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Clifford 环面所界定的 $\mathbb{S}^3$ 内域上的 Neumann问题

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Dedicated to David Jerison on the occasion of his 70th birthday

Abstract. We discuss the solution of the Neumann problem associated with the CR Yamabe operator on a subset Ω of the CR manifold $\mathbb{S}^3$ bounded by the Clifford torus Σ. We also discuss the Yamabe-type problem of finding a contact form on Ω which has zero Tanaka–Webster scalar curvature and for which Σ has constant p-mean curvature.

1. Introduction

我们将 $\mathbb{S}^3 = \{(\zeta^1, \zeta^2) \in \mathbb{C}^2 : |\zeta^1|^2 + |\zeta^2|^2 = 1\}$ 视为 $\mathbb{C}^2$中单位球的边界，并赋予其与接触形式相关联的伪厄米结构

$$\theta := i\overline{\partial}(|\zeta|^2 - 1) = \frac{i}{2}(\zeta^1 d\overline{\zeta}^1 + \zeta^2 d\overline{\zeta}^2 - \overline{\zeta}^1 d\zeta^1 - \overline{\zeta}^2 d\zeta^2).$$

设 $\Omega \subset \mathbb{S}^3$ 为该区域

$$\Omega = \{(\zeta^1, \zeta^2) \in \mathbb{S}^3 : |\zeta^1| < |\zeta^2|\}, \tag{1.1}$$

其边界由.....给出。

$$\Sigma := \partial\Omega = \{(\zeta^1, \zeta^2) \in \mathbb{S}^3 : |\zeta^1| = |\zeta^2| = \frac{1}{\sqrt{2}}\},$$

该 Clifford 环面。

在 Ω 上，我们给出 Neumann问题 的一个显式解,

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ \nabla_\nu u = h & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

这里 L 表示 $(\mathbb{S}^3, \theta)$ 上的 CR Yamabe算子 (2.3)， ∇_ν 表示函数 $u \in C^\infty(\Omega)$ 的单侧水平法向导数，即。

$$\nabla_\nu u(\zeta) := \lim_{t \to 0^+} \frac{u(\gamma_\zeta(t)) - u(\zeta)}{t}, \quad \zeta \in \partial\Omega,$$

其中， $\gamma_\zeta(t)$ 是一条曲线，使得 $\gamma_\zeta(0) = \zeta$ ，且 $\gamma'(0)$ 是 Ω 在 $\zeta \in \partial\Omega$ 处的内向水平法向量 ν 。

起初，我们假设边界数据 h 在 Σ上光滑，即 $h \in C^\infty(\partial\Omega)$，并求解(1.2)。我们的解可表示为一个适当的单层位势 $\mathcal{S}$，其对 $f \in C^1(\Sigma)$的定义为

$$\mathcal{S}f(\zeta) := \oint_{\eta \in \Sigma} G(\zeta, \eta) f(\eta) d\sigma(\eta), \quad \zeta \in \overline{\Omega},$$

其中， $G(\zeta, \eta)$ 是 CR Yamabe算子 L 的格林函数， $d\sigma$ 是 Σ 上与体积形式 $\iota_\nu\theta \wedge d\theta$ 相对应的曲面测度。

Theorem 1.1. *If $h \in C^\infty(\Sigma)$, then there exists a unique $u \in C^\infty(\overline{\Omega})$ that solves the Neumann problem (1.2). Furthermore, u is given by $\mathcal{S}f$ on $\overline{\Omega}$ for some $f \in C^\infty(\Sigma)$, where f is determined by h from*

$$f = (-\frac{1}{2}I + \mathcal{N})^{-1} h, \tag{1.3}$$

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and $\mathcal{N}$ is the singular integral operator given by

$$\mathcal{N}f(\zeta) := p.v. \oint_{\eta\in\Sigma} (\nabla_\nu)_\zeta G(\zeta,\eta) f(\eta)\, d\sigma(\eta), \quad \zeta \in \Sigma.$$

Some of the arguments follow those in [OV20], where (1.2) is solved on flag domains of $\mathbb{H}^1$. In contrast to the Euclidean case [FJR78], the operator $\mathcal{N}$ is not a compact operator on $L^2(\Sigma)$; it is only bounded on $L^2(\Sigma)$, but not smoothing of any positive order. As a result, a more careful analysis is necessary to show that $-\frac{1}{2}I + \mathcal{N}$ is invertible on $L^2(\Sigma)$ (and all higher order L^2-based Sobolev spaces).

A general result of Nhieu [Nhi01] implies the existence of a solution u to (1.2) in the Folland–Stein [FS74] space $S^{1,2}(\Omega)$ of functions $u \in L^2(\Omega)$ whose horizontal gradient is also in $L^2(\Omega)$. We recover this result using the single layer potential.

定理 1.2。如果 $h \in L^2(\Sigma)$，则存在唯一的 $u \in S^{1,\,2}(\Omega)$，使得 (1.2) 在弱意义下得到满足，即。

$$\int_\Omega \left(\langle \nabla_b u, \nabla_b \phi \rangle + \frac{R}{4}\, u\, \phi \right) \theta \wedge d\theta + \oint_\Sigma h\phi\, d\sigma = 0$$

对所有 $\phi \in C^\infty(\overline{\Omega})$。此外，$u$ 在Ω 上由 $\mathcal{S}f$ 给出，其中 $f \in L^2(\Sigma)$ 由 h 通过 (1.3) 确定。

定理 1.2 证明中的关键观察是，单层位势可延拓为有界线性映射 $\mathcal{S}\colon L^2(\Sigma) \to S^{1,\,2}(\Omega)$。

Our work is motivated in part by the desire to formulate and study the CR boundary Yamabe problem (cf. [Esc92]): Given a closed CR three-manifold with boundary $(M^3, T^{1,0})$, construct a Webster-flat contact form with respect to which the boundary has constant p-mean curvature H [CHMY05]. Given a contact form θ, the contact form $\widehat{\theta} := u^2\theta$ satisfies these properties if and only if u is a positive critical point of the functional $\mathcal{F}\colon \mathcal{V} \to \mathbb{R}$,

$$\mathcal{F}(u) := \int_M \left(|\nabla_b u|^2 + \frac{R}{4}u^2 \right) \theta \wedge d\theta + \frac{1}{3}\oint_{\partial M} H u^2\, d\sigma,$$
$$\mathcal{V} := \left\{ u \in C^\infty(M) : \oint_{\partial M} |u|^3\, d\sigma = 1 \right\}.$$

构造这种接触形式的一种方法是证明存在一个光滑的正函数，使其实现 CR 边界 Yamabe 常数

$$Y(M, T^{1,0}) := \inf \{ \mathcal{F}(u) : u \in \mathcal{V} \}.$$

If ∂M has no characteristic points, then the Sobolev trace embedding theorem [Nhi01] implies that the restriction map $C^\infty(\overline{M}) \ni u \mapsto u|_{\partial M} \in C^\infty(\partial M)$ extends to a continuous linear map $\mathrm{Tr}\colon S^{1,2}(M) \to L^2(\partial M)$; in particular, $\mathcal{F}$ is well-defined on $S^{1,2}(M)$. As in the Riemannian case [Esc94], it holds that $Y(M, T^{1,0}) > -\infty$ if and only if the Dirichlet eigenvalues of L are positive; i.e.

$$\lambda_{1,D}(L) := \inf \left\{ \mathcal{F}(u) : u|_{\partial M} = 0, \int_M |u|^2\, \theta \wedge d\theta = 1 \right\} > 0.$$

$\lambda_{1,D}(L)$ 的正性还意味着第一 Steklov 特征值

$$\mu_1(L) := \inf \left\{ \mathcal{F}(u) : \oint_{\partial M} |u|^2\, d\sigma = 1 \right\} \tag{1.4}$$

L 有限。

In Section 7 we discuss the equivalence of the signs of $\mu_1(L)$ and $Y(M, T^{1,0})$, under the assumption that minimizers of $\mu_1(L)$ are smooth up to the boundary. Such regularity assumption can be verified in the case where $M = \overline{\Omega}$, where Ω is the domain in $\mathbb{S}^3$ defined by (1.1). Note that $H = 0$ for the Clifford torus $\Sigma = \partial\Omega$.

命题 1.3. 设 $(\Omega\ \theta)$ 是带有标准球面接触形式的 Clifford 环面的内部。如果 $u \in S^{1,2}(\Omega)$ 使 $\mu_1(L)$ 取最小值，则 $u \in C^\infty(\Omega)$。

命题 1.3 w　　将通过观察到 $\mu_1(L)$ 的一个极小化子 $u \in S^{1,2}(\Omega)$ 是一个弱解来证明。

$$\begin{cases} Lu = 0, & \text{on } \Omega \\ \nabla_\nu u = \mu \operatorname{Tr}(u), & \text{in } \partial\Omega, \end{cases}$$

whe其中 $\mu = -\mathcal{F}(u)$ 为常数，然后利用为定理证明而发展出的工具　　1.2.

Since the standard contact form on $\mathbb{S}^3$ has positive Tanaka–Webster scalar curvature and is such that the Clifford torus is p-minimal, the Sobolev trace embedding theorem [Nhi01] implies that the boundary Yamabe constant $Y(\Omega\ T^{1,0})$ is positive. We identify an explicit critical point of the functional $\mathcal{F}\colon \mathcal{V} \to \mathbb{R}$.

定理 1.4. 设$(\Omega\ T^{1,0}, \theta)$ 为 Clifford 环面的内部。令

$$u := \frac{{}_2F_1\big(\frac{1}{2}, \frac{1}{2}; 1; |z^2|^2\big)}{{}_2F_1\big(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{2}\big)},$$

其中

$$\begin{aligned} {}_2F_1(a, b; c; x) &:= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \\ (a)_0 &:= 1, \\ (a)_n &:= a(a+1)\cdots(a+n-1), \qquad \text{if } n \geq 1, \end{aligned}$$

是标准的高斯超几何函数。然后， $u^2\theta$ 是Ω 上的标量平坦接触形式，相对于该形式， Σ 具有常数 p-平均曲率。

定理1.4 的解经过归一化，使得 θ 和 $u^2\theta$ 在 Σ上重合。

The article is organized as follows. In Section 2, we give some background in pseudohermitian geometry and Fourier analysis, in the specific setting of the Clifford torus. In Section 3, we compute the single layer potential in local coordinates. Theorems 1.1, 1.2, Proposition 1.3 and Theorem 1.4 will be proved in Sections 4, 5, 6 and 7, respectively.

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2. Setup and notations

2.1. **Pseudohermitian geometry.** A *CR three-manifold* is a pair $(M^3, T^{1,0})$ consisting of a real three-manifold M and a complex rank 1 distribution $T^{1,0} \subset T_{\mathbb{C}}M$. Let $H := \operatorname{Re}\big(T^{1,0} \oplus \overline{T^{0,1}}\big)$ denote the space of *horizontal vectors*. Then

$$J(Z + \overline{Z}) := iZ - i\overline{Z},$$

$Z \in T^{1,0}$, defines an integrable almost complex structure on H. We say that $(M^3, T^{1,0})$ is *nondegenerate* if locally there is a real one-form θ such that $\ker\theta = H$ and $\theta \wedge d\theta$ is nowhere-vanishing. Nondegenerate CR three-manifolds are orientable [Jac90, Lemma 19], and hence there is a global real one-form θ such that $\ker\theta = H$; in this case we call θ a *contact form*.

A *(strictly pseudoconvex) pseudohermitian manifold* is a triple $(M^3, T^{1,0}, \theta)$ consisting of a nondegenerate CR three-manifold $(M^3, T^{1,0})$ and a contact form θ such that $d\theta(Z, \overline{Z}) > 0$ for all nonzero $Z \in T^{1,0}$. The *Reeb vector field* is the unique vector field T such that $\theta(T) = 1$a and $d\theta(T, \cdot) = 0$.

An *admissible coframe* for $(M^3, T^{1,0}, \theta)$ is a nowhere-vanishing local complex-valued one-form θ^1 such that $\theta^1(T) = 0$ and $\theta^1(\overline{Z}) = 0$ for all $Z \in T^{1,0}$. Set $\theta^{\bar{1}} := \overline{\theta^1}$. Then $\{\theta, \theta^1, \theta^{\bar{1}}\}$ is a local coframe for $T^*_{\mathbb{C}}M$. It follows that

$$d\theta = ih_{1\bar{1}}\, \theta^1 \wedge \theta^{\bar{1}}.$$

注意， $h_{1\bar{1}} > 0$。令 $\{T, Z_1, Z_{\bar{1}}\}$ 为 $\{\theta, \theta^1, \theta^{\bar{1}}\}$的对偶标架。这样（整体上）在 H 上由此确定了一个正定内积。

$$\langle \operatorname{Re} a^1 Z_1, \operatorname{Re} b^1 Z_1 \rangle := \frac{1}{2} \operatorname{Re} h_{1\bar{1}} a^1 \overline{b^1}.$$

设 θ^1 是 $(M^3, T^{1,0}, \theta)$ 的一个容许余标架。那么存在唯一的复值一形式 ${\omega_1}^1$，使得

$$\begin{cases} d\theta^1 = \theta^1 \wedge {\omega_1}^1 + A^1{}_{\bar{1}}\,\theta \wedge \theta^{\bar{1}}, \\ dh_{1\bar{1}} = {\omega_1}^1 h_{1\bar{1}} + {\omega_{\bar{1}}}^{\bar{1}} h_{1\bar{1}}, \end{cases}$$

where ${\omega_{\bar{1}}}^{\bar{1}} := \overline{{\omega_1}^1}$. The *Tanaka–Webster connection* ∇ is uniquely determined from $\nabla Z_1 := {\omega_1}^1 \otimes Z_1$ and $\nabla T := 0$ by complex linearity. The *pseudohermitian torsion* is the globally-defined tensor $A_{11}\,\theta^1 \otimes \theta^1$. The *Tanaka–Webster scalar curvature* is the globally-defined function R determined by

$$d{\omega_1}^1 = R h_{1\bar{1}}\,\theta^1 \wedge \theta^{\bar{1}} \mod \theta.$$

我们称 θ 是标量平坦的，当且仅当 $R = 0$。

给定函数 $f \in C^\infty(M)$，我们用 $\nabla_b f$表示 f的次梯度；即将 df限制在 H上。次拉普拉斯算子 $\Delta_b\colon C^\infty(M) \to C^\infty(M)$ 是

$$\Delta_b := \nabla_b^* \nabla_b,$$

其中 ∇_b^* 是关于 $\theta \wedge d\theta$ 的形式 L^2-伴随算子 ∇_b。局部地，

$$\Delta_b u = -h^{\bar{1}1}\left((Z_1 Z_{\bar{1}} + Z_{\bar{1}} Z_1)u - {\omega_1}^1(Z_{\bar{1}}) Z_1 u - {\omega_{\bar{1}}}^{\bar{1}}(Z_1) Z_{\bar{1}} u\right).$$

容易计算得（参见 [Lee86，公式 (2.4)]），

$$\int_M \langle \nabla_b u, \nabla_b w\rangle\,\theta \wedge d\theta = \int_M u \Delta_b w\,\theta \wedge d\theta + 2\,\mathrm{Re}\oint_{\partial M} iu(Z_1 w)\,\theta \wedge \theta^1. \tag{2.1}$$

设 $(M^3, T^{1,0}, \theta)$ 是一个边界为 $\Sigma := \partial M$ 的伪 Hermitian 三维流形。若 $p \in \Sigma\ T_p\Sigma = H_p$，则称其为奇异点。若 Σ 不含奇异点，则称其为非奇异。

Suppose now that $(M^3, T^{1,0}, \theta)$ is a pseudohermitian three-manifold with nonsingular boundary $\Sigma := \partial M$. Assume additionally that Σ is oriented. Then there is a unique $H \cap T\Sigma$-valued unit vector field e_1 such that $\nu := Je_1$ is inward-pointing. Let Z_1 be the unique section of $T^{1,0}$ along Σ such that $e_1 = \mathrm{Re}\, Z_1$. A straightforward computation using (2.1) gives

$$\int_M \langle \nabla_b u, \nabla_b w\rangle\,\theta \wedge d\theta = \int_M u \Delta_b w\,\theta \wedge d\theta - \oint_{\partial M} u \nabla_\nu w\,d\sigma, \tag{2.2}$$

其中 $d\sigma :=\ \iota_{-\nu}(\theta \wedge d\theta)$。

由于田中–Webster联络保持接触形式和Levi形式，$\nabla_{e_1} e_1$ 位于空间中。e_2 的 n 次幂。p均值 cu曲率是由 H 定义的函数

$$\nabla_{e_1} e_1 := H\nu.$$

CRYamabe算子 $L^\theta\colon\ C^\infty(M) \to C^\infty(M)$ 是

$$L^\theta u := \Delta_b u + \frac{R}{4} u.$$

CRYamabe算子具有共形协变性 [JL87, 公式 (3.1)]：如果 $\hat{}\theta = w^2\theta$，那么

$$w^3 L^{\hat{\theta}} u = L^\theta(wu).$$

当接触形式根据上下文可明确时，我们将用 L 表示 L^θ。CR Robin 算子 $B^\theta\colon\ C^\infty(M) \to C^\infty(\Sigma)$ 是

$$B^\theta u := -\nabla_\nu u + \frac{H}{3} u.$$

Tanaka–Webster联络的共形变换法则 [JL89，第3引理4] 表明，如果 $\hat{}\theta = w^2\theta$，那么

$$w^2 B^{\hat{\theta}} u = B^\theta(wu).$$

由此可得，CR Yamabe 泛函 $\mathcal{F}\colon\ C^\infty(M) \to \mathbb{R}$，

$$\mathcal{F}^\theta(u) := \int_M u\,Lu\,\theta \wedge d\theta + \oint_\Sigma u\,Bu\,d\sigma$$

是 CR 不变量；事实上，

$$\mathcal{F}^{w^2\theta}(u) = \mathcal{F}^\theta(wu)$$

对于所有正的 $w \in C^\infty(M)$ 以及所有 $u \in C^\infty(M)$，式（2.2）意味着

$$\mathcal{F}(u) = \int_M \left(|\nabla_b u|^2 + \frac{R}{4} u^2 \right) \theta \wedge d\theta + \frac{1}{3} \oint_\Sigma H u^2 \, d\sigma.$$

2.2. 标准球面 $\mathbb{S}^3$。设

$$\theta = i\overline{\partial}(|\zeta|^2 - 1) = \frac{i}{2}(\zeta^1 d\overline{\zeta}^1 + \zeta^2 d\overline{\zeta}^2 - \overline{\zeta}^1 d\zeta^1 - \overline{\zeta}^2 d\zeta^2)$$

为 $\mathbb{S}^3$ 上的接触形式。记

$$\theta^1 := \zeta^2 d\zeta^1 - \zeta^1 d\zeta^2.$$

Then

$$d\theta = i\theta^1 \wedge \theta^{\bar{1}}$$

所以 $h_{1\bar{1}} = 1$。由于

$$d\theta^1 = -2i\theta^1 \wedge \theta,$$

我们有

$$\omega_1{}^1 = -2i\theta.$$

由此可得

$$d\omega_1{}^1 = 2\theta^1 \wedge \theta^{\bar{1}},$$

因此，Tanaka–Webster 标量曲率为 $R = 2$。于是，CR Yamabe算子为

$$Lu := -(Z_1 Z_{\bar{1}} + Z_{\bar{1}} Z_1)u + \frac{1}{2}u \tag{2.3}$$

其中

$$Z_1 := \overline{\zeta}^2 \frac{\partial}{\partial \zeta^1} - \overline{\zeta}^1 \frac{\partial}{\partial \zeta^2}.$$

