1+r^2)^{\frac{m^2}{m+1}}}dr\Big)^{\frac{m+1}{2m}}$$

$$\lesssim \begin{cases} \lambda^{-\frac{1}{2}} & m=2, \\ 1 & m=3, \\ \lambda^{\frac{m-3}{2}} & m\geq 4, \end{cases} \tag{A.4}$$

$$\begin{aligned} \Big|\operatorname{Re}\int_{|x|\leq\frac{\delta}{\lambda}} x_j x_k(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma},\varphi)d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big| &\lesssim \Big(\int_0^{\frac{\delta}{\lambda}} \frac{r^{\frac{4m}{m+1}+m-1}}{(1+r^2)^{\frac{m^2}{m+1}}}dr\Big)^{\frac{m+1}{2m}} \\ &= \Big(\int_0^1+\int_1^{\frac{\delta}{\lambda}} \frac{r^{\frac{4m}{m+1}+m-1}}{(1+r^2)^{\frac{m^2}{m+1}}}dr\Big)^{\frac{m+1}{2m}} \\ &\lesssim \begin{cases} \lambda^{\frac{m-5}{2}} & 2\leq m\leq 4, \\ |\ln\lambda|^{\frac{3}{5}} & m=5, \\ 1 & m\geq 6. \end{cases} \end{aligned} \tag{A.5}$$

$$\begin{aligned} \Big|\operatorname{Re}\int_{|x|\leq\frac{\delta}{\lambda}} (\partial_k \cdot_{g_{\mathbb{R}^m}} \psi_{1,0,\gamma},\varphi)d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big| &\lesssim \Big(\int_0^{\frac{\delta}{\lambda}} \frac{r^{m-1}}{(1+r^2)^{\frac{m(m-1)}{m+1}}}dr\Big)^{\frac{m+1}{2m}} \\ &= \Big(\int_0^1+\int_1^{\frac{\delta}{\lambda}} \frac{r^{m-1}}{(1+r^2)^{\frac{m(m-1)}{m+1}}}dr\Big)^{\frac{m+1}{2m}} \\ &\lesssim \begin{cases} \lambda^{-\frac{1}{2}} & m=2, \\ |\ln\lambda|^{\frac{2}{3}} & m=3, \\ 1 & m\geq 4, \end{cases} \end{aligned} \tag{A.6}$$

$$\begin{aligned} \Big|\operatorname{Re}\int_{\frac{\delta}{\lambda}<|x|\leq\frac{R}{\lambda}} (\partial_k \cdot_{g_{\mathbb{R}^m}} \psi_{1,0,\gamma},\varphi)d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big| &\lesssim \Big(\int_{\frac{\delta}{\lambda}}^{\frac{R}{\lambda}} \frac{r^{m-1}}{(1+r^2)^{\frac{m(m-1)}{m+1}}}dr\Big)^{\frac{m+1}{2m}} \\ &\lesssim \begin{cases} \lambda^{-\frac{1}{2}} & m=2, \\ 1 & m=3, \\ \lambda^{\frac{m-3}{2}} & m\geq 4, \end{cases} \end{aligned} \tag{A.7}$$

$$\begin{aligned} \Big|\operatorname{Re}\int_{|x|\leq\frac{\delta}{\lambda}} x_j(\partial_k \cdot_{g_{\mathbb{R}^m}} \psi_{1,0,\gamma},\varphi)d\operatorname{vol}_{g_{\mathbb{R}^m}}\Big| &\lesssim \Big(\int_0^{\frac{\delta}{\lambda}} \frac{r^{\frac{2m}{m+1}+m-1}}{(1+r^2)^{\frac{m(m-1)}{m+1}}}dr\Big)^{\frac{m+1}{2m}} \\ &= \Big(\int_0^1+\int_1^{\frac{\delta}{\lambda}} \frac{r^{\frac{2m}{m+1}+m-1}}{(1+r^2)^{\frac{m(m-1)}{m+1}}}dr\Big)^{\frac{m+1}{2m}} \\ &\lesssim \begin{cases} \lambda^{\frac{m-5}{2}} & 2\leq m\leq 4, \\ |\ln\lambda|^{\frac{3}{5}} & m=5, \\ 1 & m\geq 6. \end{cases} \end{aligned} \tag{A.8}$$

特别地，对于 $m=5$，我们有 $\frac{2m}{m+1}=\frac{5}{3}$，并且估计如下：对于固定的 $p>2$，

$$\Big(\int_{|x|\le\frac{\delta}{\lambda}}|x|^{\frac{5p}{3}}|\nabla_{\partial_i}\psi_{1,0,\gamma}|^{\frac{5}{3}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\Big)^{\frac{3}{5}}\lesssim\Big(\int_0^{\frac{\delta}{\lambda}}\frac{r^{\frac{5p}{3}+4}}{(1+r^2)^{\frac{25}{6}}}dr\Big)^{\frac{3}{5}}\lesssim\lambda^{2-p}\tag{A.9}$$

and

$$\Big(\int_{|x|\le\frac{\delta}{\lambda}}|x|^{\frac{5(p-1)}{3}}|\psi_{1,0,\gamma}|^{\frac{5}{3}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\Big)^{\frac{3}{5}}\lesssim\Big(\int_0^{\frac{\delta}{\lambda}}\frac{r^{\frac{5(p-1)}{3}+4}}{(1+r^2)^{\frac{10}{3}}}dr\Big)^{\frac{3}{5}}\lesssim\lambda^{2-p}.\tag{A.10}$$

A.3 引理 4.6 和命题 4.7 的证明

引理 4.6 的证明。设 $z\in\mathcal{M}$，注意到 K_z 表示 $\nabla^2\mathcal{J}_0(z)$ 在 $\mathcal{W}_z=T_z\mathcal{M}^\perp$ 上的逆，通过令 $\bar w_z=-K_z(\nabla\Gamma(z))$，我们得到

$$\langle K_z(\nabla\Gamma(z)),\nabla\Gamma(z)\rangle=-\langle\nabla\Gamma(z),\bar w_z\rangle=\nabla^2\mathcal{J}_0(z)[\bar w_z,\bar w_z].\tag{A.11}$$

