arXiv:2304.03564v1 [math.RA] 7 Apr 2023

环上乘法（广义）斜半导子的可加性

SK AZIZ、ARINDAM GHOSH 和 OM PRAKAS H*

Abstract. 本文介绍一类新的导子，它推广了斜导子和半导子，并将其称为“斜半导子”。此外，我们研究了这类乘法导子成为可加映射的条件。

1. Introduction

设 R为一个环。若映射 f: $R \to R$满足对所有 x、 $y \in R$，有 $f(x+y) = f(x) + f(y)$，则称其为可加映射。1937年，Jacobson [10] 正式引入了代数上导子的概念，并证明环上的半单代数的任一导子都可以延拓为底层环上的内导子。若可加映射h: $R \to R$满足对所有 x、 $y \in R$，有 $h(xy) = h(x)y + xh(y)$，则称其为导子。关于环和代数上的导子的更多细节，可参见 [16, 23, 12, 20]。

1985年，Leroy [14] 引入了环上的斜导子概念，并进一步研究了其在斜域和素环 [13]上的性质。若一个可加映射 δ: $R \to R$被称为斜导子，当且仅当 $\delta(xy) = \delta(x)y+\alpha(x)\delta(y)$对所有 x、 $y \in R$成立，其中 α 是 R上的一个自同构。感兴趣的读者可以在 [7, 6]中看到关于斜导子的更多结果。

1983年，Bergen [3] 引入了半导子的概念，将其作为环上导子的推广。回顾一下，若可加映射 D: $R \to R$ 满足 $D(xy) = D(x)g(y) + xD(y) = D(x)y + g(x)D(y)$ 且 $D(g(x)) = g(D(x))$，其中 x、 $y \in R$ 和 g 是 R上的任意映射，则称其为半导子。关于环上的半导子的基本结果，请参阅 [5, 4] 。

2. Preliminaries

上述概念引出了这样一个问题：是否存在半导子和斜导子的统一定义。对此，我们给出了肯定的回答，并在下文中引入了两个推广半导子和斜导子的新定义。

定义 2.1。若一个可加映射 d: $R \to R$ 被称为斜半导子，如果

$$d(xy) = d(x)g(y) + \alpha(x)d(y) = d(x)\alpha(y) + g(x)d(y)$$
$$\text{and } d(g(x)) = g(d(x))$$

其中 x 是一个自同构， g 是 R 上的映射。

2020 *Mathematics Subject Classification.* 16W10,16S50.*Key words and phrases.* 可加性；斜导子；半导子；（广义）斜半导子；幂等元。* 通讯作者。

若对于所有 x、$y \in R$，以及 R 上的某个导子 h，有 $\Delta(xy) = \Delta(x)y + xh(y)$，则称可加映射 Δ 为广义导子。现在，我们通过结合广义导子和斜半导子的概念，引入广义斜半导子的概念如下：

定义 2.2. 若可加映射 $f: R \to R$ 满足

$$f(xy) = f(x)g(y) + \alpha(x)d(y) = d(x)\alpha(y) + g(x)f(y)$$
$$\text{and } f(g(x)) = g(f(x))$$

对所有 x、$y \in R$，其中 d 是斜半导子，α 是自同构，并且存在 R上的某个映射 g。

注意，半导子和斜导子显然都是斜半导子的例子。然而，反过来并不成立。为此，我们给出一个斜半导子的例子，它既不是半导子，也不是具有相同伴随映射的斜导子。令 e_{ij} 为一个矩阵，其第（i，j）个元素为 1，其余元素均为 0。我们考虑实数域 $\mathbb{R}$ 上的全矩阵环 $M_n(\mathbb{R})$，其中 $n \geq 2$ 是一个整数。

例 2.3. 我们在 $M_n(\mathbb{R})$上定义三个映射 d、α 和 g 如下：

- The map $d : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is defined by
$$d\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} 2a_{22} & 2a_{21} \\ 2a_{12} & 2a_{11} \end{pmatrix},$$
其中 $a_{i\,j} \in \mathbb{R}$.
- 映射 α：$M_n(\mathbb{R}) \to M_n(\mathbb{R})$ 定义如下
$$\alpha\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}.$$
- 映射 g：$M_n(\mathbb{R}) \to M_n(\mathbb{R})$ 定义为
$$g\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

于是 α 是一个自同构，而 g 是一个简单映射。此外，简单计算可得，d 是一个斜半导子，其伴随映射为 g，在 $M_n(\mathbb{R})$ 上的自同构为α。但取 $x = e_{11}$ 和 $y = e_{12}$，可得

$$d(xy) = 2e_{21} \neq 0 = d(x)g(y) + xd(y).$$

因此，d 不是伴随映射为 g的半导子。此外，取 $x = e_{21}$和 $y = e_{22}$，可得

$$d(xy) = 0 \neq 2e_{12} = d(x)y + \alpha(x)d(y).$$

因此，d 不是伴随自同构为 α的斜导子。

需要指出的是，每个斜半导子都是广义斜半导子，但反之则不成立，以下例子对此进行了说明。

例2.4. 同样地，我们考虑如下四个映射：

- A map $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is defined by
$$f\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} 3a_{22} & 3a_{21} \\ 3a_{12} & 3a_{11} \end{pmatrix};$$

- g： $M_n(\mathbb{R}) \to M_n(\mathbb{R})$定义为

$$g\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix};$$

- d： $M_n(\mathbb{R}) \to M_n(\mathbb{R})$ 定义为

$$d\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

and

- α： $M_n(\mathbb{R}) \to M_n(\mathbb{R})$ 定义为

$$\alpha\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix},$$

其中 $a_{ij} \in \mathbb{R}$。

然后 α 是一个自同构， g 是一个映射，通过简单计算可得， d 是一个斜半导子，其伴随映射为 g ，自同构为 α。此外，容易看出， f 是一个广义斜半导子，其伴随映射为 g，自同构为 α，斜半导子为 d。然而，取 $x = e_{21}$ 和 $y = e_{22}$，我们有

$$f(x)\alpha(y) + g(x)f(y) = 3e_{12} \neq 0 = f(xy).$$

因此， f 不是具有伴随映射 g 和自同构 α的斜半导子。

现在，让我们从上述定义中去掉可加性条件。在这种情况下，我们分别称这些映射为乘法导子、乘法斜导子、乘法半导子、乘法斜半导子、乘法广义导子和乘法广义斜半导子。一个令人振奋的问题是，在哪些条件下这些乘法映射会变为可加映射。最早的相关结果由 Martindale [15]于 1969 年提出，他证明了环 R上每个具有非平凡幂等元的乘法双射都是可加的。更多相关结果可见 [1, 2, 17, 11, 8, 21, 22, 9, 18, 24]。2022 年，Siddeeque 和 Khan [19] 证明了满足某些条件的环 R 上的乘法半导子是可加的。这些结果促使我们引入斜半导子和广义斜半导子，并研究它们在环上的性质。本文试图回答以下问题：“在什么条件下，乘法斜半导子和乘法广义斜半导子会变为可加映射？”

在本文中，设 R 为一个含有非平凡幂等元 e 的环。设 $e_1 = e$ 和 $e_2 = 1 - e_1$，其中 1 是 R 的单位元。则 $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$，其中对于 i，有 $R_{ij} = e_i R e_j$，$j \in \{1, 2\}$。

假设 2.5. 设 R 是一个含零元素 0 的环，且 g： $R \to R$ 是一个映射，满足 $g(0) = 0$，并且对于 $a_{ij}, b_{ij} \in R_{ij}$)， $i, j, k = 1, 2$ 满足以下条件：

(i) 如果 $(a_{k1}+b_{k2})g(x_{ij})=0$ 且 $i\leq j$，对于所有 $x_{ij}\in R_{ij}$，那么

$$\begin{cases} a_{k1}=0 & \text{if } i=1,\\ b_{k2}=0 & \text{if } i=2. \end{cases}$$

(ii) 如果对于所有 $x_{ii}\in R_{ii}$，$g(x_{ii})(a_{1j}+b_{2j})=0$，那么

$$\begin{cases} a_{1j}=0 & \text{if } i=1,\\ b_{2j}=0 & \text{if } i=2. \end{cases}$$

(iii) $g(a_{11}+b_{12})=g(a_{11})+g(b_{12})$.
(iv) $g(a_{11}+b_{21})=g(a_{11})+g(b_{21})$.