在 $\mathbb{S}^3$ 上 L 的基本解为 [Gel80，定理 2.1]

$$G(\zeta, \eta) := \frac{1}{8\pi} |1 - \zeta \cdot \overline{\eta}|^{-1}.$$

2.3. Clifford 环面。回顾一下，区域 $\Omega \subset \mathbb{S}^3$ 由以下内容给出。

$$\Omega = \{(\zeta^1, \zeta^2) \in \mathbb{S}^3 \colon |\zeta^1| < |\zeta^2|\}.$$

Its boundary is the Clifford torus $\Sigma = \{(\zeta^1, \zeta^2) \colon |\zeta^1| = |\zeta^2| = \frac{1}{\sqrt{2}}\}$. Note that Σ is contained in the open set $\{(\zeta^1, \zeta^2) \in \mathbb{S}^3 \colon |\zeta^1| \neq 0, |\zeta^2| \neq 0\}$. On this open set, define a frame of horizontal vectors

$$\begin{aligned} e_1 &:= \frac{i}{\sqrt{2}} \left(\frac{\zeta^1 \zeta^2}{|\zeta^1 \zeta^2|} Z_1 - \frac{\overline{\zeta}^1 \overline{\zeta}^2}{|\zeta^1 \zeta^2|} Z_{\bar{1}} \right) \\ &= -\sqrt{2} \operatorname{Im} \left(|\zeta^2| \frac{\zeta^1}{|\zeta^1|} \frac{\partial}{\partial \zeta^1} - |\zeta^1| \frac{\zeta^2}{|\zeta^2|} \frac{\partial}{\partial \zeta^2} \right), \end{aligned}$$

and

$$\begin{aligned} \nu := Je_1 &= -\frac{1}{\sqrt{2}} \left(\frac{\zeta^1 \zeta^2}{|\zeta^1 \zeta^2|} Z_1 + \frac{\overline{\zeta}^1 \overline{\zeta}^2}{|\zeta^1 \zeta^2|} Z_{\bar{1}} \right) \\ &= -\sqrt{2} \operatorname{Re} \left(|\zeta^2| \frac{\zeta^1}{|\zeta^1|} \frac{\partial}{\partial \zeta^1} - |\zeta^1| \frac{\zeta^2}{|\zeta^2|} \frac{\partial}{\partial \zeta^2} \right). \end{aligned}$$

在 $\mathbb{S}^3$ 上的 Reeb 向量场是

$$\begin{aligned} T &:= i \left(\zeta^1 \frac{\partial}{\partial \zeta^1} + \zeta^2 \frac{\partial}{\partial \zeta^2} - \overline{\zeta}^1 \frac{\partial}{\partial \overline{\zeta}^1} - \overline{\zeta}^2 \frac{\partial}{\partial \overline{\zeta}^2} \right) \\ &= -2 \operatorname{Im} \left(\zeta^1 \frac{\partial}{\partial \zeta^1} + \zeta^2 \frac{\partial}{\partial \zeta^2} \right). \end{aligned}$$

对于 $t \in [0, \frac{\pi}{2\sqrt{2}})$，设

$$\Omega_t = \left\{ \zeta \in \mathbb{S}^3 \colon \frac{|\zeta^1|}{\cos(\frac{\pi}{4} + \frac{t}{\sqrt{2}})} < \frac{|\zeta^2|}{\sin(\frac{\pi}{4} + \frac{t}{\sqrt{2}})} \right\}$$

and

$$\Sigma_t = \partial\Omega_t = \left\{ \zeta \in \mathbb{S}^3 \colon |\zeta^1| = \cos\Big(\frac{\pi}{4} + \frac{t}{\sqrt{2}}\Big), |\zeta^2| = \sin\Big(\frac{\pi}{4} + \frac{t}{\sqrt{2}}\Big) \right\},$$

使得$\Omega_0 = \Omega$ 和 $\Sigma_0 = \Sigma$。那么 ν 是 $\partial\Omega_t$的内法向量，而 e_1 和 T 与 Σ_t相切。事实上，Ω_t 的一个定义函数是

$$\rho_t := \frac{|\zeta^1|^2}{\cos^2(\frac{\pi}{4} + \frac{t}{\sqrt{2}})} - \frac{|\zeta^2|^2}{\sin^2(\frac{\pi}{4} + \frac{t}{\sqrt{2}})},$$

而且， e_1 和 T 的上述表达式表明

$$\nabla_{e_1}\rho_t = \nabla_T \rho_t = 0 \quad \text{on } \Sigma_t.$$

此外，

$$\nabla_\nu \rho_t = -\frac{\sqrt{2}}{\cos(\frac{\pi}{4} + \frac{t}{\sqrt{2}})\sin(\frac{\pi}{4} + \frac{t}{\sqrt{2}})} = -\frac{2\sqrt{2}}{\cos(\sqrt{2}t)} < 0$$

所以 $\nu = Je_1$ 是指向 $\partial\Omega_t$ 内部的法向量。此外，我们很容易计算得到 $\nabla_{e_1} e_1 = 0$，因此 Clifford 环面是 p-极小的。

2.4. 傅里叶分析。为了参数化 Σ_t，考虑格点将会很方便。 e

$$\Lambda := 2\pi\{(m/2, n/2) \in \mathbb{Z}^2 \colon m \equiv n \pmod 2\},$$

and the associated abelian group $\mathbb{R}^2/\Lambda$. A fundamental domain for $\mathbb{R}^2/\Lambda$ is given by $[-\pi, \pi) \times [-\pi/2, \pi/2)$; the identity element of $\mathbb{R}^2/\Lambda$ will be written as $(0, 0)$ using this identification. Any function defined on $[-\pi, \pi) \times [-\pi/2, \pi/2)$ can be lifted to $\mathbb{R}^2/\Lambda$. For instance, we define, for $(u, v) \in [-\pi, \pi) \times [-\pi/2, \pi/2)$, the natural norm for this problem, namely

$$\|(u, v)\| := \max\{|u|, |v|^{1/2}\},$$

并由此在 $\mathbb{R}^2/\Lambda$ 上产生相应的范数函数。

Lebesgue 测度 $dudv$ 在 $[-\pi, \pi) \times [-\pi/2, \pi/2)$ 上诱导出 $\mathbb{R}^2/\Lambda$ 上的 Haar 测度，其体积为 $2\pi^2$，而 $\frac{1}{2\pi}\Lambda$ 的对偶格（$\frac{1}{2\pi}\Lambda)^*$ 给出为

$$(\frac{1}{2\pi}\Lambda)^* = \{(m, n) \in \mathbb{Z}^2 \colon m \equiv n \pmod 2\}.$$

$L^2(\mathbb{R}^2/\Lambda, dudv)$ 的一组标准正交基为

$$\Big\{ \frac{1}{\sqrt{2\pi^2}} e^{i(mu+nv)} \colon (m, n) \in (\frac{1}{2\pi}\Lambda)^* \Big\},$$

因此，帕塞瓦尔公式为

$$\iint_{\substack{|u|\le\pi \\ |v|\le\pi/2}} |F(u, v)|^2 dudv = \frac{1}{2\pi^2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\equiv n \pmod 2}} |\widehat{F}(m, n)|^2,$$

其中

$$\widehat{F}(m, n) := \iint_{\substack{|u|\le\pi \\ |v|\le\pi/2}} F(u, v) e^{-i(mu+nv)} dudv.$$

卷积 $\mathbb{R}^2/\Lambda$ 上两个函数 F 和 K 的 n 可由以下两个等价表达式中的任意一个给出 s:

$$\begin{aligned} F * K(u', v') &:= \iint_{\substack{|u|\le\pi \\ |v|\le\pi/2}} F(u' - u, v' - v) K(u, v) dudv \\ &= \iint_{\substack{|u|\le\pi \\ |v|\le\pi/2}} F(u, v) K(u' - u, v' - v) dudv, \end{aligned}$$

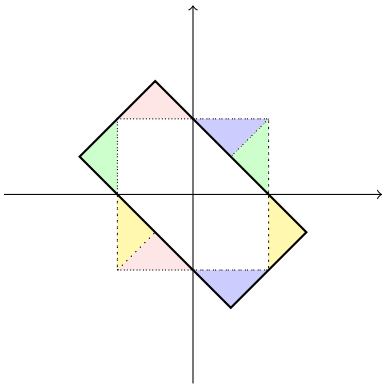

FIGURE 1. The dotted square is $[-\pi,\pi]^2$. The tilted rectangle is the image of $(u,v) \in [-\pi,\pi]\times[-\pi/2,\pi/2] \mapsto (v+u, v-u) \in \mathbb{R}^2$. By translating the colored pieces in the tilted rectangle either vertically or horizontally by 2π, we see that the tilted rectangle also parametrizes a copy of $(\mathbb{R}/2\pi\mathbb{Z})^2$.

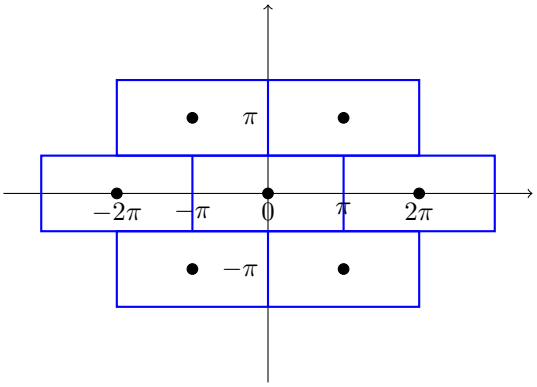

FIGURE 2. 这些点表示格子 Λ。每个蓝色矩形都是 $\mathbb{R}^2/\Lambda$ 的一个基本域。通过映射 $\Phi(t,\cdot,\cdot)$，它可以与 Σ_t 对应。

对于每个 $(u',\ v') \in \mathbb{R}^2/\Lambda$，我们有

$$\widehat{F * K}(m,n) = \widehat{F}(m,n)\widehat{K}(m,n),$$

thus

$$\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}} |F*K(u,v)|^2 du dv \le \sup_{\substack{(m,n)\in\mathbb{Z}^2\\ m\equiv n \pmod 2}} |\widehat{K}(m,n)|^2 \iint_{\substack{|u|\le\pi\\|v|\le\pi/2}} |F(u,v)|^2 du dv.$$

2.5. 对 Σ_t 进行参数化。对于 $0 \le t \le \frac{\pi}{100}$，我们将使用参数化 Σ_t

$$(u,v) \in \mathbb{R}^2/\Lambda \longmapsto \Phi(t,u,v) := \left(\cos(\frac{\pi}{4}+\frac{t}{\sqrt{2}})e^{i(v+u)}, \sin(\frac{\pi}{4}+\frac{t}{\sqrt{2}})e^{i(v-u)}\right) \in \Sigma_t.$$

注意到 $e^{i(v+u)} = e^{i(v-u)} = 1$ 当且仅当 $(u,v) \in \Lambda$，即可看出上述映射是良定义的双射；或者，也可以观察到，该映射

$$(u,v) \in [-\pi,\pi)\times[-\pi/2,\pi/2) \mapsto (v+u, v-u) \in (\mathbb{R}/2\pi\mathbb{Z})^2$$

是一个双射（见图 1）。我们有 $\Phi_*(\frac{\partial}{\partial t}) = \nu$ 和 $\Phi_*(\frac{\partial}{\partial v}) = T$。

在对 Σ_t 进行此参数化后， Σ_t 上的体积元素 $d\sigma_t$ 为

$$\begin{aligned}
d\sigma_t &= \iota_{-\nu}\theta \wedge d\theta \\
&= i\theta \wedge \iota_\nu(\theta^1 \wedge \theta^{\bar{1}}) \\
&= -\frac{1}{\sqrt{2}} i\Big(\frac{\zeta^1\zeta^2}{|\zeta^1\zeta^2|}\theta \wedge \theta^{\bar{1}} - \frac{\overline{\zeta}^1\overline{\zeta}^2}{|\zeta^1\zeta^2|}\theta \wedge \theta^1\Big) \\
&= -\frac{1}{\sqrt{2}} i\Big(-\frac{\zeta^1\zeta^2}{|\zeta^1\zeta^2|}\frac{1}{2i}d\theta^{\bar{1}} - \frac{\overline{\zeta}^1\overline{\zeta}^2}{|\zeta^1\zeta^2|}\frac{1}{2i}d\theta^1\Big) \\
&= -\frac{1}{\sqrt{2}}\Big(\frac{\zeta^1\zeta^2}{|\zeta^1\zeta^2|}d\overline{\zeta}^1 \wedge d\overline{\zeta}^2 + \frac{\overline{\zeta}^1\overline{\zeta}^2}{|\zeta^1\zeta^2|}d\zeta^1 \wedge d\zeta^2\Big) \\
&= -\sqrt{2}\cos(\frac{\pi}{4} + \frac{t}{\sqrt{2}})\sin(\frac{\pi}{4} + \frac{t}{\sqrt{2}})\operatorname{Re}\Big((-i(du+dv)) \wedge (-i(dv-du))\Big) \\
&= \sqrt{2}\cos(\sqrt{2}t)du \wedge dv.
\end{aligned}$$

特别地， 令 $t = 0$， 则 Σ 上的曲面测度 $d\sigma$ 由 $\sqrt{2}dudv$给出。如果 f 是 Σ_t上的可积函数， 我们常写 $F(u,v) := f(\Phi(0,u,v))$， 从而

$$\oint_\Sigma f d\sigma = \sqrt{2}\iint_{\substack{|u|\le\pi \\ |v|\le\pi/2}} F(u,v)dudv. \tag{2.4}$$

3. The single layer potential

对于 $f \in C^1(\Sigma)$， 单层位势定义为

$$\mathcal{S}f(\zeta) = \int_{\eta\in\Sigma} f(\eta)G(\zeta,\eta)d\sigma(\eta), \quad \zeta \in \overline{\Omega}.$$

给定 $\zeta = (\zeta^1, \zeta^2) \in \Sigma$， 曲线

$$\gamma_\zeta(t) := \left(\sqrt{2}\cos(\frac{\pi}{4} + \frac{t}{\sqrt{2}})\zeta^1, \sqrt{2}\sin(\frac{\pi}{4} + \frac{t}{\sqrt{2}})\zeta^2\right), \quad t \ge 0, \tag{3.1}$$

满足 $\gamma_\zeta(0) = \zeta$ 且 $\gamma'_\zeta(0) = \nu$ 于 ζ。

命题 3.1. 对于 $f \in C^1(\Sigma)$， 单侧法向导数

$$\nabla_\nu \mathcal{S}f(\zeta) := \lim_{t\to 0^+}\frac{\mathcal{S}f(\gamma_\zeta(t)) - \mathcal{S}f(\gamma_\zeta(0))}{t}$$

在每个 $\zeta \in \Sigma$ 处都存在，且由下式给出

$$\nabla_\nu \mathcal{S}f(\zeta) = (-\frac{1}{2}I + \mathcal{N})f(\zeta),$$

其中

$$\mathcal{N}f(\zeta) := p.v.\int_{\eta\in\Sigma}\langle \nu_\zeta, (\nabla_b)_\zeta G(\zeta,\eta)\rangle f(\eta)d\sigma(\eta), \quad \zeta \in \Sigma.$$

To prove this, first we understand the kernel $G(\zeta,\eta)$ of $\mathcal{S}$ in the (u,v) coordinates. For $t \ge 0$ and $(u',v'),(u,v) \in \mathbb{R}^2/\Lambda$, if $\zeta = \Phi(t,u',v')$ and $\eta = \Phi(0,u,v)$, then

$$\begin{aligned}
&1 - \zeta\cdot\overline{\eta} \\
&= 1 - \cos(\frac{\pi}{4})\cos(\frac{\pi}{4} + \frac{t}{\sqrt{2}})e^{i(v'-v+u'-u)} - \sin(\frac{\pi}{4})\sin(\frac{\pi}{4} + \frac{t}{\sqrt{2}})e^{i(v'-v-u'+u)} \\
&= e^{i(v'-v)}\left[e^{-i(v'-v)} - \cos(\frac{\pi}{4})\cos(\frac{\pi}{4} + \frac{t}{\sqrt{2}})e^{i(u'-u)} - \sin(\frac{\pi}{4})\sin(\frac{\pi}{4} + \frac{t}{\sqrt{2}})e^{-i(u'-u)}\right] \\
&= e^{i(v'-v)}\left[\Big(\cos(v'-v) - \cos(\frac{t}{\sqrt{2}})\cos(u'-u)\Big) - i\Big(\sin(v'-v) - \sin(\frac{t}{\sqrt{2}})\sin(u'-u)\Big)\right]
\end{aligned}$$

so from $G(\zeta,\eta)=\frac{1}{8\pi}\left|1-\zeta\cdot\overline{\eta}\right|^{-1}$ we obtain

$$G(\zeta,\eta)=\frac{1}{8\pi}k(\frac{t}{\sqrt{2}},u'-u,v'-v) \tag{3.2}$$

where for $t\geq 0$ and $(u,v)\in\mathbb{R}^2/\Lambda$ we write

$$k(t,u,v):=\Big[(\cos v-\cos t\cos u)^2+(\sin v-\sin t\sin u)^2\Big]^{-1/2}.$$

We collect a few facts we will need of $k(t,u,v)$:

Lemma 3.2. *(a) If $t\in(0,\pi/2)$, then $k(t,u,v)$ is a continuous function of $(u,v)\in\mathbb{R}^2/\Lambda$, and so is $\frac{\partial}{\partial t}k(t,u,v)$.*

(b) If $t=0$, then $k(t,u,v)$ is continuous at every $(u,v)\in\mathbb{R}^2/\Lambda$ except at $(0,0)$, and is locally integrable near $(0,0)$.

(c) In fact,

$$|k(t,u,v)|\lesssim(|t|+\|(u,v)\|)^{-2}$$

for all $t\in[0,\pi/4]$, $(u,v)\in\mathbb{R}^2/\Lambda$, where the implicit constant is independent of (t,u,v).