根据引理 4.3 中 Γ 的表达式以及 $\nabla\Gamma(z)\in\mathcal{W}_z$ 这一事实，我们得到（通过分部积分）

$$\begin{aligned}\langle\nabla\Gamma(z),v\rangle &= \mathrm{Re}\int_{\mathbb{R}^m}\frac{\mathrm{tr}\,h}{2}\Big[\frac{1}{2}\big[(D_{g_{\mathbb{R}^m}}z,v)_{g_{\mathbb{R}^m}}+(z,D_{g_{\mathbb{R}^m}}v)_{g_{\mathbb{R}^m}}\big]-|z|^{2^*-2}_{g_{\mathbb{R}^m}}(z,v)_{g_{\mathbb{R}^m}}\Big]\\ &\quad-\frac{1}{4}\sum_i h_{ii}\Big((\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}z,v)_{g_{\mathbb{R}^m}}+(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}v,z)_{g_{\mathbb{R}^m}}\Big)d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &= \mathrm{Re}\int_{\mathbb{R}^m}\frac{\mathrm{tr}\,h}{2}\Big[(D_{g_{\mathbb{R}^m}}z,v)_{g_{\mathbb{R}^m}}-|z|^{2^*-2}_{g_{\mathbb{R}^m}}(z,v)_{g_{\mathbb{R}^m}}\Big]d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &\quad+\frac{1}{4}\,\mathrm{Re}\int_{\mathbb{R}^m}(\nabla(\mathrm{tr}\,h)\cdot_{g_{\mathbb{R}^m}}z,v)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &\quad-\frac{1}{2}\sum_i\mathrm{Re}\int_{\mathbb{R}^m}h_{ii}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}z,v)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &\quad-\frac{1}{4}\sum_i\mathrm{Re}\int_{\mathbb{R}^m}\partial_i h_{ii}(\partial_i\cdot_{g_{\mathbb{R}^m}}z,v)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\end{aligned}\tag{A.12}$$

对于所有 $v\in\mathcal{W}_z$，其中我们使用了以下事实

$$\begin{aligned}\mathrm{Re}\int_{\mathbb{R}^m}(\mathrm{tr}\,h)(z,D_{g_{\mathbb{R}^m}}v)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}&=\mathrm{Re}\int_{\mathbb{R}^m}(\mathrm{tr}\,h)(D_{g_{\mathbb{R}^m}}z,v)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &\quad+\mathrm{Re}\int_{\mathbb{R}^m}(\nabla(\mathrm{tr}\,h)\cdot_{g_{\mathbb{R}^m}}z,v)_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\end{aligned}$$

and

$$\partial_i\,\mathrm{Re}(\partial_i\cdot_{g_{\mathbb{R}^m}}z,v)=\mathrm{Re}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}z,v)-\mathrm{Re}(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}v,z).$$

注意， $z \in \mathcal{M}$ 满足该方程

$$D_{g_{\mathbb{R}^m}} z = |z|_{g_{\mathbb{R}^m}}^{2^*-2} z \quad \text{on } \mathbb{R}^m,$$

我们有

$$\nabla\Gamma(z) = -\frac{1}{2}\sum_i h_{ii}\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i} z + \frac{1}{4}\nabla(\operatorname{tr} h) \cdot_{g_{\mathbb{R}^m}} z - \frac{1}{4}\sum_i \partial_i h_{ii}\partial_i \cdot_{g_{\mathbb{R}^m}} z$$

对于所有 $z \in \mathcal{M}$。此外，我们有 $\bar{w}_z$ 满足该方程

$$\nabla^2 \mathcal{J}_0(z)[\bar{w}_z] = -\nabla\Gamma(z). \tag{A.13}$$

其中

$$\nabla^2 \mathcal{J}_0(z)[\bar{w}_z] = D_{g_{\mathbb{R}^m}} \bar{w}_z - |z|_{g_{\mathbb{R}^m}}^{2^*-2}\bar{w}_z - (2^*-2)|z|_{g_{\mathbb{R}^m}}^{2^*-4}\operatorname{Re}(z, \bar{w}_z)z.$$

接下来，我们将把$z = \psi_{\lambda,\xi,\gamma}$显式代入 (A.13)。为此，令 $\bar{w}_{\lambda,\xi,\gamma} = \bar{w}_{\psi_{\lambda,\xi,\gamma}}$ 以及 $w^*_{\lambda,\xi,\gamma}(x) = \lambda^{\frac{m-1}{2}}\bar{w}_{\lambda,\xi,\gamma}(\lambda x+\xi)$。那么，通过变量替换，我们有 $w^*_{\lambda,\xi,\gamma}$ 是解。

$$\begin{aligned}\nabla^2 \mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{\lambda,\xi,\gamma}] \;=\; & \frac{1}{2}\sum_i h_{ii}(\lambda x+\xi)\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma} - \frac{\lambda}{4}\nabla(\operatorname{tr} h)(\lambda x+\xi) \cdot_{g_{\mathbb{R}^m}} \psi_{1,0,\gamma} \\ & + \frac{\lambda}{4}\sum_i \partial_i h_{ii}(\lambda x+\xi)\partial_i \cdot_{g_{\mathbb{R}^m}} \psi_{1,0,\gamma}\end{aligned} \tag{A.14}$$

