基于双重/去偏机器学习，高效估计加权累积治疗效应

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摘要：在事件发生时间结局的实证研究中，当无法获得随机对照试验数据时，研究人员通常会利用观察性数据，就暴露的效应开展因果推断。模型设定错误和缺乏重叠性是观察性研究中的常见问题，往往会导致平均治疗效应的估计量不一致且效率低下。为应对重叠较差这一挑战，已有研究提出以重叠加权效应为目标的估计量；而允许对干扰模型灵活使用机器学习的方法，则可应对模型设定错误。然而，在重叠较差的情况下，允许对干扰模型使用机器学习的方法尚未扩展至事件发生时间结局的加权平均治疗效应研究。本文提出一类单步交叉拟合双重/去偏机器学习估计量，用于估计作为限制时间函数的加权累积因果效应。我们证明，在满足正则性条件时，所提估计量是一致且渐近线性的，并达到半参数效率界。我们的模拟结果表明，使用非参数机器学习干扰模型的所提估计量，表现不逊于需要正确设定参数模型的成熟方法。

干扰模型，说明我们的估计量降低了对预言机参数干扰模型的需求。我们将提出的方法应用于来自英国初级保健数据库的真实世界观察性数据，比较抗糖尿病药物对癌症临床结局的影响。

关键词：累积治疗效应；数据自适应估计；双重/去偏机器学习；影响函数；有限重叠；限制平均生存时间；限制平均损失时间；加权平均治疗效应。

1 引言

从观察性数据估计平均处理效应（ATE）时，有两个众所周知的挑战——即便一致性、正值性和可交换性这些标准因果假设均成立：治疗组之间重叠较差，以及所谓的干扰模型设定错误。当在某些需要在分析中加以调整的协变量取值下，某一种（二元）治疗分配远比另一种更有可能发生时，就会出现重叠较差的情况。如果某些协变量取值下完全不可能出现某种治疗取值，那么正值性假设就被违反，效应估计值可能存在偏倚。但即使正值性没有被违反、只是重叠较差，平均处理效应估计量的统计性质也可能不理想。为解决重叠较差的问题，有研究提出通过“移动目标”（Crump 等人，2006年），将目标转向临床均衡人群，并以加权平均治疗效应（WATE）为目标；在这种估计目标中，倾向评分接近零或一的受试者权重较低。这一思路类似于强化学习文献中的重要性抽样（Hanna 等人，2019年），该方法由 Kloek 和 Van Dijk（1978年）引入统计学领域。一些作者甚至认为，与平均处理效应相比，这些替代估计目标在政策制定方面更具相关性（且估计量更稳定）。本文将考察 WATE 估计量。

对于存在删失的事件发生时间结局，人们定义了多种用于量化暴露效应的测度，例如需治疗人数、生存函数、中位生存时间和限制平均生存时间（RMST）（Kloecker 等人，2020）。在本文中，我们关注 RMST，即在感兴趣的随访时间 τ（即“限制时间”）内的预期无事件生存时间。这一指标具有良好特性：既能用单一数值概括生存结局，又保留了具有因果和临床意义的解释。既往研究提出了针对生存结局（包括 RMST）的 WATE 估计量。Cheng 等人（2022）提出了用于加权生存函数的逆删失概率加权（IPCW）估计量；Zeng 等人（2021）则研究了 WATE 的删一法生存函数、RMST 和平均生存时间。然而，这些方法都依赖于设定正确的参数干扰模型。

在其他情境中，人们已开发出多种方法，以解决结局、治疗及其参数干扰回归模型可能设定错误的问题，并且

计算因果效应估计量所需的删失过程。非参数机器学习模型本质上设定正确。然而，将机器学习估计的干扰参数直接代入标准效应估计量，可能会导致插件偏倚，并因收敛速率缓慢而产生不正确的置信区间。基于半参数理论，人们提出了基于有效影响函数的估计量，例如去偏机器学习（DML）和靶向学习估计量，以确保 $n^{1/2}$ 一致性和渐近正态性，前提是结局模型和治疗模型均一致（例如，二者都采用非参数机器学习进行估计），且其乘积 收敛速率至少达到 $n^{1/2}$ （Chernozhukov 等，2018；Van Der Laan 和 Rubin，2006）。这一性质称为速率双重稳健性；当所有干扰参数都得到一致估计时，可以达到非参数效率界。然而，针对生存分析情境下 WATE 的双重/去偏机器学习尚未得到发展。这正是本文要填补的主要空白。

我们还将估计量扩展到竞争风险情境。当存在多种失效原因，但我们只关注其中一部分时，其他失效原因被称为竞争事件，因为它们可能阻止所关注事件的发生。在此类情境下，研究者通常会考虑边际、原因特异性、亚分布以及复合累积发生函数（CIFs）（Young 等人，2020）。在本文中，我们将原因特异性限制平均损失时间（RMTLs）视为一种累积测度，用于概括 CIFs；它可以大致解释为截至限制时间 τ 因所关注原因而损失的预期时间（Andersen，2013）。据我们所知，目前尚无针对加权原因特异性限制平均损失时间或加权累积治疗效应的估计量，后者定义为加权治疗特异性限制平均损失时间之差。

本文的组织结构如下。第2节介绍符号，并正式定义我们所关注的加权限制平均生存时间和限制平均损失时间估计目标。第3节根据这些估计目标的影响函数，推导一类交叉拟合去偏机器学习估计量。第4节推导所提估计量的渐近线性，并证明所提估计量达到了半参数效率界。我们还提供了构造逐点置信区间以及在一系列限制时间范围内构造同时置信带的方法。第5节 प्रस्तुत了一项模拟研究，说明当干扰函数通过机器学习进行估计时，我们的去偏机器学习估计量与使用设定正确的参数干扰模型的估计量表现同样良好。第6节利用来自英国的真实世界初级保健观察性数据，应用所提估计量比较两种一线抗高血糖治疗对2型糖尿病患者癌症临床结局的影响。第7节总结全文。

2 数据与估计目标

2.1 符号说明

我们观测到治疗前混杂因素 $X \subseteq \mathcal{R}^d$、二元暴露 $A \in \{0, 1\}$。令 $T_j^a > 0$ 表示若治疗被设定为 $X \subseteq \mathcal{R}^d$，则可能与事实相反地，原因 j 的潜在失败时间

为 a，若在治疗分配 a 下不会发生原因 j 导致的失败，则为 $T_j^a = \infty$。定义 $T^a = \min_j\{T_j^a\}$ 为从 0 到首次事件 j 的未删失潜在生存时间，并将 $J^a = \text{argmin}_j\{T_j^a\}$ 定义为相应的潜在失败原因。我们将 $T = T^A$ 和 $J = J^A$ 分别定义为在 $A = a$ 情况下的失败时间和失败原因。我们将删失视为时间 C 处的外部粗化过程，将$\widetilde{}$删失失败时间定义为 $\widetilde{T} = \min\{T, C\}$，将事件指示变量定义为 $\Delta = I(T \leq C)$，用于表示$\widetilde{}$失败是否被删失，并将 $\widetilde{J} = J\Delta$ 定义为观测到的失败原因，其中 $\widetilde{J} \in \{0, 1, \ldots, j^*\}$。$\widetilde{}$ $\widetilde{}$观测数据单元的结构为 $O = (X, A, \widetilde{T}, \widetilde{J}) \sim P_0$，其中 P_0 是未知的真实数据生成分布。当只有一种失败类型，即 $j^* = 1$ 时，$\widetilde{}$观测数据变为 $O = (X, A, \widetilde{T}, \Delta)$。我们假设观测到 n 个独立同分布（i.i.d.）的观测值 O_1、.....、O_n，它们以概率 $1/n$ 从 P_0 中抽样，并将 P_n 记为经验概率分布。我们将观测到的$\widetilde{}$ $\widetilde{}$原因特异性计数过程记为 $N_j(t) = I(T \leq t, J = j)$，全因计数过程记为$N(t) = I(\widetilde{T} \leq t)$，删失过程记为 $N^C(t) = I(\widetilde{T} \leq t, \Delta = 0) = (1 - \Delta)N(t)$，并将$\widetilde{}$风险集过程记为 $R(t) = I(T \geq t)$。

2.2 估计目标与识别

在不存在竞争风险的情况下，我们将因果估计目标设定为不同治疗方案下限制平均生存时间（RMST）的加权平均差，即

$$\begin{aligned}\psi_0^{\text{RMST},h}(\tau) =& \psi_0^{\text{RMST},h}(\tau, a = 1) - \psi_0^{\text{RMST},h}(\tau, a = 0) \\ =& \frac{E_0[h_0(X)\{\min(T^{a=1}, \tau) - \min(T^{a=0}, \tau)\}]}{E_0\{h_0(X)\}}\end{aligned}$$

其中，$h_0(X)$ 是预先指定的“倾斜函数”。$h_0(X)$ 通常用于降低倾向评分接近 0 或 1 的受试者的权重（F. Li 等，2018）。如果存在多个竞争性失效原因（$j* > 1$），我们的估计目标是加权平均原因特异性限制平均损失时间（RMTL）

$$\begin{aligned}\psi_{j,0}^{\text{RMTL},h}(\tau) =& \psi_{j,0}^{\text{RMTL},h}(\tau, a = 1) - \psi_{j,0}^{\text{RMTL},h}(\tau, a = 0) \\ =& \frac{E_0[h_0(X)\{(\tau - \min(T^{a=1}, \tau))I(J^{a=1} = j) - (\tau - \min(T^{a=0}, \tau))I(J^{a=0} = j)\}]}{E_0\{h_0(X)\}}\end{aligned}$$

识别因果效应需要满足以下假设，对于 $a \in \{0, 1\}$:

假设 1. *(正值性)* $P_0(A = a \mid X) \geq 1/\epsilon$ 且 $P_0(C \geq t \mid A = a, X) \geq 1/\epsilon$几乎必然地对于 $t \in [0, \tau]$和 $\epsilon \in (1, \infty)$;

假设 2。 (*SUTVA*) $T = T^A$, $T_{(A_1, \ldots, A_n)\, i} = T_{(A'_1, \ldots, A'_n)\, i}$，若 $A_i = A'_i$、$\forall i$;

假设 3。 （条件可忽略性） $T^a \perp\!\!\!\perp A \mid X$;

假设 4。 （条件独立删失） $T^a \perp\!\!\!\perp C \mid A$、$X$。

假设 *5*。（竞争风险 *SUTVA*）$(T^a, J^a) = A(T^{a=1}, J^{a=1}) + (1-A)(T^{a=0}, J^{a=0})$，$(T_{(A_1,\ldots,A_n)}\, J_{(A_1,\ldots,A_n)}) = (T_{(A'_1,\ldots,A'_n)}\, J_{(A'_1,\ldots,A'_n)})$，若 $A_i = A'_i$, $\forall i$;

假设 6。（竞争风险条件可忽略性）$(T^a, J^a) \perp\!\!\!\perp A \mid X$;

假设 7。*(*竞争风险条件独立删失*)* $(T^a, J^a) \perp\!\!\!\perp C \mid A, X$.

在不存在竞争风险时，识别（加权）限制平均生存时间估计目标需要假设 1–4; 在存在竞争风险时，识别（加权）RMTL 估计目标则需要假设 1 和 5–7。假设 1 要求每个个体接受每种治疗并在整个随访期间保持未删失的概率均大于零。假设 2 和 5 称为稳定单位治疗值（SUTVA）假设。这些假设指出，观测到的结局等于与观测到的治疗相对应的潜在结局（一致性），且一个个体的治疗不会影响其他个体的结局（无干扰）。条件可忽略性以及条件独立删失假设 3–4 和 6–7 指出，观测到的基线协变量足以调整混杂。

在假设 1–4 下，我们有

$$\psi_0^{\mathrm{RMST},h}(\tau, a) = \frac{E_0[h_0(X)\{\mathrm{RMST}_0(\tau \mid a, X)\}]}{E_0\{h_0(X)\}}$$

在假设 1 和 5–7 下,

$$\psi_{j,0}^{\mathrm{RMTL},h}(\tau, a) = \frac{E_0[h_0(X)\{\mathrm{RMTL}_{j,0}(\tau \mid a, X)\}]}{E_0\{h_0(X)\}}$$

其中，限制平均生存时间$_0(\tau \mid a, \ X) = E_0\{\min(T, \ \tau) \mid A = a, \ X\}$，且 RMTL$_{j, \ 0}(\tau \mid a, X) = E_0[\{\tau - \min(T, \ \tau)\} I(J = j) \mid A = a, \ X]$。

3 提出的方法

3.1 效率影响函数

本节首先推导估计目标 (1) 和 (3) 的效率影响函数。然后，基于这些效率影响函数构造交叉拟合的双重去偏估计量，并介绍一种将形状约束纳入其中的简单方法。我们给出方差的夹心估计量，并确保方差随限制时间 τ 单调递增。

在继续之前，我们先介绍几个量。将倾向评分定义为$\pi(a \mid X) = P(A = a \mid X)$，平衡权重定义为 $w^h(a, X) = h(X) I(A = a)/\pi(a \mid X)$，$S(t \mid a, X) = P(T > t \mid A = a, X)$，$F_j(t \mid a, X) = P(T \leq t, J = j \mid A = a, X)$，$G(t \mid a, X) = P(C > t \mid A = a, X)$。全因鞅、特定原因鞅和删失鞅可定义为 $\mathrm{d}M(t \mid a, X) = \mathrm{d}N(t) - R(t)\mathrm{d}\Lambda(t \mid a, X)$，$\mathrm{d}M_j(t \mid$

$a, X) = \mathrm{d}N_j(t) - R(t)\mathrm{d}\Lambda_j(t \mid a, X)$，以及 $\mathrm{d}M^C(t \mid a, X) = \mathrm{d}N^C(t) - R(t)\mathrm{d}\Lambda^C(t \mid a, X)$，分别表示。此外，记 $\eta_0^\Lambda = \{\pi_0(a \mid X)\ \Lambda_0^C(t \mid a, X)\ \Lambda_0(t \mid a, X)\}, \eta^\Lambda{}_{,0}{}^h = \{h_0(X)\ \eta_0^\Lambda\}$, $\eta_{j\,0}^\Lambda = \{\pi_0\ a \mid X\ \ \Lambda_0^C\ t \mid a\ X\ \ \Lambda_0\ t \mid a\ X\ \ \Lambda_{j\,0}\ t \mid a\ X\ \}$ $\eta_{j\,0}^{\Lambda\,h} = \{h_0\ X\ \ \eta_{j\,0}^\Lambda\}$ ，()(,)(,),(,),和,,(),。

在定理 1 中，我们给出了 $\psi^{\mathrm{RMST}}{}_{,0}{}^h(\tau, a)$ 以及 $\psi_0^{\mathrm{RMTL}}{}_{,j}{}^h{}_{,}(\tau, a)$ 的效率影响函数。

定理 *1*。假设 $S_0(t \mid a, X) > 1/\epsilon$，$0 \le h_{\min} \le h_0(X) \le h_{\max} \le \infty$，其中 $\epsilon \in (1, \infty)$，并令 $\Delta(\tau) = I(C > \tau \wedge T-)$，则 $\psi^{\mathrm{RMST}}{}_{,0}{}^h(\tau, a)$ 是一个路径可微的量，其倾斜未中心化高效影响函数为 $\phi^{\mathrm{RMST}}{}_{,0}{}^h(\tau, a; \eta^\Lambda{}_{,0}{}^h) = [E_0\{h_0(X)\}]^{-1}\{h_0(X)\phi_0^{\mathrm{RMST}}(\tau, a; \eta_0^\Lambda)\}$，其倾斜中心化高效影响函数为

$$\varphi_0^{\mathrm{RMST},h}(\tau, a; \eta_0^{\Lambda,h}) = [E_0\{h_0(X)\}]^{-1}[h_0(X)\{\phi_0^{\mathrm{RMST}}(\tau, a; \eta_0^\Lambda) - \psi_0^{\mathrm{RMST},h}(\tau, a)\}] \quad (1)$$

其中，未倾斜的非参数未中心化高效影响函数为

$$\begin{aligned}\phi_0^{\mathrm{RMST}}(\tau, a; \eta_0^\Lambda) = {} & \frac{I(A=a)\Delta(\tau)\min(\widetilde{T}, \tau)}{\pi_0(a \mid X)G_0(\tau \wedge \widetilde{T}- \mid a, X)} + \left\{1 - \frac{I(A=a)}{\pi_0(a \mid X)}\right\}\mathrm{RMST}_0(\tau \mid a, X) \\ & + \frac{I(A=a)}{\pi_0(a \mid X)}\int_0^{\tau\wedge\widetilde{T}} \frac{t\mathrm{d}M_0^C(t \mid a, X)}{G_0(t- \mid a, X)} \\ & + \frac{I(A=a)}{\pi_0(a \mid X)}\int_0^{\tau\wedge\widetilde{T}} \frac{\{\mathrm{RMST}_0(\tau \mid a, X) - \mathrm{RMST}_0(t \mid a, X)\}\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)}\end{aligned} \quad (2)$$

$\tau\ a$ $_0$ $\tau\ a$ $_0$ $= \psi_0^{\mathrm{RMTL}}{}_{,j}{}^h{}_{,}([E_0\{h_0\ X\ \}]^{-1}[h_0\ X\ \phi_{j\,0}^{\mathrm{RMTL}}\ \tau\ a\ \eta_{j\,0}^\Lambda\ ]$,) 是一个路径可微的量，其倾斜未中心化高效影响函数为 $\phi_j^{\mathrm{RMTL}\,h}{}_{,}{}_{,}(,;\ \eta^\Lambda{}_{,j}{}^h{}_{,})()().(,;,,)$，其倾斜中心化高效影响函数为

$$\varphi_{j,0}^{\mathrm{RMTL},h}(\tau, a; \eta_{j,0}^{\Lambda,h}) = [E_0\{h_0(X)\}]^{-1}[h_0(X)\{\phi_{j,0}^{\mathrm{RMTL}}(\tau, a; \eta_{j,0}^\Lambda) - \psi_{j,0}^{\mathrm{RMTL},h}(\tau, a)\}] \quad (3)$$

其中，未倾斜的非参数未中心化高效影响函数为

$$\begin{aligned}\phi_{j,0}^{\mathrm{RMTL}}(\tau, a; \eta_{j,0}^\Lambda) = {} & \frac{I(A=a)(\tau - \min(\widetilde{T}, \tau))I(\widetilde{J}=j)}{\pi_0(a \mid X)G_0(\widetilde{T}- \mid a, X)} + \left\{1 - \frac{I(A=a)}{\pi_0(a \mid X)}\right\}\mathrm{RMTL}_{j,0}(\tau \mid a, X) \\ & - \frac{I(A=a)}{\pi_0(a \mid X)}\int_0^{\tau\wedge\widetilde{T}} \frac{(\tau - t)F_{j,0}(t \mid a, X)\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} \\ & + \frac{I(A=a)}{\pi_0(a \mid X)}\int_0^{\tau\wedge\widetilde{T}} \frac{\{\mathrm{RMTL}_{j,0}(\tau \mid a, X) - \mathrm{RMTL}_{j,0}(t \mid a, X)\}\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)}\end{aligned} \quad (4)$$

式（2）和（4）中的第一项是逆概率加权完整病例项，表示接受目标治疗且在限制时间之前观察到目标事件的个体，$A = a$ （~） $\Delta\tau$或 $= 1$ $J = j$。第一项的经验均值通常称为逆概率删失加权估计量（Robins 和 Rotnitzky，1992）。由于逆概率删失加权估计量仅考虑完整病例，因此这些估计量的效率会有所损失；当删失程度较高和/或治疗分配严重不平衡时，这一损失会很显著。为提高效率，式（2）和（4）中的第二项类似于连续结局双重稳健估计量中的增广

项（Robins 等人，1994），通过结局的条件期望捕捉所有参与者的信息，无论其属于哪个暴露组或删失状态如何。其余的勒贝格–斯蒂尔切斯删失鞅积分项则进一步恢复来自目标暴露组中删失观测的信息 $\Delta = 0 \;\; A = a$。

3.2 去偏机器学习估计量

我们用四个步骤简要描述估计程序：(i) 将大小为 n 的数据划分为 $K = 2$ 个互不相交集合 $\mathcal{V}_1$ 和 $\mathcal{V}_2$；(ii) 构建 $\widehat{\eta}^{\Lambda,h}_{1}$ 或 $\widehat{\eta}^{\Lambda,h}_{j1}$，在 $\mathcal{V}_2$ 上训练并在 $\mathcal{V}_1$ 上估计。交换训练集和估计集，得到 $\widehat{\eta}^{\Lambda,h}_{2}$ 或 $\widehat{\eta}^{h}_{j,2}$；(iii) 对于 $k = 1, 2$，通过代入 $\widehat{\eta}^{\Lambda,h}_{k}$ 或 $\widehat{\eta}^{\Lambda,h}_{jk}$，，计算 $\widehat{\psi}^{\mathrm{RMST},h}_{k}(\tau, a) = P_{n,k}\{\widehat{\phi}^{\mathrm{RMST},h}_{k}(\tau, a; \widehat{\eta}^{\Lambda,h}_{k})\}$ 或 $\widehat{\psi}^{\mathrm{RMTL},h}_{jk}(\tau, a) = P_{n,k}\{\widehat{\phi}^{\mathrm{RMTL},h}_{j,k}(\tau, a; \widehat{\eta}^{\Lambda,h}_{jk},)\}$，其中 $P_{n,k}$ 是子集 $\mathcal{V}_k$ 的经验概率分布，而 $\widehat{\phi}^{\mathrm{RMST},h}_{k}(\tau, a; \widehat{\eta}^{\Lambda,h}_{k})$ 和 $\widehat{\phi}^{\mathrm{RMTL},h}_{jk}(\tau, a; \widehat{\eta}^{\Lambda,h}_{jk},)$ 关于 $\psi^{\mathrm{RMST},h}_{0}(\tau, a)$ 和 $\psi^{\mathrm{RMST},h}_{0j}(\tau, a)$ 均为线性（Kennedy，2022）。

$= 1/2\{\widehat{\psi}^{\mathrm{RMST},h}_{k=1}(\tau, a) + \widehat{\psi}^{\mathrm{RMST},h}_{k=2}(\tau, a)\}$ 或 $\widehat{\psi}^{\mathrm{RMTL},h}_{j}(\tau, a) = 1/2\{\widehat{\psi}^{\mathrm{RMTL},h}_{j,k=1}(\tau, a) + \widehat{\psi}^{\mathrm{RMTL},h}_{jk=2}(\tau, a)\}$，然后校正形状约束。所提出的交叉拟合去偏机器学习估计量的具体实现程序详见补充材料。

为减少式（5）和（6）中的抽样变异，分子中使用的平衡权重取平均后作为分母，以控制数值不稳定性（哈耶克，1971）。然而，对于式（5）中的逆概率删失加权项，我们不使用 $P_{n,k}[\{\Delta(\tau)/\widehat{G}_k(\tau \wedge \widetilde{T} - \mid a, X)\}]^{-1}$，或式（6）中的 $P_{n,k}[\{\Delta/\widehat{G}_k(\widetilde{T} - \mid a, X)\}]^{-1}$ 进一步控制数值不稳定性。注意，一步估计量与估计方程估计量是等价的，因为 $\varphi^{\mathrm{RMST},h}_{0}(\tau, a; \eta^{\Lambda,h}_{0})$ 和 $\varphi^{\mathrm{RMTL},h}_{0j}(\tau, a; \eta^{\Lambda,h}_{j0},)$ 关于 $\psi^{\mathrm{RMST},h}_{0}(\tau, a)$ 和 $\psi^{\mathrm{RMST},h}_{0j}(\tau, a)$ 均为线性（Kennedy，2022）。

注 *1*。事件发生时间的干扰函数只能通过条件风险率 $(d\widehat{\Lambda}^{C}_{k}(t \mid a, X))$、$(d\widehat{\Lambda}_{k}(t \mid a, X))$、$(d\widehat{\Lambda}_{j,k}(t \mid a, X))$ 获得，其中 $(\widehat{S}_{k}(t \mid a, X) = \exp\{-\widehat{\Lambda}_{k}(t \mid a, X)\})$、$(\widehat{RMST}_{k}(\tau \mid a, X) = \int_0^\tau \widehat{S}_{k}(t \mid a, X)dt$，$\widehat{F}_{j,k}(t \mid a, X) = \int_0^t \widehat{S}_{k}(u \mid a, X)d\widehat{\Lambda}_{j,k}(u \mid a, X))$、$(\widehat{RMTL}_{j,k}(\tau \mid a, X) = \int_0^\tau \widehat{F}_{j,k}(t \mid a, X)dt)$，以及 $(\widehat{G}_{k}(t \mid a, X) = \exp\{-\widehat{\Lambda}^{C}_{k}(t \mid a, X)\})$。直接对计数过程进行建模通常会导致条件事件发生时间分布的估计不一致。详细内容见补充材料。

注释 *2*. $\psi^{\mathrm{RMST},h}_{0}(\tau, a)$ 和 $\psi^{\mathrm{RMTL},h}_{0j}(\tau, a)$ 相对于 τ 自然分别为凹函数和凸函数，因为它们的导数分别是单调非递增的生存函数和单调非递减的原因特异性累积发生函数。除点估计的形状约束外，渐近标准误 $\sigma^{\mathrm{RMST},h}_{0}(\tau, a)$、$\sigma^{\mathrm{RMST},h}_{0}(\tau)$、$\sigma^{\mathrm{RMTL},h}_{j0}(\tau, a)$ 以及 $\sigma^{\mathrm{RMTL},h}_{j0}(\tau)$ 严格单调非递增，因为相对于随访时间，删失总是单调粗化的。

然而，由于在每个时间点分别进行估计，且没有施加任何全局约束，因此无法保证所提估计量在任何有限样本中都满足形状约束。本文采用 Grenander 型估计量（Westling 和 Carone，2020）确保点估计具有凸性/凹性，并确保渐近方差具有单调性。

采用投影型估计量（Daouia 和 Park，2013）。由于形状校正后的估计量在所有有限样本中均保证不会比不受约束的估计量偏离真实值更远，且两种估计量在渐近意义下等价，因此我们重点关注未经校正的估计量的大样本性质。

注释 3。 请注意，所提估计量可以轻松扩展到多种治疗 $A \in \{0, 1, 2, \ldots\}$。只需对 $\widehat{\pi}_k(a \mid X)$ 进行多类别分类器的交叉拟合，并拟合风险模型，即可获得 $\widehat{\Lambda}_k(t \mid a, X)$、$\widehat{\Lambda}_k^C(t \mid a, X)$，以及可能的 $\widehat{\Lambda}_{j,k}(t \mid a, X)$，适用于每一种治疗 a。

注释 4. 所提出的方法适用于连续型、离散型和混合型生存时间。只需将连续时间风险函数和鞅积分替换为相应的离散型或混合型生存时间等价形式即可。

4 渐近性质

4.1 一致性与渐近线性

我们在此考察所提估计量的大样本性质。首先给出点态一致性和一致一致性的条件。

定理 2。 *(一致性)*：在补充材料中给出的正则性条件下，所提估计量 $\widehat{\psi}^{\mathrm{RMST},h}(\tau, a)$ 和 $\widehat{\psi}^{\mathrm{RMTL},h}_{\cdot,j}(\tau, a)$ 分别依概率收敛于 $\psi^{\mathrm{RMST},h}(\tau, a)$ 和 $\psi^{\mathrm{RMTL},h}_{\cdot,j}(\tau, a)$。在稍强的正则性条件下（同样见补充材料），这种收敛是一致的。