(d) For all $(t,u,v)\in[0,\pi/4]\times(\mathbb{R}^2/\Lambda)\setminus\{(0,0,0)\}$, we have

$$\frac{\partial}{\partial t}k(t,u,v)=-\frac{(\cos v-\cos t\cos u)\sin t\cos u-(\sin v-\sin t\sin u)\cos t\sin u}{\Big[(\cos v-\cos t\cos u)^2+(\sin v-\sin t\sin u)^2\Big]^{3/2}},$$

因此

$$\Big|\frac{\partial}{\partial t}k(t,u,v)\Big|\lesssim(|t|+\|(u,v)\|)^{-3},$$

其中隐含常数与 (t,u,v) 无关。

(e) 我们有

$$\lim_{t\to 0^+}\frac{1}{8\pi}\iint_{\substack{|u|\leq\pi\\|v|\leq\pi/2}}\frac{\partial k}{\partial t}(t,u,v)dudv=-\frac{1}{2}. \tag{3.3}$$

我们将此引理的证明推迟到本节末尾。

命题 3.1 的证明。利用 (2.4) 和 (3.2)，若记 $F(u,v):=f(\Phi(0,u,v))$，则有

$$\mathcal{S}f(\zeta)=\frac{\sqrt{2}}{8\pi}\iint_{\substack{|u|\leq\pi\\|v|\leq\pi/2}}F(u,v)k(\frac{t}{\sqrt{2}},u'-u,v'-v)dudv$$

whenever $\zeta=\Phi(t,u',v')$, $t\geq 0$. When $f\in L^1(\Sigma)$, the above integral converges absolutely at every point ζ by the estimate for $k(t,u,v)$ in Lemma 3.2. This integral is a convolution on the group $\mathbb{R}^2/\Lambda$, and thus we can also write

$$\mathcal{S}f(\zeta)=\frac{\sqrt{2}}{8\pi}\iint_{\substack{|u|\leq\pi\\|v|\leq\pi/2}}F(u'-u,v'-v)k(\frac{t}{\sqrt{2}},u,v)dudv. \tag{3.4}$$

类似地，当 $f\in C^1(\Sigma)$ 且 $\zeta=\Phi(0,u',v')$ 时，我们有

$$\mathcal{N}f(\zeta)=\lim_{\varepsilon\to 0^+}\frac{1}{8\pi}\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2\\\|(u'-u,v'-v)\|\geq\varepsilon}}F(u,v)\frac{\partial k}{\partial t}(0,u'-u,v'-v)dudv$$

是 F 与 $\frac{1}{8\pi}p.v.\frac{\partial k}{\partial t}(0,\cdot)$ 在 $\mathbb{R}^2/\Lambda$ 上的卷积；当 $f\in C^1(\Sigma)$ 时主值存在，因为 $\frac{\partial k}{\partial t}(0,u,v)$ 关于 u 和 v 均为奇函数，这意味着

$$\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2\\\|(u'-u,v'-v)\|\geq\varepsilon}}\frac{\partial k}{\partial t}(0,u,v)dudv=0$$

对于每个 $\varepsilon>0$，以及 $\left|\frac{\partial k}{\partial t}(0,u,v)\right|\lesssim\|(u,v)\|^{-3}$。事实上，可以将 $\mathcal{N}f$ 写成一个绝对收敛的积分：

$$\mathcal{N}f(\zeta)=\frac{1}{8\pi}\iint_{\substack{|u|\leq\pi\\|v|\leq\pi/2}}\Big[F(u'-u,v'-v)-F(u',v')\Big]\frac{\partial k}{\partial t}(0,u,v)dudv. \tag{3.5}$$

现在令 $\zeta \in \Sigma$，并回忆在（3.1）中引入的曲线 $\gamma_\zeta(t)$。利用（3.4）、引理 3.2 中关于 k 的界以及控制收敛定理，我们看到

$$\lim_{t\to 0^+} \mathcal{S}f(\gamma_\zeta(t)) = \mathcal{S}f(\gamma_\zeta(0)).$$

此外，由（3.4）以及我们对 $\frac{\partial}{\partial t}k$ 的估计可知，对于 $t > 0$,

$$\frac{d}{dt}\mathcal{S}f(\gamma_\zeta(t)) = \frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}} F(u'-u,v'-v)\frac{\partial k}{\partial t}(\frac{t}{\sqrt{2}},u,v)dudv.$$

Hence to prove Proposition 3.1, by L'Hôpital's rule and (3.5), it suffices to show, under the assumption that $f \in C^1(\Sigma)$, that if $F(u,v) = f(\Phi(0,u,v))$ and $(u',v') \in [-\pi,\pi)\times[-\pi/2,\pi/2)$, then

$$\lim_{t\to 0^+} \frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}} F(u'-u,v'-v)\frac{\partial k}{\partial t}(\frac{t}{\sqrt{2}},u,v)dudv \tag{3.6}$$

存在且等于

$$-\frac{1}{2}F(u',v') + \frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}} \Big[F(u'-u,v'-v)-F(u',v')\Big]\frac{\partial k}{\partial t}(0,u,v)dudv. \tag{3.7}$$

为此，我们将（3.6）中极限内的表达式改写为

$$\begin{aligned}\frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}} \Big[F(u'-u,v'-v)-F(u',v')\Big]\frac{\partial k}{\partial t}(t,u,v)dudv\\ +\frac{1}{8\pi}F(u',v')\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}\frac{\partial k}{\partial t}(t,u,v)dudv.\end{aligned} \tag{3.8}$$

By (3.3) in Lemma 3.2, as $t\to 0^+$, the second term in (3.8) converges to $-\frac{1}{2}F(u',v')$, which appears as the first term of (3.7). Furthermore, since $f\in C^1(\Sigma)$, our earlier bound for $\frac{\partial}{\partial t}k$ in Lemma 3.2 and the Dominated Convergence Theorem shows that as $t\to 0^+$, the first term in (3.8) converges to the second term in (3.7). This completes the proof of Proposition 3.1. □

Proof of Lemma 3.2. First, for fixed t, the function $k(t,u,v)$ is doubly 2π periodic and invariant under $(u,v)\mapsto(u+\pi,v+\pi)$. Thus $k(t,u,v)$ is a well-defined function on $\mathbb{R}^2/\Lambda$.

为进一步推进，我们证明

$$(\cos v-\cos t\cos u)^2+(\sin v-\sin t\sin u)^2=0 \iff \begin{cases}\cos t=0\\ \cos u=0\\ \sin v=\sin t\sin u\end{cases} \quad\text{or}\quad \begin{cases}\sin t=0\\ \sin u=0\\ \cos v=\cos t\cos u\end{cases}.$$

事实上，前者等价于

$$\begin{cases}\cos v=\cos t\cos u\\ \sin v=\sin t\sin u,\end{cases}$$

这意味着，由于 $\cos^2 v+\sin^2 v=1$，所以

$$\begin{aligned}1&=\cos^2 t\cos^2 u+\sin^2 t\sin^2 u\\ &=(\cos^2 t+\sin^2 t)(\cos^2 u+\sin^2 u)-\cos^2 t\sin^2 u-\sin^2 t\cos^2 u\\ &=1-\cos^2 t\sin^2 u-\sin^2 t\cos^2 u.\end{aligned}$$

因此，$\cos t\sin u=\sin t\cos u=0$。要么是$\cos t=0$，此时$\sin t=\pm1$ and$\cos u=0$；要么是$\sin t=0$，此时$\cos t=\pm1$ and $\sin u=0$。由此即可得到所声称的等价性。

如果 $t\in(0,\pi/2)$，那么 $\cos t$ 和 $\sin t$ 均不为零，因此上述结果意味着

$$(\cos v-\cos t\cos u)^2+(\sin v-\sin t\sin u)^2>0.$$

This proves (a). If $t=0$，于是 $\cos t=1$，因此 $(\cos v-\cos t\cos u)^2+(\sin v-\sin t\sin u)^2$ 的唯一零点为

$$\{(u,v)\in\mathbb{R}^2:\ \sin u=0,\cos v=\cos u\}=2\pi\mathbb{Z}^2\cup((\pi,\pi)+2\pi\mathbb{Z}^2).$$

这证明了 (b)。

为证明 (c)，注意到 $k(t,u,v)$ 在 $[0,\pi/4]\times[-\pi,\pi]\times[-\pi/2,\pi/2]\setminus\{(0,0,0)\}$ 上连续。因此我们只需证明

$$(\cos v-\cos t\cos u)^2+(\sin v-\sin t\sin u)^2\gtrsim(|t|+|u|+|v|^{1/2})^4$$

在 $(t,u,v)=(0,0,0)$ 附近。但若我们记为

$$g(t,u,v):=(\cos v-\cos t\cos u)^2+(\sin v-\sin t\sin u)^2, \tag{3.9}$$

然后通过泰勒展开

$$\begin{aligned} g(t,u,v)&=[(1-(1-\frac{t^2}{2})(1-\frac{u^2}{2})+O^{5/2}]^2+(v-tu+O^{5/2})^2\\ &=(\frac{t^2+u^2}{2})^2+(v-tu)^2+O^{9/2}\\ &=((\frac{t^2+u^2}{2})^2+(v-tu)^2)(1+O^{1/2}) \end{aligned} \tag{3.10}$$

其中，O^j是一个由 $C_j(|t|+|u|+|v|^{1/2})^j$ 界定的误差，其中 C_j 为某个绝对常数。我们的论断现在由以下事实得出：

$$(\frac{t^2+u^2}{2})^2+(v-tu)^2\simeq(|t|+|u|+|v|^{1/2})^4 \tag{3.11}$$

关于 t、u、v 一致（这可通过注意到左边是集合 $|t|+|u|+|v|^{1/2}=1$ 上的正连续函数，并利用齐次性验证）。

(d) 由.....可得

$$|\frac{\partial}{\partial t}k(t,u,v)|\le\frac{|\sin t\cos u|+|\cos t\sin u|}{(\cos v-\cos t\cos u)^2+(\sin v-\sin t\sin u)^2}\le\frac{|t|+|u|}{g(t,u,v)}$$

以及我们上面证明的关于 $[0,\pi/4]\times[-\pi,\pi]\times[-\pi/2,\pi/2]$ 的下界 $g(t,u,v)\gtrsim|t|+|u|+|v|^{1/2}$。

Finally, we prove (3.3) in (e). Since $\frac{\partial k}{\partial t}(t,u,v)$ converges uniformly to 0 on $[-\pi,\pi]\times[-\pi/2,\pi/2]\setminus[-\varepsilon,\varepsilon]\times[-\varepsilon^2,\varepsilon^2]$ for any $\varepsilon>0$, it suffices to show that there exists $\varepsilon>0$ such that

$$\lim_{t\to0^+}\iint_{\substack{|u|\le\varepsilon\\|v|\le\varepsilon^2}}\frac{\partial k}{\partial t}(t,u,v)dudv=-4\pi.$$

From

$$\frac{\partial}{\partial t}k(t,u,v)=-\frac{(\cos v-\cos t\cos u)\sin t\cos u-(\sin v-\sin t\sin u)\cos t\sin u}{g(t,u,v)^{3/2}},$$

其中 $g(t,u,v)$ 已在（3.9）中定义，因此只需证明

$$\lim_{t\to0^+}\iint_{\substack{|u|\le\varepsilon\\|v|\le\varepsilon^2}}\frac{\sin t\cos u(\cos v-\cos t\cos u)}{g(t,u,v)^{3/2}}dudv=4\pi,$$

$$\lim_{t\to0^+}\iint_{\substack{|u|\le\varepsilon\\|v|\le\varepsilon^2}}\frac{\sin u\sin v}{g(t,u,v)^{3/2}}dudv=4\pi$$

and

$$\lim_{t\to0^+}\iint_{\substack{|u|\le\varepsilon\\|v|\le\varepsilon^2}}\frac{\sin t\sin^2u}{g(t,u,v)^{3/2}}dudv=4\pi.$$

We do so by Taylor expanding the numerators and denominators of the integrands. By choosing $\varepsilon>0$ sufficiently small, we have, whenever $t\in[0,\varepsilon]$, $|u|\le\varepsilon$ and $|v|\le\varepsilon^2$, that $(1+O^{1/2})^{-3/2}=1+O^{1/2}$, and hence from (3.10)

$$g(t,u,v)^{-3/2}=[(\frac{t^2+u^2}{2})^2+(v-tu)^2]^{-3/2}(1+O^{1/2}).$$

同样，对于 $t\in[0,\varepsilon]$、$|u|\le\varepsilon$ 和 $|v|\le\varepsilon^2$，我们有

$$\sin t\cos u(\cos v-\cos t\cos u)=t\Big[\frac{(t^2+u^2)}{2}+O^{5/2}\Big]$$

and

$$\sin t \sin^2 u = t\Big[u^2 + O^{5/2}\Big].$$

Thus

$$\begin{aligned}
&\lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} \frac{\sin t\cos u(\cos v - \cos t\cos u)}{g(t,u,v)^{3/2}} dudv \\
&= \lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} \frac{t\Big[\frac{(t^2+u^2)}{2} + O^{5/2}\Big]}{[(\frac{t^2+u^2}{2})^2 + (v-tu)^2]^{3/2}} dudv.
\end{aligned}$$

涉及 $O^{5/2}$ 的误差项可利用 (3.11) 进行界定。因此

$$\begin{aligned}
&\iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} \frac{tO^{5/2}}{[(\frac{t^2+u^2}{2})^2 + (v-tu)^2]^{3/2}} dudv \\
&\le Ct \iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} (|t| + |u| + |v|^{1/2})^{-7/2} dudv \\
&= Ct \iint_{\substack{|u|\le\varepsilon/t\\ |v|\le\varepsilon^2/t^2}} (t + |tu| + |t^2 v|^{1/2})^{-7/2} t^3 dudv \\
&= Ct^{1/2} \iint_{\substack{|u|\le\varepsilon/t\\ |v|\le\varepsilon^2/t^2}} (1 + |u| + |v|^{1/2})^{-7/2} dudv
\end{aligned} \tag{3.12}$$

where we performed a change of variables $(u,v) \mapsto (ut, vt^2)$ in the second-to-last line, and the last line is $O(t^{1/2})$ as $t \to 0^+$ since the double integral converges to a finite limit (we have $\iint_{\mathbb{R}^2}(1+|u|+|v|^{1/2})^{\alpha} dudv < \infty$ if and only if $\alpha < -3$). Furthermore, the main term in our earlier computation can be identified, via the same change of variable, as

$$\begin{aligned}
&\lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon/t\\ |v|\le\varepsilon^2/t^2}} \frac{t\frac{t^2+(ut)^2}{2}}{[(\frac{t^2+(ut)^2}{2})^2 + (vt^2 - t(ut))^2]^{3/2}} t^3 dudv \\
&= \iint_{\mathbb{R}^2} \frac{\frac{1+u^2}{2}}{[(\frac{1+u^2}{2})^2 + (v-u)^2]^{3/2}} dudv = 4\pi.
\end{aligned}$$

类似地,

$$\begin{aligned}
&\lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} \frac{\sin t\sin^2 u}{g(t,u,v)^{3/2}} dudv \\
&= \lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} \frac{t\Big[u^2 + O^{5/2}\Big]}{[(\frac{t^2+u^2}{2})^2 + (v-tu)^2]^{3/2}} dudv \\
&= \iint_{\mathbb{R}^2} \frac{u^2}{[(\frac{1+u^2}{2})^2 + (v-u)^2]^{3/2}} dudv = 4\pi.
\end{aligned}$$

最后，使用 $\int_{|v|\le\varepsilon^2} f(v)dv = \frac{1}{2}\int_{|v|\le\varepsilon^2}[f(v) + f(-v)]dv$并写出

$$\begin{aligned}
&\iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} \frac{\sin u\sin v}{g(t,u,v)^{3/2}} dudv \\
&= \frac{1}{2}\iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} \frac{\sin u\sin v}{g(t,u,v)^{3/2}} - \frac{\sin u\sin v}{g(t,u,-v)^{3/2}} dudv \\
&= \frac{1}{2}\iint_{\substack{|u|\le\varepsilon\\ |v|\le\varepsilon^2}} \frac{\sin u\sin v[g(t,u,-v) - g(t,u,v)][g(t,u,-v)^2 + g(t,u,-v)g(t,u,v) + g(t,u,v)^2]}{g(t,u,v)^{3/2}g(t,u,-v)^{3/2}[g(t,u,-v)^{3/2} + g(t,u,v)^{3/2}]} dudv.
\end{aligned} \tag{3.13}$$

我们通过注意到以下事实对分子进行泰勒展开

$$\sin u \sin v = uv + O^{7/2},$$

and

$$\begin{aligned} &g(t,u,-v)^2 + g(t,u,-v)g(t,u,v) + g(t,u,v)^2 \\ &= h(t,u,-v)^2 + h(t,u,-v)h(t,u,v) + h(t,u,v)^2 + O^{17/2} \end{aligned}$$

其中我们简记了

$$h(t,u,v) := (\frac{t^2+u^2}{2})^2 + (v-tu)^2,$$

and

$$\begin{aligned} g(t,u,-v) - g(t,u,v) &= 4\sin t \sin u \sin v \\ &= 4tuv + tO^{7/2} \\ &= h(t,u,-v) - h(t,u,v) + tO^{7/2}. \end{aligned}$$

我们还对分母进行泰勒展开，从而得到

$$\begin{aligned} &g(t,u,v)^{3/2}g(t,u,-v)^{3/2}[g(t,u,-v)^{3/2} + g(t,u,v)^{3/2}] \\ &= h(t,u,v)^{3/2}h(t,u,-v)^{3/2}[h(t,u,-v)^{3/2} + h(t,u,v)^{3/2}](1+O^{1/2}). \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon \\ |v|\le\varepsilon^2}} \frac{\sin u \sin v}{g(t,u,v)^{3/2}} dudv \\ &= \lim_{t\to 0^+} \frac{1}{2} \iint_{\substack{|u|\le\varepsilon \\ |v|\le\varepsilon^2}} \frac{uv[h(t,u,-v)-h(t,u,v)][h(t,u,-v)^2 + h(t,u,-v)h(t,u,v) + h(t,u,v)^2]}{h(t,u,v)^{3/2}h(t,u,-v)^{3/2}[h(t,u,-v)^{3/2} + h(t,u,v)^{3/2}]} dudv \\ &\quad + \lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon \\ |v|\le\varepsilon^2}} \frac{tO^{29/2}}{h(t,u,v)^{3/2}h(t,u,-v)^{3/2}[h(t,u,-v)^{3/2} + h(t,u,v)^{3/2}]} dudv. \end{aligned}$$

第二个极限可由（3.12）的证明中相同的论证得到，为零。在第一个极限中，我们考虑 $t \to 0^+$ 时的极限，它是一个类似于（3.13）最后一个表达式的二重积分。通过反向推导（3.13），只是将 g 改写为 h，我们看到第一个极限等于

$$\begin{aligned} &\lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon \\ |v|\le\varepsilon^2}} \frac{uv}{h(t,u,v)^{3/2}} dudv \\ &= \lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon \\ |v+tu|\le\varepsilon^2}} \frac{(v+tu)u}{[(\frac{t^2+u^2}{2})^2 + v^2]^{3/2}} dudv \\ &= \lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon \\ -tu-\varepsilon^2\le v\le -tu+\varepsilon^2}} \frac{uv}{[(\frac{t^2+u^2}{2})^2 + v^2]^{3/2}} dudv + \lim_{t\to 0^+} \iint_{\substack{|u|\le\varepsilon \\ |v+tu|\le\varepsilon^2}} \frac{tu^2}{[(\frac{t^2+u^2}{2})^2 + v^2]^{3/2}} dudv \\ &= I + II \end{aligned}$$

我们有

$$\begin{aligned} I &= \lim_{t\to 0^+} \int_{|u|\le\varepsilon} \frac{u}{[(\frac{t^2+u^2}{2})^2 + (-tu-\varepsilon^2)^2]^{1/2}} - \frac{u}{[(\frac{t^2+u^2}{2})^2 + (-tu+\varepsilon^2)^2]^{1/2}} du \\ &= \int_{|u|\le\varepsilon} \frac{u}{(\frac{u^2}{4}+\varepsilon^4)^{1/2}} du - \int_{|u|\le\varepsilon} \frac{u}{(\frac{u^2}{4}+\varepsilon^4)^{1/2}} du = 0 - 0 = 0 \end{aligned}$$

并通过变量替换 $(u,v) \mapsto (tu, t^2v)$,

$$II = \lim_{t\to 0^+} \iint_{\substack{|u|\le\frac{\varepsilon}{t} \\ |v+u|\le\frac{\varepsilon^2}{t^2}}} \frac{u^2}{[(\frac{1+u^2}{2})^2 + v^2]^{3/2}} dudv = \iint_{\mathbb{R}^2} \frac{u^2}{[(\frac{1+u^2}{2})^2 + v^2]^{3/2}} dudv = 4\pi.$$

Thus

$$\lim_{t\to 0^+}\iint_{\substack{|u|\le\varepsilon\\|v|\le\varepsilon^2}}\frac{\sin u\sin v}{g(t,u,v)^{3/2}}dudv=4\pi,$$

如所愿。 □

4. The classical solution to the Neumann boundary value problem

在本节中，我们证明定理 1.1。

4.1. 存在性。现在我们将求取 $-\frac{1}{2}I+\mathcal{N}$ 在 L^2 上的逆，这将给出边界数据 h 属于 $C^\infty(\Sigma)$ 时 CR Yamabe 算子的诺伊曼边值问题的解。

命题 4.1. 我们有 $\|\mathcal{N}\|_{L^2\to L^2}<\frac{1}{2}$。因此，对于任意 $h\in L^2(\Sigma)$，存在唯一的 $f\in L^2(\Sigma)$ 使得

$$(-\frac{1}{2}I+\mathcal{N})f=h.$$

此外，如果 $h\in C^\infty(\Sigma)$，那么 $f\in C^\infty(\Sigma)$。

The main thrust is in showing that $\|\mathcal{N}\|_{L^2\to L^2}<\frac{1}{2}$. Once that is proved, then $-\frac{1}{2}I+\mathcal{N}$ is invertible on L^2, and is given by a multiplier operator whose coefficients are bounded below. Hence $(-\frac{1}{2}I+\mathcal{N})^{-1}$ maps $W^{k,2}(\Sigma)$ to $W^{k,2}(\Sigma)$ for any $k\ge 0$, and hence maps $C^\infty(\Sigma)$ to $C^\infty(\Sigma)$. As a result, we obtain the existence assertion in Theorem 1.1.