这项工作的精彩之处在于引入了一类新的导子，它扩展了斜导子和半导子的概念，被称为斜半导子。此外，还引入了一个更一般的概念，即广义斜半导子，并给出了恰当的例子以支持所提出的概念。

在研究这些新概念时，一个自然的问题随之产生：乘法型斜半导子和广义斜半导子是否会在某些条件下变为可加映射？我们对此问题给出了肯定的回答，并在第 2 节和第 3 节中提供了证明。此外，第 2 节证明了，在环 R 具有一个非平凡幂等元 e 的情况下，如果底环和映射满足某些假设，则该环上的任意乘法斜半导子都是可加的。第 3 节在附加假设下，将这一结果推广到了同一环上的广义斜半导子。我们的结果拓展了环上导子的理论，并为斜导子和半导子的结构提供了新的见解。

3. Skew Semi-Derivation

定理 3.1。设 R 是一个环，且 $g:\ R\to R$ 是满足假设2.5的映射。如果 $d:\ R\to R$ 是一个乘法斜半导子，其伴随映射为$g:\ R\to R$，且自同构为 $\alpha:\ R\to R$，那么 d 在 R上是可加的。

在证明定理 3.1之前，我们证明几个引理。

引理 3.2. $d(0)=0$.

证明。 $d(0)=d(0\cdot 0)=d(0)g(0)+\alpha(0)d(0)=0$(根据假设 2.5， $g(0)=0$ 且由于 α 是一个自同构， $\alpha(0)=0$）。□

引理 3.3。设 $a_{ij}\in R_{ij}$。那么

$$\begin{aligned} &(i)\ d(a_{11}+a_{12})=d(a_{11})+d(a_{12});\\ &(ii)\ d(a_{11}+a_{21})=d(a_{11})+d(a_{21});\\ &(iii)\ d(a_{22}+a_{12})=d(a_{22})+d(a_{12});\\ &(iv)\ d(a_{22}+a_{21})=d(a_{22})+d(a_{21});\\ &(v)\ d(a_{11}+a_{22})=d(a_{11})+d(a_{22}). \end{aligned}$$

证明。设 $x_{11}\in R_{11}$。那么

$$d((a_{11}+a_{12})x_{11})=d(a_{11}+a_{12})g(x_{11})+\alpha(a_{11}+a_{12})d(x_{11}). \tag{3.1}$$

另一方面,

$$
\begin{aligned}
&d((a_{11}+a_{12})x_{11})\\
&=d(a_{11}x_{11})\\
&=d(a_{11}x_{11})+d(a_{12}x_{11})\\
&=d(a_{11})g(x_{11})+\alpha(a_{11})d(x_{11})+d(a_{12})g(x_{11})+\alpha(a_{12})d(x_{11})
\end{aligned}
\tag{3.2}
$$

比较（3.1）与（3.2，可得

$$[d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]g(x_{11})=0 \text{ which implies that,}$$

$$
\begin{aligned}
&([d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{11}\\
&+[d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{12})g(x_{11})=0\\
\text{and }&([d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{21}\\
&+[d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{22})g(x_{11})=0.
\end{aligned}
\tag{3.3}
$$

同理，对于 $x_{22}\in R_{22}$,

$$
\begin{aligned}
&d(a_{11}+a_{12})g(x_{22})+\alpha(a_{11}+a_{12})d(x_{22})\\
&=d((a_{11}+a_{12})x_{22})\\
&=d(a_{12}x_{22})\\
&=d(a_{11}x_{22})+d(a_{12}x_{22})\\
&=d(a_{11})g(x_{22})+\alpha(a_{11})d(x_{22})+d(a_{12})g(x_{22})+\alpha(a_{12})d(x_{22}).
\end{aligned}
\tag{3.4}
$$

比较（3.4）两边，可得

$$[d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]g(x_{22})=0 \text{ which implies that,}$$

$$
\begin{aligned}
&([d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{11}\\
&+[d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{12})g(x_{22})=0\\
\text{and }&([d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{21}\\
&+[d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{22})g(x_{22})=0.
\end{aligned}
\tag{3.5}
$$

由假设2.5 （i）、（3.3）和（3.5），可得

$$
\begin{aligned}
&[d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{k1}=0\\
\&\ &[d(a_{11}+a_{12})-d(a_{11})-d(a_{12})]_{k2}=0
\end{aligned}
$$

对于 $k=1$，2。引理 3.3 （i）的证明到此完成。

为证明引理 3.3 的(ii)，取 $x_{11}\in R_{11}$，则

$$
\begin{aligned}
&d(x_{11})\alpha(a_{11}+a_{21})+g(x_{11})d(a_{11}+a_{21})=d(x_{11}(a_{11}+a_{21}))\\
&=d(x_{11}a_{11})\\
&=d(x_{11}a_{11})+d(x_{11}a_{21})\\
&=d(x_{11})\alpha(a_{11})+g(x_{11})d(a_{11})+d(x_{11})\alpha(a_{21})+g(x_{11})d(a_{21}).
\end{aligned}
\tag{3.6}
$$

比较（3.6）两边，得到

$$g(x_{11})[d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]=0 \text{ which implies that,}$$

$$\begin{aligned} & g(x_{11})([d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{11} \\ & +[d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{21})=0 \\ \text{and } & g(x_{11})([d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{12} \\ & +[d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{22})=0. \end{aligned} \tag{3.7}$$

同样，取 $x_{22}\in R_{22}$，可得

$$g(x_{22})[d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]=0 \text{ which implies that,}$$

$$\begin{aligned} & g(x_{22})([d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{11} \\ & +[d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{21})=0 \\ \text{and } & g(x_{22})([d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{12} \\ & +[d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{22})=0. \end{aligned} \tag{3.8}$$

利用假设 2.5 (ii), (3.7) 和 (3.8),

$$\begin{aligned} & [d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{1k}=0 \\ & \& \ [d(a_{11}+a_{21})-d(a_{11})-d(a_{21})]_{2k}=0. \end{aligned}$$

对于 $k=1$，2。因此，引理 3.3 中的（ii）得证。类似地，我们可以证明引理 3.3 中的（iii）、（iv）和（v）。□

引理 3.4. 设 $a_{ij},b_{ij},c_{ij}\in R_{ij}$。则

$$\begin{aligned} &(i)\ d(a_{12}+b_{12}c_{22})=d(a_{12})+d(b_{12}c_{22}), \\ &(ii)\ d(a_{21}+b_{22}c_{21})=d(a_{21})+d(b_{22}c_{21}). \end{aligned}$$

证明。注意，$a_{12}+b_{12}c_{22}=(e_1+b_{12})(a_{12}+c_{22})$。因此，

$$\begin{aligned} d(a_{12}+b_{12}c_{22}) &= d[(e_1+b_{12})(a_{12}+c_{22})] \\ &= d(e_1+b_{12})\alpha(a_{12}+c_{22})+g(e_1+b_{12})d(a_{12}+c_{22}) \\ &= d(e_1)\alpha(a_{12})+g(e_1)d(a_{12})+d(b_{12})\alpha(a_{12})+g(b_{12})d(a_{12}) \\ &\quad +d(e_1)\alpha(c_{22})+g(e_1)d(c_{22})+d(b_{12})\alpha(c_{22})+g(b_{12})d(c_{22}) \\ &\quad \text{(By the Assumption 2.5 (iii), Lemma 3.3 (i) and (iii))} \\ &= d(e_1a_{12})+d(b_{12}a_{12})+d(e_1c_{22})+d(b_{12}c_{22}) \\ &= d(a_{12})+d(b_{12}c_{22}), \text{ By Lemma 3.2.} \end{aligned}$$

因此，引理 3.4 的 (i) 的证明如下。类似地，为证明引理 3.4 的 (ii)，我们使用单位元 $a_{21}+b_{22}c_{21}=(a_{21}+b_{22})(e_1+c_{21})$。然后利用假设 2.5(iv)、引理 3.3 的 (ii) 和 (iv)，即可得到所需结果。□

引理 3.5. 设 a_{ij}、$b_{ij}\in R_{ij}$。则

$$\begin{aligned} &(i)\ d(a_{12}+b_{12})=d(a_{12})+d(b_{12}), \\ &(ii)\ d(a_{21}+b_{21})=d(a_{21})+d(b_{21}). \end{aligned}$$

证明。令 $x_{22} \in R_{22}$。那么

$$\begin{aligned}
&d(a_{12}+b_{12})g(x_{22})+\alpha(a_{12}+b_{12})d(x_{22})\\
&=d((a_{12}+b_{12})x_{22})\\
&=d(a_{12}x_{22}+b_{12}x_{22}) \qquad (3.9)\\
&=d(a_{12}x_{22})+d(b_{12}x_{22}) \text{ (By Lemma 3.4 (i))}\\
&=d(a_{12})g(x_{22})+\alpha(a_{12})d(x_{22})+d(b_{12})g(x_{22})+\alpha(b_{12})d(x_{22})
\end{aligned}$$