以下定理表明，所提估计量在逐点意义下和一致意义下均具有渐近线性，从而能够构造逐点置信区间和曲线置信区间。

定理 3。 （渐近线性）在补充材料中给出的正则性条件下，所提估计量 $\widehat{\psi}^{\mathrm{RMST},h}(\tau, a)$ 和 $\widehat{\psi}^{\mathrm{RMTL},h}_{\cdot,j}(\tau, a)$ 对于给定的 τ 具有渐近线性。在补充材料中给出的稍强正则性条件下，渐近线性关于 τ 一致成立。

（效率界）此外，在 $\mathrm{var}\{\phi_0^{\mathrm{RMST}}(\tau, 1, \eta_0^{\Lambda,h}) \mid X\} = \mathrm{var}\{\phi_0^{\mathrm{RMST}}(\tau, 0, \eta_0^{\Lambda,h}) \mid X\}$ 和 $\mathrm{var}\{\phi_{j,0}^{\mathrm{RMTL}}(\tau, 1, \eta_{j\,0}^{\Lambda,h}) \mid X\} = \mathrm{var}\{\phi_{j,0}^{\mathrm{RMTL}}(\tau, 0, \eta_{j\,0}^{\Lambda,h}) \mid X\}$ 下，当 $h_0(X) \propto \mathrm{var}_0(a \mid X)$ 时，在平衡逆概率处理加权估计量类中，方差达到最小。

定理 3 要求式 (5) 和 (6) 中的干扰函数以 $n^{-1/2}$ 速率收敛。当并非所有干扰函数都得到一致估计时，尽管由于双重稳健性，点估计仍是一致的，但所得方差估计量会产生方向和幅度均未知的偏倚。在这种情况下，自助法程序或许能够恢复适当的覆盖率，但代价是显著增加计算时间（Bai 等人，2013）。

4.2 渐近推断

我们可以基于定理 3 进行渐近推断。显著性水平为 α 的逐点置信区间（置信区间）定义为

$$P_0\left(\widehat{l}^{\mathrm{RMST},h}(\tau,a) \le \widehat{\psi}^{\mathrm{RMST},h,lcm}(\tau,a) \le \widehat{u}^{\mathrm{RMST},h}(\tau,a)\right) = 1-\alpha$$

这里，$\widehat{l}^{\mathrm{RMST},h}(\tau, a)$ 和 $\widehat{u}^{\mathrm{RMST},h}(\tau, a)$ 是两个随机过程，而 z_q 是标准正态分布的 q 分位数。Wald 型对称置信区间可构造为 $\widehat{\psi}^{\mathrm{RMST},\underline{h},lcm}(\tau, a) \pm z_{1-\alpha/2}\widehat{\sigma}^{\mathrm{RMST},h,+}(\tau, a)/\sqrt{n}$。与仅描述单个时间点估计曲线的置信区间不同，同时置信带涵盖用户指定随访时间内的整个估计曲线，可用于评估比较效果研究中的等效性或非劣效性主张（Zhao 等人，2016）。

在条件 4 和 11 下，可在区间 $[\tau_l, \tau_u]$ 上构造覆盖水平为 $(1-\alpha)$ 的同时置信带，使得

$$P_0\left(\widehat{l}^{\mathrm{RMST},h}(\tau,a) \le \widehat{\psi}^{\mathrm{RMST},h,lcm}(\tau,a) \le \widehat{u}^{\mathrm{RMST},h}(\tau,a) \quad \forall \tau \in [\tau_l,\tau_u]\right) \ge 1-\alpha$$

关于生存函数各种类型同时置信带的构建，我们参见 Chen 等人（2021）。这里，我们考虑递增宽度置信带 $\widehat{\psi}^{\mathrm{RMST},h,lcm}(\tau, \underline{a}) \pm \widehat{c}_\alpha\widehat{\sigma}^{\mathrm{RMST},h,+}(\tau, a)/\sqrt{n}$，其中 $\widehat{c}_\alpha$ 是零均值高斯过程样本路径绝对值上确界的 $(1-\alpha)$ 分位数，该过程定义在$[\tau_l, \tau_u]$ 上，其协方差函数采用交叉拟合方式估计

$$\widehat{\sigma}^{\mathrm{RMST},h}(u,t,a) = \frac{P_n\{\widehat{\varphi}^{\mathrm{RMST},h}(u,a;\widehat{\eta}_k^{\Lambda,h})\}\widehat{\varphi}^{\mathrm{RMST},h}(t,a;\widehat{\eta}_k^{\Lambda,h})\}}{\widehat{\sigma}^{\mathrm{RMST},h,+}(u,a)\widehat{\sigma}^{\mathrm{RMST},h,+}(t,a)}$$

在实践中，我们选择 $\tau_l \ge \mathrm{argmin}_\tau\{\widehat{\sigma}^{\mathrm{RMST},h,+}(\tau, a) > 0\}$，以避免协方差估计量中出现零分母，并且 $\tau_u \le \tau_{\max}$。关于如何处理非正半定协方差估计量，我们在补充材料中进行了讨论。

5 模拟研究

5.1 数据生成过程

我们研究了四种数据生成过程，以评估比较估计量和所提估计量在 ATE 和 WATE 上的有限样本性能。这些模拟仿照四种设置下的观察性研究：设置 1——重叠良好且无竞争风险；设置 2——重叠较差且无竞争风险；设定3——重叠良好且存在竞争风险；设置4——重叠较差且存在竞争风险。我们生成了 1000 次重复，以此。

每种设定。数据生成过程的详细信息以及模拟数据集的汇总统计量见补充材料。各暴露组真实倾向评分的分布如图 1 所示。可以看出，设定1 和 设定 3 的重叠充分，而设定 2 和 设定 4 的重叠较差。

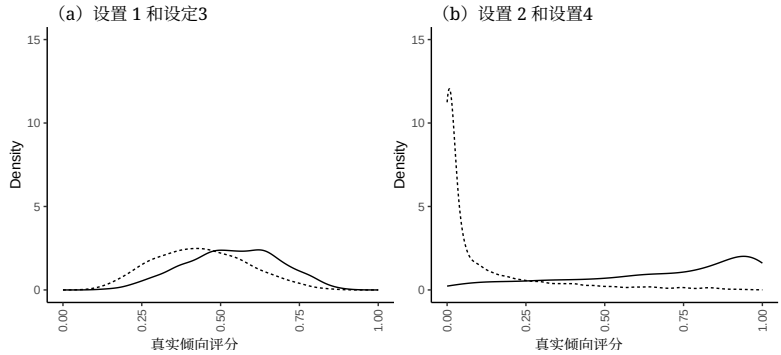


图 1：真实倾向评分的密度。请注意，良好重叠设置（设置 1 和 设置 3）的治疗模型相同，而重叠较差设置（设置 2 和 设置 4）的治疗模型也相同；虚线表示治疗组，实线表示对照组。

5.2 估计目标与估计量

我们考虑与表1中总结的四类平衡权重相对应的估计目标。 $h(x)$ 的设定定义了目标参数和权重。对于生存函数和原因特异性累积发生函数，我们考虑结局回归、逆删失概率加权（IPCW）以及双重稳健的比较估计量。比较估计量中的所有干扰模型均采用正确设定的参数模型：暴露采用逻辑回归模型，所有条件事件发生时间分布均采用 Cox 比例风险模型。比较估计量的所有标准误均通过非参数自助法获得。这些估计量的详细信息见补充材料。对于所提估计量，我们假设真实的数据生成机制未知，并通过自适应机器学习对所有干扰模型进行交叉拟合。具体而言，我们使用 R（R开发核心团队，2012）中的 SuperLearner（Polley 和 van der Laan，2017）与 survSuperLearner（Westling，2020）软件包，学习由机器学习干扰模型构成的堆叠集成。集成中使用的具体学习器见补充材料。

表 1：估计目标、倾斜函数及其来源。ATO：重叠人群平均治疗效应（F. Li 等，2018）；ATM：匹配人群平均治疗效应（L. Li 和 Greene，2013）；ATEN：熵加权人群平均治疗效应（Zhou 等，2020）

估计目标	倾斜函数 $h(x)$	起源
ATE	1	调查
ATO	$\pi(0 \mid x)\pi(1 \mid x)$	基尼指数
ATM	$\min_{a=0,\ 1}\{\pi(a \mid x)\}$	误分类误差
ATEN	$-\sum_{a=0,1}\pi(a \mid x)\log\ \pi(a \mid x)$	交叉熵或偏差

5.3 模拟结果

生存设置（设置 1 和 设置 2）的模拟结果见图2；竞争风险情境（设置 3 和设置 4）的结果见补充材料中的图5。生存设置与竞争风险设置的解释和结论相似。

对于每种比较估计量和所提估计量，我们报告了在限制时间 $\tau = 4$ 下、重叠充分和重叠较差的数据生成过程中，其有限样本性能。图 2(a) 和 (b) 按估计目标展示了加权累积治疗效应的估计值和真实效应。重叠充分时，真实的平均处理效应和加权平均处理效应相近（图 2(a)）；而重叠较差时，二者差异较大（图 2(b)）。重叠充分时，偏倚（空心与实心）通常较低（图 2(a)），但在重叠较差时偶尔会偏高（图 2(b)）。对于每种估计目标，即使干扰模型设定正确，逆概率删失加权估计量的有限样本偏倚也比其他估计量更大，且随着重叠程度下降，这种情况会加剧。重叠有限时，将逆概率处理加权截尾至第 99 百分位数会显著增加偏倚。此外，可以观察到，采用（设定正确的）结局模型的估计量比未采用结局模型的估计量偏倚更小。在所有情形下，所提估计量似乎近似无偏。

图 2 (c) 和 (d) 展示了按估计目标分组的限制时间 $\tau = 4$ 下的蒙特卡罗标准误。蒙特卡罗标准误与估计标准误非常吻合。当重叠充分时，不同估计目标和估计量的标准误相近。然而，在重叠有限时，WATE 的估计量比 ATE 的估计量更有效率，而缩尾处理提高了 ATE 的效率（代价是产生偏倚，正如我们在图2 (b) 中看到的那样）。在重叠有限的情形下，对于每个估计目标，显然结局回归估计量的效率最高，其次是双重稳健估计量和所提估计量，而逆概率删失加权估计量具有更大的有限样本方差。这表明，适当的增广项能够恢复大量信息，从而带来效率提升。

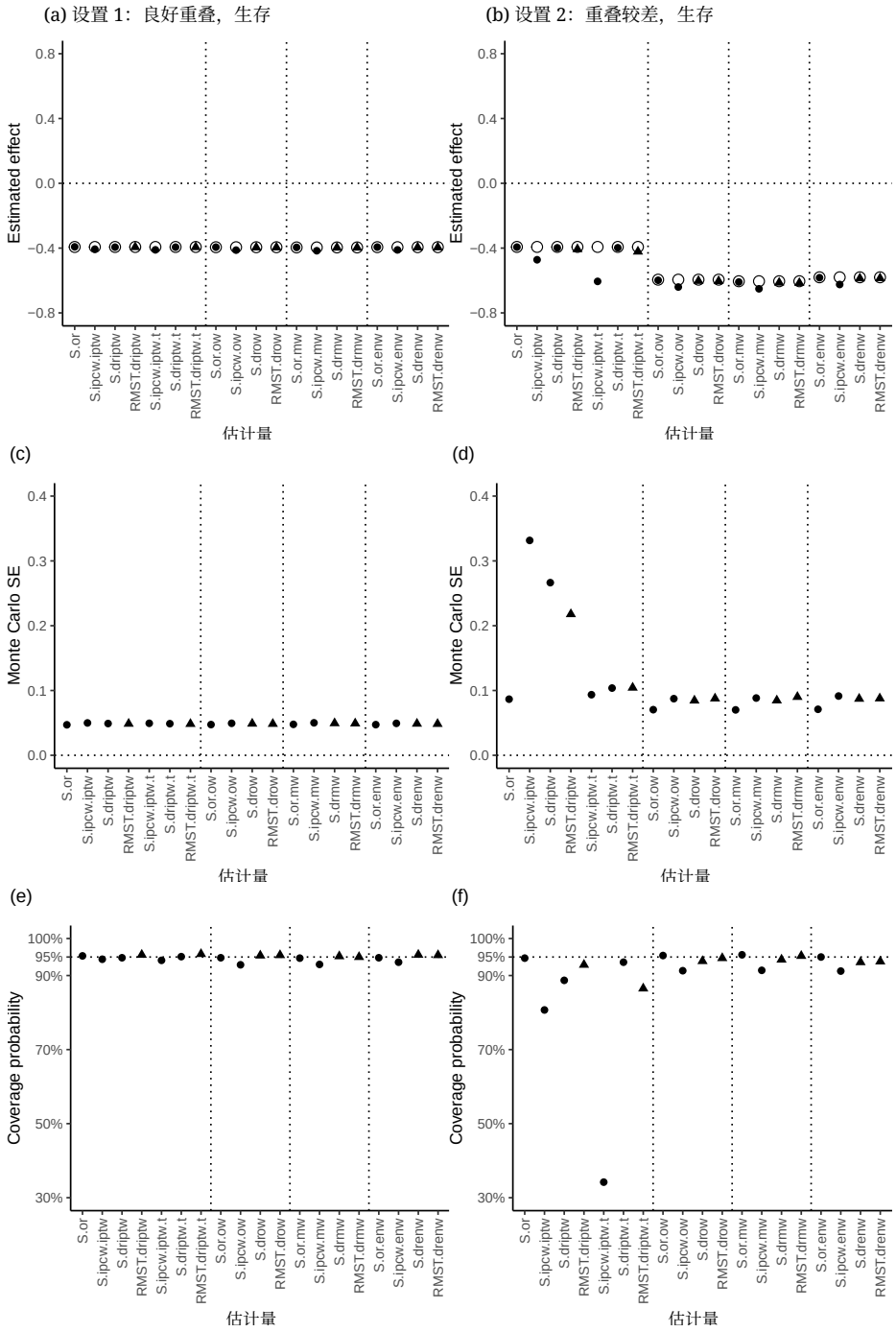


图 2：在 $\tau = 4$ 时，按估计目标分组的不同估计量所得到的估计的累积治疗效应、蒙特卡罗标准误和覆盖概率。每幅图中的估计目标由垂直虚线分隔。从左至右依次为：平均处理效应、重叠人群平均治疗效应、匹配人群平均治疗效应、治疗人群平均治疗效应。(a) 和 (b) 中的空心圆表示真实效应，实心圆表示比较估计量，实心三角形表示所提估计量。S.or：结局回归估计量；S.ipcw.iptw：逆删失概率加权估计量；S.dr：双重稳健估计量；ow：重叠权重；mw：匹配权重；enw：熵权重；前缀 RMST.：所提估计量；后缀 .t：经第99百分位数缩尾处理的逆概率治疗加权。

这是预期的，因为双重稳健估计量和所提估计量在逆概率处理加权估计量这一类中效率最高，但它们的效率略低

优于结局回归估计量（Robins 等人，1994；Bai 等人，2013）。当然，如果结局模型设定错误，这种效率提升将以偏倚为代价；而我们的估计量采用数据自适应干扰模型估计，大大缓解了这一问题。

图 2 (e) 和 (f) 展示了按估计目标分组的限制时间 $\tau = 4$处 95% 逐点置信区间的经验覆盖率。当重叠充分时，图2 (e) 显示，对比估计量与所提估计量的经验覆盖率都相当接近名义值 0.95。然而，当重叠较差时，如图2 (f) 所示，在每个估计目标下，逆概率删失加权估计量的覆盖不足程度都高于其他估计量。对逆概率处理加权进行缩尾处理会降低覆盖率，因为其偏倚更大、估计方差更小。当重叠有限时，对于加权平均处理效应，结局回归估计量、双重稳健估计量和所提估计量的覆盖率都较为理想，接近 0.95。

总而言之，我们通过模拟证明，所提估计量的表现可与比较估计量相媲美，即使比较估计量能够使用正确的参数干扰模型设定，而我们的方法必须通过机器学习来估计干扰模型。这表明，所提估计量比其他方案安全得多，因为采用我们的方法可降低模型设定错误带来的偏倚风险，同时不会牺牲效率。

6 应用

本节中，我们将所提出的方法应用于临床研究实践数据链接（CPRD）中的数据，以评估一线抗高血糖单药治疗对以下人群的影响：(a) 无癌症病史的2型糖尿病患者的癌症发病；以及 (b) 开始治疗时已患癌症的2型糖尿病患者的全因死亡。CPRD 是一个持续运行的全科初级保健数据库，涵盖英国674家诊所的6000多万名患者（Herrett 等人，2015）。研究人群包括截至2018年4月30日前年龄超过50岁的所有2型糖尿病患者。对这些患者的随访持续至以下时间中最早者：死亡、离开诊所/CPRD 数据库，或2018年4月30日数据纳入结束。独立科学咨询委员会（ISAC）根据方案 20_000207 授权访问 CPRD 数据。

为简洁起见并确保一致性，我们在定义暴露、结局和选择协变量时遵循 Tsilidis 等（2014）的做法。我们比较二甲双胍起始用药者（$A = 1$）与磺脲类药物起始用药者（$A = 0$）。当临床结局为癌症患者的全因死亡率时，只有一种失效类型，不存在竞争风险。然而，当关注的结局为癌症发病时，死亡便成为竞争事件。在两项研究中，我们将首次开具一线抗低血糖药物处方的时间作为基线。由于失访或随访于2018年4月30日行政性结束，失效时间存在右删失。治疗前混杂因素 X包括性别、处方时年龄、处方年份、体质指数（BMI）、糖化血红蛋白（HbA1c）、心力衰竭、冠心病、心房颤动、卒中、高血压】【。

病变、外周血管疾病、慢性肾病、慢性阻塞性肺疾病、吸烟、多重贫困指数（IMD）和地区。与磺脲类药物相比，二甲双胍初始用药者通常年龄更小、体重更重，且开始治疗的时间更晚。

我们应用提出的方法来估计逆概率处理加权和重叠加权限制平均生存时间 $\psi_0^{\mathrm{RMST},h}(\tau,\ a)$ 以及原因特异性 RMTL $\psi_0^{\mathrm{RMTL},j,h}(\tau,\ a)$。当假设 1–4 成立时，反事实限制平均生存时间曲线可以解释为：假设所有参与者均接受治疗 a，截至限制时间 τ 的平均生存时间。如果假设 1 和 5–7 成立，则反事实限制平均失寿时间曲线对应于：假设所有患者均具有暴露 a，截至限制时间 τ 由原因 j 导致的平均时间损失。

与我们的模拟结果一致，干扰函数再次通过超级学习器（Van der Laan et al., 2007）获得。我们在补充材料中详细说明了集成干扰学习器。图3展示了癌症发病和癌症诊断后的全因死亡率这两种结局中，按治疗组划分的交叉拟合倾向评分的非参数密度分布。在这两种情形下，治疗组之间的协变量重叠程度均有限，且癌症发病情形下的重叠程度更差。

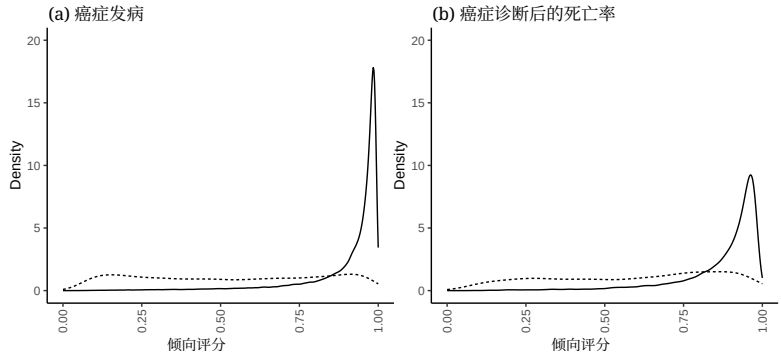


图3：估计倾向评分的密度分布；虚线：磺脲类药物；实线：二甲双胍；(a) 癌症发病：n (磺脲类药物) $= 15{,}595$ 和 n (二甲双胍)$= 88,091$；(b) 癌症诊断后的死亡率：n (磺脲类药物)$= 2{,}110$ 和 n (二甲双胍) $= 9{,}763$。

图4展示了截至限制时间 $\tau = 15$ 年的累积治疗效应。第一行显示癌症发病的原因特异性双重稳健逆概率处理加权和双重稳健重叠加权限制平均损失时间的估计值。第二行显示癌症诊断后全因死亡率的限制平均生存时间的双重稳健逆概率处理加权和双重稳健重叠加权估计值。这些效应分别是二甲双胍初始用药者与磺脲类药物初始用药者之间的差异。癌症发病的同时置信带基于高斯过程样本路径的 10,000 次实现构建，时间为 $[\tau_l, \tau_u] = [1/365.25, 7779/365.25]$ 年；另一组置信带的时间为 $[\tau_l, \tau_u] = [2/365.25, 4533/365.25]$ 年，这些时间对应于估计标准误严格为正的最长随访区间。我们的估计结果表明，在癌症患者中，开始使用二甲双胍相较于磺脲类药物可降低全因死亡率，而其在降低癌症发病方面的优势则较小；这一结果与 Morgan等（2014）和 Tsilidis等（2014）的结论一致。在每项分析中，WATE 的点估计都更接近零效应，且标准误低于 ATE 的点估计。更多结果见补充材料。

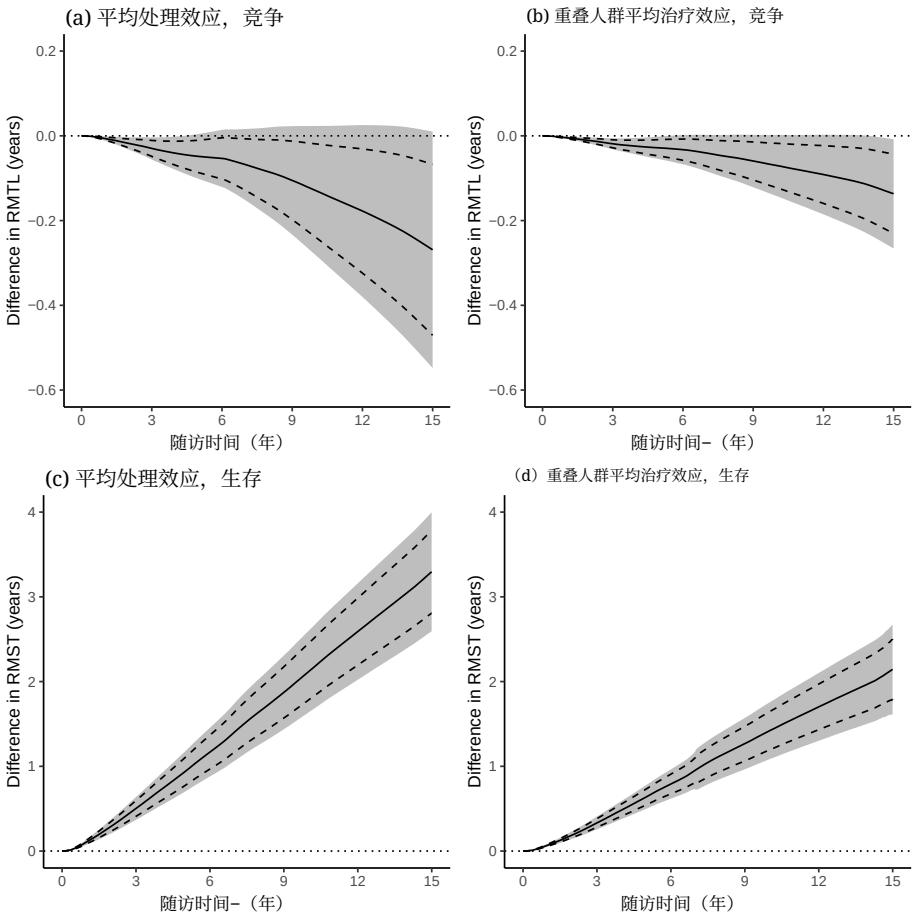


图4：CPRD 数据中估计的累积治疗效应曲线；（a）癌症发病的原因特异性平均处理效应；（b）癌症发病的原因特异性重叠人群平均治疗效应；（c）癌症诊断后死亡率的平均处理效应；以及（d）癌症诊断后死亡率的重叠人群平均治疗效应。所有图中，点线：$y = 0$；实线：点估计；虚线：95% 逐点置信区间；阴影区域：95% 均匀同时置信带。

7 讨论

除了基于因果估计目标的中心化EIFs推导出的估计量外，还有一种基于累积风险双重稳健估计量推导出的 RMST 代入估计量（Zhang 和 Schaubel，2012）。然而，这种 Nelson-Aalen 型估计量无法改造为针对 WATE 的估计量，因为平衡加权累积风险的乘积积分并不对应于我们关注的生存函数/累积发生函数。

与针对整个生存分布的单步估计量不同（Cai and van der Laan，2020；Rytgaard and van der Laan，2021），所提估计量仍可能违反这样一项约束：所有原因的 RMST 与 RMTL 之和等于 τ。未来一个可能的研究方向是，在不作参数建模假设、甚至风险模型设定错误时，确保满足这项约束和形状约束。

竞争风险情境中的其他估计目标包括生存者平均因果效应（SACE）（Tchetgen Tchetgen，2014年）和可分离效应（Stensrud 等，2021年；Stensrud、Young 等，2022年；Stensrud、Robins 等，2022年）。未来的研究可以为这些估计目标的重叠加权版本开发双重/去偏机器学习估计量。我们推测，可分离效应的正则性条件将与我们的条件等价，因为这两类估计量涉及类似的干扰函数（Martinussen 和 Stensrud，2023）。

事实上，平衡权重的一个缺点是，其一致性依赖于对倾斜函数的一致估计，而倾斜函数通常是倾向得分的有界变换。尽管这一假设并不令人满意，但对于加权平均处理效应而言，这一假设无处不在，从而可以减轻因偏离协变量分布而对因果效应进行过度外推的问题。

所提估计量及其正则性条件同样适用于目标化学习；此外，由于采用了交叉拟合，干扰参数的复杂性不受限制。然而，当某个干扰函数未被一致估计时，双重机器学习/目标最大似然估计量都可能不再是正则且渐近线性（RAL）的。为了在至多一个干扰参数估计不一致的情况下进行双重稳健推断（Benkeser 等，2017年），可以遵循 Díaz（2019年）的方法，通过对漂移项进行高斯化来构造漂移校正估计量。另一个潜在方向是在干扰函数估计不一致的情况下实现有效推断（Smucler 等，2019年）。

尽管自适应估计能够缓解模型设定错误，所提出的方法仍会受到未测量混杂的影响。未来研究可将敏感性分析（Ciocănea- Teodorescu等，2022年）、工具变量（Ding等，2017年）以及近期发展的近端推断（Tchetgen等，2020）拓展至重叠有限的事件发生时间结局。除二元暴露外，在我们的研究设定中，连续处理（Kennedy等，2017年）和多重稳健时变处理（Luedtke等，2017年）也是值得关注且有意义的未来研究课题。