下面我们证明 $\|\mathcal{N}\|_{L^2\to L^2}<\frac{1}{2}$。

Indeed, since $\mathcal{N}f(\Phi(0,u',v'))$ is the convolution of $f(\Phi(0,u,v))$ with $\frac{1}{8\pi}K(u,v)$ where $K(u,v):=p.v.\frac{\partial k}{\partial t}(0,u,v)$ on $\mathbb{R}^2/\Lambda$, we have

$$\begin{aligned}\int_\Sigma|\mathcal{N}f|^2d\sigma&=\sqrt{2}\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}|\mathcal{N}f(\Phi(0,u,v))|^2dudv\\&\le\sup_{\substack{(m,n)\in\mathbb{Z}^2\\m\equiv n\pmod 2}}|\frac{1}{8\pi}\widehat{K}(m,n)|^2\sqrt{2}\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}|f(\Phi(0,u,v))|^2dudv\\&=\sup_{\substack{(m,n)\in\mathbb{Z}^2\\m\equiv n\pmod 2}}|\frac{1}{8\pi}\widehat{K}(m,n)|^2\int_\Sigma|f|^2d\sigma,\end{aligned}$$

其中

$$\widehat{K}(m,n)=\lim_{\varepsilon\to 0^+}\iint_{\substack{|u|\le\pi,|v|\le\pi/2\\\|(u,v)\|\ge\varepsilon}}\frac{\partial k}{\partial t}(0,u,v)e^{-i(mu+nv)}dudv.$$

因此，我们需要证明的是

$$\sup_{\substack{(m,n)\in\mathbb{Z}^2\\m\equiv n\pmod 2}}|\widehat{K}(m,n)|<\frac{8\pi}{2}=4\pi\simeq 12.56. \tag{4.1}$$

由于 $\frac{\partial k}{\partial t}(0,u,v)$ 关于 u 和 v 均为奇函数，我们得到 $\widehat{K}(m,n)$ 的如下表达式：

$$\widehat{K}(m,n)=-\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}K(u,v)\sin(mu)\sin(nv)dudv. \tag{4.2}$$

为了估计这个积分并证明（4.1），将其分子和分母作泰勒展开，以近似 $K(u,v)$ 会很方便。记

$$K(u,v)=\frac{n(u,v)}{d(u,v)^6}$$

其中

$$n(u,v):=\sin u\sin v,\qquad d(u,v):=[(\cos u-\cos v)^2+\sin^2 v]^{1/4}.$$

我们将 $n(u,v)$ 和 $d(u,v)$ 近似为

$$n_0(u,v) := uv \qquad \text{and} \qquad d_0(u,v) := \Big[\frac{u^4}{4} + v^2\Big]^{1/4}$$

分别，使得齐次函数

$$K_\circ(u,v) := \frac{n_0(u,v)}{d_0(u,v)^6}$$

will be a good approximation of $K(u,v)$ near $(u,v) = (0,0)$, along with its derivatives; this homogeneous function can be integrated easily near the origin, so in (4.2) we can approximate $K(u,v)$ by $K_\circ(u,v)$, and eventually obtain (4.1). In fact, it will be convenient to note that

$$d(u,v)^4 \geq d_0(u,v)^4 - \Big(\frac{u^6}{4!} + \frac{u^2v^2}{2} + \frac{2u^8}{8!} + \frac{2v^4}{4!}\Big) \tag{4.3}$$

对于所有的 $(u,v) \in \mathbb{R}^2$，这是因为

$$d(u,v)^4 = \cos^2 u - 2\cos u\cos v + 1 = \frac{1}{2}(1+\cos 2u) - 2\cos u + 1 + 2\cos u(1-\cos v)$$

然而

$$\begin{aligned}\frac{1}{2}(1+\cos 2u) - 2\cos u + 1 &\geq \frac{1}{2}\left(1+1-\frac{(2u)^2}{2!}+\frac{(2u)^4}{4!}-\frac{(2u)^6}{6!}\right) - 2\left(1-\frac{u^2}{2!}+\frac{u^4}{4!}-\frac{u^6}{6!}+\frac{u^8}{8!}\right)+1\\ &= \frac{u^4}{4} - \frac{u^6}{4!} - \frac{2u^8}{8!}\end{aligned}$$

and

$$\begin{aligned}2\cos u(1-\cos v) &= \int_0^v\int_0^w 2\cos u\cos x\,dxdw\\ &= \int_0^v\int_0^w \cos(u+x)+\cos(u-x)\,dxdw\\ &\geq \int_0^v\int_0^w \left(1-\frac{(u+x)^2}{2}\right)+\left(1-\frac{(u-x)^2}{2}\right)\,dxdw\\ &= \int_0^v\int_0^w (2-u^2-x^2)\,dxdw\\ &= v^2 - \frac{u^2v^2}{2} - \frac{2v^4}{4!}.\end{aligned}$$

为实施上述策略，我们对 $K(u,v)$求导，并写成

$$\frac{\partial K}{\partial u} = \frac{n_u(u,v)}{d(u,v)^{10}}, \qquad \frac{\partial K}{\partial v} = \frac{n_v(u,v)}{d(u,v)^{10}}, \qquad \frac{\partial^2 K}{\partial u^2} = \frac{n_{uu}(u,v)}{d(u,v)^{14}},$$

where we have, via the half-angle formulae, the following expressions for the numerators:

$$\begin{aligned}n_u(u,v) &:= \sin(v)\left(-20\sin^4(\frac{u}{2}) + 4\sin^2(\frac{v}{2})\right)\\ &\quad + \sin(v)\left(16\sin^6(\frac{u}{2}) + 2\sin^2(u)\sin^2(\frac{v}{2})\right),\\ n_v(u,v) &:= \sin(u)\left(4\sin^4(\frac{u}{2}) - 8\sin^2(\frac{v}{2})\right)\\ &\quad + \sin(u)\sin^2(\frac{v}{2})\left[8\sin^2(\frac{u}{2})\Big(1+\cos^2(\frac{u}{2})\Big) + 4\cos(u)\sin^2(\frac{v}{2})\right]\end{aligned}$$

$$\begin{aligned} n_{uu}(u,v) := {} & 120\sin(u)\sin^2(\frac{u}{2})\sin(v)(\sin^4(\frac{u}{2})-\sin^2(\frac{v}{2})) \\ & - 64\sin(u)\sin^8(\frac{u}{2})\sin(v) \\ & - 8\Big(\cos(\frac{u}{2})+\cos(\frac{3u}{2})\Big)\sin^5(\frac{u}{2})\sin(v)\sin^2(\frac{v}{2}) \\ & + 4\Big(14+\sin^2(u)\Big)\sin(u)\sin(v)\sin^4(\frac{v}{2}). \end{aligned}$$

Similarly we write

$$\frac{\partial K_\circ}{\partial u}=\frac{n_{0u}(u,v)}{d_0(u,v)^{10}},\qquad \frac{\partial K_\circ}{\partial v}=\frac{n_{0v}(u,v)}{d_0(u,v)^{10}},\qquad \frac{\partial^2 K_\circ}{\partial u^2}=\frac{n_{0uu}(u,v)}{d_0(u,v)^{14}},$$

其中

$$n_{0u}(u,v) := v(-\frac{5}{4}u^4+v^2),\qquad n_{0v}(u,v) := u(\frac{u^4}{4}-2v^2),\qquad n_{0uu}(u,v) := \frac{15}{2}u^3v(\frac{u^4}{4}-v^2).$$

稍后我们将使用

$$|n_u(u,v)| \le |n_{0u}(u,v)| + \frac{1451|u^6v|}{3072} + \frac{|u^2v^3|}{2} + \frac{v^4}{3} \tag{4.4}$$

$$|n_v(u,v)| \le |n_{0v}(u,v)| + \frac{683|u^7|}{15360} + |u^3v^2| + \frac{7|uv^3|}{6} \tag{4.5}$$

$$|n_{uu}(u,v)| \le |n_{0uu}(u,v)| + \frac{1195|u^9v|}{2048} + \frac{|u^5v^3|}{8} + \frac{5|u^3v^4|}{2} + \frac{15|uv^5|}{4} \tag{4.6}$$

事实上，我们有

$$\left|\frac{d^6}{du^6}\sin^4(\frac{u}{2})\right| = \left|\frac{\cos(u)}{2}-8\cos(2u)\right| \le \frac{2049}{256} \quad \text{for all } u\in\mathbb{R}$$

and

$$\left|\frac{d^3}{dv^3}\sin^2(\frac{v}{2})\right| = \left|-\frac{\sin v}{2}\right| \le \frac{1}{2} \quad \text{for all } v\in\mathbb{R}.$$

因此，泰勒展开（分别展开到 5 阶和 3 阶）给出

$$\left|\sin^4(\frac{u}{2})-\frac{u^4}{16}\right| \le \frac{2049}{256}\frac{u^6}{6!}$$

and

$$\left|\sin^2(\frac{v}{2})-\frac{v^2}{4}\right| \le \frac{1}{2}\frac{|v|^3}{3!}.$$

因此，我们可以估算 n_u、 n_v 和 n_{uu} 的表达式中的主要项:

$$\begin{aligned} \left|\sin(v)\left(-20\sin^4(\frac{u}{2})+4\sin^2(\frac{v}{2})\right)\right| &\le |v|\left(\left|-20\cdot\frac{u^4}{16}+4\cdot\frac{v^2}{4}\right|+20\cdot\frac{2049}{256}\frac{u^6}{6!}+4\cdot\frac{1}{2}\frac{|v|^3}{3!}\right) \\ &= |n_{0u}(u,v)|+\frac{683}{3072}|u^6v|+\frac{1}{3}v^4, \end{aligned}$$

$$\begin{aligned} \left|\sin(u)\left(4\sin^4(\frac{u}{2})-8\sin^2(\frac{v}{2})\right)\right| &\le |u|\left(\left|4\cdot\frac{u^4}{16}-8\cdot\frac{v^2}{4}\right|+4\cdot\frac{2049}{256}\frac{u^6}{6!}+8\cdot\frac{1}{2}\frac{|v|^3}{3!}\right) \\ &= |n_{0v}(u,v)|+\frac{683}{15360}|u^7|+\frac{2}{3}|uv^3|, \end{aligned}$$

and

$$\begin{aligned} \left|120\sin(u)\sin^2(\frac{u}{2})\sin(v)\left(\sin^4(\frac{u}{2})-\sin^2(\frac{v}{2})\right)\right| &\le 30|u|^3|v|\left(\left|\frac{u^4}{16}-\frac{v^2}{4}\right|+\frac{2049}{256}\frac{u^6}{6!}+\frac{1}{2}\frac{|v|^3}{3!}\right) \\ &= |n_{0uu}(u,v)|+\frac{683}{2048}|u^9v|+\frac{5}{2}|u^3v^4|. \end{aligned}$$

此外，误差项可以通过以下方式界定

$$\begin{aligned}\Big|\sin(v)\Big(16\sin^6(\frac{u}{2})+2\sin^2(u)\sin^2(\frac{v}{2})\Big)\Big| &\le |v|\left(16\frac{|u|^6}{2^6}+2|u|^2\frac{|v|^2}{2^2}\right)\\ &=\frac{1}{4}|u^6v|+\frac{1}{2}|u^2v^3|,\end{aligned}$$

$$\begin{aligned}\Big|\sin(u)\sin^2(\frac{v}{2})\left[8\sin^2(\frac{u}{2})\Big(1+\cos^2(\frac{u}{2})\Big)+4\cos(u)\sin^2(\frac{v}{2})\right]\Big| &\le |\frac{uv^2}{4}|\left[8\frac{u^2}{2^2}(1+1)+4\cdot 1\cdot 1\cdot\frac{|v|}{2}\right]\\ &=|u^3v^2|+\frac{1}{2}|uv^3|,\end{aligned}$$

$$\Big|-64\sin(u)\sin^8(\frac{u}{2})\sin(v)\Big|\le 64|u|\frac{u^8}{2^8}|v|=\frac{1}{4}|u^9v|,$$

$$\Big|-8\Big(\cos(\frac{u}{2})+\cos(\frac{3u}{2})\Big)\sin^5(\frac{u}{2})\sin(v)\sin^2(\frac{v}{2})\Big|\le 8(1+1)\frac{|u|^5}{2^5}|v|\frac{|v|^2}{2^2}=\frac{1}{8}|u^5v^3|,$$

and

$$\Big|4\Big(14+\sin^2(u)\Big)\sin(u)\sin(v)\sin^4(\frac{v}{2})\Big|\le 4(14+1)|uv|\frac{v^4}{2^4}=\frac{15}{4}|uv^5|.$$

上述九个不等式共同蕴含 (4.4)、(4.5) 和 (4.6)。

现在令

$$b:=\frac{2}{5}.$$

对于 $0<a\le 5$，令

$$e_0(a):=\sup\left\{\frac{u^6}{4!}+\frac{u^2v^2}{2}+\frac{2u^6(ab)^2}{8!}+\frac{2|v|^3b^2}{4!}\colon d_0(u,v)=1\right\}$$

and

$$e_1(a):=1-e_0(a)d_0(ab,b^2)^2.$$

由齐次性可知，因此我们有

$$\frac{u^6}{4!}+\frac{u^2v^2}{2}+\frac{2u^6(ab)^2}{8!}+\frac{2|v|^3b^2}{4!}\le e_0(a)d_0(u,v)^6\quad\text{for all }(u,v)\in\mathbb{R}^2 \tag{4.7}$$

Furthermore, $e_1(a)$ is a decreasing function of $a\in[0,\infty)$, with $e_1(5)>0$. Thus $e_1(a)>0$ for $0<a\le 5$. We claim that if $0<a\le 5$ and $\|(u/a,v)\|\le b$, then

$$|K(u,v)|\le|K_\circ(u,v)|+\mathrm{err}_0(a,u,v), \tag{4.8}$$

其中

$$\mathrm{err}_0(a,u,v):=c(6,a)\frac{|n_0(u,v)|}{d_0(u,v)^4}$$

with

$$c(\alpha,a):=\frac{\alpha e_0(a)}{4e_1(a)^{\frac{\alpha+4}{4}}}.$$

事实上，当 $\|(u/a,v)\|\le b$ 时，我们有

$$\begin{aligned}d(u,v)^4&\ge d_0(u,v)^4-\Big(\frac{u^6}{4!}+\frac{u^2v^2}{2}+\frac{2u^8}{8!}+\frac{2v^4}{4!}\Big)\\ &\ge d_0(u,v)^4-\Big(\frac{u^6}{4!}+\frac{u^2v^2}{2}+\frac{2u^6(ab)^2}{8!}+\frac{2|v|^3b^2}{4!}\Big)\\ &\ge d_0(u,v)^4\left(1-e_0(a)d_0(u,v)^2\right),\end{aligned} \tag{4.9}$$

其中我们首先使用了 (4.3)，接着在 $\|(u/a,v)\|\le b$ 时使用了 $|u|\le ab$ 和 $|v|\le b^2$，最后使用了 (4.7)。此外，由中值定理可得

$$\frac{1}{(1-t)^{\alpha/4}}\le 1+\frac{\alpha t}{4(1-t)^{\frac{\alpha+4}{4}}}\quad\text{if }\alpha>0\text{ and }0\le t<1.$$

Ap将其与 $t = e_0(a)d_0(u,v)^2 \le e_0(a)d_0(ab,b^2)^2 = 1 - e_1(a) < 1$ 相乘，并注意到

$$\frac{1}{(1-t)^{\frac{\alpha+4}{4}}} \le \frac{1}{(1-e_0(a)d_0(ab,b^2)^2)^{\frac{\alpha+4}{4}}} = \frac{1}{e_1(a)^{\frac{\alpha+4}{4}}},$$

我们得到

$$\frac{1}{d(u,v)^\alpha} \le \frac{1}{d_0(u,v)^\alpha} + \frac{c(\alpha,a)}{d_0(u,v)^{\alpha-2}} \qquad \text{for } \alpha > 0,\ 0 < a \le 5 \text{ and } \|(u/a,v)\| \le b. \tag{4.10}$$

不等式(4.8) 现由 $|n(u,v)| \le |n_0(u,v)|$ 得出，其中

$$|K(u,v)| \le \frac{|n_0(u,v)|}{d(u,v)^6},$$

并且由于我们可以利用 (4.10) 和 $\alpha = 6$ 来界定 $d\ \frac{1}{u\ v(,)^6}$

我们还注意到

$$\frac{1}{d(u,v)} \le \frac{1}{e_1(a)^{1/4}d_0(u,v)} \qquad \text{if } 0 < a \le 5 \text{ and } \|(u/a,v)\| \le b, \tag{4.11}$$

因为这样我们就可以应用(4.9)，并使用 $1 - e_0(a)d_0(u,v)^2 \ge 1 - e_0(a)d_0(ab,b^2)^2 = e_1(a)$。

运用类似的技巧，我们可以证明，对于 $0 < a \le 5$ 和 $\|(u/a,v)\| \le b$,

$$|\frac{\partial K}{\partial u}(u,v)| \le |\frac{\partial K_\circ}{\partial u}(u,v)| + \mathrm{err}_u(a,u,v) \tag{4.12}$$

and

$$|\frac{\partial K}{\partial v}(u,v)| \le |\frac{\partial K_\circ}{\partial v}(u,v)| + \mathrm{err}_v(a,u,v), \tag{4.13}$$

其中

$$\mathrm{err}_u(a,u,v) := c(10,a)\frac{|n_{0u}(u,v)|}{d_0(u,v)^8} + \frac{\frac{1451|u^6v|}{3072} + \frac{|u^2v^3|}{2} + \frac{v^4}{3}}{e_1(a)^{10/4}d_0(u,v)^{10}},$$

and

$$\mathrm{err}_v(a,u,v) := c(10,a)\frac{|n_{0v}(u,v)|}{d_0(u,v)^8} + \frac{\frac{683|u^7|}{15360} + |u^3v^2| + \frac{7|uv^3|}{6}}{e_1(a)^{10/4}d_0(u,v)^{10}}.$$

这是因为当 $0 < a \le 5$ 和 $\|(u/a,v)\| \le b$,

$$\begin{aligned}|\frac{\partial K}{\partial u}(u,v)| &\le \frac{|n_{0u}(u,v)|}{d(u,v)^{10}} + \frac{\frac{1451|u^6v|}{3072} + \frac{|u^2v^3|}{2} + \frac{v^4}{3}}{d(u,v)^{10}} \\ &\le \frac{|n_{0u}(u,v)|}{d_0(u,v)^{10}} + c(10,a)\frac{|n_{0u}(u,v)|}{d_0(u,v)^8} + \frac{\frac{1451|u^6v|}{3072} + \frac{|u^2v^3|}{2} + \frac{v^4}{3}}{e_1(a)^{10/4}d_0(u,v)^{10}},\end{aligned}$$

where in the first inequality we applied (4.4), and in second inequality we used (4.10) with $\alpha = 10$ to estimate the first term, and used (4.11) to estimate the second term. This proves (4.12). Similarly we have (4.13). We also have, for $0 < a \le 5$ and $\|(u/a,v)\| \le b$, that

$$|\frac{\partial^2 K}{\partial u^2}(u,v)| \le |\frac{\partial^2 K_\circ}{\partial u^2}(u,v)| + \mathrm{err}_{uu}(a,u,v) \tag{4.14}$$