比较（3.9）的两边,

$$[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]g(x_{22})=0 \text{ which implies that}$$

$$\begin{aligned}
&([d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{11} \qquad (3.10)\\
&+[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{12})g(x_{22})=0\\
\text{and } &([d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{21}\\
&+[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{22})g(x_{22})=0.
\end{aligned}$$

令 $x_{11} \in R_{11}$。那么

$$\begin{aligned}
&d(a_{12}+b_{12})g(x_{11})+\alpha(a_{12}+b_{12})d(x_{11})\\
&=d((a_{12}+b_{12})x_{11})\\
&=d(0)=0 \text{ (By Lemma 3.2)} \qquad (3.11)\\
&=d(a_{12}x_{11})+d(b_{12}x_{11})\\
&=d(a_{12})g(x_{11})+\alpha(a_{12})d(x_{11})+d(b_{12})g(x_{11})+\alpha(b_{12})d(x_{11}).
\end{aligned}$$

比较两边的(3.11),

$$[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]g(x_{11})=0 \text{ which implies that,}$$

$$\begin{aligned}
&([d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{11} \qquad (3.12)\\
&+[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{12})g(x_{11})=0\\
\text{and } &([d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{21}\\
&+[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{22})g(x_{11})=0.
\end{aligned}$$

根据假设2.5 (i)、(3.10) 和 (3.12),

$$\begin{aligned}
&[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{k2}=0\\
\&\quad &[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]_{k1}=0.
\end{aligned}$$

对于 $k=1,\ 2$. 因此，我们有

$$d(a_{12}+b_{12})=d(a_{12})+d(b_{12}).$$

同样地，我们可以证明

$$d(a_{21}+b_{21})=d(a_{21})+d(b_{21}).$$

□

引理 3.6. 设 a_{ii}、$b_{ii} \in R_{ii}$。则

$$\begin{aligned}
&(i)\ d(a_{11}+b_{11})=d(a_{11})+d(b_{11}),\\
&(ii)\ d(a_{22}+b_{22})=d(a_{22})+d(b_{22}).
\end{aligned}$$

证明。令 $x_{12} \in R_{12}$。然后

$$\begin{aligned} & d(a_{11}+b_{11})g(x_{12})+\alpha(a_{11}+b_{11})d(x_{12}) \\ & = d((a_{11}+b_{11})x_{12}) \\ & = d(a_{11}x_{12}+b_{11}x_{12}) \\ & = d(a_{11}x_{12})+d(b_{11}x_{12}) \text{ (By Lemma 3.5 (i))} \\ & = d(a_{11})g(x_{12})+\alpha(a_{11})d(x_{12})+d(b_{11})g(x_{12})+\alpha(b_{11})d(x_{12}). \end{aligned} \tag{3.13}$$

比较两边3.13），

$$[d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]g(x_{12})=0 \text{ which implies that,}$$

$$\begin{aligned} & ([d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{11} \\ & +[d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{12})g(x_{12})=0 \\ \text{and } & ([d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{21} \\ & +[d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{22})g(x_{12})=0. \end{aligned} \tag{3.14}$$

同样，取 $x_{22} \in R_{22}$，我们有

$$[d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]g(x_{22})=0 \text{ which implies that,}$$

$$\begin{aligned} & ([d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{11} \\ & +[d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{12})g(x_{22})=0 \\ \text{and } & ([d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{21} \\ & +[d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{22})g(x_{22})=0. \end{aligned} \tag{3.15}$$

根据假设2.5 (i), (3.14) and (3.15)，我们有

$$\begin{aligned} [d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{k1}=0, \\ [d(a_{11}+b_{11})-d(a_{11})-d(b_{11})]_{k2}=0. \end{aligned}$$

对于 $k=1$，2。因此，我们有

$$d(a_{11}+b_{11})=d(a_{11})+d(b_{11}).$$

同样，我们可以证明

$$d(a_{22}+b_{22})=d(a_{22})+d(b_{22}).$$

□

引理 3.7。设 $a_{ij} \in R_{ij}$。则

$$d(a_{11}+a_{12}+a_{21}+a_{22})=d(a_{11})+d(a_{12})+d(a_{21})+d(a_{22}).$$

证明。设 $x_{11} \in R_{11}$。则

$$\begin{aligned} & d(a_{11}+a_{12}+a_{21}+a_{22})g(x_{11})+\alpha(a_{11}+a_{12}+a_{21}+a_{22})d(x_{11}) \\ & = d((a_{11}+a_{12}+a_{21}+a_{22})x_{11}) \\ & = d(a_{11}x_{11}+a_{21}x_{11}) \\ & = d(a_{11}x_{11})+d(a_{12}x_{11})+d(a_{21}x_{11})+d(a_{22}x_{11}) \\ & = d(a_{11})g(x_{11})+\alpha(a_{11})d(x_{11})+d(a_{12})g(x_{11})+\alpha(a_{12})d(x_{11}) \\ & +d(a_{21})g(x_{11})+\alpha(a_{21})d(x_{11})+d(a_{22})g(x_{11})+\alpha(a_{22})d(x_{11}). \end{aligned} \tag{3.16}$$

比较两边的（3.16），

$$
\begin{aligned}
&[d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})\\
&-d(a_{21})-d(a_{22})]g(x_{11})=0\\
&\text{which implies that,}\\
&([d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})-d(a_{21})-d(a_{22})]_{11}\\
&+[d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})\\
&-d(a_{21})-d(a_{22})]_{12})g(x_{11})=0\\
&\text{and } ([d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})-d(a_{21})-d(a_{22})]_{21}\\
&+[d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})\\
&-d(a_{12})-d(a_{21})-d(a_{22})]_{22})g(x_{11})=0.
\end{aligned}
\tag{3.17}
$$

类似地，令 $x_{22}\in R_{22}$，我们有

$$
\begin{aligned}
&[d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})\\
&-d(a_{21})-d(a_{22})]g(x_{22})=0 \text{ which implies that,}\\
&([d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})-d(a_{21})-d(a_{22})]_{11}\\
&+[d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})\\
&-d(a_{21})-d(a_{22})]_{12})g(x_{22})=0\\
&\text{and } ([d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})-d(a_{21})-d(a_{22})]_{21}\\
&+[d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})\\
&-d(a_{21})-d(a_{22})]_{22})g(x_{22})=0.
\end{aligned}
\tag{3.18}
$$

根据假设2.5 (i), (3.17) 和 (3.18)，我们得到

$$
\begin{aligned}
&[d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})-d(a_{21})-d(a_{22})]_{k1}=0,\\
&[d(a_{11}+a_{12}+a_{21}+a_{22})-d(a_{11})-d(a_{12})-d(a_{21})-d(a_{22})]_{k2}=0,
\end{aligned}
$$

对于 $k=1$，2. 因此，

$$
d(a_{11}+a_{12}+a_{21}+a_{22})=d(a_{11})+d(a_{12})+d(a_{21})+d(a_{22}).
$$

□

定理的证明 *3.1*。 设 a，$b\in R$。则

$$
\begin{aligned}
a&=a_{11}+a_{12}+a_{21}+a_{22},\\
b&=b_{11}+b_{12}+b_{21}+b_{22},
\end{aligned}
$$

对于某些 a_{ij}、$b_{ij} \in R_{ij}$。现在，

$$\begin{aligned} d(a+b) &= d(a_{11}+a_{12}+a_{21}+a_{22}+b_{11}+b_{12}+b_{21}+b_{22}) \\ &= d((a_{11}+b_{11})+(a_{12}+b_{12})+(a_{21}+b_{21})+(a_{22}+b_{22})) \\ &= d(a_{11}+b_{11})+d(a_{12}+b_{12})+d(a_{21}+b_{21}) \\ &\quad + d(a_{22}+b_{22}) \text{ (By Lemma 3.7)} \\ &= d(a_{11})+d(b_{11})+d(a_{12})+d(b_{12})+d(a_{21}) \\ &\quad + d(b_{21})+d(a_{22})+d(b_{22}) \text{ (By Lemma 3.5 and 3.6)} \\ &= d(a_{11}+a_{12}+a_{21}+a_{22})+d(b_{11}+b_{12}+b_{21}+b_{22}) \\ &\quad \text{(By Lemma 3.7)} \\ &= d(a)+d(b). \end{aligned}$$

因此，d 在 R上可加。此外，d 是 R上的斜半导子。 □

推论 3.8. 设 R 是含有非平凡幂等元 e的环。设 d 是 R上的乘法斜半导子，且 g 是 R 上的映射，使得 $g(0) = 0$。假设 d和 g满足以下条件，其中 i、j、$k = 1$取值于 2：

(i) 若对所有 $x_{ij} \in R_{ij}$，$(a_{k1}+b_{k2})g(x_{ij}) = 0$ 且 $i \le j$，则

$$\begin{cases} a_{k1} = 0 & \text{if } i = 1, \\ b_{k2} = 0 & \text{if } i = 2. \end{cases}$$