令 $\lambda \to 0$ 取极限，由 (A.2)、(A.6)、(A.7) 以及 $K_{\psi_{1,0,\gamma}} = \nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})^{-1}$ 的连续性，我们看到 $w^*_{\lambda,\xi,\gamma} \to w^*_{0,\xi,\gamma}$ 在 $\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m, \mathbb{S}(\mathbb{R}^m))$ 中，其中 $w^*_{0,\xi,\gamma}$ 满足方程

$$\nabla^2 \mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma}] = \frac{1}{2}\sum_i h_{ii}(\xi)\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma}. \tag{A.15}$$

方程 (A.15) 的解并不唯一。它在加上 $\ker \nabla^2\mathcal{J}_0(\psi_{1,\,0,\,\gamma}) = T_{\psi_{1,\,0,\,\gamma}}\mathcal{M}$中的元素的意义下唯一确定。然而，通过再次作变量代换，我们得到该量

$$\lim_{\lambda\to 0}\langle \nabla\Gamma(\psi_{\lambda,\xi,\gamma}), \bar{w}_{\lambda,\xi,\gamma}\rangle = -\frac{1}{2}\sum_i h_{ii}(\xi)\operatorname{Re}\int_{\mathbb{R}^m}(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma}, w^*_{0,\xi,\gamma})_{g_{\mathbb{R}^m}} d\operatorname{vol}_{g_{\mathbb{R}^m}}$$

不依赖于 $w^*_{0,\xi,\gamma}$ 在 $T_{\psi_{1,0,\gamma}}\mathcal{M}$ 的投影（因为 $\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma} \in T_{\psi_{1,0,\gamma}}\mathcal{M}^\perp$）。因此，我们只需构造出方程(A.15) 的一个解。注意到方程 (A.15) 关于 h 是线性的，容易验证

$$w^*_{0,\xi,\gamma}(x) = \frac{1}{2}\sum_i h_{ii}(\xi)x_i\nabla_{\partial_i}\psi_{1,0,\gamma}(x)$$

确实是一个解。此外，由(4.10)、(4.11)以及Clifford乘法规则可知，

$$\operatorname{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma}, x_i\nabla_{\partial_i}\psi_{1,0,\gamma}) = 0$$

对于 $i=1,\dots,m$ 和

$$\begin{aligned}
&\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma}, x_j\nabla_{\partial_j}\psi_{1,0,\gamma})\\
&\quad= \frac{m^m x_i x_j}{\left(1+|x|^2\right)^{m+1}}\,\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} (1-x)\cdot_{g_{\mathbb{R}^m}} \gamma, \partial_j \cdot_{g_{\mathbb{R}^m}} \gamma) - \frac{m^m x_j^2}{\left(1+|x|^2\right)^{m+1}}\\
&\quad= \frac{m^m x_i x_j}{\left(1+|x|^2\right)^{m+1}}\,\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \gamma, \partial_j \cdot_{g_{\mathbb{R}^m}} \gamma)\\
&\qquad- \sum_{\substack{k=1\\ k\neq i,\ k\neq j}}^{m} \frac{m^m x_i x_j x_k}{\left(1+|x|^2\right)^{m+1}}\,\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \partial_k \cdot_{g_{\mathbb{R}^m}} \gamma, \partial_j \cdot_{g_{\mathbb{R}^m}} \gamma) - \frac{m^m x_j^2}{\left(1+|x|^2\right)^{m+1}}
\end{aligned}$$

对于 $i\neq j$。由于 $\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \gamma, \partial_j \cdot_{g_{\mathbb{R}^m}} \gamma)\equiv 0$ 且 $\mathrm{Re}(\partial_i \cdot_{g_{\mathbb{R}^m}} \partial_k \cdot_{g_{\mathbb{R}^m}} \gamma, \partial_j \cdot_{g_{\mathbb{R}^m}} \gamma)$ 是常数，我们可推出

$$\begin{aligned}
\lim_{\lambda\to 0}\langle \nabla\Gamma(\psi_{\lambda,\xi,\gamma}), \bar{w}_{\lambda,\xi,\gamma}\rangle &= -\frac{1}{2}\sum_i h_{ii}(\xi)\,\mathrm{Re}\int_{\mathbb{R}^m}(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma}, w^*_{0,\xi,\gamma})_{g_{\mathbb{R}^m}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\
&= \frac{1}{4}\sum_{\substack{i,j\\ i\neq j}} h_{ii}(\xi)h_{jj}(\xi)\int_{\mathbb{R}^m}\frac{m^m x_j^2}{\left(1+|x|^2\right)^{m+1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\
&= \frac{1}{4}\int_{\mathbb{R}^m}\frac{m^m x_1^2}{\left(1+|x|^2\right)^{m+1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\left((\mathrm{tr}\,h)^2-\mathrm{tr}(h^2)\right)(\xi)
\end{aligned}$$

注意，$|x|^2 = x_1^2+\cdots+x_m^2$ 和

$$\int_{\mathbb{R}^m}\frac{x_1^2}{\left(1+|x|^2\right)^{m+1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}} = \cdots = \int_{\mathbb{R}^m}\frac{x_m^2}{\left(1+|x|^2\right)^{m+1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}$$

我们有

$$\lim_{\lambda\to 0}\langle \nabla\Gamma(\psi_{\lambda,\xi,\gamma}), \bar{w}_{\lambda,\xi,\gamma}\rangle = \frac{m^{m-1}}{4}\int_{\mathbb{R}^m}\frac{|x|^2}{\left(1+|x|^2\right)^{m+1}} d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\left((\mathrm{tr}\,h)^2-\mathrm{tr}(h^2)\right)(\xi)$$

如所需。 □

ˆ 命题 4.7 的证明。我们的步骤是求出 $\Phi(\psi_{\lambda,\xi,\gamma})$ 关于λ 在0 的邻域内的展开。为此，我们需要分别求出泛函 $\Phi(\psi_{\lambda,\xi,\gamma})$ 和 $\left\langle K_{\psi_{\lambda,\xi,\gamma}}(\nabla\Gamma(\psi_{\lambda,\xi,\gamma}))\ \nabla\Gamma(\psi_{\lambda,\xi,\gamma})\right\rangle$ 的展开。