其中

$$\mathrm{err}_{uu}(a,u,v) := c(14,a)\frac{|n_{0uu}(u,v)|}{d_0(u,v)^{12}} + \frac{\frac{1195|u^9v|}{2048} + \frac{|u^5v^3|}{8} + \frac{5|u^3v^4|}{2} + \frac{15|uv^5|}{4}}{e_1(a)^{14/4}d_0(u,v)^{14}}.$$

$e_0(a)$、$e_1(a)$和 $c(\alpha,a)$ 的数值都可以在给定 α 和 a 后使用Mathematica 进行估计。因此，err_0,err_u、err_v 和 err_{uu} 都是完全具体的函数。

在接下来的两个引理中，我们将证明关于 $K(m,n)$的两个界。第一个界在 m 较大时较好；第二个界在 n 较大时较好。回顾 $b = \frac{2}{5}$。

引理 4.2. 假设 $c_1 > 0$、 m、 n 是奇偶性相同的正整数，且 $m \geq c_1/b$。那么对于任意 $0 < a \leq 5$,

$$\begin{aligned}|\widehat{K}(m,n)| \leq c_1^3 I_0 \frac{n}{m^2} + \frac{I_1}{c_1} + \frac{I_2+I_3}{c_1^2} \\ + \frac{1}{m^2}\left(c_1^5 \frac{n}{m^2}\epsilon_{I0} + c_1\epsilon_{I1} + \epsilon_{I2} - \frac{I_3}{b^2} + \epsilon_{I3}\log\frac{bm}{c_1} + \epsilon_{I4}\right)\end{aligned}$$

其中 $I_0, ..., I_3, \epsilon_{I0}, ..., \epsilon_{I4}$ 是依赖于 a 的常数，给出为

$$I_0 := \iint_{\|(u/a,v)\|\leq 1} |uvK_\circ(u,v)|dudv, \qquad I_1 := 2\int_{-1}^{1} |K_\circ(a,v)|dv,$$
$$I_2 := 2\int_{-1}^{1} |\frac{\partial K_\circ}{\partial u}(a,v)|dv, \qquad I_3 := \iint_{\|(u/a,v)\|\geq 1} |\frac{\partial^2 K_\circ}{\partial u^2}(u,v)|dudv,$$

and

$$\epsilon_{I0} := \iint_{\|(u/a,v)\|\leq 1} |uv|\,\mathrm{err}_0(a,u,v)dudv,$$
$$\epsilon_{I1} := 2\int_{-1}^{1} \mathrm{err}_0(a,a,v)dv, \qquad \epsilon_{I2} := 2\int_{-1}^{1} \mathrm{err}_u(a,a,v)dv,$$
$$\epsilon_{I3} := 2a\int_{-1}^{1} \mathrm{err}_{uu}(a,a,v)dv + 4\int_{-a}^{a} \mathrm{err}_{uu}(a,u,1)du,$$
$$\epsilon_{I4} := \iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>b}} |\frac{\partial^2 K}{\partial u^2}(u,v)|dudv.$$

证明。定义 $R = c_1/m$。我们的假设保证 $R \leq b$。我们将其拆分。

$$\begin{aligned}-\widehat{K}(m,n) &= \iint_{\substack{|u|\leq\pi \\ |v|\leq\pi/2}} K(u,v)\sin(mu)\sin(nv)dudv \\ &= \iint_{\|(u/a,v)\|\leq R} + \iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>R}} K(u,v)\sin(mu)\sin(nv)dudv.\end{aligned}$$

我们通过将绝对值置于其中来估计第一项，并在第二项中使用 $\sin(mu) = -\frac{1}{m}\frac{d}{du}\cos(mu)$进行分部积分。注意，集合 $\{(u,v): \|(u/a,v)\| = R\}$ 是一个矩形的边界。因此

$$\begin{aligned}&\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>R}} K(u,v)\sin(mu)\sin(nv)dudv \\ &= -\frac{2}{m}\int_{-R^2}^{R^2} K(aR,v)\cos(maR)\sin(nv)dv + \frac{1}{m}\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>R}} \frac{\partial K}{\partial u}(u,v)\cos(mu)\sin(nv)dudv.\end{aligned}$$

对于最后一个二重积分，我们再次进行分部积分，使用 $\cos(mu) = \frac{1}{m}\frac{d}{du}\sin(mu)$，得到

$$\begin{aligned}&\frac{1}{m}\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>R}} \frac{\partial K}{\partial u}(u,v)\cos(mu)\sin(nv)dudv \\ &= \frac{2}{m^2}\int_{-R^2}^{R^2} \frac{\partial K}{\partial u}(aR,v)\sin(maR)\sin(nv)dv - \frac{1}{m^2}\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>R}} \frac{\partial^2 K}{\partial u^2}(u,v)\sin(mu)\sin(nv)dudv.\end{aligned}$$

取绝对值，并利用 $|\sin(mu)\sin(nv)| \leq mn|uv|$，得到

$$\begin{aligned}|\widehat{K}(m,n)| \leq\ & mn\iint_{\|(u/a,v)\|\leq R} |uvK(u,v)|dudv + \frac{2}{m}\int_{-R^2}^{R^2} |K(aR,v)|dv + \frac{2}{m^2}\int_{-R^2}^{R^2} |\frac{\partial K}{\partial u}(aR,v)|dv \\ &+ \frac{1}{m^2}\iint_{R<\|(u/a,v)\|\leq b} |\frac{\partial^2 K}{\partial u^2}(u,v)|dudv + \frac{1}{m^2}\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>b}} |\frac{\partial^2 K}{\partial u^2}(u,v)|dudv.\end{aligned}$$

前四个积分都包含在 $\{(u,v)\colon\ \|(u/a,v)\|\le b\}$ 中，因此我们之前的估计式 (4.8)、(4.12)和 (4.14)适用。这些估计使我们能够用 $K_\circ$ 及其导数来控制 K 及其导数，误差项则可加以控制。于是我们得到

$$\begin{aligned}|\widehat{K}(m,n)|\le\ & mn\iint_{\|(u/a,v)\|\le R}|uvK_\circ(u,v)|dudv+\frac{2}{m}\int_{-R^2}^{R^2}|K_\circ(aR,v)|dv+\frac{2}{m^2}\int_{-R^2}^{R^2}|\frac{\partial K_\circ}{\partial u}(aR,v)|dv\\&+\frac{1}{m^2}\iint_{R<\|(u/a,v)\|\le b}|\frac{\partial^2K_\circ}{\partial u^2}(u,v)|dudv\\&+mn\iint_{\|(u/a,v)\|\le R}|uv|\operatorname{err}_0(a,u,v)dudv+\frac{2}{m}\int_{-R^2}^{R^2}\operatorname{err}_0(a,aR,v)dv+\frac{2}{m^2}\int_{-R^2}^{R^2}\operatorname{err}_u(a,aR,v)dv\\&+\frac{1}{m^2}\iint_{R<\|(u/a,v)\|\le b}\operatorname{err}_{uu}(a,u,v)dudv+\frac{\epsilon_{I4}}{m^2}.\end{aligned}$$

上述右侧的所有积分现在都是关于 (u,v) 的齐次函数的积分。由于$|uvK_\circ(u,v)|$ 是次数为0 的齐次函数，变量替换$(u,v)\mapsto(Ru,R^2v)$ 给出

$$mn\iint_{\|(u/a,v)\|\le R}|uvK_\circ(u,v)|dudv=mnR^3\iint_{\|(u/a,v)\|\le 1}|uvK_\circ(u,v)|dudv=mnR^3I_0.$$

类似地，涉及 $K_\circ$、 $\frac{\partial K_\circ}{\partial u}$、$\operatorname{err}_0$ 和 err_u 的积分是 R的齐次函数。接下来，我们通过变量代换重写涉及 $\frac{\partial^2K_\circ}{\partial u^2}$ 的积分:

$$\begin{aligned}&\frac{1}{m^2}\iint_{R<\|(u/a,v)\|\le b}|\frac{\partial^2K_\circ}{\partial u^2}(u,v)|dudv\\&=\frac{1}{m^2}\left(\iint_{\|(u/a,v)\|>R}|\frac{\partial^2K_\circ}{\partial u^2}(u,v)|dudv-\iint_{\|(u/a,v)\|>b}|\frac{\partial^2K_\circ}{\partial u^2}(u,v)|dudv\right)\\&=I_3\left(\frac{1}{m^2R^2}-\frac{1}{m^2b^2}\right).\end{aligned}$$

最后，根据微积分基本定理和链式法则,

$$\frac{d}{dt}\iint_{1<\|(u/a,v)\|\le t}\operatorname{err}_{uu}(a,u,v)dudv=\ 2a\int_{-t^2}^{t^2}\operatorname{err}_{uu}(a,at,v)dv+4t\int_{-at}^{at}\operatorname{err}_{uu}(a,u,t^2)du,$$

根据齐次性，等于

$$\frac{1}{t}\Big(2a\int_{-1}^{1}\operatorname{err}_{uu}(a,a,v)dv+4\int_{-a}^{a}\operatorname{err}_{uu}(a,u,1)du\Big)=\frac{\epsilon_{I3}}{t}.$$

Thus

$$\begin{aligned}\iint_{R<\|(u/a,v)\|\le b}\operatorname{err}_{uu}(a,u,v)dudv&=\iint_{1<\|(u/a,v)\|\le b/R}\operatorname{err}_{uu}(a,u,v)dudv\\&=\int_1^{b/R}\frac{d}{dt}\iint_{1<\|(u/a,v)\|\le t}\operatorname{err}_{uu}(a,u,v)dudvdt\\&=\int_1^{b/R}\frac{\epsilon_{I3}}{t}dt=\epsilon_{I3}\log(\frac{b}{R}),\end{aligned}$$

综合以上各项可知,

$$\begin{aligned}|\widehat{K}(m,n)|\le\ & mnR^3I_0+\frac{I_1}{mR}+\frac{I_2+I_3}{m^2R^2}\\&+mnR^5\epsilon_{I0}+\frac{\epsilon_{I1}}{m}R+\frac{\epsilon_{I2}}{m^2}-\frac{I_3}{m^2b^2}+\frac{\epsilon_{I3}\log(b/R)}{m^2}+\frac{\epsilon_{I4}}{m^2}.\end{aligned}$$

记住 $R=c_1/m$ ，即可得到引理 4.2 中的估计。 □

引理 4.3. 假设 $c_2 > 0$、 m、 n 是奇偶性相同的正整数，并且 $n \geq (c_2/b)^2$。那么对于任意 $0 < a \leq 5$,

$$\begin{aligned}|\widehat{K}(m,n)| \leq\, & c_2^3 J_0 \frac{m}{\sqrt{n}} + \frac{J_1+J_2}{c_2^2} \\ & + \frac{1}{n}\left(c_2^5 \epsilon_{J0} \frac{m}{\sqrt{n}} + \epsilon_{J1} - \frac{J_2}{b^2} + \epsilon_{J2} \log \frac{b\sqrt{n}}{c_2} + \epsilon_{J3} \right)\end{aligned}$$

其中 J_0、.....、 J_2和 ϵ_{J0}、.....、 ϵ_{J3} 是依赖于 a 的常数，表示为

$$\begin{aligned} J_0 &:= \iint_{\|(u/a,v)\|\leq 1} |uvK_\circ(u,v)|dudv, \\ J_1 &:= 2\int_{-a}^{a} |K_\circ(u,1)|du, \qquad J_2 := \iint_{\|(u/a,v)\|\geq 1} |\frac{\partial K_\circ}{\partial v}(u,v)|dudv, \end{aligned}$$

and

$$\begin{aligned} \epsilon_{J0} &:= \iint_{\|(u/a,v)\|\leq 1} |uv|\,\mathrm{err}_0(a,u,v)dudv, \\ \epsilon_{J1} &:= 2\int_{-a}^{a} \mathrm{err}_0(a,u,1)du \\ \epsilon_{J2} &:= 2a\int_{-1}^{1} \mathrm{err}_v(a,a,v)dv + 4\int_{-a}^{a} \mathrm{err}_v(a,u,1)du, \\ \epsilon_{J3} &:= \iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>b}} |\frac{\partial K}{\partial v}(u,v)|dudv. \end{aligned}$$

Proof. The proof is similar to the previous lemma (and slightly easier); one only needs to split the integral into two parts, one where $\|(u/a,v)\| \leq R$, and another where $\|(u/a,v)\| > R$, with $R := c_2/\sqrt{n} < b$; one then integrates by parts once in v for the second integral, using $\sin(nv) = -\frac{1}{n}\frac{d}{dv}\cos(nv)$.

更准确地说，定义 $R = c_2/\sqrt{n}$。我们的假设保证了 $R \leq b$。我们将其分开

$$\begin{aligned} -\widehat{K}(m,n) &= \iint_{\substack{|u|\leq\pi \\ |v|\leq\pi/2}} K(u,v)\sin(mu)\sin(nv)dudv \\ &= \iint_{\|(u/a,v)\|\leq R} + \iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>R}} K(u,v)\sin(mu)\sin(nv)dudv. \end{aligned}$$

We estimate the first term by putting absolute value inside, and integrate by parts in the second term using $\sin(nv) = -\frac{1}{n}\frac{d}{dv}\cos(nv)$. Note that the set $\{(u,v)\colon \|(u/a,v)\| = R\}$ is the boundary of a rectangle. Thus

$$\begin{aligned} &\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>R}} K(u,v)\sin(mu)\sin(nv)dudv \\ &= -\frac{2}{n}\int_{-aR}^{aR} K(u,R^2)\sin(mu)\cos(nR^2)du + \frac{1}{n}\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>R}} \frac{\partial K}{\partial v}(u,v)\sin(mu)\cos(nv)dudv. \end{aligned}$$

Thus

$$\begin{aligned} |\widehat{K}(m,n)| \leq\, & mn\iint_{\|(u/a,v)\|\leq R} |uvK(u,v)|dudv + \frac{2}{n}\int_{-aR}^{aR} |K(u,R^2)|du \\ & + \frac{1}{n}\iint_{R<\|(u/a,v)\|\leq b} |\frac{\partial K}{\partial v}(u,v)|dudv + \frac{1}{n}\iint_{\substack{|u|\leq\pi,|v|\leq\pi/2 \\ \|(u/a,v)\|>b}} |\frac{\partial K}{\partial v}(u,v)|dudv. \end{aligned}$$

The first three integrals are all contained in $\{(u,v)\colon \|(u/a,v)\| \le b\}$, so our earlier estimates (4.8) and (4.13) apply. Thus

$$\begin{aligned}|\widehat{K}(m,n)| \le\ & mn \iint_{\|(u/a,v)\|\le R} |uvK_\circ(u,v)|dudv + \frac{2}{n}\int_{-aR}^{aR} |K_\circ(u,R^2)|du + \frac{1}{n}\iint_{R<\|(u/a,v)\le b\|} |\frac{\partial K_\circ}{\partial v}(u,v)|dudv \\ & + mn \iint_{\|(u/a,v)\|\le R} |uv|\,\mathrm{err}_0(a,u,v)dudv + \frac{2}{n}\int_{-aR}^{aR} \mathrm{err}_0(a,u,R^2)dv \\ & + \frac{1}{n}\iint_{R<\|(u/a,v)\|\le b} \mathrm{err}_v(a,u,v)dudv + \frac{\epsilon_{J3}}{m^2}.\end{aligned}$$

上面右侧的所有积分现在都是关于 (u,v) 的齐次函数积分。因此，这些积分都是关于 R 的齐次函数，唯一的例外是涉及 err_v 的积分，因为 err_v 是 -3 次齐次函数。不过，根据微积分基本定理和链式法则，

$$\frac{d}{dt}\iint_{1<\|(u/a,v)\|\le t} \mathrm{err}_v(a,u,v)dudv = 2a\int_{-t^2}^{t^2} \mathrm{err}_v(a,at,v)dv + 4t\int_{-at}^{at} \mathrm{err}_v(a,u,t^2)du,$$

根据齐次性，这等于

$$\frac{1}{t}\Big(2a\int_{-1}^{1} \mathrm{err}_v(a,a,v)dv + 4\int_{-a}^{a} \mathrm{err}_v(a,u,1)du\Big) = \frac{\epsilon_{J2}}{t}.$$

Thus

$$\begin{aligned}\iint_{R<\|(u/a,v)\|\le b} \mathrm{err}_v(a,u,v)dudv &= \iint_{1<\|(u/a,v)\|\le b/R} \mathrm{err}_v(a,u,v)dudv \\ &= \int_1^{b/R} \frac{d}{dt}\iint_{1<\|(u/a,v)\|\le t} \mathrm{err}_v(a,u,v)dudvdt \\ &= \int_1^{b/R} \frac{\epsilon_{J2}}{t}dt = \epsilon_{J2}\log(\frac{b}{R}),\end{aligned}$$

并且通过对所有其他积分进行类似的缩放，可以看出：

$$\begin{aligned}|\widehat{K}(m,n)| \le\ & mnR^3J_0 + \frac{J_1+J_2}{nR^2} \\ & + mnR^5\epsilon_{J0} + \frac{\epsilon_{J1}}{n} - \frac{J_2}{nb^2} + \frac{\epsilon_{J2}\log(b/R)}{n} + \frac{\epsilon_{J3}}{n}.\end{aligned}$$

注意， $R = c_2/\sqrt{n}$ 给出了引理 4.3 中的估计。 □

We分别将上述两个引理应用于 $a=2$, $c_1 := \frac{8}{5}$ 和 $a=1$, $c_2 := \frac{4\sqrt{2}}{5}$。 :

推论 4.4.假设 m、n 是具有相同奇偶性的正整数。如果 $m \ge 4$，那么

$$|\widehat{K}(m,n)| \le \frac{7168}{625}\frac{n}{m^2} + 5 + \frac{1}{m^2}\Big(\frac{783}{40} + \frac{98304}{3125}\frac{n}{m^2} + 192\log(\frac{m}{4})\Big). \tag{4.15}$$

证明。我们应用引理 4.2，并使用 $a=2$，因此Mathematica返回I_0、I_1、I_2、I_3、ϵ_{I0}、ϵ_{I1}、ϵ_{I2}、ϵ_{I3} 的精确值表达式，以及 ϵ_{I4} 的数值估计（见附录 A）。于是我们有

$$\begin{aligned}I_0 &= \frac{32}{3}\Big(\log(\frac{1+\sqrt{5}}{2}) - \frac{1}{32}\sqrt{\frac{2}{\pi}}\Gamma(-\frac{1}{4})^2 + {}_2F_1(-\frac{1}{4},\frac{1}{2},\frac{3}{4},-\frac{1}{4}) - {}_2F_1(\frac{1}{2},\frac{3}{4},\frac{7}{4},-4)\Big) \\ I_1 &= 4 - \frac{8}{\sqrt{5}}, \qquad I_2 = 2 - \frac{12}{5\sqrt{5}}, \qquad I_3 = 6 + 4\sqrt{2} - \frac{12}{5\sqrt{5}},\end{aligned}$$

so

$$I_0 < \frac{14}{5}, \quad I_1 < \frac{1}{2}, \quad I_2 < 1, \quad \frac{21}{2} < I_3 < 11.$$