(ii) 若对所有 $x_{ii} \in R_{ii}$，$g(x_{ii})(a_{1j}+b_{2j}) = 0$，则

$$\begin{cases} a_{1j} = 0 & \text{if } i = 1, \\ b_{2j} = 0 & \text{if } i = 2. \end{cases}$$

(iii) 若对所有 $x_{ii} \in R_{ii}$，$d(x_{ii})(a_{1j}+b_{2j}) = 0$，则

$$\begin{cases} a_{1j} = 0 & \text{if } i = 1, \\ b_{2j} = 0 & \text{if } i = 2. \end{cases}$$

则 d 是可加的。

证明。只需证明：若上述推论的条件 (iii) 成立，则它蕴含假设 2.5 的条件 (iii) 和 (iv)。我们证明假设 2.5 的条件 (iii) 和 (iv)。设 x_{11}，$a_{11} \in R_{11}$ 且 $b_{12} \in R_{12}$。则

$$d(x_{11})g(a_{11}+b_{12}) = d(x_{11}(a_{11}+b_{12})) - \alpha(x_{11})d(a_{11}+b_{12}).$$

由引理 3.3(i)，

$$\begin{aligned} d(x_{11})g(a_{11}+b_{12}) &= d(x_{11}a_{11})+d(x_{11}b_{12})-\alpha(x_{11})d(a_{11})-\alpha(x_{11})d(b_{12}) \\ &= d(x_{11})g(a_{11})+d(x_{11})g(b_{12}), \end{aligned}$$

这意味着

$$d(x_{11})[g(a_{11}+b_{12})-g(a_{11})-g(b_{12})] = 0.$$

因此，

$d(x_{11})([g(a_{11}+b_{12}) - g(a_{11})-g(b_{12})]_{11}+[g(a_{11}+b_{12}) - g(a_{11}) - g(b_{12})]_{21})$ $= 0$和

$$d(x_{11})([g(a_{11}+b_{12})-g(a_{11})-g(b_{12})]_{12}+[g(a_{11}+b_{12})-g(a_{11})-g(b_{12})]_{22}) = 0.$$

类似地，取 $x_{22} \in R_{22}$，我们有

$d(x_{22})([g(a_{11}+b_{12}) - g(a_{11}) - g(b_{12})]_{11} + [g(a_{11}+b_{12}) - g(a_{11}) - g(b_{12})]_{21})$ $= 0$和

$d(x_{22})([g(a_{11}+b_{12}) - g(a_{11}) - g(b_{12})]_{12} + [g(a_{11}+b_{12}) - g(a_{11}) - g(b_{12})]_{22})$ $= 0$.利用推论3.8的条件 (iii),

$$[g(a_{11}+b_{12}) - g(a_{11}) - g(b_{12})]_{1k} = 0,$$
$$[g(a_{11}+b_{12}) - g(a_{11}) - g(b_{12})]_{2k} = 0,$$

其中 $k = 1$，2。因此,

$$g(a_{11}+b_{12}) = g(a_{11}) + g(b_{12}).$$

同理，我们可以证明

$$g(a_{11}+b_{21}) = g(a_{11}) + g(b_{21}).$$

□

推论 3.9. 设 R 是一个含单位元 $1 \neq 0$且含有幂等元 $e \neq 0$（ $e \neq 0 \neq 1$）的素环。若 d 是 R 上的任意乘法斜半导子，其伴随映射 g 是 R 上的一个自同态，并且满足 $g(e) = e$, 则 d 可加。

推论 3.10。 设 R 是如推论 3.9中的一个环。如果 d 是 R 上的任意乘法斜半导子，其中伴随映射 g 在 R_{ij}上是不变的，即 $g(R_{ij}) \subseteq R_{ij}$ 其中 i， $j = 1$、 2 以及 g 满足假设 2.5中的条件 (iii) 和 (iv)，那么 d 是可加的。

定理3.11.设 R 是一个具有非平凡幂等元 e 的环，且 g 是 R上任意一个在零处为零并满足以下条件的映射，对于 i、 $j = 1$，2:

(i) *If* $(a_{j1} + b_{j2})g(x_{ii}) = 0$ *for all* $x_{ii} \in R_{ii}$, *then* $a_{j1} + b_{j2} = 0$.

(ii) *If* $g(x_{ii})(a_{11} + a_{21}) = 0$ *for all* $x_{ii} \in R_{ii}$, *then* $a_{11} + a_{21} = 0$.

如果 d 是 R 上的乘法斜半导子，伴随映射为 g 且自同构为 α，那么 d 是可加的。

为证明定理 3.11，我们证明如下结果。注意到，我们有$d(0) = 0$ 由引理 3.2。

引理 3.12。设 $a_{11} \in R_{11}$、 $b_{12} \in R_{12}$ 和 $c_{21} \in R_{21}$。 则

(i) $d(a_{11} + b_{12}) = d(a_{11}) + d(b_{12})$,

(ii) $d(a_{11} + c_{21}) = d(a_{11}) + d(c_{21})$.

证明。(i) 令 a_{11}、 $x_{11} \in R_{11}$ 和 $b_{12} \in R_{12}$。 则

$$\begin{aligned} d((a_{11}+b_{12})x_{11}) &= d(a_{11}x_{11}) \\ &= d(a_{11}x_{11}) + d(b_{12}x_{11}) \\ &= d(a_{11})g(x_{11}) + \alpha(a_{11})d(x_{11}) \\ &\quad + d(b_{12})g(x_{11}) + \alpha(b_{12})d(x_{11}). \end{aligned}$$

另一方面,

$$d((a_{11}+b_{12})x_{11}) = d(a_{11}+b_{12})g(x_{11}) + \alpha(a_{11}+b_{12})d(x_{11}).$$

比较这些恒等式，

$$[d(a_{11}+b_{12})-d(a_{11})-d(b_{12})]g(x_{11})=0$$

这意味着

$$d\cdot g(x_{11})=0,$$

where $d=d(a_{11}+b_{12})-d(a_{11})-d(b_{12})$. This gives

$$([d]_{11}+[d]_{12})g(x_{11})=0 \tag{3.19}$$

and

$$([d]_{21}+[d]_{22})g(x_{11})=0. \tag{3.20}$$

同样，取 $x_{22}\in R_{22}$，可得

$$\begin{aligned} d(x_{22}(a_{11}+b_{12})) &= d(x_{22}a_{11})+d(x_{22}b_{12}) \\ &= d(x_{22})\alpha(a_{11})+g(x_{22})d(a_{11}) \\ &\quad + d(x_{22})\alpha(b_{12})+g(x_{22})d(b_{12}). \end{aligned}$$

另一方面，

$$d(x_{22}(a_{11}+b_{12}))=d(x_{22})\alpha(a_{11}+b_{12})+g(x_{22})d(a_{11}+b_{12}).$$

比较，

$$g(x_{22})[d(a_{11}+b_{12})-d(a_{11})-d(b_{12})]=0$$

这意味着

$$g(x_{22})\cdot d=0.$$

我们得到

$$g(x_{22})([d]_{11}+[d]_{21})=0. \tag{3.21}$$

应用定理 3.11 (i) $-(ii)$ 于（3.19)-（3.21），可得

$$\begin{aligned} [d]_{11}+[d]_{12}&=0,\\ [d]_{21}+[d]_{22}&=0,\\ [d]_{11}+[d]_{21}&=0. \end{aligned}$$

因此，$[d]_{ij}=0$ 对任意 i、j均成立，证毕。

（ii）设 $a_{11}\in R_{11}$、$x_{22}\in R_{22}$ 和 $c_{21}\in R_{21}$。则

$$\begin{aligned} d((a_{11}+c_{21})x_{22}) &= d(a_{11}x_{22})+d(c_{21}x_{22}) \\ &= d(a_{11})g(x_{22})+\alpha(a_{11})d(x_{22}) \\ &\quad + d(c_{21})g(x_{22})+\alpha(c_{21})d(x_{22}). \end{aligned}$$

另一方面，

$$d((a_{11}+c_{21})x_{22})=d(a_{11}+c_{21})g(x_{22})+\alpha(a_{11}+c_{21})d(x_{22}).$$

比较这两个等式，可得

$$[d(a_{11}+c_{21})-d(a_{11})-d(c_{21})]g(x_{22})=0.$$

这意味着

$$d\cdot g(x_{22})=0$$

其中 $d=d(a_{11}+c_{21})-d(a_{11})-d(c_{21})$。现在，我们得到

$$([d]_{11}+[d]_{12})g(x_{22})=0 \tag{3.22}$$

and

$$([d]_{21}+[d]_{22})g(x_{22})=0. \tag{3.23}$$

类似地，取 $x_{11}\in R_{11}$，我们有

$$\begin{aligned} d(x_{11}(a_{11}+c_{21})) &= d(x_{11}a_{11}) \\ &= d(x_{11}a_{11})+d(x_{11}c_{21}) \\ &= d(x_{11})\alpha(a_{11})+g(x_{11})d(a_{11}) \\ &+ d(x_{11})\alpha(c_{21})+g(x_{11})d(c_{21}). \end{aligned}$$