对于 $m\geq 3$，不失一般性，我们可以假设 $\mathrm{h} = \mathrm{diag}(\mathrm{h}_{11},\dots,\mathrm{h}_{m-1m-1},0)$ 在点 $\xi\in\mathbb{R}^m$ 附近是 (m,p)-初等的，且 $\partial_m \mathrm{h}_{ii}(\xi)\equiv c_m\neq 0$，对于 $1\leq i\leq m-1$。其他情形也可以用非常类似的方法处理。

根据定义 1.3 并通过变量替换 $x\mapsto \lambda x+\xi$，我们得出

$$h_{ii}(\lambda x+\xi) = h_{ii}(\xi)+\lambda(c_i x_i + c_m x_m)+o(|\lambda x|^p)$$

对于 $i=1,\dots,m-1$，因此

$$\begin{aligned}\big(\operatorname{tr}(h^2)-(\operatorname{tr}h)^2\big)(\lambda x+\xi) &= -\sum_{\substack{i,j\\ i\neq j}}(h_{ii}h_{jj})(\lambda x+\xi)\\ &= -\sum_{\substack{i,j\\ i\neq j}}\Big[h_{ii}(\xi)h_{jj}(\xi)+\lambda h_{ii}(\xi)(c_jx_j+c_mx_m)+\lambda h_{jj}(\xi)(c_ix_i+c_mx_m)\\ &\qquad +\lambda^2\big(c_ic_jx_ix_j+c_ic_mx_ix_m+c_jc_mx_jx_m+c_m^2x_m^2\big)+o(|\lambda x|^p)\Big].\end{aligned}$$

阶数为 λ 的项在 $\Phi(\psi_{\lambda,\ \xi,\ \gamma})$ 中的系数形如

$$\int_{\mathbb{R}^m}\frac{x_i}{\big(1+|x|^2\big)^m}d\operatorname{vol}_{g_{\mathbb{R}^m}}$$

它们全都为零。而 λ^2 的系数属于以下类型

$$\int_{\mathbb{R}^m}\frac{x_ix_j}{\big(1+|x|^2\big)^m}d\operatorname{vol}_{g_{\mathbb{R}^m}}$$

且此类积分仅在 $i=j$ 时非零。因此，由引理 4.5 和 $p\geq 2$，我们可得

$$\Phi(\psi_{\lambda,\xi,\gamma})=C_0\big(\operatorname{tr}(h^2)-(\operatorname{tr}h)^2\big)(\xi)-Ac_m^2\lambda^2+o(\lambda^2) \tag{A.16}$$

其中

$$A=\frac{m^{m-2}(m-1)(m-2)}{16}\int_{\mathbb{R}^m}\frac{|x|^2}{\big(1+|x|^2\big)^m}d\operatorname{vol}_{g_{\mathbb{R}^m}}.$$

接下来需要求出 $\big\langle K_{\psi_{\lambda,\xi,\gamma}}(\nabla\Gamma(\psi_{\lambda,\xi,\gamma}))\ \nabla\Gamma(\psi_{\lambda,\xi,\gamma})\big\rangle$的展开式。由公式 (A.14) 和 $\mathbf{h}_{ii}$ 的局部展开，我们希望将 $w^*_{\lambda,\xi,\gamma}$ 按 λ 的幂展开。

我们首先假设 $m\geq 6$。然后，根据附录中的计算（A.2）至（A.8），可得

$$w^*_{\lambda,\xi,\gamma}=w^*_{0,\xi,\gamma}+\lambda\nu^*_{0,\xi,\gamma}+\lambda^2\mu^*_{0,\xi,\gamma}+o(\lambda^2)\quad \text{in } \mathscr{D}^{\frac{1}{2}}(\mathbb{R}^m,\mathbb{S}(\mathbb{R}^m))$$

其中 $w^*_{0,\xi,\gamma}$, $\nu^*_{0,\xi,\gamma}$ 和 $\mu^*_{0,\xi,\gamma}$ 满足以下方程

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma}]=\frac{1}{2}\sum_{i=1}^{m-1}h_{ii}(\xi)\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{1,0,\gamma}, \tag{A.17}$$

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma}]=\frac{1}{2}\sum_{i=1}^{m-1}(c_ix_i+c_mx_m)\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{1,0,\gamma}-\frac{m-1}{4}c_m\partial_m\cdot_{g_{\mathbb{R}^m}}\psi_{1,0,\gamma} \tag{A.18}$$

并且（由于 $\mathbf{h}_{ii}$ 在 ξ 处的 Hessian 矩阵恒为零）

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{0,\xi,\gamma}]=0. \tag{A.19}$$

And now we have

$$\begin{aligned}\big\langle K_{\psi_{\lambda,\xi,\gamma}}(\nabla\Gamma(\psi_{\lambda,\xi,\gamma})),\nabla\Gamma(\psi_{\lambda,\xi,\gamma})\big\rangle &= \nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{\lambda,\xi,\gamma},w^*_{\lambda,\xi,\gamma}]\\ &= \nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},w^*_{0,\xi,\gamma}]+2\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}]\lambda\\ &\quad+\big[2\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},\mu^*_{0,\xi,\gamma}]+\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}]\big]\lambda^2+o(\lambda^2).\end{aligned}$$

Next, we will calculate the coefficients in front of each power of λ:

1. 零阶项
正如引理 4.6 所示，我们有

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma}, w^*_{0,\xi,\gamma}] = C_1\big(\operatorname{tr}(h^2) - (\operatorname{tr} h)^2\big)(\xi).$$