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补充材料

补充材料包括扩展证明的技术细节、额外的模拟信息以及应用中的数值结果。

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《通过双重/去偏机器学习高效估计加权累积治疗效应》的补充材料

A 提出的方法

A.1 提出的双重/去偏机器学习估计量的实施步骤

步骤 1. 样本分割。我们将数据 $O_1, \ldots, O_n$ 随机分为 K 个互不重叠的验证集 $\mathcal{V}_1, \ldots, \mathcal{V}_K$，其大小分别为 $n_1, \ldots, n_K$，其中 $K \in \{2, 3, \ldots, \lfloor n/2 \rfloor\}$。对于每个 $k = 1, \ldots, K$，我们定义训练集 $\mathcal{V}_k = \{O_i : i \notin \mathcal{V}_k\}$ 以及治疗特异性训练集 $\mathcal{V}_{a,k} = \{O_i : i \in \mathcal{V}_k, A_i = a\}$。

步骤 *2.* 自适应估计。对于每个 $k = 1, \ldots, K$，*(1)* 在 $\mathcal{V}_k$ 上训练倾向模型，并在 $\mathcal{V}_k$ 上预测 $\widehat{\pi}_k(a \mid X)$；*(2)* 分别在 $\mathcal{V}_{a=0,k}$ 和 $\mathcal{V}_{a=1,k}$ 上训练生存时间模型，并计算 $\widehat{\Lambda}_k(t \mid a, X)$、$\widehat{\Lambda}_k^C(t \mid a, X)$，以及可能的 $\widehat{\Lambda}_{j,k}(t \mid a, X)$，用于 $a = 0, 1$；使用 $\mathcal{V}_k$ 在 $\mathcal{V}_k$ 中观测到的唯一生存时间 $\widetilde{T}$ 处提取观测计数过程 $dN(t)$、$dN^C(t)$，以及可能的 $dN_j(t)$ 中的跳跃信息。计数过程鞅可以通过直接代入相应的累积风险获得。

步骤 3。 渐近有效估计量。通过将步骤 *2* 中估计的干扰函数代入，构造 K 估计量 $\widehat{\phi}^{\mathrm{RMST},h}_{,k}(\tau, a; \widehat{\eta}^{\Lambda,h}_{,k})$ 或 $\widehat{\phi}^{\mathrm{RMTL},h}_{,j,k}(\tau, a; \widehat{\eta}^{\Lambda,h}_{,j,k})$。

$$
\begin{aligned}
\widehat{\phi}_k^{\mathrm{RMST},h}(\tau, a; \widehat{\eta}_k^{\Lambda,h}) = {} & [P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)\widehat{\mathrm{RMST}}_k(\tau \mid a, X) + [P_{n,k}\{\widehat{w}_k^h(a, X)\}]^{-1}\widehat{w}_k^h(a, X) \\
& \times \left\{ \frac{\Delta(\tau)\min(\widetilde{T}, \tau)}{\widehat{G}_k(\tau \wedge \widetilde{T}- \mid a, X)} - \widehat{\mathrm{RMST}}_k(\tau \mid a, X) + \int_0^{\tau \wedge \widetilde{T}} \frac{t\mathrm{d}\widehat{M}_k^C(t \mid a, X)}{\widehat{G}_k(t- \mid a, X)} \right. \\
& \left. \int_0^{\tau \wedge \widetilde{T}} \frac{\{\widehat{\mathrm{RMST}}_k(\tau \mid a, X) - \widehat{\mathrm{RMST}}_k(t \mid a, X)\}\mathrm{d}M_0^C(t \mid a, X)}{S_k(t \mid a, X)G_k(t- \mid a, X)} \right\}
\end{aligned} \tag{5}
$$

or

$$
\begin{aligned}
\widehat{\phi}_{j,k}^{\mathrm{RMTL},h}(\tau, a; \widehat{\eta}_{j,k}^{\Lambda,h}) = {} & [P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)\widehat{\mathrm{RMTL}}_{j,k}(\tau \mid a, X) + [P_{n,k}\{\widehat{w}_k^h(a, X)\}]^{-1}\widehat{w}_k^h(a, X) \\
& \times \left\{ \frac{(\tau - \min(\widetilde{T}, \tau))I(\widetilde{J} = j)}{\widehat{G}_k(\widetilde{T}- \mid a, X)} - \widehat{\mathrm{RMTL}}_{j,k}(\tau \mid a, X) \right. \\
& - \int_0^{\tau \wedge \widetilde{T}} \frac{(\tau - t)\widehat{F}_{j,k}(t \mid a, X)\mathrm{d}\widehat{M}_k^C(t \mid a, X)}{\widehat{S}_k(t \mid a, X)\widehat{G}_k(t- \mid a, X)} \\
& \left. + \int_0^{\tau \wedge \widetilde{T}} \frac{\{\widehat{\mathrm{RMTL}}_{j,k}(\tau \mid a, X) - \widehat{\mathrm{RMTL}}_{j,k}(t \mid a, X)\}\mathrm{d}\widehat{M}_k^C(t \mid a, X)}{\widehat{S}_k(t \mid a, X)\widehat{G}_k(t- \mid a, X)} \right\}
\end{aligned} \tag{6}
$$

将 k 折合并为 $\widehat{\phi}^{\mathrm{RMST},h}_{,}(\tau, a; \widehat{\eta}^{\Lambda,h}_{,})$ 或 $\widehat{\phi}^{\mathrm{RMTL},h}_{,j}(\tau, a; \widehat{\eta}^{\Lambda,h}_{,j})$ 其中 $\widehat{\eta}^{\Lambda,h}_{,} = \{\widehat{\eta}^{\Lambda,h}_{k=1}, \ldots, \widehat{\eta}^{\Lambda,h}_{k=K}\}$ 且 $\widehat{\eta}^{\Lambda,h}_{,j} = \{\widehat{\eta}^{\Lambda,h}_{j,k=1}, \ldots, \widehat{\eta}^{\Lambda,h}_{j,k=K}\}$。

步骤 *4*。形状校正点估计量。计算 $\widehat{\psi}^{\mathrm{RMST},h}(\tau, a) = P_n\{\widehat{\phi}^{\mathrm{RMST},h}(\tau, a; \widehat{\eta}^{\Lambda,h})\}$ 和 $\widehat{\psi}^{\mathrm{RMTL},h}_{,j}(\tau, a) = P_n\{\widehat{\phi}_j^{\mathrm{RMTL},h}(\tau, a; \widehat{\eta}^{\Lambda,h}_{,j})\}$。对于每个 $a \in \{0, 1\}$，我们使用最小凹上包络将 $\widehat{\psi}^{\mathrm{RMST},h}(\tau, a)$ 投影到凹函数空间，得到 $\widehat{\psi}^{\mathrm{RMST},h,lcm}(\tau, a)$；并使用最大凸下包络将有界的 $\widehat{\psi}^{\mathrm{RMTL},h}_{,j}(\tau, a)$ 投影到凸函数空间，得到 $\widehat{\psi}^{\mathrm{RMTL},h,gcm}_{,j}(\tau, a)$。然后

$$
\begin{aligned}
\widehat{\psi}^{\mathrm{RMST},h,lcm}(\tau) = {} & \widehat{\psi}^{\mathrm{RMST},h,lcm}(\tau, 1) - \widehat{\psi}^{\mathrm{RMST},h,lcm}(\tau, 0) \\
\widehat{\psi}_j^{\mathrm{RMTL},h,gcm}(\tau) = {} & \widehat{\psi}_j^{\mathrm{RMTL},h,gcm}(\tau, 1) - \widehat{\psi}_j^{\mathrm{RMTL},h,gcm}(\tau, 0)
\end{aligned}
$$

步骤 5。 用于渐近标准误的形状校正估计量。

$$\widehat{\varphi}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h})=\widehat{\phi}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h})-\widehat{\psi}^{\mathrm{RMST},h,lcm}(\tau,a)$$
$$\widehat{\varphi}_j^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_j^{\Lambda,h})=\widehat{\phi}_j^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_j^{\Lambda,h})-\widehat{\psi}_j^{\mathrm{RMTL},h,gcm}(\tau,a)$$
$$\widehat{\varphi}^{\mathrm{RMST},h}(\tau;\widehat{\eta}^{\Lambda,h})=\widehat{\varphi}^{\mathrm{RMST},h}(\tau,1;\widehat{\eta}^{\Lambda,h})-\widehat{\varphi}^{\mathrm{RMST},h}(\tau,0;\widehat{\eta}^{\Lambda,h})$$
$$\widehat{\varphi}_j^{\mathrm{RMTL},h}(\tau;\widehat{\eta}_j^{\Lambda,h})=\widehat{\varphi}_j^{\mathrm{RMTL},h}(\tau,1;\widehat{\eta}_j^{\Lambda,h})-\widehat{\varphi}_j^{\mathrm{RMTL},h}(\tau,0;\widehat{\eta}_j^{\Lambda,h})$$
$$\widehat{\sigma}^{\mathrm{RMST},h}(\tau,a)=[P_n\{\widehat{\varphi}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h})\}^2]^{1/2},\quad \widehat{\sigma}_j^{\mathrm{RMTL},h}(\tau,a)=[P_n\{\widehat{\varphi}_j^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_j^{\Lambda,h})\}^2]^{1/2}$$
$$\widehat{\sigma}^{\mathrm{RMST},h}(\tau)=[P_n\{\widehat{\varphi}^{\mathrm{RMST},h}(\tau;\widehat{\eta}^{\Lambda,h})\}^2]^{1/2},\quad \widehat{\sigma}_j^{\mathrm{RMTL},h}(\tau)=[P_n\{\widehat{\varphi}_j^{\mathrm{RMTL},h}(\tau;\widehat{\eta}_j^{\Lambda,h})\}^2]^{1/2}$$

当采用参数模型估计干扰参数时，所得估计量的渐近展开等于原始代入影响函数加上一个与干扰模型的影响函数相关的函数（*Ozenne* 等人，*2020*）。如果省略这一反映干扰模型不确定性的附加项，代入估计量的三明治方差可能会被低估。

我们通过取累积最大值，将 $\widehat{\sigma}^{\mathrm{RMST},h}(\tau,a)$，$\widehat{\sigma}^{\mathrm{RMST},h}(\tau)$，$\widehat{\sigma}_j^{\mathrm{RMTL},h}(\tau,a)$ 和 $\widehat{\sigma}_j^{\mathrm{RMTL},h}(\tau)$ 投影到一个非递减函数空间中，并记为

$$\widehat{\sigma}^{\mathrm{RMST},h,+}(\tau,a)=\max_{t\le\tau}\widehat{\sigma}^{\mathrm{RMST},h}(t,a),\quad \widehat{\sigma}_j^{\mathrm{RMTL},h,+}(\tau,a)=\max_{t\le\tau}\widehat{\sigma}_j^{\mathrm{RMTL},h}(t,a)$$
$$\widehat{\sigma}^{\mathrm{RMST},h,+}(\tau)=\max_{t\le\tau}\widehat{\sigma}^{\mathrm{RMST},h}(t),\quad \widehat{\sigma}_j^{\mathrm{RMTL},h,+}(\tau)=\max_{t\le\tau}\widehat{\sigma}_j^{\mathrm{RMTL},h}(t)$$

最终，$\psi^{\mathrm{RMST},h}(\tau,a)$、$\psi^{\mathrm{RMST},h}(\tau)$、$\psi^{\mathrm{RMTL},h}_j(\tau,a)$和 $\psi^{\mathrm{RMTL},h}_j(\tau)$的有限样本标准误可分别计算为 $\widehat{\sigma}^{\mathrm{RMST},h,+}(\tau,a)/\sqrt{n}$, $\widehat{\sigma}^{\mathrm{RMST},h,+}(\tau)/\sqrt{n}$, $\widehat{\sigma}^{\mathrm{RMTL},h,-}_j(\tau,a)/\sqrt{n}$和 $\widehat{\sigma}_j^{\mathrm{RMTL},h,+}(\tau)/\sqrt{n}$。

A.2 注 1

下面列出了在逆删失概率加权下，条件未删失事件发生时间分布与条件删失生存时间分布之间的一些恒等关系。

$$\begin{aligned}
S(t\mid a,x)=&\frac{P(C>t\mid A=a,X=x)P(T>t\mid A=a,X=x)}{P(C>t\mid A=a,X=x)}\\
=&\frac{P(\widetilde{T}>t\mid A=a,X=x)}{G(t\mid a,x)}=E\left\{\frac{I(\widetilde{T}>t)}{G(t\mid a,x)}\mid A=a,X=x\right\}\\
S(t\mid a,x)=&\frac{E(\Delta\mid A=a,X=x)E\{I(T>t)\mid A=a,X=x\}}{G(\widetilde{T}-\mid a,x)}\\
=&\frac{E\{\Delta I(\widetilde{T}>t)\mid A=a,X=x\}}{G(\widetilde{T}-\mid a,x)}=E\left\{\frac{\Delta I(\widetilde{T}>t)}{G(\widetilde{T}-\mid a,x)}\mid A=a,X=x\right\}\\
\mathrm{RMST}(\tau\mid a,x)=&\frac{E\{\Delta(\tau)\mid A=a,X=x\}E\{\min(T,\tau)\mid A=a,X=x\}}{G(\tau\wedge\widetilde{T}-\mid a,x)}\\
=&\frac{E\{\Delta(\tau)\min(\widetilde{T},\tau)\mid A=a,X=x\}}{G(\tau\wedge\widetilde{T}-\mid a,x)}=E\left\{\frac{\Delta(\tau)\min(\widetilde{T},\tau)}{G(\tau\wedge\widetilde{T}-\mid a,x)}\mid A=a,X=x\right\}\\
F_j(t\mid a,x)=&\frac{E(\Delta\mid A=a,X=x)E\{I(T\le t,J=j)\mid A=a,X=x\}}{G(\widetilde{T}-\mid a,x)}\\
=&\frac{E\{I(\widetilde{T}\le t,\widetilde{J}=j)\mid A=a,X=x\}}{G(\widetilde{T}-\mid a,x)}=E\left\{\frac{I(\widetilde{T}\le t,\widetilde{J}=j)}{G(\widetilde{T}-\mid a,x)}\mid A=a,X=x\right\}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{RMTL}_j(\tau \mid a, x) =& \frac{E(\Delta \mid A=a, X=x) E[\{\tau - \min(T, \tau)\} I(J=j) \mid A=a, X=x]}{G(\widetilde{T}- \mid a, x)} \\
=& \frac{E[\{\tau - \min(\widetilde{T}, \tau)\} I(\widetilde{J}=j) \mid A=a, X=x]}{G(\widetilde{T}- \mid a, x)} \\
=& E\left[\frac{\{\tau - \min(\widetilde{T}, \tau)\} I(\widetilde{J}=j)}{G(\widetilde{T}- \mid a, x)} \mid A=a, X=x \right]
\end{aligned}
$$

~ ~通过将可观测的 $I(T>t)$、 $I(T>t)/G(t \mid A, X)$、 ~ ~ ~ ~ ~ ~ ~ ~$\Delta I(T>t)$、 $\Delta I(T>t)/G(T- \mid A, X)$、 $\Delta(\tau)\min(T, \tau)$、 $\Delta(\tau)\min(T, \tau)/G(\tau \wedge T- \mid A, X)$、 $I(T \le t, J=j)$、~ ~ ~ ~ ~ ~ ~ ~$I(T \le t, J=j)/G(T- \mid A, X)$、 $\{\tau - \min(T, \tau)\} I(J=j)$以及 $\{\tau - \min(T, \tau)\} I(J=j)/G(T- \mid A, X)$作为响应变量，并将 A、 X 和 t 或 τ 作为特征，构建合并的分类或回归模型，存在三大缺点。

首先，将数据从宽格式转换为长格式，并以 t 或 τ 作为枢轴，只能处理离散时间的生存时间（Stitelman 和 van der Laan，2010）。将连续生存时间粗化为离散时间通常会导致信息和准确性的损失。其次，长格式的训练需要更多空间，并且可能非常耗时。最后但同样重要的是，大多数标准统计学习算法都要求数据满足独立同分布（i.i.d.）。然而，这种“计数过程学习”会保留主体离开风险集后的所有行，从而使长格式中的各行之间存在很强的依赖关系。另一方面，离散时间条件风险函数类型的分类满足独立同分布假设。根据定义，

$$
\begin{aligned}
\lambda(t \mid a, x) &= P(T=t \mid T \ge t, A=a, X=x) = \frac{P(T=t, T \ge t, A=a, X=x)}{P(T \ge t, A=a, X=x)} \\
&= \frac{P(T=t, A=a, X=x) P(C \ge t, A=a, X=x)}{P(T \ge t, A=a, X=x) P(C \ge t, A=a, X=x)} = \frac{P(\widetilde{T}=t, T \le C, A=a, X=x)}{P(\widetilde{T} \ge t, A=a, X=x)} \\
&= \frac{P(\widetilde{T}=t, \Delta, \widetilde{T} \ge t, A=a, X=x)}{P(\widetilde{T} \ge t, A=a, X=x)} = P(\widetilde{T}=t, \Delta \mid \widetilde{T} \ge t, A=a, X=x)
\end{aligned}
$$

上面最后一个概率中的条件确保离开风险集的个体不再接受训练，因此对于所有处于风险集中的个体，在每个时间点该个体是否发生事件变得相互独立同分布。删失风险函数和特定原因风险函数遵循类似的推导和原理。

B 模拟研究

B.1 数据生成过程

在每次重复实验中，按照 Cheng 等人（2022年）的方法，考虑六个协变量 X_1-X_6。协变量 X_4、X_5 和 X_6 独立地从成功概率为 50% 的伯努利分布中抽取。协变量 X_1、 X_2 和 X_3 则由均值为零、方差为 1、两两相关系数为 0.5 的多元正态分布生成。

设定 1（良好重叠、无竞争风险）：我们考虑样本量为 $n = 4000$。暴露根据具有倾向评分的伯努利分布生成

$$
\mathrm{expit}(-(0.3 + 0.2X_1 + 0.3X_2 + 0.3X_3 - 0.2X_4 - 0.3X_5 - 0.2X_6))
$$

我们生成生成潜在生存时间 $T^{a=0}$和 $T^{a=1}$以及潜在删失时间 $C^{a=0}$an d

$C^{a=1}$ 服从风险函数为

$$\begin{aligned}
\lambda^{a=0}(t \mid X) &= 0.12 \exp\{0.1 + 0.1X_1 - 0.2X_2 + 0.2X_3 + 0.1X_4 + 0.8X_5 - 0.2X_6\} \\
\lambda^{a=1}(t \mid X) &= 0.15 \exp\{0.17 + 0.2X_1 - 0.1X_2 + 0.4X_3 + 0.2X_4 + 0.3X_5 + 0.4X_6\} \\
\lambda^{C}(t \mid A=0, X) &= 0.06 \exp\{0.1 + 0.4X_1 - 0.7X_2 - 0.4X_3 - 0.5X_4 + 0.8X_5 - 0.6X_6\} \\
\lambda^{C}(t \mid A=1, X) &= 0.08 \exp\{0.5X_1 - 0.6X_2 + 0.2X_3 + 0.6X_4 + 0.9X_5 - 0.5X_6\}
\end{aligned}$$

设定 2（重叠较差，无竞争风险）：我们考虑样本量为 $n = 4000$。暴露由伯努利分布生成，其倾向评分

$$\text{expit}(-(-1 + X_1 + 1.5X_2 + 1.5X_3 - X_4 - 1.5X_5 - X_6))$$

潜在生存时间和删失时间的生成方式与设定 1相同。

设定 3（良好重叠，有竞争风险）：我们考虑样本量为 $n = 4000$。暴露的模拟方式与设定 1相同。两个原因的潜在失效时间和删失时间根据风险函数生成

$$\begin{aligned}
\lambda_{j=1}^{a=0}(t \mid X) &= 0.12 \exp\{0.1 + 0.1X_1 - 0.2X_2 + 0.2X_3 + 0.1X_4 + 0.8X_5 - 0.2X_6\} \\
\lambda_{j=1}^{a=1}(t \mid X) &= 0.15 \exp\{0.17 + 0.2X_1 - 0.1X_2 + 0.4X_3 + 0.2X_4 + 0.3X_5 + 0.4X_6\} \\
\lambda_{j=2}^{a=0}(t \mid X) &= 0.1 \exp\{0.12 - 0.1X_1 + 0.3X_2 + 0.1X_3 + 0.2X_4 - 0.4X_5 + 0.5X_6\} \\
\lambda_{j=2}^{a=1}(t \mid X) &= 0.08 \exp\{0.1 - 0.2X_1 - 0.1X_2 + 0.2X_3 + 0.3X_4 + 0.3X_5 - 0.3X_6\} \\
\lambda^{C}(t \mid A=0, X) &= 0.12 \exp\{0.1 + 0.4X_1 - 0.7X_2 - 0.4X_3 - 0.5X_4 + 0.8X_5 - 0.6X_6\} \\
\lambda^{C}(t \mid A=1, X) &= 0.14 \exp\{0 + 0.5X_1 - 0.6X_2 + 0.2X_3 + 0.6X_4 + 0.9X_5 - 0.5X_6\}
\end{aligned}$$

我们关注原因1。

设定 4（重叠较差，有竞争风险）：我们考虑样本量为 $n = 4000$。暴露的模拟方式与设定 2相同。两个原因的潜在失效时间和删失时间的生成方式与设定 3相同。我们关注原因1。

B.2 汇总统计量

表2汇总了设置1-4的统计摘要，包括治疗比例、删失概率、1000次模拟中比例风险假设的平均违背率，以及 $\tau = 4$之前的事件发生率。

表2：数据生成机制摘要——设置1至设置4。%PH violation：比例风险（PH）假设的违背百分比；设置1和设置2采用4种风险模型（$\lambda^{a=0}$，$\lambda^{a=1}$，$\lambda^{a=0,\,C}$，$\lambda^{a=1,\,C}$），设置3和设置4采用6种风险模型（$\lambda_{j=1}^{a=0}$，$\lambda_{j=1}^{a=1}$，$\lambda^{a=0}$，$\lambda^{a=1}$，$\lambda^{a=0,\,C}$，$\lambda^{a=1,\,C}$）。

$\tau = 4$之前的事件发生率：$\tau = 4$ 之前发生的关注事件数量除以 n。

	设置 1	设置 2	设置 3	设置 4
治疗比例	49%	25%	49%	25%
删失概率	35.7%	35.4%	35.9%	33.7%
比例风险假设违反率	33.3%	29.8%	8.2%	13.7%
$\tau = 4$ 前的事件发生率	48.7%	50.2%	36.2%	37.6%

B.3 比较估计量

其中一些比较估计量尚未被明确提出，为求完整，我们在此详细说明这些估计量。在生存分析情境中，使用结局回归的 g公式估计量是

$$\widehat{\psi}^{S,\mathrm{OR},h}(t,a) = [P_n\{\widehat{h}(X)\}]^{-1} P_n[\widehat{h}(X)\{\widehat{S}(t \mid a, X)\}]$$

以及逆概率删失加权估计量（Cheng 等人，2022年）是

$$\widehat{\psi}^{S,\mathrm{IPCW},h}(t,a) = [P_n\{\widehat{w}^h(a,X)\}]^{-1} P_n \left\{ \frac{\widehat{w}^h(a,X) I(\widetilde{T} > t)}{\widehat{G}(t \mid a, X)} \right\}$$

另一种逆概率删失加权估计量可以构造为

$$\widehat{\psi}^{S,\mathrm{IPCW},h}(t,a) = [P_n\{\widehat{w}^h(a,X)\}]^{-1} P_n \left\{ \frac{\widehat{w}^h(a,X) \Delta I(\widetilde{T} > t)}{\widehat{G}(\widetilde{T}- \mid a, X)} \right\}$$

双重稳健估计量的形式为

$$\begin{aligned}\widehat{\psi}^{S,\mathrm{DR},h}(t,a) = {} & [P_n\{\widehat{h}(X)\}]^{-1} P_n[\widehat{h}(X)\widehat{S}(t \mid a, X)] + [P_n\{\widehat{w}^h(a,X)\}]^{-1} P_n \left[\widehat{w}^h(a,X) \right. \\ & \left. \times \left\{ \frac{I(\widetilde{T} > t)}{\widehat{G}(t \mid a, X)} - \widehat{S}(t \mid a, X) + \widehat{S}(t \mid a, X) \int_0^{t \wedge \widetilde{T}} \frac{\mathrm{d}\widehat{M}^C(u \mid a, X)}{\widehat{S}(u \mid a, X)\widehat{G}(u- \mid a, X)} \right\} \right]\end{aligned}$$

这是关于 t 的右连续函数，而 Bai 等人（2013）提供了一个左连续版本。由于加权组别特异生存函数是单调非增的，我们取其累积最小值，并记为 $\widehat{\psi}^{S,h}(t,a) = \min_{t \le \tau} \widehat{\psi}^{S,h}(t,a)$。因果对比为 $\widehat{\psi}^{S,h}(t) = \widehat{\psi}^{S,h,-}(t,1) - \widehat{\psi}^{S,h,-}(t,0)$。然后，可通过对上述每个估计量进行积分得到 $\widehat{\psi}^{\mathrm{RMST},h}(\tau)$，即 $\int_0^\tau \widehat{\psi}^{S,h}(t)\,\mathrm{d}t$。

在竞争风险情境中，使用原因特异累积发生函数结局回归的 g公式估计量是，

$$\widehat{\psi}_j^{F,\mathrm{OR},h}(t,a) = [P_n\{\widehat{h}(X)\}]^{-1} P_n[\widehat{h}(X)\{\widehat{F}_j(t \mid a, X)\}]$$

且逆概率删失加权估计量为

$$\widehat{\psi}_j^{F,\mathrm{IPCW},h}(t,a) = [P_n\{\widehat{w}^h(a,X)\}]^{-1} P_n \left\{ \frac{\widehat{w}^h(a,X) I(\widetilde{T} \le t, \widetilde{J} = j)}{\widehat{G}(\widetilde{T}- \mid a, X)} \right\}$$

双重稳健估计量的形式为

$$\begin{aligned}\widehat{\psi}_j^{F,\mathrm{DR},h}(t,a) = {} & [P_n\{\widehat{h}(X)\}]^{-1} P_n[\widehat{h}(X)\widehat{F}_j(t \mid a, X)] + [P_n\{\widehat{w}^h(a,X)\}]^{-1} P_n \left[\widehat{w}^h(a,X) \right. \\ & \left. \times \left\{ \frac{I(\widetilde{T} \le t, \widetilde{J} = j)}{\widehat{G}(\widetilde{T}- \mid a, X)} - \widehat{F}_j(t \mid a, X) + \widehat{F}_j(t \mid a, X) \int_0^{t \wedge \widetilde{T}} \frac{\mathrm{d}\widehat{M}^C(u \mid a, X)}{\widehat{S}(u \mid a, X)\widehat{G}(u- \mid a, X)} \right. \right. \\ & \left. \left. - \int_0^{t \wedge \widetilde{T}} \frac{\widehat{F}_j(u \mid a, X)\mathrm{d}\widehat{M}^C(u \mid a, X)}{\widehat{S}(u \mid a, X)\widehat{G}(u- \mid a, X)} \right\} \right]\end{aligned}$$