接下来，回顾 $b=2/5$，因此

$$e_0(2)=\frac{1}{75}\sqrt{\frac{68935125533+1312025\sqrt{1102101}}{55105058}},$$
$$e_1(2)=1-\frac{4}{5\sqrt{5}}e_0(2),$$
$$\epsilon_{I0}=\frac{3}{5}\frac{e_0(2)}{e_1(2)^{5/2}}\Big(16+\pi+4\arctan(3)-32\arctan(\frac{1}{2})-2\log 5\Big),$$
$$\epsilon_{I1}=\frac{6e_0(2)}{e_1(2)^{5/2}}\log\frac{5}{4},$$
$$\epsilon_{I2}=(6+5\log\frac{4}{5})\frac{e_0(2)}{e_1(2)^{7/2}}+\frac{1}{e_1(2)^{5/2}}\Big(\frac{2219}{288}-\frac{307}{20\sqrt{5}}+\frac{4}{3}\log\frac{1+\sqrt{5}}{2}\Big),$$
$$\epsilon_{I3}=105\frac{e_0(2)}{e_1(2)^{9/2}}+\frac{9473}{240\,e_1(2)^{7/2}}.$$

结合 ϵ_{I4} 的数值，我们得到

$$\epsilon_{I0}<3,\quad \epsilon_{I1}<2,\quad \epsilon_{I2}<7,\quad \epsilon_{I3}<192,\quad \epsilon_{I4}<75.$$

最后我们令$c_1=\frac{8}{5}$，使得 $c_1/b=4\le m$。 □

推论 4.5.假设 m、n 是具有相同奇偶性的正整数。如果 $n\ge 8$，那么

$$|\widehat{K}(m,n)|\le\frac{4832\sqrt{2}}{3125}\frac{m}{\sqrt{n}}+\frac{245}{32}+\frac{1}{n}\Big(\frac{53}{8}+\frac{16384\sqrt{2}}{15625}\frac{m}{\sqrt{n}}+19\log(\frac{n}{8})\Big).\tag{4.16}$$

证明。我们应用引理4.3与 $a=1$，因此Mathematica返回 J_0、J_1、J_2、ϵ_{J0}、ϵ_{J1}和 ϵ_{J2} 的精确值表达式，以及 ϵ_{J3} 的数值估计（见附录 A）。因此我们有：

$$J_0=\frac{1}{3}\Big(4\log(2+\sqrt{5})-\sqrt{\frac{2}{\pi}}\Gamma(-\frac{1}{4})^2+16\,{}_2F_1(-\frac{1}{4},\frac{1}{2},\frac{3}{4},-4)-4\,{}_2F_1(\frac{1}{2},\frac{3}{4},\frac{7}{4},-\frac{1}{4})\Big),$$
$$J_1=\frac{4}{\sqrt{5}},\qquad J_2=\frac{32}{3\sqrt{3}}+\frac{4}{\sqrt{5}},$$

so

$$J_0<\frac{151}{100},\quad J_1<\frac{9}{5},\quad \frac{79}{10}<J_2<8.$$

接下来，回顾 $b=2/5$，因此

$$e_0(1)=\frac{1}{75}\sqrt{\frac{137909641801+2625550\sqrt{1102731}}{110273101}}$$
$$e_1(1)=1-\frac{2}{5\sqrt{5}}e_0(1)$$
$$\epsilon_{J0}=\frac{3}{5}\frac{e_0(1)}{e_1(1)^{5/2}}\Big(2+3\arctan(2)-2\log 5\Big)$$
$$\epsilon_{J1}=6\arctan(\frac{1}{2})\frac{e_0(1)}{e_1(1)^{5/2}}$$
$$\epsilon_{J2}=5\Big(4\sqrt{2}+\pi-4\arctan(\frac{1}{\sqrt{2}})\Big)\frac{e_0(1)}{e_1(1)^{7/2}}+\frac{4843}{360\,e_1(1)^{5/2}}.$$

连同 ϵ_{J3} 的数值，我们得到

$$\epsilon_{J0}<\frac{4}{5},\quad \epsilon_{J1}<2,\quad \epsilon_{J2}<38,\quad \epsilon_{J3}<54.$$

最后，我们设定 $c_2=\frac{4\sqrt{2}}{5}$，使得 $(c_2/b)^2=8\le n$。 □

备注 4.6。推论 4.4 和 4.5 中使用的 Mathematica 代码收录于附录 A，供有兴趣的读者参考。

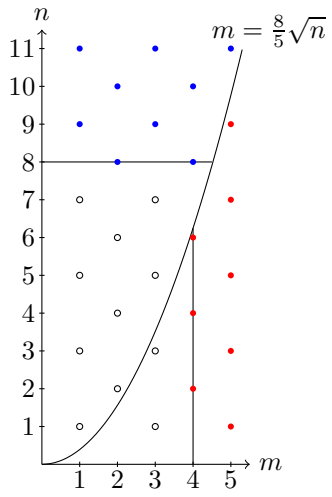

FIGURE 3. 其中 $|K(m,n)|$ 可由推论 4.4 和 4.5 分别界定的点，分别标为红色和蓝色。

由推论 4.4 可知，若 $m \geq 4$ 和 $\frac{8}{5}\sqrt{n} \leq m$，则

$$|\widehat{K}(m,n)| \leq \frac{237}{25} + \frac{1}{m^2}\Big(\frac{31863}{1000} + 192\log(\frac{m}{4})\Big). \tag{4.17}$$

The right hand side is an increasing function of m on $[4,5]$ and a decreasing function of m on $[6,\infty)$, and is $< 4\pi$ at both $m = 5$ and $m = 6$. Thus we have $|\widehat{K}(m,n)| < 4\pi$ if $m \geq 4$ and $m \geq \frac{8}{5}\sqrt{n}$. This shows $|\widehat{K}(m,n)| < 4\pi$ for (m,n) coloured in red in Figure 3.

由推论 4.5 可知，若 $n \geq 8$ 且 $m \leq \frac{8}{5}\sqrt{n}$，则

$$|\widehat{K}(m,n)| \leq \frac{38656\sqrt{2}}{15625} + \frac{245}{32} + \frac{1}{n}\Big(\frac{131072\sqrt{2}}{78125} + \frac{53}{8} + 19\log(\frac{n}{8})\Big) \tag{4.18}$$

The right hand side is an increasing function of n on $[8,13]$ and decreasing on $[14,\infty)$. Since this function of n is $< 4\pi$ at both $n = 13$ and $n = 14$, we have $|\widehat{K}(m,n)| < 4\pi$ if $n \geq 8$ and $m \leq \frac{8}{5}\sqrt{n}$. Such pairs (m,n) are pictured in blue in Figure 3.

还需验证 $|\widehat{K}(m,n)| < 4\pi$ for the 11 points in Figure 3 that are not coloured, i.e. when $1 \leq m \leq 3$, $1 \leq n \leq 7$ and (m,n) are of the same parity. We achieved this numerically using Mathematica (in fact, $|\widehat{K}(m,n)| < 5.8$ for all (m,n) that is not coloured). For interested readers, see Appendix B for the Mathematica codes.

5. The weak solution to the Neumann problem

在本节中，我们证明定理 1.2。

5.1. **Functional Analysis.** Recall that the Folland–Stein space $S^{1,2}(\Omega)$ is the space of all functions $u \in L^2(\Omega)$ whose weak derivatives $\nabla_b u$ are in $L^2(\Omega)$. It is known [Nhi01, Theorem 1.3] that $C^\infty(\overline{\Omega}) \subset S^{1,2}(\Omega)$ is a dense subspace and that the restriction map

$$C^\infty(\overline{\Omega}) \ni u \mapsto u|_\Sigma \in C^\infty(\Sigma)$$

是连续的。因此，迹算子 $\mathrm{Tr}\colon S^{1,2}(\Omega) \to L^2(\Sigma)$ 已定义。已知 [Nhi01,定理 1.4]$\mathrm{Tr}\big(S^{1,2}(\Omega)\big) \subset L^3(\Sigma)$。

由于 Σ 是 p-极小的，我们可以看到

$$\mathcal{F}(u) := \int_\Omega \left(|\nabla_b u|^2 + \frac{R}{4}|u|^2\right) \theta \wedge d\theta$$

是连续的 $\mathcal{F}$从 $C^\infty(\Omega)$ 到 $S^{1,\ 2}(\Omega)$ 的延拓 $\to \mathbb{R}$。此外，存在一个极小化子 $u_0 \in S^{1,\ 2}(\Omega)$ of

$$\mu_1 := \left\{ \mathcal{F}(u) : u \in S^{1,2}(\Omega),\ \oint_\Sigma |\mathrm{Tr}(u)|^2\, d\sigma = 1 \right\}.$$

必然地 u_0 是一个弱解

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ \nabla_\nu u = -\mu_1 \,\mathrm{Tr}\, u, & \text{on } \Sigma; \end{cases}$$

i.e.

$$0 = \int_\Omega \left(\langle \nabla_b u_0, \nabla_b \phi \rangle + \frac{1}{2} u_0 \phi \right) \theta \wedge d\theta + \oint_\Sigma \mu_1 \,\mathrm{Tr}(u_0)\phi \, d\sigma$$

对所有 $\phi \in C^\infty(\overline{\Omega})$。本节中我们证明 $u_0 \in C^\infty(\Omega)$。

5.2. 位势论。设 $f \in L^2(\Sigma, d\sigma)$。回顾单层位势

$$\mathcal{S}f(\zeta) = \int_{\eta \in \Sigma} f(\eta) G(\zeta, \eta) d\sigma(\eta), \quad \zeta \in \overline{\Omega}.$$

我们有 $\mathcal{S}f \in C^\infty(\Omega)$。由 (3.4)，我们还得到

$$\mathcal{S}f(\zeta) = \frac{\sqrt{2}}{8\pi} \iint_{\substack{|u| \le \pi \\ |v| \le \pi/2}} F(u'-u, v'-v) k(\frac{t}{\sqrt{2}}, u, v) du dv$$

如果 $\zeta = \Phi(t, u', v')$ 且 $F(u, v) := f(\Phi(0, u, v))$。

命题 5.1。若 $f \in L^2(\Sigma, d\sigma)$，则 $\mathcal{S}f \in S^{1,\ 2}(\Omega)$。

证明。注意到

$$\sup_{t \in [0, \pi/100]} \iint_{\substack{|u| \le \pi \\ |v| \le \pi/2}} |k(\frac{t}{\sqrt{2}}, u, v)| du dv \lesssim 1$$

由该娱乐主管 引理 3.2 中 k 的界。对 $\mathbb{R}^2/\Lambda$ 使用杨卷积不等式，我们可得 th at

$$\sup_{t \in [0, \pi/100]} \|\mathcal{S}f\|_{L^2(\Sigma_t)} \lesssim \|f\|_{L^2(\Sigma)}.$$

因此 $\mathcal{S}f \in L^2(\Omega)$。此外，对于 $\zeta \in \Omega$，我们有

$$Z_{\bar{1}} \mathcal{S}f(\zeta) = \int_{\eta \in \Sigma} f(\eta) Z_{\bar{1}} G(\zeta, \eta) d\sigma(\eta).$$

But

$$Z_{\bar{1}} G(\zeta, \eta) = \frac{1}{16\pi} \frac{(\zeta^2 \eta^1 - \zeta^1 \eta^2)(1 - \zeta \cdot \overline{\eta})}{|1 - \zeta \cdot \overline{\eta}|^3}.$$

如果 $\zeta = \Phi(t, u', v')$ 且 $\eta = \Phi(0, u, v)$，那么

$$\begin{aligned} \zeta^2 \eta^1 - \zeta^1 \eta^2 =& \cos(\frac{\pi}{4}) \sin(\frac{\pi}{4} + \frac{t}{\sqrt{2}}) e^{i(v'-u'+v+u)} - \sin(\frac{\pi}{4}) \cos(\frac{\pi}{4} + \frac{t}{\sqrt{2}}) e^{i(v'+u'+v-u)} \\ =& e^{i(v'+v)} \Big[\sin(\frac{t}{\sqrt{2}}) \cos(u'-u) - i \cos(\frac{t}{\sqrt{2}}) \sin(u'-u) \Big], \end{aligned}$$

so

$$\begin{aligned} Z_{\bar{1}} G(\zeta, \eta) =& \frac{e^{2iv'}}{16\pi} \Big[\sin(\frac{t}{\sqrt{2}}) \cos(u'-u) - i \cos(\frac{t}{\sqrt{2}}) \sin(u'-u) \Big] \\ & \times \frac{\Big[\Big(\cos(v'-v) - \cos(\frac{t}{\sqrt{2}}) \cos(u'-u) \Big) - i \Big(\sin(v'-v) - \sin(\frac{t}{\sqrt{2}}) \sin(u'-u) \Big) \Big]}{g(t, u'-u, v'-v)^{3/2}} \\ =& -\frac{e^{2iv'}}{16\pi} i \cos(\frac{t}{\sqrt{2}}) \frac{\sin(u'-u) \Big[\Big(\cos(v'-v) - \cos(\frac{t}{\sqrt{2}}) \cos(u'-u) \Big) - i \sin(v'-v) \Big]}{g(t, u'-u, v'-v)^{3/2}} \\ & + O\Big(\frac{t}{g(t, u'-u, v'-v)} \Big) \end{aligned}$$

第一项的分子关于 t是偶函数，但分母不是。因此，我们通过使第一项关于 t 为偶函数来对其进行对称化，这会产生一个同样属于 $O\Big(\frac{t}{g\ t\ u'-u(,,v'-v)}\Big)$ 的误差。事实上，误~~差项 $\mathcal{E}$ 由下~~式给出

$$\begin{aligned}\mathcal{E} =\ & C\sin(u'-u)[(\cos(v'-v)-\cos t\cos(u'-u))-i\sin(v'-v)]\\ &\times\Big(\frac{1}{g(t,u'-u,v'-v)^{3/2}}-\frac{1}{g(-t,u'-u,v'-v)^{3/2}}\Big)\\ \le\ & C|u'-u|[(\cos(v'-v)-\cos t\cos(u'-u))^2+\sin^2(v'-v)]^{1/2}\\ &\times\frac{1}{g(t^*,u'-u,v'-v)^{5/2}}\frac{\partial g}{\partial t}(t^*,u'-u,v'-v)\cdot t.\end{aligned}$$

代入

$$\frac{\partial g}{\partial t}(t^*,u'-u,v'-v)=g^{1/2}(t^*,u'-u,v'-v)\cdot O(|u'-u|+|t|),$$

其中，将 $t^*\in[-t,t]$ 代入误差项，我们得到

$$\mathcal{E}=O\Big(\frac{t}{g(t,u'-u,v'-v)}\Big).$$

总而言之，如果 $\zeta=\Phi(t,u',v')$ 且 $\eta=\Phi(0,u,v)$，我们得到

$$Z_{\bar{1}}G(\zeta,\eta)=-\frac{e^{2iv'}}{32\pi}i\cos(\frac{t}{\sqrt{2}})K_{t/\sqrt{2}}(u'-u,v'-v)+O\Big(\frac{t}{g(t,u'-u,v'-v)}\Big)$$

其中

$$K_t(u,v):=\sin u[(\cos v-\cos t\cos u)-i\sin v]\Big(\frac{1}{g(t,u,v)^{3/2}}+\frac{1}{g(-t,u,v)^{3/2}}\Big).$$

由此可得

$$\begin{aligned}Z_{\bar{1}}\mathcal{S}f(\Phi(t,u',v'))=&-\frac{e^{2iv'}}{32\pi}i\cos(\frac{t}{\sqrt{2}})\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}F(u'-u,v'-v)K_{t/\sqrt{2}}(u,v)dudv\\ &+O\left(\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}|F(u'-u,v'-v)|\frac{t}{g(t,u,v)}dudv\right)\end{aligned}$$

本质上是 $F(u,v):=f(\Phi(0,u,v))$ 与 $\mathbb{R}^2/\Lambda$ 上两个核的卷积之和。第二项可以用.....处理。

$$\left\|\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}F(u'-u,v'-v)\frac{t}{g(t,u,v)}dudv\right\|_{L^2(du'dv')}\lesssim\|F\|_{L^2},$$

其中我们在 $\mathbb{R}^2/\Lambda$ 上应用了杨氏卷积不等式和估计

$$\begin{aligned}\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}\frac{t}{g(t,u,v)}dudv&\lesssim\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}\frac{t}{(|t|+|u|+|v|^{1/2})^4}dudv\\ &\le\iint_{\mathbb{R}^2}\frac{1}{(1+|u|+|v|^{1/2})^4}dudv<\infty\end{aligned}$$

在 $t\in(0,\pi/100]$ 上一致。我们声称，

$$\left\|\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}}F(u'-u,v'-v)K_t(u,v)dudv\right\|_{L^2(du'dv')}\lesssim\|F\|_{L^2}$$

uniformly in $t\in(0,\pi/100]$. Once this is shown, we see that $Z_{\bar{1}}\mathcal{S}f\in L^2(\Sigma_t)$ uniformly in $t\in(0,\pi/100]$, so $Z_{\bar{1}}\mathcal{S}f\in L^2(\Omega)$. Since $\mathcal{S}$ is real, we also have $Z_1\mathcal{S}f\in L^2(\Omega)$. Thus $\mathcal{S}f\in S^{1,2}(\Omega)$, as desired.