2. 一阶项
利用方程 (A.18)，我们得到

$$\begin{aligned}\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma}, \nu^*_{0,\xi,\gamma}] = \frac{1}{2}\sum_{i=1}^{m-1}\operatorname{Re}\int_{\mathbb{R}^m}(c_ix_i + c_mx_m)(\partial_i \cdot_{g_{\mathbb{R}^m}} \nabla_{\partial_i}\psi_{1,0,\gamma}, w^*_{0,\xi,\gamma})_{g_{\mathbb{R}^m}} d\operatorname{vol}_{g_{\mathbb{R}^m}}\\ - \frac{(m-1)c_m}{4}\operatorname{Re}\int_{\mathbb{R}^m}(\partial_m \cdot_{g_{\mathbb{R}^m}} \psi_{1,0,\gamma}, w^*_{0,\xi,\gamma})_{g_{\mathbb{R}^m}} d\operatorname{vol}_{g_{\mathbb{R}^m}}.\end{aligned}$$

注意，$w^*_{0,\xi,\gamma}$ 可以明确表述为

$$w^*_{0,\xi,\gamma} = \frac{1}{2}\sum_{i=1}^{m-1} h_{ii}(\xi)x_i\nabla_{\partial_i}\psi_{1,0,\gamma},$$

我们发现 $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma}, \nu^*_{0,\xi,\gamma}]$ 中的积分属于这一类型

$$\int_{\mathbb{R}^m}\frac{x_ix_j}{\big(1+|x|^2\big)^m}d\operatorname{vol}_{g_{\mathbb{R}^m}}, \quad \int_{\mathbb{R}^m}\frac{x_i^2x_j}{\big(1+|x|^2\big)^{m+1}}d\operatorname{vol}_{g_{\mathbb{R}^m}}, \quad \int_{\mathbb{R}^m}\frac{x_i^2x_kx_j}{\big(1+|x|^2\big)^{m+1}}d\operatorname{vol}_{g_{\mathbb{R}^m}},$$

$$\int_{\mathbb{R}^m}\frac{x_i^3}{\big(1+|x|^2\big)^m}d\operatorname{vol}_{g_{\mathbb{R}^m}} \quad \text{and} \quad \int_{\mathbb{R}^m}\frac{x_ix_jx_kx_l}{\big(1+|x|^2\big)^m}d\operatorname{vol}_{g_{\mathbb{R}^m}}$$

其中 $i \neq j$、$i \neq k$、$j \neq k$ 和 $k \neq l$ 均为零。因此，一阶项恒等于零。

3. 二阶项
根据 (A.19)，我们只需计算 $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma}, \nu^*_{0,\xi,\gamma}]$。为此，只需解方程(A.18)。与求解方程(A.15) 类似，我们只需构造 (A.18) 的一个解。特别地，可以验证

$$\nu^*_{0,\xi,\gamma} = \frac{1}{4}\Big(\sum_{i=1}^{m-1} c_ix_i^2\nabla_{\partial_i}\psi_{1,0,\gamma} - c_mx_m^2\nabla_{\partial_m}\psi_{1,0,\gamma}\Big) - \frac{m-1}{4}c_mx_m\psi_{1,0,\gamma}$$

求解方程 (A.18)。因此，根据 $\psi_{1,\ 0,\ \gamma}$ 的显式表达式，我们有

$$\begin{aligned}\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}] &= \frac{1}{2}\sum_{i=1}^{m-1}\mathrm{Re}\int_{\mathbb{R}^m}(c_ix_i+c_mx_m)(\partial_i\cdot_{g_{\mathbb{R}^m}}\nabla_{\partial_i}\psi_{1,0,\gamma},\nu^*_{0,\xi,\gamma})_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &\quad-\frac{(m-1)c_m}{4}\mathrm{Re}\int_{\mathbb{R}^m}(\partial_m\cdot_{g_{\mathbb{R}^m}}\psi_{1,0,\gamma},\nu^*_{0,\xi,\gamma})_{g_{\mathbb{R}^m}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &=\frac{1}{2}\sum_{i=1}^{m-1}\int_{\mathbb{R}^m}\left(\frac{m^mc_m^2x_m^4}{4(1+|x|^2)^{m+1}}-\frac{m^{m-1}(m-1)c_m^2x_m^2}{4(1+|x|^2)^m}\right)d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &\quad-\frac{(m-1)c_m^2}{4}\int_{\mathbb{R}^m}\frac{m^{m-1}x_m^2}{4(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\\ &=-\frac{5m^{m-2}(m-1)(m-2)c_m^2}{64}\int_{\mathbb{R}^m}\frac{|x|^2}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}},\end{aligned}$$

其中最后一个等式由这一事实得出

$$\int_{\mathbb{R}^m}\frac{x_m^4}{(1+|x|^2)^{m+1}}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}=\frac{3(m+2)}{8m}\int_{\mathbb{R}^m}\frac{x_m^2}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}\ \text{ for } m\geq 3.$$

现在，结合（A.16），我们得到

$$\hat{\Phi}(\psi_{\lambda,\xi,\gamma})=-\frac{3m^{m-2}(m-1)(m-2)c_m^2}{128}\lambda^2\int_{\mathbb{R}^m}\frac{|x|^2}{(1+|x|^2)^m}d\,\mathrm{vol}_{g_{\mathbb{R}^m}}+o(\lambda^2)$$

如所期望的那样。

对于 $m=5$，根据 (A.2)-(A.10)，我们得到

$$w^*_{\lambda,\xi,\gamma}=w^*_{0,\xi,\gamma}+\lambda\nu^*_{0,\xi,\gamma}+\mu^*_{\lambda,\xi,\gamma}+\rho^*_{\lambda,\xi,\gamma}\quad\text{in }\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^5,\mathbb{S}(\mathbb{R}^5)) \tag{A.20}$$

where

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma}]=\frac{1}{2}\sum_{i=1}^4\boldsymbol{h}_{ii}(\xi)\partial_i\cdot_{g_{\mathbb{R}^5}}\nabla_{\partial_i}\psi_{1,0,\gamma}$$