另一方面，

$$d(x_{11}(a_{11}+c_{21}))=d(x_{11})\alpha(a_{11}+c_{21})+g(x_{11})d(a_{11}+c_{21})$$

比较上述等式，得到

$$g(x_{11})[d(a_{11}+c_{21})-d(a_{11})-d(c_{21})]=0$$

这意味着

$$g(x_{11})\cdot d=0.$$

此外，我们有

$$g(x_{11})([d]_{11}+[d]_{21})=0. \tag{3.24}$$

利用定理3.11 (i)-(ii) in (3.22)-(3.24),

$$\begin{aligned} [d]_{11}+[d]_{12}&=0 \\ [d]_{21}+[d]_{22}&=0 \\ [d]_{11}+[d]_{21}&=0 \end{aligned}$$

因此，$[d]_{ij}=0$ 对所有 i，j。因此，我们得到所需结果。 □

引理 3.13。以下陈述成立：
(i) d 在 R_{12}上可加，(ii) d 在 R_{21}上可加，(iii) d在 R_{11} 上可加，(iv) d在 R_{22}上可加。

证明。设 $x_{11}\in R_{11}$ 和 $a_{12}, b_{12}\in R_{12}$。那么

$$\begin{aligned} d((a_{12}+b_{12})x_{11}) &= d(a_{12}x_{11})+d(b_{12}x_{11}) \\ &= d(a_{12})g(x_{11})+\alpha(a_{12})d(x_{11}) \\ &+ d(b_{12})g(x_{11})+\alpha(b_{12})d(_{11}). \end{aligned}$$

另一方面，

$$d((a_{12}+b_{12})x_{11})=d(a_{12}+b_{12})g(x_{11})+\alpha(a_{12}+b_{12})d(x_{11}).$$

比较上述恒等式，

$$[d(a_{12}+b_{12})-d(a_{12})-d(b_{12})]g(x_{11})=0$$

这意味着，

$$d\cdot g(x_{11})=0,$$

其中 $d=d(a_{12}+b_{12})-d(a_{12})-d(b_{12})$。现在，我们得到

$$([d]_{11}+[d]_{12})g(x_{11})=0 \tag{3.25}$$

and

$$([d]_{21} + [d]_{22})g(x_{11}) = 0. \tag{3.26}$$

同样，取 $x_{22} \in R_{22}$,

$$\begin{aligned} d(x_{22}(a_{12} + b_{12})) &= d(x_{22}a_{12}) + d(x_{22}b_{12}) \\ &= d(x_{22})\alpha(a_{12}) + g(x_{22})d(a_{12}) \\ &\quad + d(x_{22})\alpha(b_{12}) + g(x_{22})d(b_{12}). \end{aligned}$$

另一方面,

$$d(x_{22}(a_{12} + b_{12})) = d(x_{22})\alpha(a_{12} + b_{12}) + g(x_{22})d(a_{12} + b_{12}).$$

比较上述等式,

$$g(x_{22})[d(a_{12} + b_{12}) - d(a_{12}) - d(b_{12})] = 0$$

这意味着,

$$g(x_{22}) \cdot d = 0.$$

我们得到

$$g(x_{22})([d]_{11} + [d]_{21}) = 0. \tag{3.27}$$

利用定理 3.11(i)-(ii) 于 (3.25)-(3.27),

$$\begin{aligned} [d]_{11} + [d]_{12} &= 0, \\ [d]_{21} + [d]_{22} &= 0, \\ [d]_{11} + [d]_{21} &= 0. \end{aligned}$$

因此，我们得到 $[d]_{ij} = 0$ 对所有 i, j。这就完成了引理（i）的3.13的证明。同样地，我们可以证明（ii），（iii）和（iv）的引理 3.13。□

引理 3.14。设 $a_{ij} \in R_{ij}$。那么

$$d(a_{11} + a_{12} + a_{21} + a_{22}) = d(a_{11}) + d(a_{12}) + d(a_{21}) + d(a_{22}).$$

证明。设 $x_{11} \in R_{11}$。那么

$$\begin{aligned} &d((a_{11} + a_{12} + a_{21} + a_{22})x_{11}) \\ &= d(a_{11} + a_{12} + a_{21} + a_{22})g(x_{11}) + \alpha(a_{11} + a_{12} + a_{21} + a_{22})d(x_{11}). \end{aligned}$$

另一方面，由引理 3.12 (ii)，可得

$$\begin{aligned} &d((a_{11} + a_{12} + a_{21} + a_{22})x_{11}) \\ =&d(a_{11}x_{11} + a_{21}x_{11}) \\ =&d(a_{11}x_{11}) + d(a_{21}x_{11}) \\ =&d(a_{11}x_{11}) + d(a_{12}x_{11}) + d(a_{21}x_{11}) + d(a_{22}x_{11}) \\ =&d(a_{11})g(x_{11}) + \alpha(a_{11})d(x_{11}) + d(a_{12})g(x_{11}) + \alpha(a_{12})d(x_{11}) \\ &+ d(a_{21})g(x_{11}) + \alpha(a_{21})d(x_{11}) + d(a_{22})g(x_{11}) + \alpha(a_{22})d(x_{11}). \end{aligned}$$

比较上述等式，我们得到

$$[d(a_{11} + a_{12} + a_{21} + a_{22}) - d(a_{11}) - d(a_{12}) - d(a_{21}) - d(a_{22})]g(x_{11}) = 0,$$

这意味着

$$d \cdot g(x_{11}) = 0,$$

其中 $d = d(a_{11} + a_{12} + a_{21} + a_{22}) - d(a_{11}) - d(a_{12}) - d(a_{21}) - d(a_{22})$。因此,

$$([d]_{11} + [d]_{12})g(x_{11}) = 0 \tag{3.28}$$

and

$$([d]_{21} + [d]_{22})g(x_{11}) = 0. \tag{3.29}$$

同理,

$$\begin{aligned} &d(x_{11}(a_{11} + a_{12} + a_{21} + a_{22})) \\ &= d(x_{11})\alpha(a_{11} + a_{12} + a_{21} + a_{22}) + g(x_{11})d(a_{11} + a_{12} + a_{21} + a_{22}). \end{aligned}$$

另一方面，由引理3.12 (i)，我们有

$$\begin{aligned} &d(x_{11}(a_{11} + a_{12} + a_{21} + a_{22})) \\ =&d(x_{11}a_{11} + x_{11}a_{12}) \\ =&d(x_{11}a_{11}) + d(x_{11}a_{12}) \\ =&d(x_{11}a_{11}) + d(x_{11}a_{12}) + d(x_{11}a_{21}) + d(x_{11}a_{22}) \\ =&d(x_{11})\alpha(a_{11}) + g(x_{11})d(a_{11}) + d(x_{11})\alpha(a_{12}) + g(x_{11})d(a_{12}) \\ &+ d(x_{11})\alpha(a_{21}) + g(x_{11})d(a_{21}) + d(x_{11})\alpha(a_{22}) + g(x_{11})d(a_{22}). \end{aligned}$$

比较上述恒等式，我们得到

$$g(x_{11})[d(a_{11} + a_{12} + a_{21} + a_{22}) - d(a_{11}) - d(a_{12}) - d(a_{21}) - d(a_{22})] = 0$$

这意味着

$$g(x_{11}) \cdot d = 0,$$

其中 $d = d(a_{11} + a_{12} + a_{21} + a_{22}) - d(a_{11}) - d(a_{12}) - d(a_{21}) - d(a_{22})$。此外,

$$g(x_{11})([d]_{11} + [d]_{21}) = 0. \tag{3.30}$$

利用定理 3.11 (i) $-(ii)$ 在 (3.28)-(3.30)

$$\begin{aligned} [d]_{11} + [d]_{12} &= 0 \\ [d]_{21} + [d]_{22} &= 0 \\ [d]_{11} + [d]_{21} &= 0. \end{aligned}$$