为证明该论断，根据 $L^2(\mathbb{R}^2/\Lambda)$ 上的帕塞瓦尔等式，我们只需证明

$$\sup_{(m,n)\in\mathbb{Z}^2}\left|\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}} K_t(u,v)e^{-i(mu+nv)}dudv\right|\lesssim 1 \tag{5.1}$$

在 $t\in(0,\pi/100]$中一致地。但我们有 $\operatorname{Re} K_t(-u,-v) = -\operatorname{Re} K_t(u,v)$，以及 $\operatorname{Im} K_t(-u,v) = -\operatorname{Im} K_t(u,v)$。因此可得消去条件

$$\iint_{\|(u,v)\|\le r} K_t(u,v)dudv = 0$$

对所有 r。接下来，对于 $|u|\le\pi$ 和 $|v|\le\pi/2$，直接计算可得

$$|K_t(u,v)|\lesssim\|(u,v)\|^{-3},$$
$$|\frac{\partial K_t}{\partial u}(u,v)|\lesssim\|(u,v)\|^{-4},$$
$$|\frac{\partial K_t}{\partial v}(u,v)|\lesssim\|(u,v)\|^{-5},$$

在 $t\in(0,\pi/100]$中均一致地成立。结合这一点与消去条件，可以按照标准方法得到 (5.1)。□

为进一步讨论，定义

$$\mathfrak{M}f(\zeta) := \sup_{t\in(0,\pi/100]}|\frac{d}{dt}\mathcal{S}f(\gamma_\zeta(t))|,\quad \zeta\in\Sigma.$$

命题 5.2. $\mathfrak{M}$ 在 $L^2(\Sigma,d\sigma)$上有界。

为证明这一点，记 M 为适用于 Σ 上非各向同性几何的 Hardy–Littlewood 极大算子:

$$Mf(\zeta) := \sup_{t>0}\frac{1}{|Q_t(\zeta)|}\int_{Q_t(\zeta)}|f(\eta)|d\sigma(\eta)$$

其中 $Q_t(\zeta)$ 是给定的非各向同性球大小规律

$$Q_t(\zeta) := \{\Phi(0,u,v)\in\Sigma\colon \|(u'-u,v'-v)\|\le t\}\quad \text{if } \zeta=\Phi(0,u',v').$$

（回顾 $\|(u,v)\| := \max\{|u|,|v|^{1/2}\}$。）已知 M在 $L^2(\Sigma)$ 上有界。由于我们还证明了 $\mathcal{N}$在 $L^2(\Sigma)$ 上有界，命题 5.2 是以下结果的推论

命题 5.3. 对于 $f\in L^2(\Sigma)$，我们有

$$\mathfrak{M}f(\zeta)\le C(M\mathcal{N}f(\zeta)+Mf(\zeta))$$

对于 $d\sigma$-几乎每个 $\zeta\in\Sigma$。

命题 5.3 的证明。事实上，如果 $\zeta=\Phi(0,u',v')$，那么 $\gamma_\zeta(t) = \Phi(t,u',v')$，因此对于 $t>0$我们可以在积分号下对 (3.4) 求导并得到

$$\frac{d}{dt}\mathcal{S}f(\gamma_\zeta(t)) = \frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\|v|\le\pi/2}} F(u'-u,v'-v)\frac{\partial k}{\partial t}(\frac{t}{\sqrt{2}},u,v)dudv,$$

其中 $F(u,v) := f(\Phi(0,u,v))$。由 (3.5) 我们还得到,

$$\mathcal{N}f(\zeta) = \lim_{\varepsilon\to0^+}\frac{1}{8\pi}\iint_{\substack{|u|\le\pi,|v|\le\pi/2\\\|(u,v)\|\ge\varepsilon}} F(u'-u,v'-v)\frac{\partial k}{\partial t}(0,u,v)dudv$$

and

$$Mf(\zeta)\simeq\sup_{0<t<100}\frac{1}{t^3}\iint_{\|(u,v)\|\le t}|F(u'-u,v'-v)|dudv.$$

Using the bound

$$|\frac{\partial k}{\partial t}(t,u,v)|\lesssim(|t|+|u|+|v|^{1/2})^{-3},$$

it is easy to see that

$$\left|\frac{1}{8\pi}\iint_{\|(u,v)\|\le 10t} F(u'-u,v'-v)\frac{\partial k}{\partial t}(\frac{t}{\sqrt{2}},u,v)dudv\right| \le CMf(\zeta); \tag{5.2}$$

事实上,

$$\left|\frac{1}{8\pi}\iint_{\|(u,v)\|\le 10t} F(u'-u,v'-v)\frac{\partial k}{\partial t}(\frac{t}{\sqrt{2}},u,v)dudv\right| \le CMf(\zeta)\iint_{\|(u,v)\|\le 10t}\frac{1}{(|t|+|u|+|v|^{1/2})^3}dudv$$

and

$$\iint_{\|(u,v)\|\le 10t}\frac{1}{(|t|+|u|+|v|^{1/2})^3}dudv \le \iint_{\|(u,v)\|\le 10t}\frac{1}{t^3}dudv = C.$$

此外，利用该界

$$|\frac{\partial k}{\partial t}(t,u,v)-\frac{\partial k}{\partial t}(0,u,v)| \lesssim \frac{t}{(|u|+|v|^{1/2})^4} \quad \text{when } \|(u,v)\|\ge t,$$

可以证明

$$\left|\frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\ |v|\le\pi/2\\ \|(u,v)\|>10t}} F(u'-u,v'-v)\left(\frac{\partial k}{\partial t}(\frac{t}{\sqrt{2}},u,v)-\frac{\partial k}{\partial t}(0,u,v)\right)dudv\right| \le CMf(\zeta). \tag{5.3}$$

这是因为

$$\begin{aligned}&\left|\frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\ |v|\le\pi/2\\ \|(u,v)\|>10t}} F(u'-u,v'-v)\left(\frac{\partial k}{\partial t}(\frac{t}{\sqrt{2}},u,v)-\frac{\partial k}{\partial t}(0,u,v)\right)dudv\right|\\ &\qquad\lesssim Mf(\zeta)\iint_{\|(u,v)\|>10t}\frac{t}{(|u|+|v|^{1/2})^4}dudv \lesssim Mf(\zeta).\end{aligned}$$

最后，我们断言，对于几乎所有的 $\zeta\in\Sigma$,

$$\left|\frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\ |v|\le\pi/2\\ \|(u,v)\|>10t}} F(u'-u,v'-v)\frac{\partial k}{\partial t}(0,u,v)dudv\right| \le C(M\mathcal{N}f+Mf)(\zeta). \tag{5.4}$$

In fact, the operators on both sides of the inequality are bounded on $L^2(\Sigma)$, and $C^\infty(\Sigma)$ is dense in $L^2(\Sigma)$, we may assume, without loss of generality, that $f\in C^\infty(\Sigma)$. Suppose now $\psi\in C_c^\infty(\Sigma)$ is supported in $Q_1(\Phi(0,0,0))$, with $\int_\Sigma \psi d\sigma = 1$. Let $\Psi_t(u,v) = \sqrt{2}^{-1}t^{-3}\psi(\Phi(0,t^{-1}u,t^{-2}v))$ for $t\in(0,\pi/100]$ and $(u,v)\in\mathbb{R}^2$, so that

$$\iint_{|u|\le\pi,|v|\le\pi/2}\Psi_t(u,v)dudv = 1.$$

回顾 $K(u,v)=p.v.\frac{\partial k}{\partial t}(0,u,v)$，并令

$$\tilde{K}_t(u,v) = K*\Psi_t(u,v)$$

为 K 与 Ψ_t 在 $\mathbb{R}^2/\Lambda$上的卷积。我们将证明

$$|\frac{\partial k}{\partial t}(0,u,v)1_{\|(u,v)\|>10t}-\tilde{K}_t(u,v)| \lesssim \frac{t}{(|t|+|u|+|v|^{1/2})^4}. \tag{5.5}$$

事实上，如果 $\|(u,v)\|>10t$，那么由中值定理可得

$$\begin{aligned}|\frac{\partial k}{\partial t}(0,u,v)-\tilde{K}_t(u,v)| &\le \iint_{\|(u,v)\|\le t}\left|\frac{\partial k}{\partial t}(0,u',v')-\frac{\partial k}{\partial t}(0,u'-u,v'-v)\right||\Psi_t(u,v)|dudv\\ &\lesssim \frac{t}{(|u|+|v|^{1/2})^4}.\end{aligned}$$

另一方面，如果 $\|(u',v')\| \le 10t$，那么

$$\begin{aligned}\tilde{K}_t(u',v') = &\iint_{\|(u,v)\|\le 100t} \Big[\Psi_t(u'-u,v'-v) - \Psi_t(u',v')\Big]\frac{\partial k}{\partial t}(0,u,v)dudv \\ &+ \iint_{\substack{|u|\le\pi,|v|\le\pi/2\\ \|(u,v)\|>100t}} \Psi_t(u'-u,v'-v)\frac{\partial k}{\partial t}(0,u,v)dudv,\end{aligned}$$

其中最后一个积分为零，因为 $\Phi_t(u'-u,v'-v)=0$ 在那里。因此，如果 $\|(u',v')\| \le 10t$，那么

$$|\tilde{K}_t(u',v')| \le \iint_{\|(u,v)\|\le 100t} \frac{1}{t^3}\Big(\frac{|u|}{t}+\frac{|v|}{t^2}\Big)\frac{1}{\|(u,v)\|^3}dudv \lesssim \frac{1}{t^3}.$$

这就证明了 (5.5)，从而

$$\begin{aligned}&\left|\frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\ |v|\le\pi/2\\ \|(u,v)\|>10t}} F(u'-u,v'-v)\frac{\partial k}{\partial t}(0,u,v)dudv\right| \\ &\qquad\le \left|\frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\ |v|\le\pi/2}} F(u'-u,v'-v)\tilde{K}_t(u,v)dudv\right| + Mf(\zeta).\end{aligned}$$

But

$$\begin{aligned}\frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\ |v|\le\pi/2}} F(u'-u,v'-v)\tilde{K}_t(u,v)dudv &= F * \frac{1}{8\pi}K * \Psi_t(u,v) \\ &= NF * \Psi_t(u,v),\end{aligned}$$

其中 $NF(u',v') = \mathcal{N}f(\Phi(0,u',v'))$。因此

$$\left|\frac{1}{8\pi}\iint_{\substack{|u|\le\pi\\ |v|\le\pi/2}} F(u'-u,v'-v)\tilde{K}_t(u,v)dudv\right| \lesssim M\mathcal{N}f(\zeta).$$

这就证明了 (5.4)。

□

现在我们准备完成定理 1.2的证明。我们需要证明，如果 $f \in L^2(\Sigma)$ 且 $u = \mathcal{S}f$，那么

$$\begin{cases} Lu = 0 \quad \text{on } \Omega \\ \nabla_\nu u = (-\frac{1}{2}I + \mathcal{N})f \quad \text{on } \Sigma \end{cases}$$

在弱意义下，即

$$0 = \int_\Omega (\langle \nabla_b u, \nabla_b \phi\rangle + \frac{R}{4}u\phi) + \int_\Sigma (-\frac{1}{2}I + \mathcal{N})f\phi$$

对于所有 $\phi \in \overline{C}^\infty(\Omega)$。为此，请注意，

$$\int_\Omega (\langle \nabla_b u, \nabla_b \phi\rangle + \frac{R}{4}u\phi) = \lim_{t\to 0^+}\int_{\Omega_t} (\langle \nabla_b u, \nabla_b \phi\rangle + \frac{R}{4}u\phi)$$

并且对于 $t > \overline{0,}$ 有 $u \in C^\infty(\Omega_t)$。因此，对于 $\overline{\phi} \in C^\infty(\Omega)$，我们现在可以进行分部积分，并利用 Ω_t 内的 $Lu = 0$，可见上式等于

$$-\lim_{t\to 0^+}\int_{\Sigma_t} \nabla_\nu u\, \phi\, d\sigma_t = -\lim_{t\to 0^+}\int_\Sigma \frac{d}{dt}u(\gamma_\zeta(t))\phi(\gamma_\zeta(t))\frac{d\sigma_t}{d\sigma}d\sigma.$$

我们先前对 $\mathfrak{M}$ 的估计足以证明

$$\frac{d}{dt}u(\gamma_\zeta(t)) \to (-\frac{1}{2}I + \mathcal{N})f(\zeta) \tag{5.6}$$

对于几乎所有的 $\zeta \in \Sigma$，当 $t \to 0^+$时：如果 $f \in L^2(\Sigma)$，那么选择 $g \in C^\infty(\Sigma)$，使得$\|f-g\|_{L^2(\Sigma)} < \varepsilon_0$。然后对于任意 $\alpha > 0$,

$$\begin{aligned}
&\{\zeta \in \Sigma\colon \limsup_{t\to 0^+} |\frac{d}{dt}u(\gamma_\zeta(t)) - (-\frac{1}{2}I + \mathcal{N})f(\zeta)| > \alpha\} \\
&\subseteq \{\zeta \in \Sigma\colon \limsup_{t\to 0^+} |\frac{d}{dt}\mathcal{S}(f-g)(\gamma_\zeta(t)) - (-\frac{1}{2}I + \mathcal{N})(f-g)(\zeta)| > \alpha\} \\
&\subseteq \{\zeta \in \Sigma\colon \mathfrak{M}(f-g)(\zeta) > \alpha/2\} \cup \{\zeta \in \Sigma\colon |(-\frac{1}{2}I + \mathcal{N})(f-g)(\zeta)| > \alpha/2\}.
\end{aligned}$$

最后一个集合关于 $d\sigma$ 的测度至多为

$$C\alpha^{-2}\Big(\|\mathfrak{M}(f-g)\|^2_{L^2(\Sigma)} + \|(-\frac{1}{2}I + \mathcal{N})(f-g)\|^2_{L^2(\Sigma)}\Big) \lesssim \alpha^{-2}\|f-g\|^2_{L^2(\Sigma)} \lesssim \alpha^{-2}\varepsilon_0^2.$$

令 $\varepsilon_0 \to 0^+$，我们看到 (5.6) 在 Σ 上几乎处处成立。

我们先前的估计在更一般的几何情形下仍然成立，即区域 Ω 具有一般边界，其边界是部分 Lipschitz 图形的，事实上，只需对 Ω 作平移和伸缩，使得对任意足够小的 r，$\mathrm{Bridge}(r) \cap \Omega$ 包含在以原点为中心的半径为 $2r$ 的圆内，随后便可用上述估计代替 (1.3) 中的等式。适用于 $\mathfrak{M}$ 也使我们能够进一步应用支配收敛定理并证明 that

$$-\lim_{t\to 0^+}\int_\Sigma \frac{d}{dt}u(\gamma_\zeta(t))\phi(\gamma_\zeta(t))\frac{d\sigma_t}{d\sigma}d\sigma = -\int_\Sigma (-\frac{1}{2}I + \mathcal{N})f\phi.$$

这表明 u 以弱意义满足所需方程。

对于唯一性，我们需要证明，如果 $u \in S^{1,2}(\Omega)$ 且

$$0 = \int_\Omega \langle \nabla_b u, \nabla_b \phi\rangle + \frac{R}{4}u\phi \quad \text{for all } \phi \in C^\infty(\overline{\Omega}),$$

则在 Ω 中 $u = 0$ 几乎处处成立。由稠密性，上述条件实际上意味着

$$0 = \int_\Omega \langle \nabla_b u, \nabla_b \phi\rangle + \frac{R}{4}u\phi \quad \text{for all } \phi \in S^{1,2}(\Omega).$$

令 $\phi = u$ 并使用 $R > 0$，我们得到 $u = 0$，在 Ω 上几乎处处成立。

6. Proof of Proposition 1.3

Let Ω be the domain in $\mathbb{S}^3$ defined by (1.1). Let $u \in S^{1,2}(\Omega)$ be a minimizer of $\mathcal{F}(u)$ under the condition that $\oint_M |u|^2 d\sigma = 1$. We want to show that $u \in C^\infty(\overline{\Omega})$. In fact this will be the case as long as u is a critical point.

首先，因为

$$0 = \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0}\frac{\mathcal{F}(u+t\phi)}{\oint_M |\operatorname{Tr}(u+t\phi)|^2 d\sigma} = \int_\Omega \Big(\langle \nabla_b u, \nabla_b \phi\rangle + \frac{R}{4}u\phi\Big) - \mathcal{F}(u)\oint_M \operatorname{Tr}(u)\phi d\sigma$$

对于所有 $\phi \in C_c^\infty(\overline{\Omega})$，我们有

$$\begin{cases} Lu = 0, & \text{in } \Omega, \\ \nabla_\nu u = \mu \operatorname{Tr}(u), & \text{on } \partial\Omega, \end{cases}$$

其中 $\mu := -\mathcal{F}(u)$。根据定理1.2，我们有 $u = \mathcal{S}f$其中 $f := \mu(-\frac{1}{2}I + \mathcal{N})^{-1}\operatorname{Tr}(u)$.

引理 6.1. 对任意 $s \geq 0$，有

$$\|\operatorname{Tr}\mathcal{S}f\|_{W^{s+\frac{1}{2},2}(\Sigma)} \leq C_s\|f\|_{W^{s,2}(\Sigma)} \quad \textit{for all } f \in W^{s,2}(\Sigma).$$

在 Σ 上，我们将一个函数与李群 $\mathbb{R}^2/\Lambda$上的一个函数对应起来，并且 $W^{s,2}(\Sigma)$ 是 $\mathbb{R}^2/\Lambda$上相应的（各向同性）Sobolev空间，从而

$$\|f\|_{W^{s,2}(\Sigma)} := \Big(\frac{1}{2\pi^2}\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\equiv n \pmod 2}}(1+|m|^2+|n|^2)^s|\widehat{F}(m,n)|^2\Big)^{1/2}$$

如果 $F(u,v) := f(\Phi(0,u,v))$。

引理 6.1 的证明。为证明该引理，注意到，如果 $f \in L^2(\Sigma)$，并且我们将 $F(u',v') = f(\Phi(0,u',v'))$ 写成 $SF(u', v') := (\mathrm{Tr}\ \mathcal{S}f)(\Phi(0,u',v'))$，那么

$$SF = F * \frac{\sqrt{2}}{8\pi} K_0 \quad \text{a.e. on } \Sigma$$

其中 $K_0(u',v') = k(0,u',v')$。之所以这对 $f \in L^2(\Sigma)$ 成立，是因为该恒等式对$f \in C^\infty(\Sigma)$成立，并且两边都是定义在 $L^2(\Sigma)$ 上的有界线性映射。因此

$$\widehat{SF}(m,n) = \widehat{F}(m,n)\frac{\sqrt{2}}{8\pi}\widehat{K_0}(m,n)$$

对所有$(m,n) \in (\frac{1}{2\pi}\Lambda)^*$。容易看出

$$|\widehat{K_0}(m,n)| \lesssim (1+|m|+|n|^{1/2})^{-1}.$$

事实上，由于 $K_0 \in L^1(\Sigma)$，我们有 $|\widehat{K_0}(m,n)| \lesssim 1$。此外，由于

$$|K_0(u,v)| \lesssim \|(u,v)\|^{-2}, \quad |\frac{\partial^2 K_0}{\partial u^2}| + |\frac{\partial K_0}{\partial v}| \lesssim \|(u,v)\|^{-4},$$

通过书写

$$\widehat{K_0}(m,n) = \iint_{\|(u,v)\| \le (|m|+|n|^{1/2})^{-1}} + \iint_{\substack{|u|\le\pi, |v|\le\pi/2 \\ \|(u,v)\| \ge (|m|+|n|^{1/2})^{-1}}} K_0(u,v)e^{-i(mu+nv)}dudv,$$

and integrating by parts in the second integral using either $e^{-imu} = -\frac{1}{m^2}\frac{\partial^2}{\partial u^2}e^{-imu}$ or $e^{-inv} = \frac{i}{n}\frac{\partial}{\partial v}e^{-inv}$, we obtain $|\widehat{K_0}(m,n)| \lesssim (|m|+|n|^{1/2})^{-1}$. This proves our bound for $\widehat{K_0}(m,n)$; in fact, we will only use the weaker bound

$$|\widehat{K_0}(m,n)| \lesssim (1+|m|+|n|)^{-1/2}.$$

现在如果 $f \in W^{s,2}(\Sigma)$，那么对于 $\widehat{K}(m,n)$ 的这个较弱估计使我们能够对其进行界定

$$\begin{aligned}
\|\operatorname{Tr}\mathcal{S}f\|_{W^{s+1/2,2}(\Sigma)} &\simeq \Big(\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\equiv n \pmod 2}} (1+|m|^2+|n|^2)^{s+\frac{1}{2}} |\widehat{SF}(m,n)|^2 \Big)^{1/2} \\
&\simeq \Big(\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\equiv n \pmod 2}} (1+|m|^2+|n|^2)^{s+\frac{1}{2}} |\widehat{F}(m,n)\widehat{K_0}(m,n)|^2 \Big)^{1/2} \\
&\lesssim \Big(\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\equiv n \pmod 2}} (1+|m|^2+|n|^2)^{s} |\widehat{F}(m,n)|^2 \Big)^{1/2} \\
&\simeq \|f\|_{W^{s,2}(\Sigma)}.
\end{aligned}$$

□

现在回忆我们的临界点 $u = \mathcal{S}f$，其中 $f := \mu(-\frac{1}{2}I + \mathcal{N})^{-1}\operatorname{Tr}(u)$。根据引理 6.1，对于所有 $s \ge 0$，我们有

$$\|\operatorname{Tr} u\|_{W^{s+\frac{1}{2},2}(\Sigma)} \lesssim \|f\|_{W^{s,2}(\Sigma)}.$$

利用第4节中的估计，我们有

$$\|f\|_{W^{s,2}(\Sigma)} \lesssim \mu \|\operatorname{Tr}(u)\|_{W^{s,2}(\Sigma)}.$$

将这些结果结合起来，我们得到的是

$$\|\operatorname{Tr}(u)\|_{W^{k,2}(\Sigma)} \lesssim \mu^{2k}\|\operatorname{Tr}(u)\|_{L^2(\Sigma)} \lesssim \mu^{2k}\|u\|_{S^{1,2}(\Omega)}$$