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma}]=\frac{1}{2}\sum_{i=1}^4(c_ix_i+c_5x_5)\partial_i\cdot_{g_{\mathbb{R}^5}}\nabla_{\partial_i}\psi_{1,0,\gamma}-c_5\partial_5\cdot_{g_{\mathbb{R}^5}}\psi_{1,0,\gamma}$$

$$\begin{aligned}\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{\lambda,\xi,\gamma}]=(1-\eta_\lambda)\Bigg[&\frac{1}{2}\sum_{i=1}^5\boldsymbol{h}_{ii}(\lambda x+\xi)\partial_i\cdot_{g_{\mathbb{R}^5}}\nabla_{\partial_i}\psi_{1,0,\gamma}\\ &-\frac{\lambda}{4}(\nabla\,\mathrm{tr}\,\boldsymbol{h})(\lambda x+\xi)\cdot_{g_{\mathbb{R}^5}}\psi_{1,0,\gamma}+\frac{\lambda}{4}\sum_{i=1}^5\partial_i\boldsymbol{h}_{ii}(\lambda x+\xi)\partial_i\cdot_{g_{\mathbb{R}^5}}\psi_{1,0,\gamma}\\ &-\frac{1}{2}\sum_{i=1}^4\Big[\boldsymbol{h}_{ii}(\xi)+\lambda(c_ix_i+c_5x_5)\Big]\partial_i\cdot_{g_{\mathbb{R}^5}}\nabla_{\partial_i}\psi_{1,0,\gamma}+\lambda c_5\partial_5\cdot_{g_{\mathbb{R}^5}}\psi_{1,0,\gamma}\Bigg]\end{aligned}$$

and

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\rho^*_{\lambda,\xi,\gamma}]=o(\lambda^p)\eta_\lambda\big(|x|^p|\nabla_{\partial_i}\psi_{1,0,\gamma}|+|x|^{p-1}|\psi_{1,0,\gamma}|\big).$$

在上述表达式中， η_λ 是一个光滑截断函数，使得 $\eta_\lambda(x)\equiv 1$ 当 $|x|\le \delta/\lambda$，并且 $\eta_\lambda(x)\equiv 0$ 当 $|x|>2\delta/\lambda$，其中 $\delta>0$ 取为适当小的固定值。我们强调，根据 (A.4)、(A.7)、(A.9)、(A.10) 以及 $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{\lambda,\xi,\gamma}]=O(\lambda^2)$ 和 $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\rho^*_{\lambda,\xi,\gamma}]=o(\lambda^2)$ (A.19) 以及 (A.20)，我们有

利用展开式 (A.20)，我们有

$$\begin{aligned}\big\langle K_{\psi_{\lambda,\xi,\gamma}}(\nabla\Gamma(\psi_{\lambda,\xi,\gamma})),\nabla\Gamma(\psi_{\lambda,\xi,\gamma})\big\rangle &= \nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{\lambda,\xi,\gamma},w^*_{\lambda,\xi,\gamma}]\\ &=\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},w^*_{0,\xi,\gamma}]+2\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}]\lambda\\ &\quad+2\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},\mu^*_{\lambda,\xi,\gamma}]+\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}]\lambda^2+o(\lambda^2).\end{aligned}$$

由于 $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{\lambda,\xi,\gamma}]=O(\lambda^2)$，$2^*=\frac{5}{2}$并且 $\int_{|x|>\frac{\delta}{\lambda}}|w^*_{0,\xi,\gamma}|^{\frac{5}{2}}d$ 体积$_{g_{\mathbb{R}^5}}\to 0$作为 $\lambda\to 0$，我们发现

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},\mu^*_{\lambda,\xi,\gamma}]=\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{\lambda,\xi,\gamma},w^*_{0,\xi,\gamma}]=o(\lambda^2)$$

然后我们看到，一阶项的系数与 $m\ge 6$ 的情形一样为零，而 λ^2 前面的系数为 $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}]$。因此，我们可以像$m\ge 6$ 的情形一样继续推导，从而得到结论。

对于 $m=4$，通过利用 h 上更强的渐近条件，我们发现

$$w^*_{\lambda,\xi,\gamma}=w^*_{0,\xi,\gamma}+\lambda\nu^*_{0,\xi,\gamma}+\mu^*_{\lambda,\xi,\gamma}\quad\text{in }\mathscr{D}^{\frac{1}{2}}(\mathbb{R}^4,\mathbb{S}(\mathbb{R}^4)) \tag{A.21}$$

where

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma}]=\frac{1}{2}\sum_{i=1}^3 h_{ii}(\xi)\partial_i\cdot_{g_{\mathbb{R}^4}}\nabla_{\partial_i}\psi_{1,0,\gamma}$$

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma}]=\frac{1}{2}\sum_{i=1}^3(c_ix_i+c_4x_4)\partial_i\cdot_{g_{\mathbb{R}^4}}\nabla_{\partial_i}\psi_{1,0,\gamma}-\frac{3}{4}c_4\partial_4\cdot_{g_{\mathbb{R}^4}}\psi_{1,0,\gamma}$$

$$\begin{aligned}\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{\lambda,\xi,\gamma}]=(1-\eta_\lambda)\Bigg[&\frac{1}{2}\sum_{i=1}^4 h_{ii}(\lambda x+\xi)\partial_i\cdot_{g_{\mathbb{R}^4}}\nabla_{\partial_i}\psi_{1,0,\gamma}\\ &-\frac{\lambda}{4}\nabla\,\mathrm{tr}(h)(\lambda x+\xi)\cdot_{g_{\mathbb{R}^4}}\psi_{1,0,\gamma}+\frac{\lambda}{4}\sum_{i=1}^4\partial_i h_{ii}(\lambda x+\xi)\partial_i\cdot_{g_{\mathbb{R}^4}}\psi_{1,0,\gamma}\\ &-\frac{1}{2}\sum_{i=1}^3\Big[h_{ii}(\xi)+\lambda(c_ix_i+c_4x_4)\Big]\partial_i\cdot_{g_{\mathbb{R}^4}}\nabla_{\partial_i}\psi_{1,0,\gamma}+\frac{3\lambda c_4}{4}\partial_4\cdot_{g_{\mathbb{R}^4}}\psi_{1,0,\gamma}\Bigg],\end{aligned}$$