因此，对于所有 i, j，我们有 $[d]_{ij} = 0$。 □

定理 *3.11* 的证明。令 $a, b \in R$。那么 $a = a_{11} + a_{12} + a_{21} + a_{22}$ 且 $b = b_{11} + b_{12} + b_{21} + b_{22}$ 对于 $a_{ij}, b_{ij} \in R_{ij}$。利用引理 3.13 和 3.14,

$$\begin{aligned} d(a + b) &= d(a_{11} + a_{12} + a_{21} + a_{22} + b_{11} + b_{12} + b_{21} + b_{22}) \\ &= d((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22})) \\ &= d(a_{11} + b_{11}) + d(a_{12} + b_{12}) + d(a_{21} + b_{21}) + d(a_{22} + b_{22}) \\ &= d(a_{11}) + d(b_{11}) + d(a_{12}) + d(b_{12}) + d(a_{21}) \\ &\quad + d(b_{21}) + d(a_{22}) + d(b_{22}) \\ &= d(a_{11} + a_{12} + a_{21} + a_{22}) + d(b_{11} + b_{12} + b_{21} + b_{22}) \\ &= d(a) + d(b). \end{aligned}$$

因此 d 是可加的。 □

推论 3.15. 设 R 为含有非平凡幂等元 e 的环，且 d 为 R 上的乘法斜半导子。设 g 为 R 上的映射，满足 $g(0) = 0$，并且对 $i = 1$、2 满足以下三个条件：

(i) 如果 $(a_{11} + b_{12})g(x_{ii}) = 0$ 对所有 $x_{ii} \in R_{ii}$都成立，则 $a_{11} + b_{12} = 0$。(ii) 如果 $(a_{21} + b_{22})g(x_{ii}) = 0$ 对所有 $x_{ii} \in R_{ii}$ 都成立，则 $a_{21} + b_{22} = 0$。(iii) 如果 $g(x_{ii})(a_{11} + b_{21}) = 0$ 对所有 $x_{ii} \in R_{ii}$都成立，则 $a_{11} + b_{21} = 0$。(iv) 如果 $g(x_{ii})($ $a_{12} + b_{22}) = 0$ 对所有 $x_{ii} \in R_{ii}$都成立，则 $a_{12} + b_{22} = 0$。

那么 d 是可加的。

4. 广义斜半导子

定理 4.1。 设 R 是一个环，且 $g : R \to R$ 是满足假设2.5的映射。如果 $f :$ $R \to R$ 是一个乘法广义斜半导子，其伴随映射为 $g : R \to R$，是一个斜半导子 d，且是一个自同构 α:$R \to R$，那么 f 在 R上是可加的。

在证明定理 4.1之前，我们先证明几个引理。

引理 4.2. $f(0) = 0$。

证明。 $f(0) = f(0 \cdot 0) = f(0)g(0) + \alpha(0)d(0) = 0$(根据假设 2.5， $g(0) = 0$并且由于 α 是一个自同构， $\alpha(0) = 0$)。 □

引理 4.3。设 $a_{ij} \in R_{ij}$。 则

$$\begin{aligned}
(i)\ & f(a_{11} + a_{12}) = f(a_{11}) + f(a_{12}),\\
(ii)\ & f(a_{11} + a_{21}) = f(a_{11}) + f(a_{21}),\\
(iii)\ & f(a_{22} + a_{12}) = f(a_{22}) + f(a_{12}),\\
(iv)\ & f(a_{22} + a_{21}) = f(a_{22}) + f(a_{21}),\\
(v)\ & f(a_{11} + a_{22}) = f(a_{11}) + f(a_{22}).
\end{aligned}$$

证明。令 $x_{11} \in R_{11}$。于是

$$f((a_{11} + a_{12})x_{11}) = f(a_{11} + a_{12})g(x_{11}) + \alpha(a_{11} + a_{12})d(x_{11}). \tag{4.1}$$

另一方面，我们有

$$\begin{aligned}
f((a_{11} + a_{12})x_{11}) &= f(a_{11}x_{11}) + f(a_{12}x_{11})\\
&= f(a_{11})g(x_{11}) + \alpha(a_{11})d(x_{11})\\
&\quad + f(a_{12})g(x_{11}) + \alpha(a_{12})d(x_{11})
\end{aligned} \tag{4.2}$$

比较 (4.1) 和 (4.2),

$$[f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]g(x_{11})=0 \text{ which implies that,}$$

$$\begin{aligned}&([f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{11}\\&\quad+[f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{12})g(x_{11})=0\\&\text{and } ([f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{21}\\&\quad+[f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{22})g(x_{11})=0.\end{aligned}\tag{4.3}$$

同样，对于 $x_{22}\in R_{22}$，我们有

$$\begin{aligned}&f(a_{11}+a_{12})g(x_{22})+\alpha(a_{11}+a_{12})d(x_{22})\\&=f((a_{11}+a_{12})x_{22})\\&=f(a_{12}x_{22})\\&=f(a_{11}x_{22})+f(a_{12}x_{22})\\&=f(a_{11})g(x_{22})+\alpha(a_{11})d(x_{22})+f(a_{12})g(x_{22})+\alpha(a_{12})d(x_{22}).\end{aligned}\tag{4.4}$$

比较等式（4.4），我们得到

$$[f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]g(x_{22})=0 \text{ which implies that,}$$

$$\begin{aligned}&([f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{11}\\&\quad+[f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{12})g(x_{22})=0\\&\text{and } ([f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{21}\\&\quad+[f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{22})g(x_{22})=0.\end{aligned}\tag{4.5}$$

根据假设 2.5 (i), (4.3) 和 (4.5),

$$\begin{aligned}&[f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{k1}=0\\&\&\ [f(a_{11}+a_{12})-f(a_{11})-f(a_{12})]_{k2}=0\end{aligned}$$

对于 $k=1$，这就完成了引理4.3中（i）的证明。

为证明引理4.3的（ii），取 $x_{11}\in R_{11}$。则

$$\begin{aligned}&d(x_{11})\alpha(a_{11}+a_{21})+g(x_{11})f(a_{11}+a_{21})=f(x_{11}(a_{11}+a_{21}))\\&=f(x_{11}a_{11})\\&=f(x_{11}a_{11})+f(x_{11}a_{21})\\&=d(x_{11})\alpha(a_{11})+g(x_{11})f(a_{11})+d(x_{11})\alpha(a_{21})+g(x_{11})f(a_{21}).\end{aligned}\tag{4.6}$$

比较等式（4.6）两边,

$$g(x_{11})[f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]=0 \text{ which implies that,}$$

$$\begin{aligned}&g(x_{11})([f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{11}\\&\quad+[f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{21})=0\\&\text{and } g(x_{11})([f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{12}\\&\quad+[f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{22})=0.\end{aligned}\tag{4.7}$$

同理，取 $x_{22} \in R_{22}$，可得

$$g(x_{22})[f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]=0 \text{ which implies that,}$$

$$\begin{aligned} & g(x_{22})([f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{11} \\ & +[f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{21})=0 \\ \text{and } & g(x_{22})([f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{12} \\ & +[f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{22})=0. \end{aligned} \tag{4.8}$$

利用假设2.5 (ii), (4.7) 和 (4.8),

$$\begin{aligned} & [f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{1k}=0 \\ & \& \ [f(a_{11}+a_{21})-f(a_{11})-f(a_{21})]_{2k}=0. \end{aligned}$$

对于 $k=1$, 2. 因此，引理 4.3 的 (ii) 的证明随之得出。类似地，我们可以证明引理 4.3 的 (iii)、(iv) 和 (v)。□

引理 4.4. 设 $a_{ij}, b_{ij}, c_{ij} \in R_{ij}$. 则

$$\begin{aligned} & (i)\ f(a_{12}+b_{12}c_{22})=f(a_{12})+f(b_{12}c_{22}), \\ & (ii)\ f(a_{21}+b_{22}c_{21})=f(a_{21})+f(b_{22}c_{21}). \end{aligned}$$

证明。注意，$a_{12}+b_{12}c_{22}=(e_1+b_{12})(a_{12}+c_{22})$。因此，

$$\begin{aligned} f(a_{12}+b_{12}c_{22}) &= f[(e_1+b_{12})(a_{12}+c_{22})] \\ &= f(e_1+b_{12})g(a_{12}+c_{22})+\alpha(e_1+b_{12})d(a_{12}+c_{22}) \\ &= f(e_1)g(a_{12})+\alpha(e_1)d(a_{12})+f(b_{12})g(a_{12})+\alpha(b_{12})d(a_{12}) \\ &\quad +f(e_1)g(c_{22})+\alpha(e_1)d(c_{22})+f(b_{12})g(c_{22})+\alpha(b_{12})d(c_{22}) \\ &\quad \text{(By the Assumption 2.5 (iii), Lemma 4.3 (i) and (iii))} \\ &= f(e_1a_{12})+f(b_{12}a_{12})+f(e_1c_{22})+f(b_{12}c_{22}) \\ &= f(a_{12})+f(b_{12}c_{22}), \text{ By Lemma 4.2.} \end{aligned}$$