这与 Ozenne 等人（2020）的做法类似。由于加权组别特异原因特异累积发生函数单调不减，我们取其累积最大值，并记为 $\widehat{\psi}^{F,h+}_{\cdot,j}(t,a) = \max_{t \le \tau} \widehat{\psi}^{F,h}_{\cdot,j}(t,a)$。因果对比为 $\widehat{\psi}^{F,h}_{\cdot,j}(t) = \widehat{\psi}^{F,h+}_{\cdot,j}(t,1) - \widehat{\psi}^{F,h+}_{\cdot,j}(t,0)$。随后，可通过对上述各估计量进行积分得到 $\widehat{\psi}^{\mathrm{RMTL},h}_{\cdot,j}(\tau)$，即 $\int_0^\tau \widehat{\psi}^{F,h}_{\cdot,j}(t)\mathrm{d}t$。

B.4 补充模拟结果

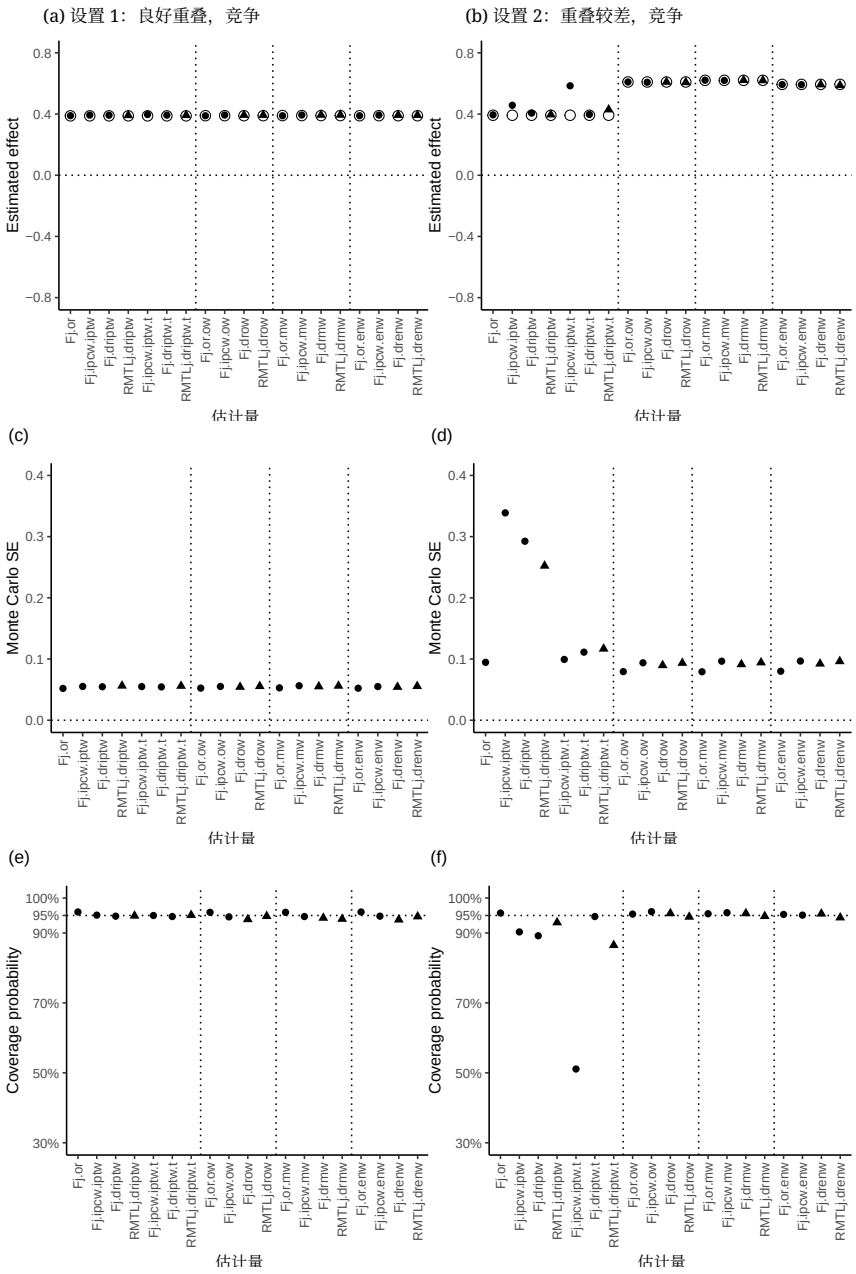


图 5： $\tau = 4$ 处不同估计量按估计目标分组得到的估计累积治疗效应、蒙特卡罗标准误和覆盖概率。每个图中，各估计目标由垂直虚线分隔。从左到右的顺序为：ATE、ATO、ATM、ATEN。(a) 和 (b) 中的空心圆表示真实效应，实心圆表示比较估计量，实心三角形表示所提估计量。S.or 表示结局回归估计量；S.ipcw.iptw 表示逆删失概率加权估计量；S.dr 表示双重稳健估计量；.t 表示 IPTW 在其第99百分位数处进行了缩尾处理；RMST. 表示所提估计量；ow、mw 和 enw 分别表示重叠权重、匹配权重和熵权重。

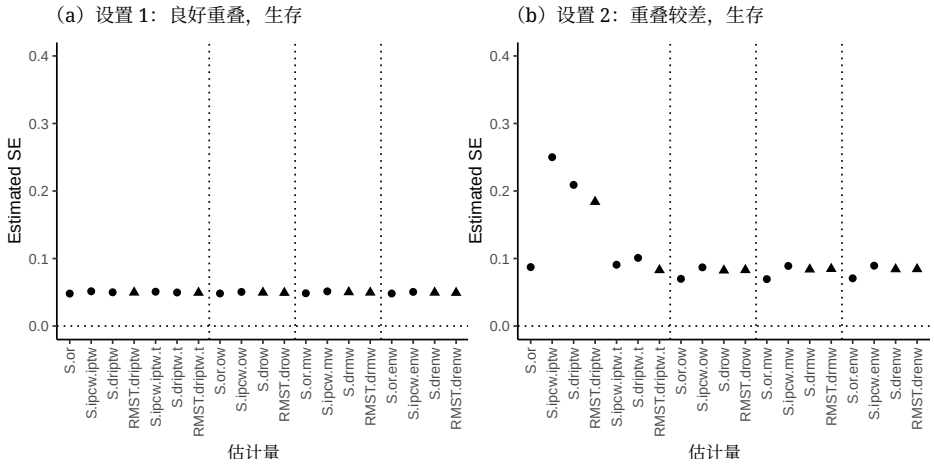


图 6：在 $\tau = 4$ 处按估计目标分组的不同估计量的平均标准误。实心圆表示比较估计量的平均自助法标准误；实心三角形表示所提估计量的平均估计标准误。其他符号和符号说明与图2类似。

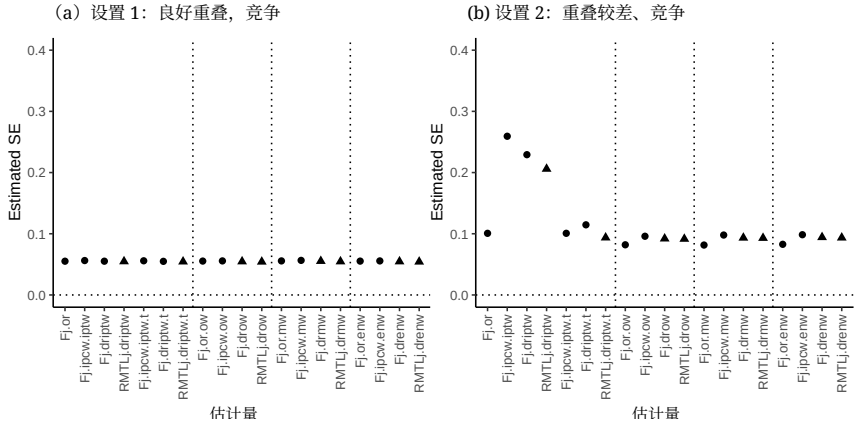


图 7： $\tau = 4$处按估计目标分组的不同估计量的平均标准误。实心圆表示比较估计量的平均自助法标准误；实心三角形表示所提估计量的平均估计标准误。其他符号和符号说明与图5类似。

B.5 候选学习器

为加快训练速度，本文始终采用二折交叉验证，并从所有可能的候选学习器中选择相对较快的学习器。对于倾向评分，我们采用了边际均值（SL.mean）、前馈神经网络（SL.nnet）、核 k 近邻（SL.kernelKnn）、分类树（SL.rpartPrune）、极端梯度提升（SL.xgboost）、随机森林（SL.ranger）、逻辑回归（SL.glm）、前向/后向逐步逻辑回归（SL.step）、广义加性模型（SL.gam）、正则化逻辑回归（SL.glmnet）以及多变量自适应回归样条（SL.earth）。对于条件失效时间分布，我们考虑了 Kaplan-Meier 估计量（survSL.km）、分段常数风险回归（survSL.pchreg）、生存随机森林（survSL.rfsrc）、Cox比例风险回归（survSL.coxph），以及假设事件时间和删失时间服从条件指数分布（survSL.expreg）、威布尔分布（survSL.weibreg）和对数逻辑斯蒂分布（survSL.loglogreg）的回归方法。

C 应用

C.1 处理非正半定协方差估计量

当样本量较大时，在较早的生存时间点，观测到的生存时间可能彼此接近。在这种情况下，协方差估计量可能不是正半定的（PSD），因此无法直接生成样本路径。另一种方法是用标准正态随机变量 ξ as 扰动中心化EIF，从而生成样本路径。

$$\xi \frac{\widehat{\varphi}^{\mathrm{RMST},h}(t,a;\widehat{\eta}^{\Lambda,h})}{\widehat{\sigma}^{\mathrm{RMST},h,+}(t,a)} \sim \mathcal{N}\left(0, \frac{\widehat{\varphi}^{\mathrm{RMST},h}(t,a;\widehat{\eta}^{\Lambda,h})\{\widehat{\varphi}^{\mathrm{RMST},h}(t,a;\widehat{\eta}^{\Lambda,h})\}^T}{\{\widehat{\sigma}^{\mathrm{RMST},h,+}(t,a)\}^2}\right)$$

从而可以避免交叉拟合协方差函数满足半正定（PSD）要求。除了对中心化EIF进行扰动外，Lin 等人（1993）还提出，通过扰动相应的计数过程，即 $\mathrm{d}M(t \mid A,\ X) = \xi I(T = t,\ \Delta)$、$\mathrm{d}M^C(t \mid A,\ X) = \xi I(\widetilde{T} = t,\ \Delta = 0)$ 和 $\mathrm{d}M_j(t \mid A,\ X) = \xi I(\widetilde{T} = t,\ \widetilde{J} = j)$，来近似~某些鞅积分的极限分布。除了扰动方法外，还可以将非半正定的协方差估计量转换为与其最近的半正定矩阵（Higham，1988），但这种变换的渐近等效性尚未得到证明。通过替换相应的形状校正估计量，可以类似地建立治疗特异性 RMST 与病因特异性 RMTL 之间因果对比的推断。

C.2 结果

由于分段常数风险回归（survSL.pchreg）在大型数据集上耗时过长，我们将其从所有条件事件发生时间分布的模拟候选库中移除。我们使用不包含核 k 近邻（SL.kernelKnn）和广义加性模型（SL.gam）的模拟候选库来估计倾向评分。

表 3：在 $\tau = 10$年时，基于 CPRD 数据估计的二甲双胍初始治疗者与磺脲类药物初始治疗者之间反事实 RMTL/RMST 和累积治疗效应结果（单位：年）。SE：标准误；CI：置信区间；CB：置信带。

	点估计（标准误）	95% 逐点置信区间	95% 同时置信带
癌症发病 平均处理效应			
磺脲类药物（RMTL）	0.859 (0.056)	(0.750, 0.968)	(0.716, 1.002)
二甲双胍 (RMTL)	0.730 (0.008)	(0.713, 0.746)	(0.706, 0.754)
二甲双胍-磺脲类药物	-0.129 (0.056)	(-0.239, -0.019)	(-0.282, 0.023)
癌症发病 重叠人群平均治疗效应			
磺脲类药物（RMTL）	0.855 (0.023)	(0.809, 0.901)	(0.787, 0.922)
二甲双胍 (RMTL)	0.785 (0.013)	(0.758, 0.811)	(0.748, 0.822)
二甲双胍-磺脲类药物（RMTL差异）	-0.070 (0.027)	(-0.122, -0.018)	(-0.142, 0.002)
癌症后的死亡率平均处理效应			
磺脲类药物（限制平均生存时间）	5.998 (0.166)	(5.672, 6.324)	(5.539, 6.457)
二甲双胍（限制平均生存时间）	8.114（0.039）	(8.037, 8.191)	（8.003，8.226）
二甲双胍-磺脲类药物限制平均生存时间差	2.116（0.17）	(1.782, 2.451)	(1.636, 2.597)
癌症诊断后死亡 ATO			
磺脲类药物（限制平均生存时间）	6.260 (0.112)	(6.041, 6.479)	(5.945, 6.574)
二甲双胍（限制平均生存时间）	7.679 (0.068)	(7.545, 7.813)	(7.495, 7.863)
二甲双胍-磺脲类药物（限制平均生存时间之差）	1.419 (0.115)	(1.194, 1.644)	(1.084, 1.754)

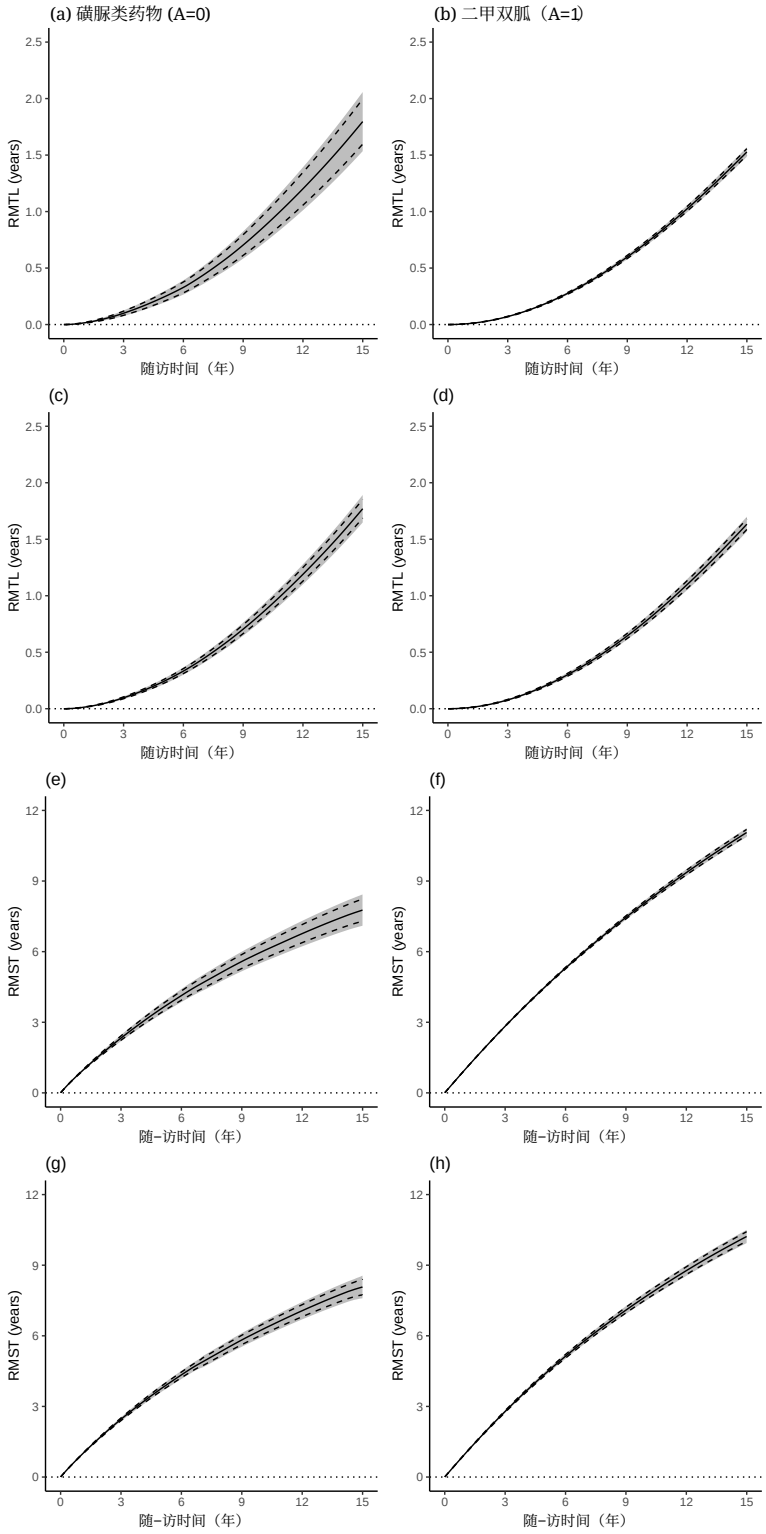


图 8：CPRD 数据中估计的反事实 RMTL/RMST 曲线；第一列（a）、（c）、（e）和（g）：磺脲类药物起始者；第二列（b）、（d）、（f）和（h）：二甲双胍起始者；第一行（a）和（b）：基于 IPTW 的治疗特异性癌症发病原因特异性 RMTL；第二行（c）和（d）：基于 OW 的治疗特异性癌症发病原因特异性 RMTL；（e）和（f）：基于 IPTW 的癌症诊断后死亡率治疗特异性 RMST；（g）和（h）：基于 OW 的癌症诊断后死亡率治疗特异性 RMST。在所有图中，点线：$y = 0$；实线：点估计；虚线：95% 逐点置信区间；阴影区域：95% 均匀同时置信带。

D 定理证明

D.1 定理 1

定理 1 的证明：根据 Robins 和 Rotnitzky（1992）；Tsiatis（2006），无需对事件发生时间分布作出额外假设，$\psi_{\cdot 0}^{\mathrm{RMST}\ \ h}(\tau,\ a)$ 可构造为增广逆概率加权完整病例（AIPWCC）估计量，其估计函数为

$$
\begin{aligned}
&m^{\mathrm{AIPWCC}}\{\psi_0^{\mathrm{RMST},h}(\tau,a);\eta_0^{\Lambda,h}\}\\
=&\frac{w_0^h(a,X)\Delta}{G_0(\widetilde{T}-\mid a,X)}\{\min(\widetilde{T},\tau)-\psi_0^{\mathrm{RMST},h}(\tau,a)\}+\{h_0(X)-w_0^h(a,X)\}\\
&\times E_0\{\min(T,\tau)-\psi_0^{\mathrm{RMST},h}(\tau,a)\mid A=a,X\}+w_0^h(a,X)\int_0^\infty\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\\
&\times E_0\{\min(T,\tau)-\psi_0^{\mathrm{RMST},h}(\tau,a)\mid A=a,X,T>t\}
\end{aligned}
$$

注意

$$
\begin{aligned}
&\int_0^\infty\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}E_0\{\min(T,\tau)-\psi_0^{\mathrm{RMST},h}(\tau,a)\mid A=a,X,T>t\}\\
=&\int_0^\infty\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}[(t\le\tau)[E_0\{\min(T,\tau)\mid A=a,X,T>t\}-\psi_0^{\mathrm{RMST},h}(\tau,a)\\
&+(t>\tau)\{\tau-\psi_0^{\mathrm{RMST},h}(\tau,a)\}]\\
=&\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}E_0\{\min(T,\tau)\mid A=a,X,T>t\}+\tau\int_{\tau+}^\infty\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\\
&-\psi_0^{\mathrm{RMST},h}(\tau,a)\int_0^\infty\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}
\end{aligned}
$$

其中

$$
\begin{aligned}
&E_0\{\min(T,\tau)\mid A=a,X,T>t\}=\int_t^\infty\min(t,\tau)\mathrm{d}P_0(T\le u\mid T>t,A=a,X)\\
=&\int_t^\infty\min(t,\tau)\mathrm{d}\left\{\frac{P_0(T>t,A=a,X,T\le u)}{P_0(T>t,A=a,X)}\right\}=\int_t^\infty\min(t,\tau)\mathrm{d}\left\{\frac{P_0(t<T\le u\mid A=a,X)}{P_0(T>t\mid A=a,X)}\right\}\\
=&\int_t^\infty\min(t,\tau)\mathrm{d}\left\{\frac{S_0(t\mid a,X)-S_0(u\mid a,X)}{S_0(t\mid a,X)}\right\}=\frac{\int_t^\infty\min(t,\tau)\{-\mathrm{d}S_0(u\mid a,X)\}}{S_0(t\mid a,X)}\\
=&\frac{-\int_t^\tau u\mathrm{d}S_0(u\mid a,X)+\tau S_0(\tau\mid a,X)}{S_0(t\mid a,X)}=t+\frac{\int_t^\tau S_0(u\mid a,X)\mathrm{d}t}{S_0(t\mid a,X)}\\
=&t+\frac{\mathrm{RMST}_0(\tau\mid a,X)-\mathrm{RMST}_0(t\mid a,X)}{S_0(t\mid a,X)}
\end{aligned}
$$

请注意

$$
\begin{aligned}
&\frac{\Delta}{G_0(\widetilde{T}-\mid a,X)}+\frac{I(\widetilde{T}>\tau)}{G_0(\tau\mid a,X)}-\frac{\Delta I(\widetilde{T}>\tau)}{G_0(\widetilde{T}-\mid a,X)}=\frac{\Delta I(\widetilde{T}\le\tau)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}+\frac{I(\widetilde{T}>\tau)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}\\
=&\frac{I(C>\tau\wedge T-)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}=\frac{\Delta(\tau)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}
\end{aligned}
$$

因此,

$$
\begin{aligned}
&m^{\mathrm{AIPWCC}}\{\psi_0^{\mathrm{RMST},h}(\tau,a);\eta_0^{\Lambda,h}\}\\
=&\frac{w_0^h(a,X)\Delta\min(\widetilde{T},\tau)}{G_0(\widetilde{T}-\mid a,X)}-\frac{w_0^h(a,X)\Delta\psi_0^{\mathrm{RMST},h}(\tau,a)}{G_0(\widetilde{T}-\mid a,X)}+\{h_0(X)-w_0^h(a,X)\}\mathrm{RMST}_0(\tau\mid a,X)\\
&-\{h_0(X)-w_0^h(a,X)\}\psi_0^{\mathrm{RMST},h}(\tau,a)+w_0^h(a,X)\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\\
&\times\left\{t+\frac{\mathrm{RMST}_0(\tau\mid a,X)-\mathrm{RMST}_0(t\mid a,X)}{S_0(t\mid a,X)}\right\}+\frac{w_0^h(a,X)\tau I(\widetilde{T}>\tau)}{G_0(\tau\mid a,X)}\\
&-\frac{w_0^h(a,X)\tau\Delta I(\widetilde{T}>\tau)}{G_0(\widetilde{T}-\mid a,X)}-w_0^h(a,X)\psi_0^{\mathrm{RMST},h}(\tau,a)+\frac{w_0^h(a,X)\Delta\psi_0^{\mathrm{RMST},h}(\tau,a)}{G_0(\widetilde{T}-\mid a,X)}\\
=&\frac{w_0^h(a,X)\Delta\min(\widetilde{T},\tau)}{G_0(\widetilde{T}-\mid a,X)}+\frac{w_0^h(a,X)I(\widetilde{T}>\tau)\min(\widetilde{T},\tau)}{G_0(\tau\mid a,X)}-\frac{w_0^h(a,X)\Delta I(\widetilde{T}>\tau)\min(\widetilde{T},\tau)}{G_0(\widetilde{T}-\mid a,X)}\\
&+\{h_0(X)-w_0^h(a,X)\}\mathrm{RMST}_0(\tau\mid a,X)+w_0^h(a,X)\\
&\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\left\{t+\frac{\mathrm{RMST}_0(\tau\mid a,X)-\mathrm{RMST}_0(t\mid a,X)}{S_0(t\mid a,X)}\right\}-h_0(X)\psi_0^{\mathrm{RMST},h}(\tau,a)\\
=&\frac{w_0^h(a,X)\Delta(\tau)\min(\widetilde{T},\tau)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}+\{h_0(X)-w_0^h(a,X)\}\mathrm{RMST}_0(\tau\mid a,X)+w_0^h(a,X)\\
&\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\left\{t+\frac{\mathrm{RMST}_0(\tau\mid a,X)-\mathrm{RMST}_0(t\mid a,X)}{S_0(t\mid a,X)}\right\}-h_0(X)\psi_0^{\mathrm{RMST},h}(\tau,a)
\end{aligned}
$$

完全增广 M 估计方程的影响函数是有效影响函数（中心化EIF），

$$
\begin{aligned}
\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta_0^{\Lambda,h})=&-\left\{E_0\left[\frac{\partial m^{\mathrm{AIPWCC}}\{\psi_0^{\mathrm{RMST},h}(\tau,a);\eta_0^{\Lambda,h}\}}{\partial\psi_0^{\mathrm{RMST},h}(\tau,a)\}}\right]\right\}^{-1}m^{\mathrm{AIPWCC}}\{\psi_0^{\mathrm{RMST},h}(\tau,a);\eta_0^{\Lambda,h}\}\\
=&[E_0\{h_0(X)\}]^{-1}\left[w_0^h(a,X)\left\{\frac{\Delta(\tau)\min(\widetilde{T},\tau)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}-\mathrm{RMST}_0(\tau\mid a,X)\right.\right.\\
&\left.+\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}+\int_0^{\tau\wedge\widetilde{T}}\frac{\{\mathrm{RMST}_0(\tau\mid a,X)-\mathrm{RMST}_0(t\mid a,X)\}\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\right\}\\
&\left.+h_0(X)\mathrm{RMST}_0(\tau\mid a,X)-h_0(X)\psi_0^{\mathrm{RMST},h}(\tau,a)\right]
\end{aligned}
$$

在竞争风险情境下， ψ_0^{RMTL}、$_j$ ^{h}、(τ、 a) 的双重稳健估计函数为

$$
\begin{aligned}
&m^{\mathrm{AIPWCC}}\{\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a);\eta_{j,0}^{\Lambda,h}\}\\
=&\frac{w_0^h(a,X)}{G_0(\widetilde{T}-\mid a,X)}[\{\tau-\min(\widetilde{T},\tau)\}I(\widetilde{J}=j)-\Delta\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)]+\{h_0(X)-w_0^h(a,X)\}\\
&\times E_0[\{\tau-\min(T,\tau)\}I(J=j)-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\mid A=a,X]+w_0^h(a,X)\int_0^{\infty}\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\\
&\times E_0[\{\tau-\min(T,\tau)\}I(J=j)-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\mid A=a,X,T>t]
\end{aligned}
$$