并且 η_λ与之前一样是一个光滑截断函数。由 (A.2)、(A.4) 和 (A.7) 可知， $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{\lambda,\xi,\gamma}]=O(\lambda^{\frac{3}{2}})$。然后，利用展开式 (A.21)，我们得到

$$\begin{aligned}\big\langle K_{\psi_{\lambda,\xi,\gamma}}(\nabla\Gamma(\psi_{\lambda,\xi,\gamma})),\nabla\Gamma(\psi_{\lambda,\xi,\gamma})\big\rangle &= \nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{\lambda,\xi,\gamma},w^*_{\lambda,\xi,\gamma}]\\ &=\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},w^*_{0,\xi,\gamma}]+2\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}]\lambda\\ &\quad+2\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},\mu^*_{\lambda,\xi,\gamma}]+\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}]\lambda^2+o(\lambda^2).\end{aligned}$$

由于 $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{\lambda,\xi,\gamma}]=O(\lambda^{\frac{3}{2}})$、$2^*=\frac{8}{3}$以及 $\big(\int_{|x|>\frac{\delta}{\lambda}}|w^*_{0,\xi,\gamma}|^{\frac{8}{3}}d\mathrm{vol}_{g_{\mathbb{R}^4}}\big)^{\frac{3}{8}}=O(\lambda^{\frac{3}{2}})$ 当 $\lambda\to$ 趋于0时，我们发现

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{0,\xi,\gamma},\mu^*_{\lambda,\xi,\gamma}]=\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\mu^*_{\lambda,\xi,\gamma},w^*_{0,\xi,\gamma}]=o(\lambda^2)$$

因此，除零阶项外，第一个非零系数仍位于λ^2之前，且等于$\nabla^2\mathcal{J}_0(\psi_{1,\ 0,\ \gamma})[\nu^*_{0,\xi,\gamma},\nu^*_{0,\xi,\gamma}]$。
接下来我们可以像处理高维情形那样继续。这就完成了整个证明。

□

Remark A.1. In 3-dimension, as is computed in (A.3) and (A.6) in the Appendix, the $L^{\frac{3}{2}}$-norm of the functions $x_j\partial_i \cdot_{g_{\mathbb{R}^3}} \nabla_{\partial_i}\psi_{1,0,\gamma}$ and $\partial_k \cdot_{g_{\mathbb{R}^3}} \psi_{1,0,\gamma}$ become infinite. This causes that $w^*_{\lambda,\xi,\gamma}$ has no more of order λ. Precisely, we see that the equation

$$\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[\nu^*_{0,\xi,\gamma}] = \frac{1}{2}\sum_{i=1}^{2}(c_ix_i + c_3x_3)\partial_i \cdot_{g_{\mathbb{R}^3}} \nabla_{\partial_i}\psi_{1,0,\gamma} - \frac{1}{2}c_3\partial_3 \cdot_{g_{\mathbb{R}^3}} \psi_{1,0,\gamma}$$

不再定义于 $\mathscr{D}^{1/2}(\mathbb{R}^3,\mathbb{S}(\mathbb{R}^3))$ 的对偶空间中，因此我们只能将 $w^*_{\lambda,\xi,\gamma}$展开为以下形式

$$w^*_{\lambda,\xi,\gamma} = w^*_{0,\xi,\gamma} + \nu^*_{\lambda,\xi,\gamma}$$

with $\|\nu^*_{\lambda,\xi,\gamma}\| = O(\lambda|\ln\lambda|^{\frac{2}{3}})$. And hence $\nabla^2\mathcal{J}_0(\psi_{1,0,\gamma})[w^*_{\lambda,\xi,\gamma}, w^*_{\lambda,\xi,\gamma}]$ has no more of order λ^2. Comparing with the asymptotic expansion of $w^*_{\lambda,\xi,\gamma}$ for $m \geq 4$, it becomes more complicated to determine the exact asymptotic expansion of $w^*_{\lambda,\xi,\gamma}$ in dimension 3. Our approach in this regard up to now have failed.

参考文献

[1] R. Adams, 《索博列夫空间》, Academic Press, 纽约, 1975年。

[2] A. D. Alexandrov, Uniqueness theorems for surfaces in the large, Amer. Math. Soc. Transl. Ser. 2, 21, 441-519, (1962).

[3] A. Ambrosetti, M. Badiale, Homoclinics: Poincaré-Melnikov type results via a variational approach, Ann. Inst. H. Poincaré Anal. Nonlinéaire 15 (1998), 233-252.

[4] A. Ambrisetti, M. Badiale, Variational perturbative methods and bifurcation of bound states from the essential spectrum, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998), 1131-1161.

[5] A. Ambrosetti, J. Garcia Azorero, I. Peral, Perturbation of $\Delta u + u^{\frac{N+2}{N-2}} = 0$, the scalar curvature problem in $\mathbb{R}^N$, and related topics, J. Funct. Anal. 165 (1999), 117-149.

[6] A. Ambrosetti, A. Malchiodi, A multiplicity result for the Yamabe problem on S^n, J. Funct. Anal. 168 (1999), 529-561.

[7] A. Ambrosetti, A. Malchiodi, Perturbation Methods and Semilinear Elliptic Problems on $\mathbb{R}^n$, Progress in Math. 240, Birkhäuser, Basel-Boston-Berlin (2006).