因此，引理 4.4 的 (i) 的证明随之得出。类似地，为证明引理 4.4 的 (ii)，我们使用恒等式 $a_{21}+b_{22}c_{21}=(a_{21}+b_{22})(e_1+c_{21})$。然后利用假设 2.5(iv)、引理 4.3 (ii) 和 (iv)，即可得到所需结果。□

引理 4.5. 设 $a_{ij}, b_{ij} \in R_{ij}$。则

$$\begin{aligned} & (i)\ f(a_{12}+b_{12})=f(a_{12})+f(b_{12}), \\ & (ii)\ f(a_{21}+b_{21})=f(a_{21})+f(b_{21}). \end{aligned}$$

证明。令 $x_{22} \in R_{22}$。然后

$$\begin{aligned} & f(a_{12}+b_{12})g(x_{22})+\alpha(a_{12}+b_{12})d(x_{22}) \\ & =f((a_{12}+b_{12})x_{22}) \\ & =f(a_{12}x_{22}+b_{12}x_{22}) \\ & =f(a_{12}x_{22})+f(b_{12}x_{22}) \text{ (By Lemma 4.4 (i))} \\ & =f(a_{12})g(x_{22})+\alpha(a_{12})d(x_{22})+f(b_{12})g(x_{22})+\alpha(b_{12})d(x_{22}) \end{aligned} \tag{4.9}$$

比较两边的(4.9),可得

$$[f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]g(x_{22})=0 \text{ which implies that,}$$

$$\begin{aligned}&([f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{11}\\&+[f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{12})g(x_{22})=0\\\text{and }&([f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{21}\\&+[f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{22})g(x_{22})=0.\end{aligned}\tag{4.10}$$

设 $x_{11}\in R_{11}$。则

$$\begin{aligned}&f(a_{12}+b_{12})g(x_{11})+\alpha(a_{12}+b_{12})d(x_{11})\\&=f((a_{12}+b_{12})x_{11})\\&=f(0)=0 \text{ (By Lemma 4.2)}\\&=f(a_{12}x_{11})+f(b_{12}x_{11})\\&=f(a_{12})g(x_{11})+\alpha(a_{12})d(x_{11})+f(b_{12})g(x_{11})+\alpha(b_{12})d(x_{11}).\end{aligned}\tag{4.11}$$

比较等式两边（4.11），

$$[f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]g(x_{11})=0 \text{ which implies that,}$$

$$\begin{aligned}&([f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{11}\\&+[f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{12})g(x_{11})=0\\\text{and }&([f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{21}+\\&[f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{22})g(x_{11})=0.\end{aligned}\tag{4.12}$$

根据假设2.5 (i), (4.10) 和 (4.12),

$$[f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{k1}=0$$
$$\&\quad [f(a_{12}+b_{12})-f(a_{12})-f(b_{12})]_{k2}=0.$$

对于 $k=1$, 2. 因此，我们有

$$f(a_{12}+b_{12})=f(a_{12})+f(b_{12}).$$

类似地，我们可以证明

$$f(a_{21}+b_{21})=f(a_{21})+f(b_{21}).$$

□

引理 4.6. 设 a_{ii}、$b_{ii}\in R_{ii}$。则

$$(i)\ f(a_{11}+b_{11})=f(a_{11})+f(b_{11}),$$
$$(ii)\ f(a_{22}+b_{22})=f(a_{22})+f(b_{22}).$$

证明。令 $x_{12}\in R_{12}$。然后

$$\begin{aligned}&f(a_{11}+b_{11})g(x_{12})+\alpha(a_{11}+b_{11})d(x_{12})\\&=f((a_{11}+b_{11})x_{12})\\&=f(a_{11}x_{12}+b_{11}x_{12})\\&=f(a_{11}x_{12})+f(b_{11}x_{12}) \text{ (By Lemma 4.5 (i))}\\&=f(a_{11})g(x_{12})+\alpha(a_{11})d(x_{12})+f(b_{11})g(x_{12})+\alpha(b_{11})d(x_{12}).\end{aligned}\tag{4.13}$$

比较等式两边（4.13），

$$[f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]g(x_{12})=0 \text{ which implies that,}$$

$$\begin{aligned}&([f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{11}\\&+[f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{12})g(x_{12})=0\\\text{and }&([f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{21}\\&+[f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{22})g(x_{12})=0.\end{aligned}\tag{4.14}$$

同样，取 $x_{22}\in R_{22}$，我们有

$$[f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]g(x_{22})=0 \text{ which implies that,}$$

$$\begin{aligned}&([f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{11}\\&+[f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{12})g(x_{22})=0\\\text{and }&([f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{21}\\&+[f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{22})g(x_{22})=0.\end{aligned}\tag{4.15}$$

根据假设2.5 (i), (4.14) 和 (4.15), 我们有

$$\begin{aligned}[f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{k1}=0,\\[f(a_{11}+b_{11})-f(a_{11})-f(b_{11})]_{k2}=0.\end{aligned}$$

对于 $k=1$，2. 因此，我们有

$$f(a_{11}+b_{11})=f(a_{11})+f(b_{11}).$$

同理，我们可以证明:

$$f(a_{22}+b_{22})=f(a_{22})+f(b_{22}).$$

□

引理 4.7. 令 $a_{ij}\in R_{ij}$。那么

$$f(a_{11}+a_{12}+a_{21}+a_{22})=f(a_{11})+f(a_{12})+f(a_{21})+f(a_{22}).$$

证明。令 $x_{11}\in R_{11}$。于是

$$\begin{aligned}&f(a_{11}+a_{12}+a_{21}+a_{22})g(x_{11})+\alpha(a_{11}+a_{12}+a_{21}+a_{22})d(x_{11})\\&=f((a_{11}+a_{12}+a_{21}+a_{22})x_{11})\\&=f(a_{11}x_{11}+a_{21}x_{11})\\&=f(a_{11}x_{11})+f(a_{12}x_{11})+f(a_{21}x_{11})+f(a_{22}x_{11})\\&=f(a_{11})g(x_{11})+\alpha(a_{11})d(x_{11})+f(a_{12})g(x_{11})+\alpha(a_{12})d(x_{11})\\&+f(a_{21})g(x_{11})+\alpha(a_{21})d(x_{11})+f(a_{22})g(x_{11})+\alpha(a_{22})d(x_{11}).\end{aligned}\tag{4.16}$$

比较等式两边（4.16），

$$\begin{aligned}&[f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})\\&-f(a_{21})-f(a_{22})]g(x_{11})=0 \text{ which implies that,}\end{aligned}$$

$$\begin{aligned}&([f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})-f(a_{21})-f(a_{22})]_{11}\\&+[f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})\\&-f(a_{21})-f(a_{22})]_{12})g(x_{11})=0\end{aligned}\tag{4.17}$$

$$\begin{aligned}\text{and }&([f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})-f(a_{21})-f(a_{22})]_{21}\\&+[f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})\\&-f(a_{21})-f(a_{22})]_{22})g(x_{11})=0.\end{aligned}$$

同理，取 $x_{22}\in R_{22}$，可得

$$\begin{aligned}&[f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})\\&-f(a_{21})-f(a_{22})]g(x_{22})=0 \text{ which implies that,}\end{aligned}$$

$$\begin{aligned}&([f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})-f(a_{21})-f(a_{22})]_{11}\\&+[f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})\\&-f(a_{21})-f(a_{22})]_{12})g(x_{22})=0\end{aligned}\tag{4.18}$$

$$\begin{aligned}\text{and }&([f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})-f(a_{21})-f(a_{22})]_{21}\\&+[f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})\\&-f(a_{21})-f(a_{22})]_{22})g(x_{22})=0.\end{aligned}$$

根据假设2.5 (i), (4.17) and (4.18)，我们得到

$$\begin{aligned}&[f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})-f(a_{21})-f(a_{22})]_{k1}=0,\\&[f(a_{11}+a_{12}+a_{21}+a_{22})-f(a_{11})-f(a_{12})-f(a_{21})-f(a_{22})]_{k2}=0,\end{aligned}$$

对于 $k=1$，2。因此，我们有

$$f(a_{11}+a_{12}+a_{21}+a_{22})=f(a_{11})+f(a_{12})+f(a_{21})+f(a_{22}).$$

□

定理的证明 *4.1*。 设 a， $b\in R$。则

$$\begin{aligned}a&=a_{11}+a_{12}+a_{21}+a_{22},\\b&=b_{11}+b_{12}+b_{21}+b_{22},\end{aligned}$$