对所有 $k \in \mathbb{N}$ 均成立。因此 $\operatorname{Tr}(u) \in C^\infty(\Sigma)$，从而 $\nabla_\nu u \in C^\infty(\Sigma)$。结果是，$f = \mu(-\frac{1}{2}I + \mathcal{N})^{-1}\nabla_\nu u \in C^\infty(\Sigma)$。因此 $u = \mathcal{S}f \in \overline{C}^\infty(\Omega)$，如所需。

7. The boundary Yamabe contact form on Σ

我们通过证明关于边界 Yamabe 常数的结果来结束本文。

首先我们证明，边界 Yamabe常数 $Y(M,T^{1,0})$是有限的，当且仅当 CR Yamabe算子的第一个狄利克雷特征值为正。

命题7.1. 设 $(M^3,T^{1,0},\theta)$是一个边界无特征点的闭伪厄米流形。那么 $Y(M,T^{1,0})>-\infty$ 当且仅当 $\lambda_{1,D}(L)>0$。

证明。令 $\lambda:=\lambda_{1,D}(L)$，并令 u 为第一狄利克雷特征值；也就是说， u 是 $S^{1,2}(M)$ 中唯一的非负函数，满足 $\int_M u^2\,\theta\wedge d\theta=1$ 且

$$\begin{cases} Lu=\lambda u, & \text{in } M,\\ u=0, & \text{on } \partial M. \end{cases}$$

Jerison proved [Jer81, Theorem 7.1] that $u\in C^\infty(M)$. Bony's maximum principle [Bon69, Corollaire 3.1] then implies that $u>0$. Note that $\mathcal{F}(u)=\lambda$. Additionally, for any fixed $v\in C^\infty(M)$, it holds that

$$\begin{aligned} \mathcal{F}(v+tu)&=\lambda t^2+\lambda t\int_M uv\,\theta\wedge d\theta+t\oint_{\partial M} v\,Bu\,d\sigma+\mathcal{F}(v),\\ \oint_{\partial M}|v+tu|^3\,d\sigma&=\oint_{\partial M}|v|^3\,d\sigma. \end{aligned} \tag{7.1}$$

首先假设 $\lambda<0$。对于任意固定的 $v\in C^\infty(M)$，由(7.1)我们推得 $\mathcal{F}(v+tu)\to-\infty$ 当$t\to\infty$。因此 $Y(M,T^{1,0})=-\infty$。

接下来假设 $\lambda=0$。取一个 $v\in C^\infty(M)$，使得 $v|_{\partial M}=-Bu$。由 (7.1) 可推出$\mathcal{F}(v+tu)\to-\infty$当 $t\to\infty$。因此 $Y(M,T^{1,0})=-\infty$。

最后，假设 $\lambda>0$。固定常数 $c\in(0,\lambda)$。标准的变分论证表明，存在弱解 $u\in S^{1,2}(M)$，满足

$$\begin{cases} Lu=cu, & \text{in } M,\\ u=1, & \text{on } \partial M. \end{cases}$$

Combining Jerison's regularity results [Jer81, Theorem 7.1] with Bony's maximum principle [Bon69, Corollaire 3.1] implies that $u\in C^\infty(M)$ is positive. Set $\widehat{\theta}:=u^2\theta$. By the conformal covariance of the CR Yamabe operator,

$$R^{\widehat{\theta}}=4L^{u^2\theta}(1)=4cu^{-2}>0.$$

特别地，存在常数 C_1、$C_2>0$，使得

$$\mathcal{F}^{\widehat{\theta}}(v)\geq\int_M\left(|\nabla_b v|^2+C_1|v|^2\right)\widehat{\theta}\wedge d\widehat{\theta}-C\oint_{\partial M}|v|^2\,d\sigma_{\widehat{\theta}}$$

for all $v\in C^\infty(M)$. On the one hand, Hölder's inequality implies that $\oint|v|^2\lesssim\left(\oint|v|^3\right)^{2/3}$. On the other hand, the Sobolev trace inequality [Nhi01, Theorem 1.4] and a standard partition of unity argument implies that

$$\int_M\left(|\nabla_b v|^2+C_1|v|^2\right)\widehat{\theta}\wedge d\widehat{\theta}\gtrsim\left(\oint_{\partial M}|v|^3\,d\widehat{\sigma}\right)^{\frac{2}{3}}.$$

因此 $Y(M,T^{1,0})>-\infty$。 □

其次，我们证明，如果前者的极小化子是光滑的，那么在有限的情况下， $\mu_1(L)$ 和 $Y(M,T^{1,0})$ 具有相同的符号。

命题7.2. 设 $(M^3,T^{1,0},\theta)$ 是一个带边界且无特征点的闭伪厄米流形。进一步假设 (1.4)的极小化子是光滑的。则 $\mu_1(L)$ 和 $Y(M,T^{1,0})$ 同号。

证明。记 $Y := Y(M, T^{1,0})$ 和 $\mu_1 := \mu_1(L)$。

首先假设 $Y > 0$ （分别为 $Y \geq 0$）。H¨older 不等式表明，如果 $u \in C^\infty$（M），那么

$$\mathcal{F}(u) \geq Y \left(\oint_{\partial M} |u|^3 \, d\sigma \right)^{\frac{2}{3}} \geq Y \mathrm{Vol}(\partial M)^{-\frac{1}{3}} \oint_{\partial M} |u|^2 \, d\sigma.$$

因此 $\mu_1 > 0$ （分别为 $\mu_1 \geq 0$）。

Suppose next that $\mu_1 > 0$ (resp. $\mu_1 \geq 0$). By restricting to nonnegative functions and applying the Sobolev trace embedding theorem [Nhi01, Theorem 1.4], we deduce that there is a nonnegative minimizer $u \in S^{1,2}(M)$ of μ_1. By assumption, u is smooth. A straightforward computation implies that u is a weak solution of

$$\begin{cases} Lu = 0, & \text{in } M, \\ Bu = \mu_1 u, & \text{on } \partial M. \end{cases} \tag{7.2}$$

The strong maximum principle [Bon69, Corollaire 3.1] implies that u is positive in the interior of Ω. The Hopf Lemma [Mon10, Corollary 2.1] implies that u is also positive on the boundary. Therefore $u > 0$. Set $\widehat{\theta} := u^2 \theta$. Equation (7.2) implies that $H^{\widehat{\theta}} > 0$ (resp. $H^{\widehat{\theta}} \geq 0$). By conformal covariance,

$$\mathcal{F}^\theta(u^{-1} v) = \mathcal{F}^{\widehat{\theta}}(v) = 2 \int_M |\nabla_b v|^2 \, \widehat{\theta} \wedge d\widehat{\theta} + \frac{1}{3} \oint_M H^{\widehat{\theta}} |v|^2 \, \widehat{d\sigma}.$$

由 Sobolev 迹嵌入定理可得 $Y > 0$ （分别地， $Y \geq 0$）。 □

最后，我们在 Ω 上构造一个标量平坦接触形式，使得 Σ 具有常数 p平均曲率。

定理 1.4 的证明。在 Ω 上引入新坐标 $(r, u, v) \in [0, \pi] \times [0, 2\pi] \times [0, 2\pi]$，使得

$$\zeta^1 = e^{i(u+v)} \sin(r/2),$$
$$\zeta^2 = e^{i(u-v)} \cos(r/2),$$

作出显然的识别后，直接计算可得。

$$Z_1 = e^{-2iu} \left(\partial_r + \frac{1}{2i} \cos(r) \partial_u + \frac{1}{2i} \csc(r) \partial_v \right).$$

由此可轻易得出ನ್ನಡ

$$L = -2\partial_r^2 - \frac{1}{2} \cot^2(r) \partial_u^2 - \frac{1}{2} \csc^2(r) \partial_v^2 - \cot(r) \csc(r) \partial_{uv}^2 - 2 \cot(r) \partial_r + \frac{1}{2}.$$

我们得出结论：Ω 上的光滑函数 $u = u(r)$ 是 $Lu = 0$ 的解，当且仅当

$$\begin{cases} \partial_r^2 u + \cot(r) \partial_r u - \frac{1}{4} u = 0, \\ \lim_{r \to \pi} u(r) < \infty. \end{cases} \tag{7.3}$$

通过变量代换 $x = \cos r$，我们看到 (7.3) 等价于

$$\begin{cases} (1 - x^2) \partial_x^2 u - 2x \partial_x u - \frac{1}{4} u = 0, \\ \lim_{x \to -1} u(x) < \infty. \end{cases} \tag{7.4}$$

众所周知 [DLMF，第 14.2 节和第 14.8] 节，（7.4）的解空间由勒让德函数张成。

$$P_{-\frac{1}{2}}(x) = {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; \frac{1+x}{2} \right).$$

Rewriting this in terms of $|z^2|^2 = \cos^2 \frac{r}{2}$ and normalizing so that $u = 1$ along Σ yields the claimed solution of $Lu = 0$. In particular, $u^2\theta$ is Webster-flat. Since u depends only on r, we see that both u and $Bu = \partial_r u$ are constant on Σ. Therefore Σ has constant p-mean curvature with respect to $u^2\theta$. □

8. Appendix

8.1. **Appendix A.** The Mathematica codes used in Corollaries 4.4 and 4.5 are reproduced below for reference. In order to simplify expressions and outputs in Mathematica, some definitions in the codes are different from the definitions taken in the rest of the paper. We first point out the relations between these two sets of definitions and then list the codes:

$\mathrm{err}_0(a,u,v) = \frac{e_0(a)}{e_1(a)^{5/2}}\texttt{error0[u, v]}, \mathrm{err}_u(a,u,v) = \frac{e_0(a)}{e_1(a)^{7/2}}\texttt{erroru1[u, v]} + \frac{1}{e_1\ a\ ()^{5/2}}$错误u2[u, v],

$$\mathrm{err}_v(a,u,v) = \frac{e_0(a)}{e_1(a)^{7/2}}\texttt{errorv1[u, v]} + \frac{1}{e_1(a)^{5/2}}\texttt{errorv2[u, v]},$$

$$\mathrm{err}_{uu}(a,u,v) = \frac{e_0(a)}{e_1(a)^{9/2}}\texttt{erroruu1[u, v]} + \frac{1}{e_1(a)^{7/2}}\texttt{erroruu2[u, v]},$$

$$\epsilon_{I0} = \frac{e_0(2)}{e_1(2)^{5/2}}\texttt{epsiloni[0]}, \qquad \epsilon_{I1} = \frac{e_0(2)}{e_1(2)^{5/2}}\texttt{epsiloni[1]},$$

$$\epsilon_{I2} = \frac{e_0(2)}{e_1(2)^{7/2}}\texttt{epsiloni[2, 1]} + \frac{1}{e_1(2)^{5/2}}\texttt{epsiloni[2, 2]},$$

$$\epsilon_{I3} = \frac{e_0(2)}{e_1(2)^{9/2}}\texttt{epsiloni[3, 1]} + \frac{1}{e_1(2)^{7/2}}\texttt{epsiloni[3, 2]},$$

$$\epsilon_{J0} = \frac{e_0(1)}{e_1(1)^{5/2}}\texttt{epsilonj[0]}, \qquad \epsilon_{J1} = \frac{e_0(1)}{e_1(1)^{5/2}}\texttt{epsilonj[1]},$$

$$\epsilon_{J2} = \frac{e_0(1)}{e_1(1)^{7/2}}\texttt{epsilonj[2, 1]} + \frac{1}{e_1(1)^{5/2}}\texttt{epsilonj[2, 2]}.$$

```
d[u_, v_] = ((Cos[u] - Cos[v])^2 + Sin[v]^2)^(1/4);
n[u_, v_] = Sin[u]*Sin[v];
K[u_, v_] = n[u, v]/d[u, v]^6;
nu[u_, v_] = D[K[u, v], u]*d[u, v]^(10);
nv[u_, v_] = D[K[u, v], v]*d[u, v]^(10);
nuu[u_, v_] = D[K[u, v], u, u]*d[u, v]^(14);
d0[u_, v_] = (u^4/4 + v^2)^(1/4);
n0[u_, v_] = u*v;
K0[u_, v_] = n0[u, v]/(d0[u, v])^6;
n0u[u_, v_] = v (-5/4 u^4 + v^2);
K0u[u_, v_] = n0u[u, v]/d0[u, v]^(10);
n0v[u_, v_] = u (u^4/4 - 2 v^2);
K0v[u_, v_] = n0v[u, v]/d0[u, v]^(10);
n0uu[u_, v_] = u^3 (-15/2*v^3 + 15/8*u^4*v);
K0uu[u_, v_] = n0uu[u, v]/d0[u, v]^(14);
error0[u_, v_] := (6/4)*Abs[n0[u, v]]/d0[u, v]^4;
erroru1[u_, v_] := (10/4)*Abs[n0u[u, v]]/d0[u, v]^8;
erroru2[u_, v_] := (1451/3072*Abs[u^6*v] + Abs[u^2 v^3]/2 + Abs[v^4]/3)/d0[u, v]^(10);
errorv1[u_, v_] := (10/4)*Abs[n0v[u, v]]/d0[u, v]^8;
errorv2[u_, v_] := (683/15360 Abs[u^7] + Abs[u^3 v^2] + 7/6 Abs[u v^3])/d0[u, v]^(10);
erroruu1[u_, v_] := (14/4)*Abs[n0uu[u, v]]/d0[u, v]^(12);
erroruu2[u_, v_] := (1195/2048 Abs[u^9 v] + Abs[u^5 v^3]/8 + 5 Abs[u^3 v^4]/2 +
     15 Abs[u v^5]/4)/d0[u, v]^(14);
norm[a_, u_, v_] := Max[Abs[u]/a, Sqrt[Abs[v]]];
region0[a_] := ImplicitRegion[norm[a, u, v] < 1, {u, v}];
region1[a_] := ImplicitRegion[norm[a, u, v] > 1, {u, v}];
b = 2/5; a1 = 2;
i[0] = FullSimplify[Integrate[Abs[u*v*K0[u, v]], {u, v} \[Element] region0[a1]]]
i[1] = 完全化简[2*积分[绝对值[K0[a1, v]], {v, -1, 1}]]
```

```
i[2] = FullSimplify[2*Integrate[Abs[K0u[a1, v]], {v, -1, 1}]]
i[3] = FullSimplify[Integrate[Abs[K0uu[u, v]], {u, v} \[Element] region1[a1]]]
region0b[a1] = ImplicitRegion[norm[a1, u, v] < b, {u, v}];
e0[a1] = ToRadicals[MaxValue[(u^6/24 + u^2 v^2/2 + 2 u^6*(a1*b)^2/Factorial[8] +
      Abs[v]^3*b^2/12)/d0[u, v]^(6), {u, v} \[Element] region0b[a1]]]
d0[a1*b, b^2]^2
epsiloni[0] = FullSimplify[Integrate[Abs[u*v]*error0[u, v], {u, v} \[Element] region0[a1]]]
epsiloni[1] = FullSimplify[2*Integrate[error0[a1, v], {v, -1, 1}]]
epsiloni[2, 1] = FullSimplify[2*Integrate[erroru1[a1, v], {v, -1, 1}]]
epsiloni[2, 2] = FullSimplify[2*Integrate[erroru2[a1, v], {v, -1, 1}]]
epsiloni[3, 1] = FullSimplify[2*a1*Integrate[erroruu1[a1, v], {v, -1, 1}] +
    4*Integrate[erroruu1[u, 1], {u, -a1, a1}]]
epsiloni[3, 2] = FullSimplify[2*a1*Integrate[erroruu2[a1, v], {v, -1, 1}] +
    4*Integrate[erroruu2[u, 1], {u, -a1, a1}]]
epsiloni[4] = SetAccuracy[
 2*NIntegrate[Abs[nuu[u, v]]/d[u, v]^(14), {u, -a1*b, a1*b}, {v, b^2, Pi/2},
    Method -> "LocalAdaptive", AccuracyGoal -> 8,
    IntegrationMonitor :> ((errors1 = Through[#1@"Error"]) &)]
+2*NIntegrate[Abs[nuu[u, v]]/d[u, v]^(14), {u, a1*b, Pi}, {v, -Pi/2, Pi/2},
    Method -> "LocalAdaptive", AccuracyGoal -> 8,
    IntegrationMonitor :> ((errors2 = Through[#1@"Error"]) &)], 10]
SetAccuracy[Total@errors1 + Total@errors2, 10]
a2 = 1;
j[0] = FullSimplify[Integrate[Abs[u*v*K0[u, v]], {u, v} \[Element] region0[a2]]]
j[1] = 2*Integrate[Abs[K0[u, 1]], {u, -a2, a2}]
j[2] = Integrate[Abs[K0v[u, v]], {u, v} \[Element] region1[a2]]
region0b[a2] = ImplicitRegion[norm[a2, u, v] < b, {u, v}];
e0[a2] = ToRadicals[MaxValue[(u^6/24 + u^2 v^2/2 + 2 u^6*(a2*b)^2/Factorial[8] +
      Abs[v]^3*b^2/12)/d0[u, v]^(6), {u, v} \[Element] region0b[a2]]]
d0[a2*b, b^2]^2
epsilonj[0] = FullSimplify[Integrate[Abs[u*v]*error0[u, v], {u, v} \[Element] region0[a2]]]
epsilonj[1] = FullSimplify[2*Integrate[error0[u, 1], {u, -a2, a2}]]
epsilonj[2, 1] = FullSimplify[2*a2*Integrate[errorv1[a2, v], {v, -1, 1}] +
   4*Integrate[errorv1[u, 1], {u, -a2, a2}]]
epsilonj[2, 2] = FullSimplify[2*a2*Integrate[errorv2[a2, v], {v, -1, 1}] +
   4*Integrate[errorv2[u, 1], {u, -a2, a2}]]
SetAccuracy[
 2*NIntegrate[Abs[nv[u, v]]/d[u, v]^(10), {u, -a2*b, a2*b}, {v, b^2, Pi/2},
    Method -> "LocalAdaptive", AccuracyGoal -> 8,
    IntegrationMonitor :> ((errors1 = Through[#1@"Error"]) &)]
+2*NIntegrate[Abs[nv[u, v]]/d[u, v]^(10), {u, a2*b, Pi}, {v, -Pi/2, Pi/2},
    Method -> "LocalAdaptive", AccuracyGoal -> 8,
    IntegrationMonitor :> ((errors2 = Through[#1@"Error"]) &)], 10]
SetAccuracy[Total@errors1 + Total@errors2, 10]
```

8.2. **Appendix B.** We use the following codes in the Mathematica to verify the inequality $|\widehat{K}(m,n)| < 4\pi$ when $1 \le m \le 3$, $1 \le n \le 7$, $m, n \in \mathbb{Z}$:

```
m := 1; For[n = 1, n <= 7, n += 2,
 Print[NIntegrate[K[u, v] Sin[m*u] Sin[n*v], {u, -Pi, Pi}, {v, -Pi/2, Pi/2},
   Method -> "LocalAdaptive", AccuracyGoal -> 8,
   IntegrationMonitor :> ((errors = Through[#1@"Error"]) &)]];
 打印[总计@错误]]
```

```
m := 2; For[n = 2, n <= 6, n += 2,
 Print[NIntegrate[K[u, v] Sin[m*u] Sin[n*v], {u, -Pi, Pi}, {v, -Pi/2, Pi/2},
   Method -> "LocalAdaptive", AccuracyGoal -> 8,
   IntegrationMonitor :> ((errors = Through[#1@"Error"]) &)]];
 Print[Total@errors]]
m := 3; For[n = 1, n <= 7, n += 2,
 Print[NIntegrate[K[u, v] Sin[m*u] Sin[n*v], {u, -Pi, Pi}, {v, -Pi/2, Pi/2},
   Method -> "LocalAdaptive", AccuracyGoal -> 8,
   IntegrationMonitor :> ((errors = Through[#1@"Error"]) &)]];
 Print[Total@errors]]
```

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