其中

$$
\begin{aligned}
&E_0[\{\tau-\min(T,\tau)\}(J=j)\mid A=a,X,T>t]\\
=&\int_t^\infty\{\tau-\min(u,\tau)\}\mathrm{d}P_0(T\le u,J=j\mid A=a,X,T>t)\\
=&\int_t^\infty\{\tau-\min(u,\tau)\}\mathrm{d}\left\{\frac{P_0(t<T\le u,J=j,A=a,X)}{P_0(A=a,X,T>t)}\right\}\\
=&\int_t^\infty\{\tau-\min(u,\tau)\}\mathrm{d}\left\{\frac{P_0(t<T\le u,J=j\mid A=a,X)}{P_0(T>t\mid A=a,X)}\right\}\\
=&\int_t^\infty\{\tau-\min(u,\tau)\}\mathrm{d}\left\{\frac{F_{j,0}(u\mid a,X)-F_{j,0}(t\mid a,X)}{S_0(t\mid a,X)}\right\}=\frac{\int_t^\infty\{\tau-\min(u,\tau)\}\mathrm{d}F_{j,0}(u\mid a,X)}{S_0(t\mid a,X)}\\
=&\frac{\int_t^\tau(\tau-u)\mathrm{d}F_{j,0}(u\mid a,X)}{S_0(t\mid a,X)}=\frac{(t-\tau)F_{j,0}(t\mid a,X)+\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)}{S_0(t\mid a,X)}
\end{aligned}
$$

请注意

$$
\begin{aligned}
&\int_0^\infty\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}E_0[\{\tau-\min(T,\tau)\}I(J=j)-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\mid A=a,X,T>t]\\
=&\int_0^\infty\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}[(t\le\tau)(E_0[\{\tau-\min(T,\tau)\}I(J=j)\mid A=a,X,T>t]-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a))\\
&+(t>\tau)\{-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\}]\\
=&\int_0^{\tau\wedge\widetilde{T}}\frac{\{(t-\tau)F_{j,0}(t\mid a,X)+\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
&-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)+\frac{\Delta\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)}{G_0(\widetilde{T}-\mid a,X)}
\end{aligned}
$$

因此，估计函数变为

$$
\begin{aligned}
&m^{\mathrm{AIPWCC}}\{\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a);\eta_{j,0}^{\Lambda,h}\}\\
=&\frac{w_0^h(a,X)\{\tau-\min(\widetilde{T},\tau)\}I(\widetilde{J}=j)}{G_0(\widetilde{T}-\mid a,X)}-\frac{w_0^h(a,X)\Delta\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)}{G_0(\widetilde{T}-\mid a,X)}+\{h_0(X)-w_0^h(a,X)\}\\
&\times\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\{h_0(X)-w_0^h(a,X)\}\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)+w_0^h(a,X)\\
&\times\int_0^{\tau\wedge\widetilde{T}}\frac{\{(t-\tau)F_{j,0}(t\mid a,X)+\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
&-w_0^h(a,X)\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)+\frac{w_0^h(a,X)\Delta\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)}{G_0(\widetilde{T}-\mid a,X)}\\
=&\frac{w_0^h(a,X)\{\tau-\min(\widetilde{T},\tau)\}I(\widetilde{J}=j)}{G_0(\widetilde{T}-\mid a,X)}+\{h_0(X)-w_0^h(a,X)\}\mathrm{RMTL}_{j,0}(\tau\mid a,X)+w_0^h(a,X)\\
&\times\int_0^{\tau\wedge\widetilde{T}}\frac{\{(t-\tau)F_{j,0}(t\mid a,X)+\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
&-h_0(X)\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)
\end{aligned}
$$

中心化EIF为

$$
\begin{aligned}
&\varphi_{j,0}^{\mathrm{RMTL},h}(\tau,a;\eta_{j,0}^{\Lambda,h}) \\
=&-\left\{E_0\left[\frac{\partial m^{\mathrm{AIPWCC}}\{\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a);\eta_{j,0}^{\Lambda,h}\}}{\partial \psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)}\right]\right\}^{-1} \partial m^{\mathrm{AIPWCC}}\{\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a);\eta_{j,0}^{\Lambda,h}\} \\
=&[E_0\{h_0(X)\}]^{-1}\left(w_0^h(a,X)\left[\frac{\{\tau-\min(\widetilde{T},\tau)\}I(\widetilde{J}=j)}{G_0(\widetilde{T}-\mid a,X)}-\mathrm{RMTL}_{j,0}(\tau\mid a,X)\right.\right. \\
&\left.+\int_0^{\tau\wedge\widetilde{T}}\frac{\{(t-\tau)F_{j,0}(t\mid a,X)+\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\right] \\
&\left.+h_0(X)\mathrm{RMTL}_{j,0}(\tau\mid a,X)-h_0(X)\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\right)
\end{aligned}
$$

D.2 引理1

引理1。

$$
\begin{aligned}
\phi_0^{\mathrm{RMST}}(\tau,a;\eta_0^{\Lambda}) =&\mathrm{RMST}_0(\tau\mid a,X)-\frac{I(A=a)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)} \\
&+\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t\mid a,X)\mathrm{d}M_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_{j,0}^{\mathrm{RMTL}}(\tau,a;\eta_{j,0}^{\Lambda}) =&\mathrm{RMTL}_{j,0}(\tau\mid a,X)+\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{(\tau-t)\mathrm{d}M_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)} \\
&-\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\{\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)} \\
&+\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{(\tau-t)F_{j,0}(t\mid a,X)\mathrm{d}M_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}
\end{aligned}
$$

引理1的证明：我们从上面第一个方程的右端开始

$$
\begin{aligned}
&\mathrm{RMST}_0(\tau\mid a,X)-\frac{I(A=a)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)} \\
&+\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0(t\mid a,X)\mathrm{RMST}_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}-h_0(x)\psi_0^{\mathrm{RMST},h}(\tau,a) \\
=&\mathrm{RMST}_0(\tau\mid a,X)-\frac{I(A=a)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)}\int_0^{\tau}\frac{\mathrm{d}N(t)}{S_0(t\mid a,X)G_0(t-\mid a,X)} \\
&+\frac{I(A=a)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\Lambda_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}+\frac{I(A=a)}{\pi_0(a\mid X)} \\
&\times\int_0^{\tau}\frac{\mathrm{RMST}_0(t\mid a,X)\mathrm{d}N(t)}{S_0(t\mid a,X)G_0(t-\mid a,X)}-\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t\mid a,X)\mathrm{d}\Lambda_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}
\end{aligned}
$$

其中

$$
\int_0^{\tau}\frac{\mathrm{d}N(t)}{S_0(t\mid a,X)G_0(t-\mid a,X)}=\frac{\Delta I(\widetilde{T}\le\tau)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}
$$

and

$$\int_0^{\tau} \frac{\mathrm{RMST}_0(t \mid a, X)\mathrm{d}N(t)}{S_0(t \mid a, X)G_0(t- \mid a, X)} = \frac{\Delta I(\widetilde{T} \le \tau)\mathrm{RMST}_0(\tau \wedge \widetilde{T} \mid a, X)}{S_0(\tau \wedge \widetilde{T} \mid a, X)G_0(\tau \wedge \widetilde{T}- \mid a, X)}$$

请注意，$\mathrm{d}\Lambda_0^C(u- \mid a, X) = P_0(u- \le T < (u+\mathrm{d}u)- \mid C \ge u-, A = a, X) = P_0(u \le T < u+\mathrm{d}u \mid C \ge u, A = a, X) = \mathrm{d}\Lambda_0^C(u \mid a, X)$，$\mathrm{d}\Lambda_0(u- \mid a, X) = \mathrm{d}\Lambda_0(u \mid a, X)$，以及 $\mathrm{d}\Lambda_j0(u- \mid a, X) = \mathrm{d}\Lambda_j0(u \mid a, X)$。因此

$$\begin{aligned}
&\int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{d}\Lambda_0(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} = \int_0^{\tau\wedge\widetilde{T}} \frac{1}{G_0(t- \mid a, X)}\mathrm{d}\left\{\frac{1}{S_0(t \mid a, X)}\right\} \\
=& \frac{1}{S_0(t \mid a, X)G_0(t- \mid a, X)}\Big|_0^{\tau\wedge\widetilde{T}} - \int_0^{\tau\wedge\widetilde{T}} \frac{1}{S_0(t \mid a, X)}\mathrm{d}\left\{\frac{1}{G_0(t- \mid a, X)}\right\} \\
=& \frac{1}{S_0(\tau \wedge \widetilde{T} \mid a, X)G_0(\tau \wedge \widetilde{T}- \mid a, X)} - 1 - \int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{d}\Lambda_0^C(t- \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} \\
=& \frac{1}{S_0(\tau \wedge \widetilde{T} \mid a, X)G_0(\tau \wedge \widetilde{T}- \mid a, X)} - 1 + \int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} - \int_0^{\tau} \frac{\mathrm{d}N^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} \\
=& \frac{1}{S_0(\tau \wedge \widetilde{T} \mid a, X)G_0(\tau \wedge \widetilde{T}- \mid a, X)} - 1 + \int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} \\
&- \frac{(1-\Delta)I(\widetilde{T} \le \tau)}{S_0(\tau \wedge \widetilde{T}- \mid a, X)G_0(\tau \wedge \widetilde{T}- \mid a, X)} \\
=& \frac{\Delta(\tau)}{S_0(\tau \wedge \widetilde{T} \mid a, X)G_0(\tau \wedge \widetilde{T}- \mid a, X)} - 1 + \int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)}
\end{aligned}$$

由于

$$\begin{aligned}
\mathrm{d}\left\{\frac{\mathrm{RMST}_0(t \mid a, X)}{G_0(t- \mid a, X)}\right\} &= \frac{S_0(t \mid a, X)G_0(t- \mid a, X)\mathrm{d}t + G_0(t- \mid a, X)\mathrm{d}\Lambda_0^C(t- \mid a, X)\mathrm{RMST}_0(t \mid a, X)}{G_0^2(t- \mid a, X)} \\
&= \frac{S_0(t \mid a, X)\mathrm{d}t}{G_0(t- \mid a, X)} + \frac{\mathrm{RMST}_0(t \mid a, X)\mathrm{d}\Lambda_0^C(t \mid a, X)}{G_0(t- \mid a, X)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{RMST}_0(t \mid a, X)\mathrm{d}\Lambda_0(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} = \int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{RMST}_0(t \mid a, X)}{G_0(t- \mid a, X)}\mathrm{d}\left\{\frac{1}{S_0(t \mid a, X)}\right\} \\
=& \frac{\mathrm{RMST}_0(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)}\Big|_0^{\tau\wedge\widetilde{T}} - \int_0^{\tau\wedge\widetilde{T}} \frac{1}{S_0(t \mid a, X)}\mathrm{d}\left\{\frac{\mathrm{RMST}_0(t \mid a, X)}{G_0(t- \mid a, X)}\right\} \\
=& \frac{\mathrm{RMST}_0(\tau \wedge \widetilde{T} \mid a, X)}{G_0(\tau \wedge \widetilde{T}- \mid a, X)S_0(\tau \wedge \widetilde{T} \mid a, X)} - \int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{d}t}{G_0(t- \mid a, X)} - \int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{RMST}_0(t \mid a, X)\mathrm{d}\Lambda_0^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} \\
=& \frac{\mathrm{RMST}_0(\tau \wedge \widetilde{T} \mid a, X)}{G_0(\tau \wedge \widetilde{T}- \mid a, X)S_0(\tau \wedge \widetilde{T} \mid a, X)} - \frac{\tau \wedge \widetilde{T}}{G_0(\tau \wedge \widetilde{T}- \mid a, X)} + \int_0^{\tau\wedge\widetilde{T}} \frac{t\mathrm{d}\Lambda_0^C(t \mid a, X)}{G_0(t- \mid a, X)} \\
&+ \int_0^{\tau\wedge\widetilde{T}} \frac{\mathrm{RMST}_0(t \mid a, X)\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X)G_0(t- \mid a, X)} - \int_0^{\tau} \frac{\mathrm{RMST}_0(t \mid a, X)\mathrm{d}N^C(t)}{S_0(t \mid a, X)G_0(t- \mid a, X)}
\end{aligned}$$

$$
\begin{aligned}
=&\frac{\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)S_0(\tau\wedge\widetilde{T}\mid a,X)}-\frac{\tau\wedge\widetilde{T}}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\\
&+\int_0^{\tau}\frac{t\mathrm{d}N^C(t\mid a,X)}{G_0(t-\mid a,X)}+\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}-\frac{\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)I(\widetilde{T}\le\tau)(1-\Delta)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)S_0(\tau\wedge\widetilde{T}\mid a,X)}\\
=&\frac{\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)S_0(\tau\wedge\widetilde{T}\mid a,X)}-\frac{\tau\wedge\widetilde{T}}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\\
&+\frac{(1-\Delta)I(\widetilde{T}\le\tau)(\tau\wedge\widetilde{T})}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}+\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}-\frac{(1-\Delta)I(\widetilde{T}\le\tau)\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)S_0(\tau\wedge\widetilde{T}\mid a,X)}\\
=&\frac{\Delta(\tau)\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)S_0(\tau\wedge\widetilde{T}\mid a,X)}-\frac{\Delta(\tau)\min(\widetilde{T},\tau)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\\
&+\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}
\end{aligned}
$$

将前面的方程代入引理1中第一个方程的右端可得,

$$
\begin{aligned}
&\mathrm{RMST}_0(\tau\mid a,X)-\frac{I(A=a)\Delta I(\widetilde{T}\le\tau)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}+\frac{I(A=a)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)}\\
&\times\left\{\frac{\Delta(\tau)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}-1+\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\right\}\\
&+\frac{I(A=a)\Delta I(\widetilde{T}\le\tau)\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}-\frac{I(A=a)}{\pi_0(a\mid X)}\left\{\frac{\Delta(\tau)\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}\right.\\
&\left.-\frac{\Delta(\tau)\min(\widetilde{T},\tau)}{G_0(\tau\wedge\widetilde{T}-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}+\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
=&\mathrm{RMST}_0(\tau\mid a,X)-\frac{I(A=a)\Delta I(\widetilde{T}\le\tau)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}+\frac{I(A=a)\Delta(\tau)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}\\
&-\frac{I(A=a)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)}+\frac{I(A=a)\mathrm{RMST}_0(\tau\mid a,X)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
&+\frac{I(A=a)\Delta I(\widetilde{T}\le\tau)\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}-\frac{I(A=a)\Delta(\tau)\mathrm{RMST}_0(\tau\wedge\widetilde{T}\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}\\
&+\frac{I(A=a)\Delta(\tau)\min(\widetilde{T},\tau)}{\pi_0(a\mid X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}+\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\\
&-\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
=&\frac{I(A=a)\Delta(\tau)\min(\widetilde{T},\tau)}{\pi_0(a\mid X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}+\left\{1-\frac{I(A=a)}{\pi_0(a\mid X)}\right\}\mathrm{RMST}_0(\tau\mid a,X)+\frac{I(A=a)}{\pi_0(a\mid X)}\\
&\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0^C(t\mid a,X)}{G_0(t-\mid a,X)}\left\{t+\frac{\mathrm{RMST}_0(\tau\mid a,X)-\mathrm{RMST}_0(t\mid a,X)}{S_0(t\mid a,X)}\right\}\\
=&\phi_0^{\mathrm{RMST},h}(\tau,a;\eta_0^{\Lambda})
\end{aligned}
$$

~注意，限制平均生存时间$_0(\tau \wedge T \mid a, X) \neq$限制平均生存时间$_0(\tau \mid a, X)$，且

~$-\Delta I(T \leq \tau)$限制平均生存时间$_0(\tau \mid a, X) + \Delta(\tau)$限制平均生存时间$_0(\tau \mid a, X) +$限制平均生存时间$(\tau \wedge T \mid a, X) - \Delta(\tau)$限制平均生存时间$_0(\tau \wedge T \mid a, X)$~

$= -\Delta I(T \leq \tau)$限制平均生存时间$_0(\tau \mid a, X) + \Delta I(T \leq \tau)$限制平均生存时间$_0(\tau \mid a, X) + I(T > \tau)$限制平均生存时间$_0(\tau \mid a, X)$

~$+\Delta I(T \leq \tau)$限制平均生存时间$_0(\tau \wedge T \mid a, X) - \Delta I(T \leq \tau)$限制平均生存时间$_0(\tau \wedge T \mid a, X) - I(T > \tau)$限制平均生存时间$_0(\tau \wedge T \mid a, X) = 0$

在竞争风险情境中，我们转换了若干条件事件鞅项。

$$
\begin{aligned}
&\int_0^{\tau \wedge \widetilde{T}} \frac{(\tau - t)\mathrm{d}M_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)} = \tau \int_0^{\tau} \frac{\mathrm{d}M_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)} - \int_0^{\tau} \frac{t\mathrm{d}M_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)} \\
=&\frac{\tau I(\widetilde{T} \leq \tau, \widetilde{J} = j)}{G_0(\widetilde{T}- \mid a, X)} - \tau \int_0^{\tau \wedge \widetilde{T}} \frac{\mathrm{d}\Lambda_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)} - \frac{\widetilde{T} I(\widetilde{T} \leq \tau, \widetilde{J} = j)}{G_0(\widetilde{T}- \mid a, X)} + \int_0^{\tau \wedge \widetilde{T}} \frac{t\mathrm{d}\Lambda_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)} \\
=&\frac{\{\tau - \min(\widetilde{T}, \tau)\} I(\widetilde{J} = j)}{G_0(\widetilde{T}- \mid a, X)} - \tau \int_0^{\tau \wedge \widetilde{T}} \frac{\mathrm{d}\Lambda_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)} + \int_0^{\tau \wedge \widetilde{T}} \frac{t\mathrm{d}\Lambda_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{d}\left\{\frac{\mathrm{RMTL}_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)}\right\} = \frac{F_{j,0}(t \mid a, X) G_0(t- \mid a, X)\mathrm{d}t + \mathrm{RMTL}_{j,0}(t \mid a, X) G_0(t- \mid a, X)\mathrm{d}\Lambda_0^C(t- \mid a, X)}{G_0^2(t- \mid a, X)} \\
=&\frac{F_{j,0}(t \mid a, X)\mathrm{d}t}{G_0(t- \mid a, X)} + \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)\mathrm{d}\Lambda_0^C(t \mid a, X)}{G_0(t- \mid a, X)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{\tau \wedge \widetilde{T}} \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)\mathrm{d}M_0(t \mid a, X)}{S_0(t \mid a, X) G_0(t- \mid a, X)} \\
=&\int_0^{\tau} \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)\mathrm{d}N(t)}{S_0(t \mid a, X) G_0(t- \mid a, X)} - \int_0^{\tau \wedge \widetilde{T}} \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)\mathrm{d}\Lambda_0(t \mid a, X)}{S_0(t \mid a, X) G_0(t- \mid a, X)} \\
=&\frac{\Delta I(\widetilde{T} \leq \tau)\mathrm{RMTL}_{j,0}(\widetilde{T} \mid a, X)}{S_0(\widetilde{T} \mid a, X) G_0(\widetilde{T}- \mid a, X)} - \int_0^{\tau \wedge \widetilde{T}} \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)}\mathrm{d}\left\{\frac{1}{S_0(t \mid a, X)}\right\} \\
=&\frac{\Delta I(\widetilde{T} \leq \tau)\mathrm{RMTL}_{j,0}(\tau \wedge \widetilde{T} \mid a, X)}{S_0(\tau \wedge \widetilde{T} \mid a, X) G_0(\tau \wedge \widetilde{T}- \mid a, X)} - \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)}{S_0(t \mid a, X) G_0(t- \mid a, X)}\Bigg|_0^{\tau \wedge \widetilde{T}} + \int_0^{\tau \wedge \widetilde{T}} \frac{1}{S_0(t \mid a, X)}\mathrm{d}\left\{\frac{\mathrm{RMTL}_{j,0}(t \mid a, X)}{G_0(t- \mid a, X)}\right\} \\
=&\frac{\Delta I(\widetilde{T} \leq \tau)\mathrm{RMTL}_{j,0}(\tau \wedge \widetilde{T} \mid a, X)}{S_0(\tau \wedge \widetilde{T} \mid a, X) G_0(\tau \wedge \widetilde{T}- \mid a, X)} - \frac{\mathrm{RMTL}_{j,0}(\tau \wedge \widetilde{T} \mid a, X)}{S_0(\tau \wedge \widetilde{T} \mid a, X) G_0(\tau \wedge \widetilde{T}- \mid a, X)} + \int_0^{\tau \wedge \widetilde{T}} \frac{F_{j,0}(t \mid a, X)\mathrm{d}t}{S_0(t \mid a, X) G_0(t- \mid a, X)} \\
&+ \int_0^{\tau \wedge \widetilde{T}} \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)\mathrm{d}\Lambda_0^C(t \mid a, X)}{S_0(t \mid a, X) G_0(t- \mid a, X)} \\
=&\frac{\Delta I(\widetilde{T} \leq \tau)\mathrm{RMTL}_{j,0}(\tau \wedge \widetilde{T} \mid a, X)}{S_0(\tau \wedge \widetilde{T} \mid a, X) G_0(\tau \wedge \widetilde{T}- \mid a, X)} - \frac{\mathrm{RMTL}_{j,0}(\tau \wedge \widetilde{T} \mid a, X)}{S_0(\tau \wedge \widetilde{T} \mid a, X) G_0(\tau \wedge \widetilde{T}- \mid a, X)} + \int_0^{\tau \wedge \widetilde{T}} \frac{F_{j,0}(t \mid a, X)\mathrm{d}t}{S_0(t \mid a, X) G_0(t- \mid a, X)} \\
&- \int_0^{\tau \wedge \widetilde{T}} \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X) G_0(t- \mid a, X)} + \frac{(1 - \Delta) I(\widetilde{T} \leq \tau)\mathrm{RMTL}_{j,0}(\tau \wedge \widetilde{T} \mid a, X)}{S_0(\tau \wedge \widetilde{T} \mid a, X) G_0(\tau \wedge \widetilde{T}- \mid a, X)} \\
=&-\frac{I(\widetilde{T} > \tau)\mathrm{RMTL}_{j,0}(\tau \wedge \widetilde{T} \mid a, X)}{S_0(\tau \wedge \widetilde{T} \mid a, X) G_0(\tau \wedge \widetilde{T}- \mid a, X)} + \int_0^{\tau \wedge \widetilde{T}} \frac{F_{j,0}(t \mid a, X)\mathrm{d}t}{S_0(t \mid a, X) G_0(t- \mid a, X)} \\
&- \int_0^{\tau \wedge \widetilde{T}} \frac{\mathrm{RMTL}_{j,0}(t \mid a, X)\mathrm{d}M_0^C(t \mid a, X)}{S_0(t \mid a, X) G_0(t- \mid a, X)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{\tau\wedge\widetilde{T}} \frac{tF_{j,0}(t\mid a,X)\mathrm{d}M_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
=&\int_0^{\tau\wedge\widetilde{T}} \frac{tF_{j,0}(t\mid a,X)\mathrm{d}N(t)}{S_0(t\mid a,X)G_0(t-\mid a,X)} - \int_0^{\tau\wedge\widetilde{T}} \frac{tF_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)}\mathrm{d}\left\{\frac{1}{S_0(t\mid a,X)}\right\}\\
=&\frac{\Delta I(\widetilde{T}\le\tau)(\tau\wedge\widetilde{T})F_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)} - \frac{tF_{j,0}(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\Big|_0^{\tau\wedge\widetilde{T}} + \int_0^{\tau\wedge\widetilde{T}} \frac{1}{S_0(t\mid a,X)}\mathrm{d}\left\{\frac{tF_{j,0}(t\mid a,X)}{G_{j,0}(t-\mid a,X)}\right\}\\
=&\frac{\Delta I(\widetilde{T}\le\tau)(\tau\wedge\widetilde{T})F_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)} - \frac{(\tau\wedge\widetilde{T})F_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}\\
&+\int_0^{\tau\wedge\widetilde{T}} \frac{1}{S_0(t\mid a,X)}\mathrm{d}\left\{\frac{tF_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)}\right\}\\
=&\frac{\Delta I(\widetilde{T}\le\tau)(\tau\wedge\widetilde{T})F_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)} - \frac{(\tau\wedge\widetilde{T})F_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)} + \int_0^{\tau\wedge\widetilde{T}} \frac{F_{j,0}(t\mid a,X)\mathrm{d}t}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
&+\int_0^{\tau\wedge\widetilde{T}} \frac{t\mathrm{d}\Lambda_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)} - \int_0^{\tau\wedge\widetilde{T}} \frac{tF_{j,0}(t\mid a,X)\mathrm{d}M_0^C(t-\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)} + \frac{(1-\Delta)I(\widetilde{T}\le\tau)(\tau\wedge\widetilde{T})F_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}\\
=&-\frac{I(\widetilde{T}>\tau)(\tau\wedge\widetilde{T})F_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)} + \int_0^{\tau\wedge\widetilde{T}} \frac{F_{j,0}(t\mid a,X)\mathrm{d}t}{S_0(t\mid a,X)G_0(t-\mid a,X)} + \int_0^{\tau\wedge\widetilde{T}} \frac{t\mathrm{d}\Lambda_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)}\\
&-\int_0^{\tau\wedge\widetilde{T}} \frac{tF_{j,0}(t\mid a,X)\mathrm{d}M_0^C(t-\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{d}\left\{\frac{tF_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)}\right\}\\
=&\frac{G_0(t-\mid a,X)\{F_{j,0}(t\mid a,X)+tS_0(t\mid a,X)\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}+tF_{j,0}(t\mid a,X)G_0(t-\mid a,X)\mathrm{d}\Lambda_0^C(t-\mid a,X)}{G_0^2(t-\mid a,X)}\\
=&\frac{F_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)} + \frac{tS_0(t\mid a,X)\mathrm{d}\Lambda_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)} + \frac{tF_{j,0}(t\mid a,X)\mathrm{d}\Lambda_0^C(t-\mid a,X)}{G_0(t-\mid a,X)}
\end{aligned}
$$