对于某些 a_{ij}、$b_{ij} \in R_{ij}$。现在，

$$\begin{aligned} f(a+b) &= f(a_{11}+a_{12}+a_{21}+a_{22}+b_{11}+b_{12}+b_{21}+b_{22}) \\ &= f((a_{11}+b_{11})+(a_{12}+b_{12})+(a_{21}+b_{21})+(a_{22}+b_{22})) \\ &= f(a_{11}+b_{11})+f(a_{12}+b_{12})+f(a_{21}+b_{21})+f(a_{22}+b_{22}) \\ &\quad \text{(By Lemma 4.7)} \\ &= f(a_{11})+f(b_{11})+f(a_{12})+f(b_{12})+f(a_{21})+f(b_{21}) \\ &\quad + f(a_{22})+f(b_{22}) \text{ (By Lemma 4.5 and 4.6)} \\ &= f(a_{11}+a_{12}+a_{21}+a_{22})+f(b_{11}+b_{12}+b_{21}+b_{22}) \\ &\quad \text{(By Lemma 4.7)} \\ &= f(a)+f(b). \end{aligned}$$

因此，f 在 R上是可加的。此外，f 是R上的广义斜半导子。□

推论4.8. 设 R 是一个含有非平凡幂等元 e 的环，f 是 R 上的乘法广义斜半导子，g 是 R 上的映射，满足 $g(0) = 0$，并且 f、g 对于 i、j、$k = 1$、2 满足以下条件：

(i) 若对于所有 $x_{ij} \in R_{ij}$，$(a_{k1} + b_{k2})g(x_{ij}) = 0$ 且 $i \leq j$，则

$$\begin{cases} a_{k1} = 0 & if\ i = 1, \\ b_{k2} = 0 & if\ i = 2. \end{cases}$$

(ii)如果 $g(x_{ii})(a_{1j} + b_{2j}) = 0$ 对所有 $x_{ii} \in R_{ii}$，那么

$$\begin{cases} a_{1j} = 0 & if\ i = 1, \\ b_{2j} = 0 & if\ i = 2. \end{cases}$$

(iii)如果 $f(x_{ii})(a_{1j} + b_{2j}) = 0$ 对所有 $x_{ii} \in R_{ii}$，则

$$\begin{cases} a_{1j} = 0 & if\ i = 1, \\ b_{2j} = 0 & if\ i = 2. \end{cases}$$

然后 f 是可加的。

推论4.9.设 R 是一个带单位元 $1 \neq 0$ 且具有非平凡幂等元 e的素环。若 f 是 R 上的乘法广义斜半导子，其中伴随映射 g 是 R 上的自同态，且 $g(e) = e$，则 f 是可加的。

推论 4.10. 设 R 是推论 *4.9*中的环。如果 f 是定义在 R上的乘法广义斜半导子，其伴随映射 g 在R_{ij}上不变，即 $g(R_{ij}) \subseteq R_{ij}$ 其中 $i, j = 1, 2$ 且满足假设(iii) 和 (iv) 2.5中的条件，则 f 是可加的。

定理 4.11. 设 R 是一个含有非平凡幂等元 e的环，g 是 R上的一个映射，在零处取值为零，并且对 i、$j = 1$、2 满足以下条件：

(i) 若对于所有 $x_{ii} \in R_{ii}$, $(a_{j1} + b_{j2})g(x_{ii}) = 0$，则 $a_{j1} + b_{j2} = 0$。

(ii) If $g(x_{ii})(a_{11} + a_{21}) = 0$ *for all* $x_{ii} \in R_{ii}$, *then* $a_{11} + a_{21} = 0$.

如果 f 是 R上的乘法广义斜半导子，那么 f 是可加的。

推论 4.12. 设 R 是含有非平凡幂等元 e的环， f 是 R上的乘法广义斜半导子， g 是 R 上的映射，满足$g(0) = 0$，并且满足以下条件中关于 $i = 1, 2$ 的三个条件:

(i) 若 $(a_{11} + b_{12})g(x_{ii}) = 0$ 对所有 $x_{ii} \in R_{ii}$成立，则 $a_{11} + b_{12} = 0$;
(ii) 若 $(a_{21} + b_{22})g(x_{ii}) = 0$ 对所有 $x_{ii} \in R_{ii}$成立，则 $a_{21} + b_{22} = 0$,

(iii) 若 $g(x_{ii})(a_{11} + b_{21}) = 0$ 对所有 $x_{ii} \in R_{11}$成立，则 $a_{11} + b_{21} = 0$,

(iv)如果 $g(x_{ii})(a_{12} + b_{22}) = 0$ 对所有 $x_{ii} \in R_{ii}$，则 $a_{12} + b_{22} = 0$。

则 f 是可加的。

Acknowledgement

第一作者感谢印度政府大学拨款委员会（UGC）通过档案编号 16-9(June 2019)/2019(NET/CSIR) 和 UGC 参考编号 1256/(CSIR-UGC NET JUNE 2019)（日期：16/12/2019）提供的经费支持，所有作者均感谢印度理工学院巴特那分校提供的研究设施。

Declarations

数据可用性声明：作者声明， [本研究结果所依据的所有其他]数据均可在本文中获得。

利益冲突：作者声明，就本文的发表而言，不存在利益冲突。

References

[1] Sk. Aziz、A. Ghosh 和 O. Prakash，三角环上的某些广义 Jordan 映射迫使其具有可加性，arXiv:2301.08093v1。[2] Sk. Aziz、A. Ghosh 和 O. Prakash，环上乘法（广义）映射的可加性，arXiv:2302.14571v1。[3] J. Bergen，素环上的导子， *Canad. Math. Bull.*，26(3)（1983年），267-270。[4] M. Bre˘sar，素环上的半导子， *Proc. Amer. Math. Soc.* **108(4)**（1990）859–860。[5] J.-C. Chang，论素环上的半导子， *Chin. J. Math.* **12(4)**（1984）255–262。[6] M.-C. Chou 和 C.-K. Liu，带斜导子的 Engel 条件， *Monatsh. Math.* **158(3)**（2009）259–270。[7] C.-L. Chuang，带斜导子的恒等式， *J. Algebra* **224(2)**（2000）292–335。[8] M. N. Daif，乘法导子何时是可加的？， *Int. J. Math. Math. Sci.* **14(3)**（1991）615–618。[9] B. L. M. Ferreira，三角 n 矩阵环上的乘法映射，

Int. J. Math. Game ( ) ( )

Theory Algebr. **23** 2 2014 1 14
–。[10] N. Jacobson，抽象导子与李代数， *Trans. Amer. Math. Soc.* **42(2)**（1937 年） 206–224。

[11] R. E. Johnson，具有唯一加法的环， *Proc. Amer. Math. Soc.* **9(1)**（1958）57–61。[12] R. V. Kadison 和 J. R. Ringrose，算子代数的导子与自同构，*Comm. Math. Phys.* **4(1)**（1967）32–63。[13] V. K. Kharchenko 和 A. Z. Popov，素环上的斜导子， *Comm. Algebra* **20(11)**（1992）3321-3345。[14] A. Leroy，S-代数导子的对数导数， *Comm. Algebra* **13(1)**（1985年）85–99。[15] W. S. Martindale III，乘法映射何时是可加的？，

Proc. Amer. Math. Soc. **( ) ( )** **21** 3 1969 695 698

– 。[16] C. E. Posner，素环上的导子， *Proc. Amer. Math. Soc.* **8(6)**（ 1957 ） 1093–1100。[17] C. E. Rickart，环与格的一一映射， *Bull. Amer. Math. Soc.* **54(8)**（1948）758–764。[18] M. S. T. E. -Sayiad、M. N. Daif 和 V. D. Filippis，左中心化子乘法性迫使其具有可加性，

Bol. Soc. Parana. Mat. **32(1)**（2014）61–69。[19] M. A. Siddeeque 和 N. Khan，乘法半导子何时是可加的？，

Georgian Math. J. **29** 1 2022 123 137 **( ) ( )**

—。[20] J. Vukman，半素环上的导子， *Bull. Aust. Math. Soc.* **53(3)**（ 1996 ） 353–359。[21] Y. Wang，环上乘法映射的可加性， *Comm. Algebra* **37(7)**（2009）2351– 2356。[22] Y. Wang，三角环上乘法映射的可加性， *Linear Algebra Appl.* **434(3)**（2011）625–635。[23] M. Weisfeld，除环上的导子， *Pacific J. Math.* **10(1)**（1960）335–343。[24] V. K. Yadav 和 R. K. Sharma，环上乘法广义 Jordan 导子的可加性，Palestine. J. Math. 6(II)（2017），253-258。

Department of Mathematics、 Indian Institute of Technology Patna、 Patna-801106 *Email address*: E-mail: aziz_2021ma22@iitp.ac.inDepartment of Mathematics、 Government Polytechnic Kishanganj、 Kishanganj-855116*Email address*: E-mail: arindam.rkmrc@gmail.comDepartment of Mathematics、 Indian Institute of Technology Patna、 Patna-801106*Email address*: om@iitp.ac.in