[8] B. Ammann, A variational problem in conformal spin geometry, Habilitationsschrift, Universität Hamburg 2003.

[9] B. Ammann, The smallest Dirac eigenvalue in a spin-conformal class and cmc immersions, Comm. Anal. Geom. 17 (2009), no. 3, 429-479.

[10] B. Ammann, M. Dahl, A. Hermann, E. Humbert, Mass endomorphism, surgery and perturbations, Ann. Inst. Fourier. 64 (2014), no. 2, 467-487.

[11] B. Ammann, J.-F. Grossjean, E. Humbert, B. Morel, A spinorial analogue of Aubin's inequality, Math. Z. 260 (2008), 127-151.

[12] B. Ammann, E. Humbert, B. Morel, Mass endomorphism and spinorial Yamabe type problems on conformally flat manifolds, Comm. Anal. Geom. 14 (2006), no. 1, 163-182.

[13] A. Bahri, J. M. Coron, The scalar curvature problem on the standard three-dimensional sphere, J. Funct. Anal. 95 (1991), 106-172.

[14] J.-P. Bourguignon, P. Gauduchon, Spineurs, opérateurs de Dirac et variations de métriques, Commun. Math. Phys. 144 (1992), 581-599.

[15] K.-C. Chang, Infinite Dimensional Morse Theory and Multiple Solution Problems, Birkhäuser Boston (1993).

[16] S.-Y. A. Chang, P. Yang, A perturbation result in prescribing scalar curvature on S^n, Duke Math. J. 64 (1991), 27-69.

[17] S.-Y. A. Chang, M. Gursky, P. Yang, The scalar curvature equation on 2- and 3-spheres, Calc. Var. 1 (1993), 205-229.

[18] T. Friedrich, On the spinor representation of surfaces in Euclidean 3-space, J. Geom. Phys. 28:1-2 (1998), 143-157.

[19] T. Friedrich, Dirac Operators in Riemannian Geometry, Graduate Studies in Mathematics 25 (2000).

[20] N. Ginoux, The Dirac Spectrum, Lecture Notes in Mathematics, vol. 1976. Springer, Berlin 2009.

[21] K. Grosse-Brauckmann, Complete embedded constant mean curvature surfaces, Habilitationsschrift, Universität Bonn, 1998.

[22] A. Hermann, Generic metrics and the mass endomorphism on spin three-manifolds, Ann. Global Anal. Geom. 37 (2010), 163-171.

[23] O. Hijazi, A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors, Comm. Math. Phys. 104 (1986), 151-162.

[24] N. Hitchin，调和旋量，Adv. Math. 14（1974），1 -55.

[25] H. Hopf, Differential geometry in the large. In: Lecture Notes in Mathematics, vol. 1000. Springer, Berlin. Notes taken by Peter Lax and John Gray. With a preface by S.S. Chern (1983).

[26] T. Isobe, Nonlinear Dirac equations with critical nonlinearities on compact spin manifolds, J. Funct. Anal. 260 (2011), 253-307.

[27] T. Isobe, A perturbation method for spinorial Yamabe type equations on S^m and its applications, Math. Ann. 355 (2013), 1255-1299.

[28] T. Isobe, Spinorial Yamabe type equations on S^3 via Conley index, Adv. Nonlinear. Stud. 15(2015), 39-60.

[29] K. Kenmotsu, Weierstrass formula for surfaces of prescribed mean curvature, Math. Ann. 245:2 (1979), 89-99.

[30] B. G. Konopelchenko, Induced surfaces and their integrable dynamics, Stud. Appl. Math. 96:1 (1996), 9-51.

[31] R. Kusner, N. Schmitt, The spinor representation of surfaces in space, dg-ga/9610005.

[32] H. B. Lawson, M. L. Michelson, Spin Geometry, Princeton University Press (1989).

[33] P. Li, S.T. Yau, A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue on compact surfaces, Invent. Math. 69 (1982), 269-291.

[34] A. Maalaoui, Infinitely many solutions for the spinorial Yamabe problem on the round sphere, Nonlinear Differ. Equ. Appl. 23, 25 (2016).

[35] A. Malchiodi, The scalar curvature problem on Sn: an approach via Morse theory, Calc. Var. 14 (2002), 429-445

[36] S. Matsutani, Immersion anomaly of Dirac operator on surface in $\mathbb{R}^3$, Rev. Math. Phys. 11:2 (1999), 171-186.

[37] J. Mawhin, M.Willem, Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences 74 (1989), Springer-Verlag.

[38] A. Mondinoa, J. Schygulla, Existence of immersed spheres minimizing curvature functionals in non-compact 3-manifolds, Ann. I.H.Poincaré 31:4 (2014), 707-724.

[39] S. Raulot, A Sobolev-like inequality for the Dirac operator, J. Funct. Anal. 26 (2009), 1588-1617.

[40] Y. Sire, T. Xu, A variational analysis of the spinorial Yamabe equation on product manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 24:5 (2023), 205-248.

[41] M. Struwe, Variational Methods, Ergeb. der Math. u. Grenzgeb. Vol. 34, Springer 1996.

[42] I. A. Taimanov, Modified Novikov-Veselov equation and differential geometry of surfaces, Amer. Math. Soc. Transl. (2) 179 (1997), 133-151; http://arxiv.org/dg-ga/9511005.

[43] I. A. Taimanov, The Weierstrass representation of closed surfaces in $\mathbb{R}^3$, Funktsional. Anal. i Prilozhen. 32:4 (1998), 49-62; English transl., Funct. Anal. Appl. 32 (1998), 258-267.

[44] I. A. Taimanov, The Weierstrass representation of spheres in $\mathbb{R}^3$, the Willmore numbers, and soliton spheres, Trudy Mat. Inst. Steklov. 225 (1999), 339-361; English transl., Proc. Steklov Inst. Math. 225 (1999), 225-243.