将所有准备等式代入引理1第二个等式的右端，得到,

$$
\begin{aligned}
&\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}} \frac{(\tau-t)\mathrm{d}M_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)} - \frac{I(A=a)}{\pi_0(a\mid X)}\\
&\times\int_0^{\tau\wedge\widetilde{T}} \frac{\{\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)} + \frac{I(A=a)}{\pi_0(a\mid X)}\\
&\times\int_0^{\tau\wedge\widetilde{T}} \frac{(\tau-t)F_{j,0}(t\mid a,X)\mathrm{d}M_0(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)} + \mathrm{RMTL}_{j,0}(\tau\mid a,X)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{I(A=a)\{\tau-\min(\widetilde{T},\tau)\}I(\widetilde{J}=j)}{\pi_0(a\mid X)G_0(\widetilde{T}-\mid a,X)}-\frac{I(A=a)\tau}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\Lambda_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)}\\
&+\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}\Lambda_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)}+\frac{I(A=a)I(\widetilde{T}>\tau)\mathrm{RMTL}_{j,0}(\tau\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}-\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}\\
&-\frac{I(A=a)\mathrm{RMTL}_{j,0}(\tau\mid a,X)}{\pi_0(a\mid X)}+\frac{I(A=a)\mathrm{RMTL}_{j,0}(\tau\mid a,X)}{\pi_0(a\mid X)}\\
&\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}-\frac{I(A=a)I(\widetilde{T}>\tau)\mathrm{RMTL}_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}\\
&+\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{F_{j,0}(t\mid a,X)\mathrm{d}t}{S_0(t\mid a,X)G_0(t-\mid a,X)}-\frac{I(A=a)}{\pi_0(a\mid X)}\\
&\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMTL}_{j,0}(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}-\frac{I(A=a)\tau I(\widetilde{T}>\tau)F_{j,0}(\tau\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}\\
&+\frac{I(A=a)\tau}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\Lambda_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)}-\frac{I(A=a)\tau}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{F_{j,0}(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
&+\frac{I(A=a)I(\widetilde{T}>\tau)(\tau\wedge\widetilde{T})F_{j,0}(\tau\wedge\widetilde{T}\mid a,X)}{\pi_0(a\mid X)S_0(\tau\wedge\widetilde{T}\mid a,X)G_0(\tau\wedge\widetilde{T}-\mid a,X)}-\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{F_{j,0}(t\mid a,X)\mathrm{d}t}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
&-\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}\Lambda_{j,0}(t\mid a,X)}{G_0(t-\mid a,X)}+\frac{I(A=a)}{\pi_0(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{tF_{j,0}(t\mid a,X)\mathrm{d}M_0^C(t-\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
&+\mathrm{RMTL}_{j,0}(\tau\mid a,X)\\
=&\frac{I(A=a)\{\tau-\min(\widetilde{T},\tau)\}I(\widetilde{J}=j)}{\pi_0(a\mid X)G_0(\widetilde{T}-\mid a,X)}+\left\{1-\frac{I(A=a)}{\pi_0(a\mid X)}\right\}\mathrm{RMTL}_{j,0}(\tau\mid a,X)+\frac{I(A=a)}{\pi_0(a\mid X)}\\
&\times\int_0^{\tau\wedge\widetilde{T}}\frac{\{(t-\tau)F_{j,0}(t\mid a,X)+\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0^C(t\mid a,X)}{S_0(t\mid a,X)G_0(t-\mid a,X)}\\
=&\phi_{j,0}^{\mathrm{RMTL},h}(\tau,a;\eta_{j,0}^{\Lambda})
\end{aligned}
$$

D.3 引理 2

引理 2。

$$
\begin{aligned}
&P_0\{\widehat{\phi}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h})\}-\psi_0^{\mathrm{RMST},h}(\tau,a)\\
=&[P_n\{\widehat{w}^h(a,X)\}]^{-1}\left(E_0\left[\widehat{h}(X)\widehat{\mathrm{RMST}}(\tau\mid a,X)\int_0^{\tau}\frac{S_0(t\mid a,X)}{\widehat{S}(t\mid a,X)}\left\{\frac{\pi_0(a\mid X)G_0(t-\mid a,X)}{\widehat{\pi}(a\mid X)\widehat{G}(t-\mid a,X)}-1\right\}\right.\right.\\
&\left.\times\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}\right]\\
&\left.-E_0\left[\widehat{h}(X)\int_0^{\tau}\frac{\widehat{\mathrm{RMST}}(t\mid a,X)S_0(t\mid a,X)}{\widehat{S}(t\mid a,X)}\left\{\frac{\pi_0(a\mid X)G_0(t-\mid a,X)}{\widehat{\pi}(a\mid X)\widehat{G}(t-\mid a,X)}-1\right\}\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}\right]\right)\\
&+([P_n\{\widehat{h}(X)\}]^{-1}-[P_n\{\widehat{w}^h(a,X)\}]^{-1})E_0\{\widehat{h}(X)\widehat{\mathrm{RMST}}(\tau\mid a,X)\}\\
&+([P_n\{\widehat{w}^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1})E_0\{\widehat{h}(X)\mathrm{RMST}_0(\tau\mid a,X)\}\\
&+[E\{h_0(x)\}]^{-1}E_0[\{\widehat{h}(X)-h_0(X)\}\mathrm{RMST}_0(\tau\mid a,X)]
\end{aligned}
$$

$$
\begin{aligned}
&P\{\widehat{\phi}_j^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_j^{\Lambda,h})\}-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\\
=&[P_n\{\widehat{w}^h(a,X)\}]^{-1}\left(-E_0\left[\widehat{h}(X)\tau\int_0^\tau S_0(t\mid a,X)\left\{\frac{\pi_0(a\mid X)G_0(t-\mid a,X)}{\widehat{\pi}(a\mid X)\widehat{G}(t-\mid a,X)}-1\right\}\right.\right.\\
&\times\{\mathrm{d}\widehat{\Lambda}_j(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\Big]+E_0\left[\widehat{h}(X)\int_0^\tau tS_0(t\mid a,X)\left\{\frac{\pi_0(a\mid X)G_0(t-\mid a,X)}{\widehat{\pi}(a\mid X)\widehat{G}(t-\mid a,X)}-1\right\}\right.\\
&\times\{\mathrm{d}\widehat{\Lambda}_j(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\Big]+E_0\left[\widehat{h}(X)\widehat{\mathrm{RMTL}}_j(\tau\mid a,X)\int_0^\tau\frac{S_0(t\mid a,X)}{\widehat{S}(t\mid a,X)}\right.\\
&\times\left.\left\{\frac{\pi(a\mid X)G_0(t-\mid a,X)}{\widehat{\pi}(a\mid X)\widehat{G}(t-\mid a,X)}-1\right\}\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}\right]\\
&-E_0\left[\widehat{h}(X)\int_0^\tau\frac{\widehat{\mathrm{RMTL}}_j(t\mid a,X)S_0(t\mid a,X)}{\widehat{S}(t\mid a,X)}\left\{\frac{\pi_0(a\mid X)G_0(t-\mid a,X)}{\widehat{\pi}(a\mid X)\widehat{G}(t-\mid a,X)}-1\right\}\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}\right]\\
&-E_0\left[\widehat{h}(X)\tau\int_0^\tau\frac{\widehat{F}_j(t\mid a,X)S_0(t\mid a,X)}{\widehat{S}(t\mid a,X)}\left\{\frac{\pi_0(a\mid X)G_0(t-\mid a,X)}{\widehat{\pi}(a\mid X)\widehat{G}(t-\mid a,X)}-1\right\}\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}\right]\\
&+E_0\left[\widehat{h}(X)\int_0^\tau\frac{t\widehat{F}_j(t\mid a,X)S_0(t\mid a,X)}{\widehat{S}(t\mid a,X)}\left\{\frac{\pi_0(a\mid X)G_0(t-\mid a,X)}{\widehat{\pi}(a\mid X)\widehat{G}(t-\mid a,X)}-1\right\}\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}\right]\Bigg)\\
&+([P_n\{\widehat{h}(X)\}]^{-1}-[P_n\{\widehat{w}^h(a,X)\}]^{-1})E_0\{\widehat{h}(X)\widehat{\mathrm{RMTL}}_j(\tau\mid a,X)\}\\
&+([P_n\{\widehat{w}^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1})E_0\{\widehat{h}(X)\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}\\
&+[E\{h_0(X)\}]^{-1}E_0[\{\widehat{h}(X)-h_0(X)\}\mathrm{RMTL}_{j,0}(\tau\mid a,X)]
\end{aligned}
$$

引理 2 的证明:

$$
\begin{aligned}
&P\{\widehat{\phi}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h})\}-\psi_0^{\mathrm{RMST},h}(\tau,a)\\
=&[P_n\{\widehat{w}^h(a,X)\}]^{-1}E_0\left[\frac{\widehat{h}(X)\pi_0(a\mid x)}{\widehat{\pi}(a\mid X)}E_0\left\{-\widehat{\mathrm{RMST}}(\tau\mid a,X)\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\right.\right.\\
&\left.\left.+\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{\mathrm{RMST}}(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\right]\\
&+[P_n\{\widehat{h}(X)\}]^{-1}E_0[\widehat{h}(X)\{\widehat{\mathrm{RMST}}(\tau\mid a,X)\}]-[E_0\{h_0(X)\}]^{-1}E_0[h_0(X)\mathrm{RMST}_0(\tau\mid a,X)]\\
=&[P_n\{\widehat{w}^h(a,X)\}]^{-1}E_0\left[\frac{\widehat{h}(X)\pi_0(a\mid X)}{\widehat{\pi}(a\mid X)}E_0\left\{-\widehat{\mathrm{RMST}}(\tau\mid a,X)\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\right.\right.\\
&\left.\left.+\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{\mathrm{RMST}}(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}+\widehat{h}(X)\{\widehat{\mathrm{RMST}}(\tau\mid a,X)-\mathrm{RMST}_0(\tau\mid a,X)\}\right]\\
&+[P_n\{\widehat{h}(X)\}]^{-1}E_0[\widehat{h}(X)\{\widehat{\mathrm{RMST}}(\tau\mid a,X)\}]-[E_0\{h_0(X)\}]^{-1}E_0[h_0(X)\mathrm{RMST}_0(\tau\mid a,X)]\\
&+([P_n\{\widehat{h}(X)\}]^{-1}-[P_n\{\widehat{w}^h(a,X)\}]^{-1})E_0\{\widehat{h}(X)\widehat{\mathrm{RMST}}(\tau\mid a,X)\}\\
&+([P_n\{\widehat{w}^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1})E_0\{\widehat{h}(X)\mathrm{RMST}_0(\tau\mid a,X)\}\\
&+[E_0\{h_0(X)\}]^{-1}E_0[\{\widehat{h}(X)-h_0(X)\}\mathrm{RMST}_0(\tau\mid a,X)]
\end{aligned}
$$

对于第一个条件事件鞅积分项,

$$
\begin{aligned}
&E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\\
=&-\int_0^\tau\frac{S_0(t-\mid a,X)G_0(t-\mid a,X)\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}
\end{aligned}
$$

以及第二个条件事件鞅积分项

$$
\begin{aligned}
&E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{\mathrm{RMST}}(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\\
=&E_0\left\{\frac{\Delta I(\widetilde{T}\le\tau)\widehat{\mathrm{RMST}}(\widetilde{T}\mid a,X)}{\widehat{S}(\widetilde{T}\mid a,X)\widehat{G}(\widetilde{T}-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{\mathrm{RMST}}(t\mid a,X)\mathrm{d}\widehat{\Lambda}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\\
=&-\int_0^{\tau}\frac{S_0(t-\mid a,X)G_0(t-\mid a,X)\widehat{\mathrm{RMST}}(t\mid a,X)\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}
\end{aligned}
$$

杜阿梅尔方程（Gill 和 Johansen，1990）给出

$$
S_0(t\mid a,X)-\widehat{S}(t\mid a,X)=\widehat{S}(t\mid a,X)\int_0^t\frac{S_0(u-\mid a,X)}{\widehat{S}(u\mid a,X)}\{\mathrm{d}\widehat{\Lambda}(u\mid a,X)-\mathrm{d}\Lambda_0(u\mid a,X)\}
$$

使得

$$
\begin{aligned}
&\widehat{\mathrm{RMST}}(t\mid a,X)-\mathrm{RMST}_0(t\mid a,X)=\int_0^{\tau}\{\widehat{S}(t\mid a,X)-S_0(t\mid a,X)\}\mathrm{d}t\\
=&-\int_0^{\tau}\left[\widehat{S}(t\mid a,X)\int_0^t\frac{S_0(u-\mid a,X)}{\widehat{S}(u\mid a,X)}\{\mathrm{d}\widehat{\Lambda}(u\mid a,X)-\mathrm{d}\Lambda_0(u\mid a,X)\}\right]\mathrm{d}t\\
=&-\widehat{\mathrm{RMST}}(\tau\mid a,X)\int_0^{\tau}\frac{S_0(t-\mid a,X)}{\widehat{S}(t\mid a,X)}\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}\\
&+\int_0^{\tau}\frac{\widehat{\mathrm{RMST}}(t\mid a,X)S_0(t-\mid a,X)}{\widehat{S}(t\mid a,X)}\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}
\end{aligned}
$$

$\widehat{\ }$将条件事件鞅积分项和限制平均生存时间$(t\mid a,\ X)$ − 限制平均生存时间$_0(t\mid a,\ X)$代入$P_0\{\widehat{\phi}^{\mathrm{RMST},h}(\tau,\ a;\ \widehat{\eta}^{\Lambda,\ h})\}-\psi^{\mathrm{RMST}}{}_{,\ 0}{}^{h}(\tau,\ a)$，即可得到引理 2 中所示的等式。

接下来，我们考虑竞争风险情境

$$
\begin{aligned}
&P_0\{\widehat{\phi}_j^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_j^{\Lambda,h})\}-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\\
=&[P_n\{\widehat{w}^h(a,X)\}]^{-1}E_0\left[\frac{\widehat{h}(X)\pi_0(a\mid X)}{\widehat{\pi}(a\mid X)}E_0\left\{\tau\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\widehat{M}_j(t\mid a,X)}{\widehat{G}(t-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}\widehat{M}_j(t\mid a,X)}{\widehat{G}(t-\mid a,X)}\right.\right.\\
&-\widehat{\mathrm{RMTL}}_j(\tau\mid a,X)\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}+\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{\mathrm{RMTL}}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\\
&\left.\left.+\tau\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{F}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{t\widehat{F}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\right]\\
&+[P_n\{\widehat{h}(X)\}]^{-1}E[\widehat{h}(X)\{\widehat{\mathrm{RMTL}}_j(\tau\mid a,X)\}]-[E_0\{h_0(X)\}]^{-1}E_0[h_0(X)\mathrm{RMTL}_{j,0}(\tau\mid a,X)]
\end{aligned}
$$

$$
\begin{aligned}
=&[P_n\{\widehat{w}^h(a,X)\}]^{-1}E_0\left[\frac{\widehat{h}(X)\pi_0(a\mid X)}{\widehat{\pi}(a\mid X)}E_0\left\{\tau\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\widehat{M}_j(t\mid a,X)}{\widehat{G}(t-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}\widehat{M}_j(t\mid a,X)}{\widehat{G}(t-\mid a,X)}\right.\right.\\
&-\widehat{\mathrm{RMTL}}_j(\tau\mid a,X)\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}+\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{\mathrm{RMTL}}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\\
&\left.+\tau\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{F}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}-\int_0^{\tau\wedge\widetilde{T}}\frac{t\widehat{F}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\\
&+\widehat{h}(X)\{\widehat{\mathrm{RMST}}(\tau\mid a,X)-\mathrm{RMST}(\tau\mid a,X)\}\Big]\\
&+([P_n\{\widehat{h}(X)\}]^{-1}-[P_n\{\widehat{w}^h(a,X)\}]^{-1})E_0\{\widehat{h}(X)\widehat{\mathrm{RMTL}}_j(\tau\mid a,X)\}\\
&+([P_n\{\widehat{w}^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1})E_0\{\widehat{h}(X)\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}\\
&+[E_0\{h(X)\}]^{-1}E_0[\{\widehat{h}(X)-h_0(X)\}\mathrm{RMTL}_{j,0}(\tau\mid a,X)]
\end{aligned}
$$

我们推导出关于条件原因特异性鞅积分的等式

$$
\begin{aligned}
E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}\widehat{M}_j(t\mid a,X)}{\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}&=-\int_0^{\tau}\frac{S_0(t\mid a,X)G_0(t-\mid a,X)\{\mathrm{d}\widehat{\Lambda}_j(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}}{\widehat{G}(t-\mid a,X)}\\
E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}\widehat{M}_j(t\mid a,X)}{\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}&=-\int_0^{\tau}\frac{tS_0(t\mid a,X)G_0(t-\mid a,X)\{\mathrm{d}\widehat{\Lambda}_j(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}}{\widehat{G}(t-\mid a,X)}
\end{aligned}
$$

条件事件鞅积分也类似,

$$
\begin{aligned}
&E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{\mathrm{RMTL}}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\\
=&-\int_0^{\tau}\frac{S_0(t\mid a,X)G_0(t-\mid a,X)\widehat{\mathrm{RMTL}}_j(t\mid a,X)\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\\
&E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\widehat{F}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\\
=&-\int_0^{\tau}\frac{S_0(t\mid a,X)G_0(t-\mid a,X)\widehat{F}_j(t\mid a,X)\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\\
&E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{t\widehat{F}_j(t\mid a,X)\mathrm{d}\widehat{M}(t\mid a,X)}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}\mid A=a,X\right\}\\
=&-\int_0^{\tau}\frac{tS_0(t\mid a,X)G_0(t-\mid a,X)\widehat{F}_j(t\mid a,X)\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}}{\widehat{S}(t\mid a,X)\widehat{G}(t-\mid a,X)}
\end{aligned}
$$

$\widehat{\ }$我们希望用条件风险差推导 $\mathrm{RMTL}_{j,\,0}(\tau\mid a,\ \ X)-\mathrm{RMTL}_j(\tau\mid a,\ \ X)$。

首先，我们考虑

$$
\begin{aligned}
&\widehat{F}_j(t \mid a, X) - F_{j,0}(t \mid a, X) = \int_0^t \widehat{S}(u- \mid a, X)\mathrm{d}\widehat{\Lambda}_j(u \mid a, X) - \int_0^t S_0(u- \mid a, X)\mathrm{d}\Lambda_{j,0}(u \mid a, X) \\
=&\int_0^t \{\widehat{S}(u- \mid a, X) - S_0(u- \mid a, X)\}\mathrm{d}\widehat{\Lambda}_j(u \mid a, X) + \int_0^t S_0(u- \mid a, X)\{\mathrm{d}\widehat{\Lambda}_j(u \mid a, X) - \mathrm{d}\Lambda_{j,0}(u \mid a, X)\} \\
=&\int_0^t \left[-\widehat{S}(u- \mid a, X)\int_0^u \frac{S_0(s- \mid a, X)}{\widehat{S}(s- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(s- \mid a, X) - \mathrm{d}\Lambda_0(s- \mid a, X)\}\right]\mathrm{d}\widehat{\Lambda}_j(u \mid a, X) \\
&+ \int_0^t S_0(u- \mid a, X)\{\mathrm{d}\widehat{\Lambda}_j(u \mid a, X) - \mathrm{d}\Lambda_{j,0}(u \mid a, X)\} \\
=&\int_0^t \left[-\int_0^u \frac{S_0(s- \mid a, X)}{\widehat{S}(s- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(s \mid a, X) - \mathrm{d}\Lambda_0(s \mid a, X)\}\right]\mathrm{d}\widehat{F}_j(u \mid a, X) \\
&+ \int_0^t S_0(u- \mid a, X)\{\mathrm{d}\widehat{\Lambda}_j(u \mid a, X) - \mathrm{d}\Lambda_{j,0}(u \mid a, X)\} \\
=&-\widehat{F}_j(t \mid a, X)\int_0^t \frac{S_0(u- \mid a, X)}{\widehat{S}(u- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(u \mid a, X) - \mathrm{d}\Lambda_0(u \mid a, X)\} + \int_0^t \frac{\widehat{F}_j(u \mid a, X)S_0(u- \mid a, X)}{\widehat{S}(u- \mid a, X)} \\
&\times \{\mathrm{d}\widehat{\Lambda}(u \mid a, X) - \mathrm{d}\Lambda_0(u \mid a, X)\} + \int_0^t S_0(u- \mid a, X)\{\mathrm{d}\widehat{\Lambda}_j(u \mid a, X) - \mathrm{d}\Lambda_{j,0}(u \mid a, X)\}
\end{aligned}
$$

由于第一项的积分

$$
\begin{aligned}
&\int_0^\tau \left[\widehat{F}_j(t \mid a, X)\int_0^t \frac{S_0(u- \mid a, X)}{\widehat{S}(u- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(u \mid a, X) - \mathrm{d}\Lambda_0(u \mid a, X)\}\right]\mathrm{d}t \\
=&\widehat{\mathrm{RMTL}}_j(\tau \mid a, X)\int_0^\tau \frac{S_0(t- \mid a, X)}{\widehat{S}(t- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(t \mid a, X) - \mathrm{d}\Lambda_0(t \mid a, X)\} \\
&- \int_0^\tau \frac{\widehat{\mathrm{RMTL}}_j(t \mid a, X)S_0(t- \mid a, X)}{\widehat{S}(t- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(t \mid a, X) - \mathrm{d}\Lambda_0(t \mid a, X)\}
\end{aligned}
$$

第二项的积分

$$
\begin{aligned}
&\int_0^\tau \left[\int_0^t \frac{\widehat{F}_j(u \mid a, X)S_0(u- \mid a, X)}{\widehat{S}(u- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(u \mid a, X) - \mathrm{d}\Lambda_0(u \mid a, X)\}\right]\mathrm{d}t \\
=&\tau\int_0^\tau \frac{\widehat{F}_j(t \mid a, X)S_0(t- \mid a, X)}{\widehat{S}(t- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(t \mid a, X) - \mathrm{d}\Lambda_0(t \mid a, X)\} \\
&- \int_0^\tau \frac{t\widehat{F}_j(t \mid a, X)S_0(t- \mid a, X)}{\widehat{S}(t- \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(t \mid a, X) - \mathrm{d}\Lambda_0(t \mid a, X)\}
\end{aligned}
$$

第三项的积分

$$
\begin{aligned}
&\int_0^\tau \left[\int_0^t S_0(u- \mid a, X)\{\mathrm{d}\widehat{\Lambda}_j(u \mid a, X) - \mathrm{d}\Lambda_{j,0}(u \mid a, X)\}\right]\mathrm{d}t \\
=&\int_0^\tau tS_0(t- \mid a, X)\{\mathrm{d}\Lambda_{j,0}(t \mid a, X) - \mathrm{d}\widehat{\Lambda}_j(t \mid a, X)\} - \tau\int_0^\tau S_0(t- \mid a, X)\{\mathrm{d}\Lambda_{j,0}(t \mid a, X) - \mathrm{d}\widehat{\Lambda}_j(t \mid a, X)\}
\end{aligned}
$$

为使 $\widehat{\quad}$ $\widehat{\quad}$ $\widehat{\quad}$利用杜阿梅尔方程，我们还假设 S 连续，即 $S(t- \mid a, X) = S(t \mid a, X$),

我们有

$$\begin{aligned}
&\widehat{\mathrm{RMTL}}_j(\tau \mid a, X) - \mathrm{RMTL}_{j,0}(\tau \mid a, X) = \int_0^\tau \{\widehat{F}_j(t \mid a, X) - F_{j,0}(t \mid a, X)\}\mathrm{d}t \\
=& -\int_0^\tau \frac{S_0(t- \mid a, X)\{\widehat{\mathrm{RMTL}}_j(\tau \mid a, X) - \widehat{\mathrm{RMTL}}_j(t \mid a, X)\}}{\widehat{S}(t \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(t \mid a, X) - \mathrm{d}\Lambda_0(t \mid a, X)\} \\
&+ \int_0^\tau \frac{(\tau - t)\widehat{F}_j(t \mid a, X)S_0(t- \mid a, X)}{\widehat{S}(t \mid a, X)}\{\mathrm{d}\widehat{\Lambda}(t \mid a, X) - \mathrm{d}\Lambda_0(t \mid a, X)\} \\
&+ \int_0^\tau (\tau - t)S_0(t- \mid a, X)\{\mathrm{d}\widehat{\Lambda}_j(t \mid a, X) - \mathrm{d}\Lambda_{j,0}(t \mid a, X)\}
\end{aligned}$$

将条件鞅积分项和 $\mathrm{RMTL}_j(t \mid a, X) - \mathrm{RMTL}_{j,0}(t \mid a, X)$ 替换为条件风险差后，我们得到所示的 $P_0\{\widehat{\phi}_j^{\mathrm{RMTL},h}(\tau, a; \widehat{\eta}^{\Lambda,h}_{\cdot,j})\} - \psi_0^{\mathrm{RMTL},h}(\tau, a)$ 等式。

D.4 引理 3

引理 3. 如果条件2成立，那么对于每个 a、τ 和 k，存在两个通用常数向量 $C^{\mathrm{RMST},h}_\iota$ 和 $C^{\mathrm{RMTL},h}_{j\iota}$，使得

$$\begin{aligned}
&[P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau, a; \widehat{\eta}_k^{\Lambda,h}) - \phi_\infty^{\mathrm{RMST},h}(\tau, a; \eta_\infty^{\Lambda,h})\}^2]^{1/2} \le \sum_{\iota=1}^{6} C_\iota^{\mathrm{RMST},h}\mathcal{B}_{\iota,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h}) \\
&[P_0\{\sup_{t\in[0,\tau]} |\widehat{\phi}_k^{\mathrm{RMST},h}(t, a; \widehat{\eta}_k^{\Lambda,h}) - \phi_\infty^{\mathrm{RMST},h}(t, a; \eta_\infty^{\Lambda,h})|\}^2]^{1/2} \\
&\le C_1^{\mathrm{RMST},h}\widetilde{\mathcal{B}}_{1,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h}) + \sum_{\iota=2}^{6} C_\iota^{\mathrm{RMST},h}\mathcal{B}_{\iota,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h}) \\
&[P_0\{\widehat{\phi}_{j,k}^{\mathrm{RMTL},h}(\tau, a; \widehat{\eta}_{j,k}^{\Lambda,h}) - \phi_{j,\infty}^{\mathrm{RMTL},h}(\tau, a; \eta_\infty^{\Lambda,h})\}^2]^{1/2} \le \sum_{\iota=1}^{9} C_{j,\iota}^{\mathrm{RMTL},h}\mathcal{B}_{j,\iota,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h}, \eta_{j,\infty}^{\Lambda,h}) \\
&[P_0\{\sup_{t\in[0,\tau]} |\widehat{\phi}_{j,k}^{\mathrm{RMTL},h}(t, a; \widehat{\eta}_{j,k}^{\Lambda,h}) - \phi_{j,\infty}^{\mathrm{RMTL},h}(t, a; \eta_\infty^{\Lambda,h})|\}^2]^{1/2} \\
\le& C_{j,1}^{\mathrm{RMTL},h}\widetilde{\mathcal{B}}_{j,1,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h}, \eta_{j,\infty}^{\Lambda,h}) + \sum_{\iota=2}^{9} C_{j,\iota}^{\mathrm{RMTL},h}\mathcal{B}_{j,\iota,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h}, \eta_{j,\infty}^{\Lambda,h})
\end{aligned}$$

其中

$$\begin{aligned}
\{\mathcal{B}_{1,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h})\}^2 &= E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(\tau \mid a, X)}{\widehat{S}_k(t \mid a, X)} - \frac{\mathrm{RMST}_\infty(\tau \mid a, X)}{S_\infty(t \mid a, X)}\right|\right]^2 \\
\{\widetilde{\mathcal{B}}_{1,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h})\}^2 &= E_0\left[\sup_{t\in[0,\tau]}\sup_{u\in[0,t]}\left|\frac{\widehat{\mathrm{RMST}}_k(t \mid a, X)}{\widehat{S}_k(u \mid a, X)} - \frac{\mathrm{RMST}_\infty(t \mid a, X)}{S_\infty(u \mid a, X)}\right|\right]^2 \\
\{\mathcal{B}_{2,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h})\}^2 &= E_0|[P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X) - [E_0\{h_\infty(X)\}]^{-1}h_\infty(X)|^2 \\
\{\mathcal{B}_{3,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h})\}^2 &= ([P_{n,k}\left\{\widehat{w}_k^h(a, X)\right\}]^{-1} - [E_0\{h_\infty(X)\}]^{-1})^2 \\
\{\mathcal{B}_{4,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h})\}^2 &= E_0\left|\widehat{w}_k^h(a, X) - w_\infty^h(a, X)\right|^2 \\
\{\mathcal{B}_{5,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h}, \eta_\infty^{\Lambda,h})\}^2 &= E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t- \mid a, X)} - \frac{1}{G_\infty(t- \mid a, X)}\right|\right]^2
\end{aligned}$$

$$\{\mathcal{B}_{6,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2 = E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2$$

and

$$\{\mathcal{B}_{j,1,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2 \xrightarrow{P} 0$$

$$\{\mathcal{B}_{j,2,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k E_0|[P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)-[E_0\{h_\infty(X)\}]^{-1}h_\infty(X))|^2 \xrightarrow{P} 0$$

$$\{\mathcal{B}_{j,3,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2 \xrightarrow{P} 0$$

$$\{\mathcal{B}_{j,4,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2 \xrightarrow{P} 0$$

$$\{\mathcal{B}_{j,5,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2 \xrightarrow{P} 0$$

$$\{\mathcal{B}_{j,6,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k E_0\left[\sup_{t\in[0,\tau]}\left|\widehat{\Lambda}_{j,k}(t\mid a,X)-\Lambda_{j,\infty}(t\mid a,X)\right|\right]^2 \xrightarrow{P} 0$$

$$\{\mathcal{B}_{j,7,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMTL}_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2 \xrightarrow{P} 0$$

$$\{\mathcal{B}_{j,8,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2 \xrightarrow{P} 0$$

$$\{\mathcal{B}_{j,9,k}^{\mathrm{RMTL},h}(\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 = \max_k E_0\left[\sup_{t\in[0,\tau]}\left|\widehat{\Lambda}_k(t\mid a,X)-\Lambda_\infty(t\mid a,X)\right|\right]^2 \xrightarrow{P} 0$$

引理3的证明：我们将 $\widehat{\phi}^{\mathrm{RMST},h}(\tau,a;\ \widehat{\eta}^{\Lambda,h}_k)-\phi^{\mathrm{RMST},h}(\tau,a;\ \eta^{\Lambda,h}_\infty)=\sum_{\iota=1}^{14}\mathcal{Q}_\iota^{\mathrm{RMST},h}(\tau,a;\ \widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)$展开为若干项，这些项的期望平方有界，其中

$\mathcal{Q}_1^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)=[P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)\{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)-$限制平均生存时间$_\infty(\tau\mid a,X)\}$

$\mathcal{Q}_2^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)=([P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)-[E_0\{h_\infty(X)\}]^{-1}h_\infty(X))$限制平均生存时间$_\infty(\tau\mid a,X)$

$\mathcal{Q}_3^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)=-([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})w_\infty^h(a,X)$限制平均生存时间$_\infty(\tau\mid a,X)$

$$\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_\infty(t\mid a,X)}{S_\infty(t\mid a,X)G_\infty(t-\mid a,X)}$$

$\mathcal{Q}_4^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\{\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\}$限制平均生存时间$_\infty(\tau\mid a,X)$

$$\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_\infty(t\mid a,X)}{S_\infty(t\mid a,X)G_\infty(t-\mid a,X)}$$

$$\mathcal{Q}_5^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\frac{\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\Delta=1)}{G_\infty(\widetilde{T}-\mid a,X)}$$

$$\times\left\{\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(\widetilde{T}\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(\widetilde{T}\mid a,X)}\right\}$$

$$\mathcal{Q}_{6,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h}) = -[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\frac{\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\Delta=1)\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(\widetilde{T}\mid a,X)}$$

$$\times\left\{\frac{1}{\widehat{G}_k(\widetilde{T}-\mid a,X)}-\frac{1}{G_\infty(\widetilde{T}-\mid a,X)}\right\}$$

$$\mathcal{Q}^{\mathrm{RMST},\ 7\ k}{}^{h},(\tau,a;\widehat{\eta}^{\Lambda}{}_{,k}{}^{h},\eta^{\Lambda}{}_{,\infty}{}^{h}) = [P_{n,k}\{\widehat{w}_k^h(a,\ X)\}]^{-1}\widehat{w}_k^h(a,\ X)\text{限制平均生存时间}_\infty(\tau\mid a,\ X)\int_0^\tau \mathrm{d}\Lambda_\infty(t\mid a,\ X)\,S_\infty(t\mid a,\ X)$$

$$\times\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right\}$$

$$\mathcal{Q}^{\mathrm{RMST},8\ k}{}^{h},(\tau,a;\widehat{\eta}^{\Lambda}{}_{,k}{}^{h},\eta^{\Lambda}{}_{,\infty}{}^{h}) = [P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\Big\{\widehat{\mathrm{R}}\mathrm{MST}_k(\tau\mid a,X)$$

$$\times\int_0^\tau\frac{\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}-\mathrm{RMST}_\infty(\tau\mid a,X)\int_0^\tau\frac{\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\Big\}$$

$$\mathcal{Q}^{\mathrm{RMST},9\ k}{}^{h},(\tau,a;\widehat{\eta}^{\Lambda}{}_{,k}{}^{h},\eta^{\Lambda}{}_{,\infty}{}^{h}) = ([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E\{h_\infty(X)\}]^{-1})w_\infty^h(a,X)$$

$$\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_\infty(t\mid a,X)\mathrm{d}M_\infty(t\mid a,X)}{S_\infty(t\mid a,X)G_\infty(t-\mid a,X)}$$

$$\mathcal{Q}^{\mathrm{RMST},\ 10\ k}{}^{h},(\tau,\ a;\widehat{\eta}^{\Lambda}{}_{,k}{}^{h},\ \eta^{\Lambda}{}_{,\infty}{}^{h}) = [P_{n,k}\{\widehat{w}_k^h(a,\ X)\}]^{-1}\{\widehat{w}_k^h(a,\ X)-w_\infty^h(a,\ X)\}\int_0^{\tau\wedge\widetilde{T}}\frac{\text{限制平均生存时间}_\infty(t\mid a,X)\mathrm{d}M_\infty(t\mid a,X)}{S_\infty(t\mid a,\ X)G_\infty(t-\mid a,\ X)}$$

$$\mathcal{Q}_{11,k}^{\mathrm{RMST},h}(\tau,\ a;\widehat{\eta}_k^{\Lambda,h}\ \eta^{\Lambda,\infty}{}^{h}) = [P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\frac{\widehat{w}_k^h(a,X)I(\ \widetilde{T}\le\tau,\Delta=1)}{G_\infty(\widetilde{T}-\mid a,X)}$$

$$\times\left\{\frac{\widehat{\mathrm{RMST}}_k(\widetilde{T}\mid a,X)}{\widehat{S}_k(\widetilde{T}\mid a,X)}-\frac{\mathrm{RMST}_\infty(\widetilde{T}\mid a,X)}{S_\infty(\widetilde{T}\mid a,X)}\right\}$$

$$\mathcal{Q}^{\mathrm{RMST},12\ k}{}^{h},(\tau,a;\widehat{\eta}^{\Lambda}{}_{,k}{}^{h},\eta^{\Lambda}{}_{,\infty}{}^{h}) = [P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\Delta=1)\widehat{\mathrm{R}}\text{限制平均生存时间}_k(\widetilde{T}\mid a,X)\,\widehat{S}_k(\widetilde{T}\mid a,X)$$

$$\times\left\{\frac{1}{\widehat{G}_k(\widetilde{T}-\mid a,X)}-\frac{1}{G_\infty(\widetilde{T}-\mid a,X)}\right\}$$

$$\mathcal{Q}^{\mathrm{RMST},13\ k}{}^{h},(\tau,a;\widehat{\eta}^{\Lambda}{}_{,k}{}^{h},\eta^{\Lambda}{}_{,\infty}{}^{h}) = -[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\int_0^\tau\frac{\text{限制平均生存时间}_\infty(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)}$$

$$\times\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right\}$$

$$\mathcal{Q}^{\mathrm{RMST},14\ k}{}^{h},(\tau,a;\widehat{\eta}^{\Lambda}{}_{,k}{}^{h},\eta^{\Lambda}{}_{,\infty}{}^{h}) = -[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\Big\{\int_0^\tau\frac{\widehat{\text{限制平均生存时间}}_k(t\mid a,X)\mathrm{d}\Lambda_k}{\widehat{S}_k(t\mid a,X)\,\widehat{G}_k(t-\mid a,\ X)}$$

$$-\int_0^\tau\frac{\text{限制平均生存时间}_\infty(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)\,S_\infty(t\mid a,X)\,\widehat{G}_k(t-\mid a,X)}\Big\}$$

根据三角不等式,

$$P_0\{\widehat{\phi}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2\le\left(\sum_{\iota=1}^{14}[P_0\{\mathcal{Q}_{\iota,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2]^{1/2}\right)^2$$

因此，我们需要对 $P_0\{\mathcal{Q}_{\iota,k}^{\mathrm{RMST},h}(\tau,\ a;\ \widehat{\eta}^{\Lambda,\ h}_{\cdot,\ k},\ \eta^{\Lambda,\ h}_{\cdot,\ \infty})\}^2,\ \forall \iota$ 进行界定。

$$
\begin{aligned}
P_0\{\mathcal{Q}_{1,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2 =& E_0\left|[P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)\{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)-\mathrm{RMST}_\infty(\tau\mid a,X)\}\right|^2\\
\le& \epsilon^2 h_{\max}^2 E_0\left|\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(\tau\mid a,X)}\right|^2\\
\le& \epsilon^2 h_{\max}^2 E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2\\
P_0\{\mathcal{Q}_{2,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2 =& E_0\left|([P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)-[E_0\{h_\infty(X)\}]^{-1}h_\infty(X))\mathrm{RMST}_\infty(\tau\mid a,X)\right|^2\\
\le& \tau^2 E_0\left|([P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)-[E_0\{h_\infty(X)\}]^{-1}h_\infty(X))\right|^2
\end{aligned}
$$

Gill 和 Johansen（1990）定理 5 中的反向方程表明，$\int_0^t \frac{S}{S(t|a,X)}(u|,a,X)\mathrm{d}\Lambda(u\mid a,X) = 1-S(t\mid a,X)$ 且

$$
\begin{aligned}
&E_0\left|\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_\infty(t\mid a,X)}{S_\infty(t\mid a,X)G_\infty(t-\mid a,X)}\right|^2 = E_0\left|\frac{I(\widetilde{T}\le\tau,\Delta=1)}{S_\infty(\widetilde{T}\mid a,X)G_\infty(\widetilde{T}-\mid a,X)}-\int_0^\tau\frac{\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)G_\infty(t-\mid a,X)}\right|^2\\
\le& E_0\left|\epsilon^2+\epsilon\int_0^\tau \mathrm{d}\left\{\frac{1}{S_\infty(t\mid a,X)}\right\}\right|^2 \le E_0\left|\epsilon^2+\epsilon(\epsilon-1)\right|^2\le 4\epsilon^4
\end{aligned}
$$

$$
\begin{aligned}
P_0\{\mathcal{Q}_{3,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,\ h}_{\cdot,k},\eta^{\Lambda,\ h}_{\cdot,\infty})\}^2 =& E_0\left|([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})\right.\\
&\left.\frac{h_\infty(X)I(A=a)\mathrm{RMST}_\infty(\tau\mid a,X)}{\pi_\infty(a\mid X)}\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_\infty(t\mid a,X)}{S_\infty(t\mid a,X)G_\infty(t-\mid a,X)}\right|^2\\
\le&([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2\epsilon^2 h_{\max}^2\tau^2\times 4\epsilon^4
\end{aligned}
$$

由于

$$
P_{n,k}\{\widehat{w}_k^h(a,X)\} = P_{n,k}\left\{\frac{\widehat{h}_k(X)I(A=a)}{\widehat{\pi}_k(a\mid X)}\right\}\ge P_{n,k}\{\widehat{h}_k(X)I(A=a)\}\ge P_{n,k}\{\widehat{h}_k(X)\}\ge h_{\min}
$$

我们有

$$
\begin{aligned}
P_0\{\mathcal{Q}_{4,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,\ h}_{\cdot,k},\eta^{\Lambda,\ h}_{\cdot,\infty})\}^2 =& E_0\left|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\{\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\}\text{限制平均生存时间}_\infty(\tau\mid a,X)\right.\\
&\left.\times\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_\infty(t\mid a,X)}{S_\infty(t\mid a,X)G_\infty(t-\mid a,X)}\right|^2 \le h^{-2}{}_{\min}\tau^2\times 4\epsilon^4 E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2\\
P_0\{\mathcal{Q}_{5,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2 =& E_0\left|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\frac{\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\Delta=1)}{G_\infty(\widetilde{T}-\mid a,X)}\right.\\
&\left.\times\left\{\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(\widetilde{T}\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(\widetilde{T}\mid a,X)}\right\}\right|^2\\
\le& h_{\min}^{-2}h_{\max}^2\epsilon^4 E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2
\end{aligned}
$$

$$
\begin{aligned}
P_0\{\mathcal{Q}_{6,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2 =& E_0\left|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\frac{\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\Delta=1)\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(\widetilde{T}\mid a,X)}\right.\\
&\left.\times\left\{\frac{1}{\widehat{G}_k(\widetilde{T}-\mid a,X)}-\frac{1}{G_\infty(\widetilde{T}-\mid a,X)}\right\}\right|^2\\
\le& h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^4E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2\\
P_0\{\mathcal{Q}_{7,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda}{}_{,k}{}^h,\eta^{\Lambda}{}_{,\infty}{}^h)\}^2 =& E_0\left|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\text{限制平均生存时间}_\infty(\tau\mid a,X)\right.\\
&\left.\int_0^\tau \mathrm{d}\Lambda_\infty(t\mid a,X)\,S_\infty(t\mid a,X)\right.\\
&\left.\times\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right\}\right|^2\\
\le& h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^2E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\int_0^\tau\frac{\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)}\right]^2\\
\le& h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^2(\epsilon-1)^2E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2
\end{aligned}
$$

As

$$
\begin{aligned}
&E_0\left|\widehat{\mathrm{RMST}}_k(\tau\mid a,X)\int_0^\tau\frac{\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}-\mathrm{RMST}_\infty(\tau\mid a,X)\int_0^\tau\frac{\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|^2\\
=&E_0\left|\widehat{\mathrm{RMST}}_k(\tau\mid a,X)\left[\frac{1}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|_0^\tau-\int_0^\tau\frac{1}{\widehat{S}_k(t\mid a,X)}\mathrm{d}\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}\right\}\right]\\
&-\mathrm{RMST}_\infty(\tau\mid a,X)\left[\frac{1}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|_0^\tau-\int_0^\tau\frac{1}{S_\infty(t\mid a,X)}\mathrm{d}\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}\right\}\Bigg]\Bigg|^2\\
=&E_0\left|\left\{\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(\tau\mid a,X)}\right\}\frac{1}{\widehat{G}_k(\tau-\mid a,X)}\right.\\
&\left.-\int_0^\tau\left\{\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right\}\mathrm{d}\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}\right\}\right|^2\\
\le&\epsilon^2E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2+E_0\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}\right|_0^\tau\\
&\times\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right|\Bigg\}^2\\
\le&\{\epsilon^2+(\epsilon-1)^2\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2
\end{aligned}
$$

我们有

$$
\begin{aligned}
&P_0\{\mathcal{Q}_{8,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2\\
=&E_0\left|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\left\{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)\right.\right.\\
&\left.\left.\times\int_0^\tau\frac{\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}-\mathrm{RMST}_\infty(\tau\mid a,X)\int_0^\tau\frac{\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right\}\right|^2\\
\le&h_{\min}^{-2}h_{\max}^2\epsilon^2\{\epsilon^2+(\epsilon-1)^2\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2
\end{aligned}
$$

与前面的项类似,

$$
\begin{aligned}
&P_0\{\mathcal{Q}_{9,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2\le([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2\epsilon^2h_{\max}^2\tau^2\times4\epsilon^4\\
&P_0\{\mathcal{Q}_{10,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2\le h_{\min}^{-2}\tau^2\times4\epsilon^4E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2\\
&P_0\{\mathcal{Q}_{11,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2\le h_{\min}^{-2}h_{\max}^2\epsilon^4E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2\\
&P_0\{\mathcal{Q}_{12,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2\le h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^4E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2\\
&P_0\{\mathcal{Q}_{13,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2\le h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^2(\epsilon-1)^2E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2
\end{aligned}
$$

As

$$
\begin{aligned}
&E_0\left|\int_0^\tau\frac{\widehat{\mathrm{RMST}}_k(t\mid a,X)\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}-\int_0^\tau\frac{\mathrm{RMST}_\infty(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|^2\\
=&E_0\left|\frac{\widehat{\mathrm{RMST}}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|_0^\tau-\int_0^\tau\frac{1}{\widehat{S}_k(t\mid a,X)}\mathrm{d}\left\{\frac{\widehat{\mathrm{RMST}}_k(t\mid a,X)}{\widehat{G}_k(t-\mid a,X)}\right\}\right.\\
&\left.-\frac{\mathrm{RMST}_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|_0^\tau+\int_0^\tau\frac{1}{S_\infty(t\mid a,X)}\mathrm{d}\left\{\frac{\mathrm{RMST}_\infty(t\mid a,X)}{\widehat{G}_k(t-\mid a,X)}\right\}\Bigg|^2\\
\le&\left\{\epsilon^2+(\epsilon-1)^2\right\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2
\end{aligned}
$$

Thus

$$
P_0\{\mathcal{Q}_{14,k}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})\}^2\le h_{\min}^{-2}h_{\max}^2\epsilon^2\{\epsilon^2+(\epsilon-1)^2\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMST}}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMST}_\infty(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2
$$

在竞争风险情境下，我们展开 $\widehat{\phi}^{\mathrm{RMTL}}{}_{,j\,k}{}^{h}{}_{,}(\tau,a;\,\widehat{\eta}^{\Lambda}{}_{,j\,k}{}^{h}{}_{,},\eta^{\Lambda}{}_{,j\,\infty}{}^{h}{}_{,})-\phi^{\mathrm{RMTL}}{}_{,j\,\infty}{}^{h}{}_{,}(\tau,a;\,\widehat{\eta}^{\Lambda}{}_{,j\,k}{}^{h}{}_{,},\eta^{\Lambda}{}_{,j\,\infty}{}^{h}{}_{,})=$

$\sum_{\iota=1}^{36}\mathcal{Q}^{\mathrm{RMTL},j}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)$，其期望平方有界，其中$\mathcal{Q}^{\mathrm{RMTL},j\,1\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)$
$=[\widehat{P}_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)\{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\;-\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\}\mathcal{Q}^{\mathrm{RMTL},j\,2\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=($
$[P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)\;-[E_0\{h_\infty(X)\}]^{-1}h_\infty(X))\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,3\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=($
$[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})w_\infty^h(a,X)\tau\int_0^{\tau\wedge T}\mathrm{d}M_{j,\infty}(t\mid a,X)\,G_\infty(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,4\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},$
$,\eta^{\Lambda},{}_{j\,\infty}^{h},)=[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\{\widehat{w}_k^h(a,X)\;-w_\infty^h(a,X)\}\tau\int_0^{\tau\wedge T}\mathrm{d}M_{j,\infty}(t\mid a,X)\,G_\infty(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,5\,k}{}^{h},,(\tau,a;$
$\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\widetilde{J}=j)\tau\left\{\widehat{G}_k{}^{1}(\widetilde{T}-\mid a,X)\;-G_\infty{}^{1}(\widetilde{T}-\mid a,X)\right\}$
$\mathcal{Q}^{\mathrm{RMTL},j\,6\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\tau\int_0^{\tau}\left\{\widehat{G}_k{}^{1}(t-\mid a,X)\;-G_\infty{}^{1}(\right.$
$\left.t-\mid a,X)\right\}\mathrm{d}\Lambda_{j,\infty}(t\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,7\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\tau\int_0^{\tau}\mathrm{d}\Lambda_{j,k}($
$t\mid a,X)\;-\mathrm{d}\Lambda_{j,\infty}(t\mid a,X)\,\widehat{G}_k(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,8\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=-([P_{n,k}\{\widehat{w}_k^h(a,X)$
$\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})w_\infty^h(a,X)\int_0^{\tau\wedge T}\mathrm{d}M_{j,\infty}(t\mid a,X)\,G_\infty(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,9\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)$
$=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\{\widehat{w}_k^h(a,X)\;-w_\infty^h(a,X)\}\int_0^{\tau\wedge T}t\,\mathrm{d}M_{j,\infty}(t\mid a,X)\,G_\infty(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,10\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,$
$\eta^{\Lambda},{}_{j\,\infty}^{h},)=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\widetilde{J}=j)\,\widetilde{T}\left\{\widehat{G}_k{}^{1}(\widetilde{T}-\mid a,X)\;-G_\infty{}^{1}(\widetilde{T}-\mid a,X)\right\}$
$\mathcal{Q}^{\mathrm{RMTL},j\,11\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\int_0^{\tau}\left\{\widehat{G}_k{}^{1}(t-\mid a,X)\;-G_\infty{}^{1}(t-\mid a,\right.$
$\left.X)\right\}t\,\mathrm{d}\Lambda_{j,\infty}(t\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,12\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)\int_0^{\tau}t\{\mathrm{d}\Lambda_{j,k}(t\mid a,$
$X\;-\;\Lambda_{j\,\infty}\;t\mid a\;X\;\})\,\mathrm{d}\,,(\,,)\,(\,,)\mathcal{Q}^{\mathrm{RMTL},j\,13\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=-([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)$
$\widehat{G}_k\;t-\mid a\;X$
$\}]^{-1})w_\infty^h(a,X)\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\times\int_0^{\tau\wedge T}\mathrm{d}M_\infty(t\mid a,X)\,S_\infty(t\mid a,X)G_\infty(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,14\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},$
$_{j\,\infty}^{h},)=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\{\widehat{w}_k^h(a,X)\;-w_\infty^h(a,X)\}\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\times\int_0^{\tau\wedge T}\mathrm{d}M_\infty(t\mid a,X)\,S_\infty(t\mid a,X)G_\infty($
$t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\,15\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\Delta=1)\,G_\infty(\widetilde{T}-\mid a,X)$
$\times\left\{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\,\widehat{S}_k(\widetilde{T}\mid a,X)\;-\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\,S_\infty(\widetilde{T}\mid a,X)\right\}\mathcal{Q}^{\mathrm{RMTL},j\,16\,k}{}^{h},,(\tau,a;\,\widehat{\eta}^{\Lambda},{}_{j\,k}^{h},,\eta^{\Lambda},{}_{j\,\infty}^{h},)$
$=-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\Delta=1)\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\,\widehat{S}_k(\widetilde{T}\mid a,X)\times\left\{\widehat{G}_k{}^{1}(\widetilde{T}-\mid a,X)\right.$
$\left.-\;G_\infty{}^{1}(\widetilde{T}-\mid a,X)\right\}$

$$\mathcal{Q}^{\mathrm{RMTL},j\ 17\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\widehat{w}^{h}_{k}(a,X)\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\int_{0}^{\tau}\mathrm{d}\Lambda_{\infty}(t\mid a,X)\,S_{\infty}(t\mid a,X)\times\left\{\frac{1}{\widehat{G}_{k}\quad(t-\mid a,X)}-\frac{1}{G_{\infty}\quad(t-\mid a,X)}\right\}\mathcal{Q}^{\mathrm{RMTL},j\ 18\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\widehat{w}^{h}_{k}(a,X)\left\{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\int_{0}^{\tau}\mathrm{d}\widehat{\Lambda}_{k}(t\mid a,X)\,\widehat{S}_{k}(t\mid a,X)\,\widehat{G}_{k}(t-\mid a,X)-\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\int_{0}^{\tau}\mathrm{d}\Lambda_{\infty}(t\mid a,X)\,S_{\infty}(t\mid a,X)\,\widehat{G}_{k}(t-\mid a,X)\right\}_{\sim}\mathcal{Q}^{\mathrm{RMTL},j\ 19\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=([P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}-[E_{0}\{h_{\infty}(X)\}]^{-1})w^{h}_{\infty}(a,X)\int_{0}^{\tau\wedge T}\mathrm{RMTL}_{j,\infty}(t\mid a,X)\mathrm{d}M_{\infty}(t\mid a,X)\,S_{\infty}(t\mid a,X)G_{\infty}(t-\mid a,X)_{\sim}\mathcal{Q}^{\mathrm{RMTL},j\ 20\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\{\widehat{w}^{h}_{k}(a,X)-w^{h}_{\infty}(a,X)\}\int_{0}^{\tau\wedge T}\mathrm{RMTL}_{j,\infty}(t\mid a,X)\mathrm{d}M_{\infty}(t\mid a,X)\,S_{\infty}(t\mid a,X)G_{\infty}(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\ 21\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\widehat{w}^{h}_{k}(a,X)I(\widetilde{T}\le\tau,\Delta=1)\,G_{\infty}(\widetilde{T}-\mid a,X)\times\left\{\widehat{\mathrm{RMTL}}_{j,k}(\widetilde{T}\mid a,X)\,\widehat{S}_{k}(\widetilde{T}\mid a,X)-\mathrm{RMTL}_{j,\infty}(\widetilde{T}\mid a,X)\,S_{\infty}(\widetilde{T}\mid a,X)\right\}\mathcal{Q}^{\mathrm{RMTL},j\ 22\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\widehat{w}^{h}_{k}(a,X)I(\widetilde{T}\le\tau,\Delta=1)\widehat{\mathrm{R}}\mathrm{MTL}_{j,k}(\widetilde{T}\mid a,X)\,\widehat{S}_{k}(\widetilde{T}\mid a,X)\times\left\{\frac{1}{\widehat{G}_{k}\quad(\widetilde{T}-\mid a,X)}-\frac{1}{G_{\infty}\quad(\widetilde{T}-\mid a,X)}\right\}\mathcal{Q}^{\mathrm{RMTL},j\ 23\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=-[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\widehat{w}^{h}_{k}(a,X)\int_{0}^{\tau}\mathrm{RMTL}_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_{\infty}(t\mid a,X)\,S_{\infty}(t\mid a,X)\times\left\{\frac{1}{\widehat{G}_{k}\quad(t-\mid a,X)}-\frac{1}{G_{\infty}\quad(t-\mid a,X)}\right\}\mathcal{Q}^{\mathrm{RMTL},j\ 24\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=-[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\widehat{w}^{h}_{k}(a,X)\left\{\int_{0}^{\tau}\widehat{\mathrm{R}}\mathrm{MTL}_{j,k}(t\mid a,X)\mathrm{d}\widehat{\Lambda}_{k}(t\mid a,X)\,\widehat{S}_{k}(t\mid a,X)\,\widehat{G}_{k}(t-\mid a,X)-\int_{0}^{\tau}\mathrm{RMTL}_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_{\infty}(t\mid a,X)\,S_{\infty}(t\mid a,X)\,\widehat{G}_{k}(t-\mid a,X)\right\}_{\sim}\mathcal{Q}^{\mathrm{RMTL},j\ 25\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=([P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}-[E_{0}\{h_{\infty}(X)\}]^{-1})w^{h}_{\infty}(a,X)\tau\int_{0}^{\tau\wedge T}F_{j,\infty}(t\mid a,X)\mathrm{d}M_{\infty}(t\mid a,X)\,S_{\infty}(t\mid a,X)G_{\infty}(t-\mid a,X)_{\sim}\mathcal{Q}^{\mathrm{RMTL},j\ 26\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\{\widehat{w}^{h}_{k}(a,X)-w^{h}_{\infty}(a,X)\}\tau\int_{0}^{\tau\wedge T}F_{j,\infty}(t\mid a,X)\mathrm{d}M_{\infty}(t\mid a,X)\,S_{\infty}(t\mid a,X)G_{\infty}(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\ 27\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\widehat{w}^{h}_{k}(a,X)I(\widetilde{T}\le\tau,\Delta=1)G_{\infty}(\widetilde{T}-\mid a,X)\left\{\widehat{F}_{j,k}(\widetilde{T}\mid a,X)\,\widehat{S}_{k}(\widetilde{T}\mid a,X)-F_{j,\infty}(\widetilde{T}\mid a,X)\,S_{\infty}(\widetilde{T}\mid a,X)\right\}\mathcal{Q}^{\mathrm{RMTL},j\ 28\ k}{}^{h}_{,,}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k}{}^{h}_{,,}\eta^{\Lambda,j\ \infty}{}^{h}_{,})=[P_{n,k}\{\widehat{w}^{h}_{k}(a,X)\}]^{-1}\widehat{w}^{h}_{k}(a,X)I(\widetilde{T}\le\tau,\Delta=1)\tau F_{j,\infty}(\widetilde{T}\mid a,X)\,\widehat{S}_{k}(\widetilde{T}\mid a,X)\times\left\{\frac{1}{\widehat{G}_{k}\quad(\widetilde{T}-\mid a,X)}-\frac{1}{G_{\infty}\quad(\widetilde{T}-\mid a,X)}\right\}$$

$$\mathcal{Q}_{j,29,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h}) = -\left[P_{n,k}\{\widehat{w}_k^h(a,X)\}\right]^{-1}\widehat{w}_k^h(a,X)\tau\int_0^\tau \frac{F_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)}$$
$$\times\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right\}$$

$$\mathcal{Q}_{j,30,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h}) = -\left[P_{n,k}\{\widehat{w}_k^h(a,X)\}\right]^{-1}\widehat{w}_k^h(a,X)\tau\left\{\int_0^\tau \frac{\widehat{F}_{j,k}(t\mid a,X)\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right.$$
$$\left.-\int_0^\tau \frac{F_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right\}$$

$$\sim\mathcal{Q}^{\mathrm{RMTL},j\ 31\ k\ \ h}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k\ \ h},\eta^{\Lambda,j\ \infty\ h},) = -\left([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1}\right)w_\infty^h(a,X)\int_0^{\tau\wedge T} tF_{j,\infty}($$

$$t\mid a,X)\mathrm{d}M_\infty(t\mid a,X)\ S_\infty(t\mid a,X)G_\infty(t-\mid a,X)\sim\mathcal{Q}^{\mathrm{RMTL},j\ 32\ k\ \ h}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k\ \ h},\eta^{\Lambda,j\ \infty\ h},) = -\left[P_{n,k}\{\widehat{w}_k^h(a,X)\right.$$
$$\left.\}\right]^{-1}\{\widehat{w}_k^h(a,X)\ -w_\infty^h(a,X)\}\int_0^{\tau\wedge T} tF_{j,\infty}(t\mid a,X)\mathrm{d}M_\infty(t\mid a,X)\ S_\infty(t\mid a,X)G_\infty(t-\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\ 33\ k\ \ h}(\tau,a;\ \widehat{\eta}^{\Lambda},$$

$$_{j\ k\ \ h},\eta^{\Lambda,j\ \infty\ h},) = -\left[P_{n,k}\{\widehat{w}_k^h(a,X)\}\right]^{-1}\widehat{w}_k^h(a,X\ I\ \widetilde{T}\le\tau\ \Delta=1\ \widetilde{T})(\ ,)\ (\ ,)\left\{\begin{array}{l}F_j\\ ,k(\widetilde{T}\mid a,X)\ \widehat{S}_k(\widetilde{T}\mid a,X)\ _\ F_{j,\infty}(\widetilde{T}\mid a,\\ G_\infty\ \widetilde{T}-\mid a\ X\end{array}\right.$$
$$\left.X)\ S_\infty(\widetilde{T}\mid a,X)\right\}\mathcal{Q}^{\mathrm{RMTL},j\ 34\ k\ \ h}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k\ \ h},\eta^{\Lambda,j\ \infty\ h},) = -\left[P_{n,k}\{\widehat{w}_k^h(a,X)\}\right]^{-1}\widehat{w}_k^h(a,X)I(\widetilde{T}\le\tau,\Delta=1)\widetilde{T}F_{j,\infty}$$
$$(\widetilde{T}\mid a,X)\ \widehat{S}_k(\widetilde{T}\mid a,X)^{\times}\left\{\widehat{G}_k\ \ \frac{1}{(\widetilde{T}-\mid a,X)}\ -\ \frac{1}{G_\infty\ \ (\widetilde{T}-\mid a,X)}\right\}\mathcal{Q}^{\mathrm{RMTL},j\ 35\ k\ \ h}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k\ \ h},\eta^{\Lambda,j\ \infty\ h},)$$

$$=\left[P_{n,k}\{\widehat{w}_k^h(a,X)\}\right]^{-1}\widehat{w}_k^h(a,X)\int_0^\tau tF_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)\ S_\infty(t\mid a,X)\mathcal{Q}^{\mathrm{RMTL},j\ 36\ k\ \ h}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k\ \ h},\eta^{\Lambda,j\ \infty\ h},)$$
$$=\left[P_{n,k}\{\widehat{w}_k^h(a,X)\}\right]^{-1}\widehat{w}_k^h(a,X)\left\{\int_0^\tau \frac{t\widehat{F}_{j,k}(t\mid a,X)\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)\ (t\mid a,X)\ \widehat{G}_k(t-\mid a,X)_\int_0^\tau tF_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,}{\widehat{S}_k}\right.$$
$$X)\ S_\infty(t\mid a,X)\ \widehat{G}_k(t-\mid a,X)^{\times}\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right\}$$

类似于 $P_0\{\mathcal{Q}_{\iota,k}^{\mathrm{RMST},h}(\tau,a;\ \widehat{\eta}^{\Lambda,j\ k\ \ h},,\eta^{\Lambda,j\ \infty\ h},)\}^2$

$$P_0\{\mathcal{Q}_{j,1,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 \le\epsilon^2h_{\max}^2E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2$$
$$P_0\{\mathcal{Q}_{j,2,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 \le\tau^2E_0\left|([P_{n,k}\{\widehat{h}_k(X)\}]^{-1}\widehat{h}_k(X)-[E_0\{h_\infty(X)\}]^{-1}h_\infty(X))\right|^2$$

由于

$$E_0\left|\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_{j,\infty}(t\mid a,X)}{G_\infty(t-\mid a,X)}\right|^2 = E_0\left|\frac{I(\widetilde{T}\le\tau,\widetilde{J}=j)}{G_\infty(\widetilde{T}-\mid a,X)}-\int_0^\tau\frac{\mathrm{d}F_{j,\infty}(t\mid a,X)}{S_\infty(t-\mid a,X)G_\infty(t-\mid a,X)}\right|^2$$
$$\le\epsilon^2+\epsilon^4E_0\left|F_{j,\infty}(\tau\mid a,X)\right|^2\le\epsilon^2+\epsilon^4$$

我们有

$$P_0\{\mathcal{Q}_{j,3,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2\epsilon^2 h_{\max}^2\tau^2(\epsilon^2+\epsilon^4)P_0\{\mathcal{Q}_{j,4,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值}\tau^2(\epsilon^2+\epsilon^4)E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2 P_0\{\mathcal{Q}_{j,5,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^2 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{G}_k{}^{-1}(t-\mid a,X)-G_\infty{}^{-1}(t-\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,6,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^2 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{G}_k{}^{-1}(t-\mid a,X)-G_\infty{}^{-1}(t-\mid a,X)\right|\right]^2 E_0\left|\int_0^\tau \mathrm{d}F_{j,\infty}(t\mid a,X)\,S_\infty(t-\mid a,X)\right|^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^2\epsilon^2 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{G}_k{}^{-1}(t-\mid a,X)-G_\infty{}^{-1}(t-\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,7,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^2\epsilon^2 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{\Lambda}_{j,k}(t\mid a,X)-\Lambda_{j,\infty}(t\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,8,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2\epsilon^2 h_{\max}^2\tau^2(\epsilon^2+\epsilon^4)P_0\{\mathcal{Q}_{j,9,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值}\tau^2(\epsilon^2+\epsilon^4)E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2 P_0\{\mathcal{Q}_{j,10,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^2 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{G}_k{}^{-1}(t-\mid a,X)-G_\infty{}^{-1}(t-\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,11,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^2\epsilon^2 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{G}_k{}^{-1}(t-\mid a,X)-G_\infty{}^{-1}(t-\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,12,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^2\epsilon^2 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{\Lambda}_{j,k}(t\mid a,X)-\Lambda_{j,\infty}(t\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,13,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2\epsilon^2 h_{\max}^2\tau^2\times 4\epsilon^4 P_0\{\mathcal{Q}_{j,14,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值}\tau^2\times 4\epsilon^4 E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2 P_0\{\mathcal{Q}_{j,15,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\epsilon^4 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\,\widehat{S}_k(t\mid a,X)-\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\,S_\infty(t\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,16,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^4 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{G}_k{}^{-1}(t-\mid a,X)-G_\infty{}^{-1}(t-\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,17,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\tau^2\epsilon^2(\epsilon-1)^2 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{G}_k{}^{-1}(t-\mid a,X)-G_\infty{}^{-1}(t-\mid a,X)\right|\right]^2 P_0\{\mathcal{Q}_{j,18,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\epsilon^2\{\epsilon^2+(\epsilon-1)^2\}E_0\left\{\text{上确界}_{t\in[0,\tau]}\left|\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\,\widehat{S}_k(t\mid a,X)-\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)\,S_\infty(t\mid a,X)\right|\right\}^2$$

$$P_0\{\mathcal{Q}_{j,19,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2\epsilon^2 h_{\max}^2\tau^2\times 4\epsilon^4 P_0\{\mathcal{Q}_{j,20,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值}\tau^2\times 4\epsilon^4 E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2 P_0\{\mathcal{Q}_{j,21,k}^{\mathrm{RMTL},h}(\tau,a;\,\widehat{\eta}^{\Lambda,h}_{j\,k},\eta^{\Lambda,h}_{j\,\infty},)\}^2 \le h^{-2}\text{最小值} h_{\max}^2\epsilon^4 E_0\left[\text{上确界}_{t\in[0,\tau]}\left|\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)\,\widehat{S}_k(t\mid a,X)-\mathrm{RMTL}_{j,\infty}(t\mid a,X)\,S_\infty(t\mid a,X)\right|\right]$$

$$P_0\{\mathcal{Q}_{j,22,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}^{\Lambda,h}_{j,k},\eta^{\Lambda,h}_{j,\infty})\}^2 \le h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^4 E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2$$

$$P_0\{\mathcal{Q}_{j,23,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}^{\Lambda,h}_{j,k},\eta^{\Lambda,h}_{j,\infty})\}^2 \le h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^2(\epsilon-1)^2 E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2$$

As

$$
\begin{aligned}
&E_0\left|\int_0^\tau \frac{\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}-\int_0^\tau \frac{\mathrm{RMTL}_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|^2\\
=&E_0\left|\frac{\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|_0^\tau-\int_0^\tau \frac{1}{\widehat{S}_k(t\mid a,X)}\mathrm{d}\left\{\frac{\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)}{\widehat{G}_k(t-\mid a,X)}\right\}\right.\\
&\left.-\frac{\mathrm{RMTL}_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|_0^\tau+\int_0^\tau \frac{1}{S_\infty(t\mid a,X)}\mathrm{d}\left\{\frac{\mathrm{RMTL}_{j,\infty}(t\mid a,X)}{\widehat{G}_k(t-\mid a,X)}\right\}\Bigg|^2\\
=&E_0\left|\left\{\frac{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-\frac{\mathrm{RMTL}_{j,\infty}(\tau\mid a,X)}{S_\infty(\tau\mid a,X)}\right\}\frac{1}{\widehat{G}_k(\tau-\mid a,X)}-\int_0^\tau\left\{\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\right.\\
&\left.\frac{\mathrm{d}t}{\widehat{G}_k(t-\mid a,X)}-\int_0^\tau\left\{\frac{\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMTL}_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\mathrm{d}\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}\right\}\right|^2\\
\le&\left\{\epsilon^2+(\epsilon-1)^2\right\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMTL}_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2\\
&+\epsilon^2\tau^2E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2
\end{aligned}
$$

Thus

$$
\begin{aligned}
P_0\{\mathcal{Q}_{j,24,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2\le&h_{\min}^{-2}h_{\max}^2\epsilon^2\{\epsilon^2+(\epsilon-1)^2\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{RMTL}_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2\\
&+h_{\min}^{-2}h_{\max}^2\epsilon^2\epsilon^2\tau^2E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2\\
P_0\{\mathcal{Q}_{j,25,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}^{\Lambda,h}_{j,k},\eta^{\Lambda,h}_{j,\infty})\}^2\le&([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2\epsilon^2h_{\max}^2\tau^2\times4\epsilon^4\\
P_0\{\mathcal{Q}_{j,26,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}^{\Lambda,h}_{j,k},\eta^{\Lambda,h}_{j,\infty})\}^2\le&h^{-2}_{\text{最小值}}\tau^2\times4\epsilon^4E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2\\
P_0\{\mathcal{Q}_{j,27,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2\le&h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^4E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2\\
P_0\{\mathcal{Q}_{j,28,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2\le&h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^4E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2\\
P_0\{\mathcal{Q}_{j,29,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2\le&h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^2(\epsilon-1)^2E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2
\end{aligned}
$$

请注意,

$$
\begin{aligned}
&E_0\left|\int_0^\tau\left\{\frac{\mathrm{d}\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{d}F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\right|^2\\
=&E_0\left|\left\{\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}\right\}\Bigg|_0^\tau-\int_0^\tau\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\Bigg|_0^\tau+\int_0^\tau\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\mathrm{d}\Lambda_\infty(t\mid a,X)\right|^2\\
=&E_0\left|\frac{\widehat{F}_{j,k}(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-\frac{F_{j,\infty}(\tau\mid a,X)}{S_\infty(\tau\mid a,X)}-\int_0^\tau\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}\{\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)-\mathrm{d}\Lambda_\infty(t\mid a,X)\}\right.\\
&\left.-\int_0^\tau\left\{\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\mathrm{d}\Lambda_\infty(t\mid a,X)\right|^2\\
\le&E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2+\epsilon^2E_0\left\{\sup_{t\in[0,\tau]}\left|\widehat{\Lambda}_k(t\mid a,X)-\Lambda_\infty(t\mid a,X)\right|\right\}^2\\
&+\log^2\epsilon E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2
\end{aligned}
$$

使得

$$
\begin{aligned}
&E_0\left|\int_0^\tau\frac{\widehat{F}_{j,k}(t\mid a,X)\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}-\int_0^\tau\frac{F_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|^2\\
=&E_0\left|\left\{\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right\}\Bigg|_0^\tau-\int_0^\tau\frac{1}{\widehat{S}_k(t\mid a,X)}\mathrm{d}\left\{\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{G}_k(t-\mid a,X)}\right\}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\Bigg|_0^\tau\right.\\
&\left.+\int_0^\tau\frac{1}{S_\infty(t\mid a,X)}\mathrm{d}\left\{\frac{F_{j,\infty}(t\mid a,X)}{\widehat{G}_k(t-\mid a,X)}\right\}\right|^2\\
=&E_0\left|\left\{\frac{\widehat{F}_{j,k}(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-\frac{F_{j,\infty}(\tau\mid a,X)}{S_\infty(\tau\mid a,X)}\right\}\frac{1}{\widehat{G}_k(\tau-\mid a,X)}-\int_0^\tau\left\{\frac{\mathrm{d}\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{d}F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\right.\\
&\left.\times\frac{1}{\widehat{G}_k(t-\mid a,X)}-\int_0^\tau\left\{\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\mathrm{d}\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}\right\}\right|^2\\
\le&\{\epsilon^2+(\epsilon-1)^2\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2\\
&+\epsilon^2E_0\left[\int_0^\tau\left\{\frac{\mathrm{d}\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{d}F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\right]^2\\
=&\{2\epsilon^2+(\epsilon-1)^2+\epsilon^2\log^2\epsilon\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2\\
&+\epsilon^4E_0\left\{\sup_{t\in[0,\tau]}\left|\widehat{\Lambda}_k(t\mid a,X)-\Lambda_\infty(t\mid a,X)\right|\right\}^2
\end{aligned}
$$

$$P_0\{\mathcal{Q}_{j,30,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 \le h_{\min}^{-2}h_{\max}^2\epsilon^2\tau^2\left[\{2\epsilon^2+(\epsilon-1)^2+\epsilon^2\log^2\epsilon\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}\right.\right.\right.$$
$$\left.\left.\left.-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2+\epsilon^4E_0\left\{\sup_{t\in[0,\tau]}\left|\widehat{\Lambda}_k(t\mid a,X)-\Lambda_\infty(t\mid a,X)\right|\right\}^2\right]$$

$$P_0\{\mathcal{Q}_{j,31,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{,j,k}^{\Lambda,h},\eta_{,j,\infty}^{\Lambda,h})\}^2 \le([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_\infty(X)\}]^{-1})^2\epsilon^2h_{\max}^2\tau^2\times 4\epsilon^4$$

$$P_0\{\mathcal{Q}_{j,32,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{,j,k}^{\Lambda,h},\eta_{,j,\infty}^{\Lambda,h})\}^2 \le h^{-2}\text{最小值}\tau^2\times 4\epsilon^4E_0\left|\widehat{w}_k^h(a,X)-w_\infty^h(a,X)\right|^2$$

$$P_0\{\mathcal{Q}_{j,33,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 \le h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^4E_0\left[\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right]^2$$

$$P_0\{\mathcal{Q}_{j,34,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 \le h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^4E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2$$

$$P_0\{\mathcal{Q}_{j,35,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2 \le h_{\min}^{-2}h_{\max}^2\tau^2\epsilon^2(\epsilon-1)^2E_0\left[\sup_{t\in[0,\tau]}\left|\frac{1}{\widehat{G}_k(t-\mid a,X)}-\frac{1}{G_\infty(t-\mid a,X)}\right|\right]^2$$

由于

$$E_0\left|\int_0^\tau\frac{t\widehat{F}_{j,k}(t\mid a,X)\mathrm{d}\widehat{\Lambda}_k(t\mid a,X)}{\widehat{S}_k(t\mid a,X)\widehat{G}_k(t-\mid a,X)}-\int_0^\tau\frac{tF_{j,\infty}(t\mid a,X)\mathrm{d}\Lambda_\infty(t\mid a,X)}{S_\infty(t\mid a,X)\widehat{G}_k(t-\mid a,X)}\right|^2$$
$$=E_0\left|\frac{\tau}{\widehat{G}_k(\tau-\mid a,X)}\left\{\frac{\widehat{F}_{j,k}(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-\frac{F_{j,\infty}(\tau\mid a,X)}{S_\infty(\tau\mid a,X)}\right\}-\int_0^\tau\frac{t}{\widehat{G}_k(t-\mid a,X)}\right.$$
$$\times\left\{\frac{\mathrm{d}\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{d}F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}-\int_0^\tau\left\{\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\frac{\mathrm{d}t}{\widehat{G}_k(t-\mid a,X)}$$
$$\left.-\int_0^\tau\left\{\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}t\mathrm{d}\left\{\frac{1}{\widehat{G}_k(t-\mid a,X)}\right\}\right|^2$$
$$\le\tau^2\epsilon^2E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2+\tau^2\epsilon^2E_0\left|\int_0^\tau\left\{\frac{\mathrm{d}\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{\mathrm{d}F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right\}\right|^2$$
$$+\tau^2\epsilon^2E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2$$
$$+\tau^2(\epsilon-1)^2E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2$$
$$=\tau^2(3\epsilon^2+(\epsilon-1)^2+\epsilon^2\log^2\epsilon)E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2$$
$$+\tau^2\epsilon^4E_0\left\{\sup_{t\in[0,\tau]}\left|\widehat{\Lambda}_k(t\mid a,X)-\Lambda_\infty(t\mid a,X)\right|\right\}^2$$

因此

$$
\begin{aligned}
&P_0\{\mathcal{Q}_{j,36,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h},\eta_{j,\infty}^{\Lambda,h})\}^2\\
\leq& h_{\min}^{-2}h_{\max}^2\epsilon^2\tau^2\left[\{3\epsilon^2+(\epsilon-1)^2+\epsilon^2\log^2\epsilon\}E_0\left\{\sup_{t\in[0,\tau]}\left|\frac{\widehat{F}_{j,k}(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-\frac{F_{j,\infty}(t\mid a,X)}{S_\infty(t\mid a,X)}\right|\right\}^2\right.\\
&\left.+\epsilon^4E_0\left\{\sup_{t\in[0,\tau]}\left|\widehat{\Lambda}_k(t\mid a,X)-\Lambda_\infty(t\mid a,X)\right|\right\}^2\right]
\end{aligned}
$$

D.5 引理 4

引理 4。 在条件 *1-2* 下，

$$
\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(\tau,a;\eta_\infty^{\Lambda,h})\}=o_P(n^{-1/2})
$$

若还满足条件 *4*，

$$
\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\sup_{t\in[0,\tau]}\left|\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(t,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(t,a;\eta_\infty^{\Lambda,h})\}\right|=o_P(n^{-1/2})
$$

在条件*2*和 *5* 下，

$$
\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\mathcal{G}_{n,k}\{\widehat{\phi}_{j,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h})-\phi_{j,\infty}^{\mathrm{RMTL},h}(\tau,a;\eta_{j,\infty}^{\Lambda,h})\}=o_P(n^{-1/2})
$$

若还满足条件 *7*，

$$
\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\sup_{t\in[0,\tau]}\left|\mathcal{G}_{n,k}\{\widehat{\phi}_{j,k}^{\mathrm{RMTL},h}(t,a;\widehat{\eta}_{j,k}^{\Lambda,h})-\phi_{j,\infty}^{\mathrm{RMTL},h}(t,a;\eta_{j,\infty}^{\Lambda,h})\}\right|=o_P(n^{-1/2})
$$

引理 4 的证明：沿用 Westling 等人（2021年）引理 6 的结论，经验过程项有如下界

$$
\begin{aligned}
&\left|\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(\tau,a;\eta_\infty^{\Lambda,h})\}\right|\\
\leq& O_P(n^{-1/2})\frac{1}{K}\sum_{k=1}^{K}\left|\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(\tau,a;\eta_\infty^{\Lambda,h})\}\right|
\end{aligned}
$$

由引理 3 可知，

$$
\frac{1}{K}\sum_{k=1}^{K}\left|\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(\tau,a;\eta_\infty^{\Lambda,h})\}\right|\leq\max_k\sum_{\iota=1}^{6}C_\iota^{\mathrm{RMST},h}\mathcal{B}_{\iota,k}^{\mathrm{RMST},h}(\widehat{\eta}_k^{\Lambda,h},\eta_\infty^{\Lambda,h})
$$

其中， $\mathcal{B}^{\mathrm{RMST},h}_{\iota,k}(\widehat{\eta}^{\Lambda,h}_k, \eta^{\Lambda,h}_\infty)$ 在引理 3 中定义。

$$
\begin{aligned}
&[P_0\{\widehat{\phi}^{\mathrm{RMST},h}_k(\tau,a;\widehat{\eta}^{\Lambda,h}_k)-\phi^{\mathrm{RMST},h}_\infty(\tau,a;\eta^{\Lambda,h}_\infty)\}^2]^{1/2} \le \sum_{\iota=1}^{6} C^{\mathrm{RMST},h}_\iota \mathcal{B}^{\mathrm{RMST},h}_{\iota,k}(\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)\\
&[P_0\{\sup_{t\in[0,\tau]}|\widehat{\phi}^{\mathrm{RMST},h}_k(t,a;\widehat{\eta}^{\Lambda,h}_k)-\phi^{\mathrm{RMST},h}_\infty(t,a;\eta^{\Lambda,h}_\infty)|\}^2]^{1/2}\\
&\le C^{\mathrm{RMST},h}_1 \widetilde{\mathcal{B}}^{\mathrm{RMST},h}_{1,k}(\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)+\sum_{\iota=2}^{6} C^{\mathrm{RMST},h}_\iota \mathcal{B}^{\mathrm{RMST},h}_{\iota,k}(\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)\\
&[P_0\{\widehat{\phi}^{\mathrm{RMTL},h}_{j,k}(\tau,a;\widehat{\eta}^{\Lambda,h}_{j,k})-\phi^{\mathrm{RMTL},h}_{j,\infty}(\tau,a;\eta^{\Lambda,h}_{j,\infty})\}^2]^{1/2} \le \sum_{\iota=1}^{9} C^{\mathrm{RMTL},h}_{j,\iota} \mathcal{B}^{\mathrm{RMTL},h}_{j,\iota,k}(\widehat{\eta}^{\Lambda,h}_{j,k},\eta^{\Lambda,h}_{j,\infty})\\
&[P_0\{\sup_{t\in[0,\tau]}|\widehat{\phi}^{\mathrm{RMTL},h}_{j,k}(t,a;\widehat{\eta}^{\Lambda,h}_{j,k})-\phi^{\mathrm{RMTL},h}_{j,\infty}(t,a;\eta^{\Lambda,h}_{j,\infty})|\}^2]^{1/2}\\
&\le C^{\mathrm{RMTL},h}_{j,1} \widetilde{\mathcal{B}}^{\mathrm{RMTL},h}_{j,1,k}(\widehat{\eta}^{\Lambda,h}_{j,k},\eta^{\Lambda,h}_{j,\infty})+\sum_{\iota=2}^{9} C^{\mathrm{RMTL},h}_{j,\iota} \mathcal{B}^{\mathrm{RMTL},h}_{j,\iota,k}(\widehat{\eta}^{\Lambda,h}_{j,k},\eta^{\Lambda,h}_{j,\infty})
\end{aligned}
$$

根据条件 1，$\max_k \sum_{\iota=1}^{6} C^{\mathrm{RMST},h}_\iota \mathcal{B}^{\mathrm{RMST},h}_{\iota,k}(\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty) \xrightarrow{P} 0$，使得

$$
\left|\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\mathcal{G}_{n,k}\{\widehat{\phi}^{\mathrm{RMST},h}_k(\tau,a;\widehat{\eta}^{\Lambda,h}_k)-\phi^{\mathrm{RMST},h}_\infty(\tau,a;\eta^{\Lambda,h}_\infty)\}\right| = o_P(n^{-1/2})
$$

对于同时推断，我们对经验随机过程进行界定。

$$
\begin{aligned}
&\left|\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\sup_{t\in[0,\tau]}\mathcal{G}_{n,k}\{\widehat{\phi}^{\mathrm{RMST},h}_k(t,a;\widehat{\eta}^{\Lambda,h}_k)-\phi^{\mathrm{RMST},h}_\infty(t,a;\eta^{\Lambda,h}_\infty)\}\right|\\
\le& o_P(n^{-1/2})\frac{1}{K}\sum_{k=1}^{K}\left|\sup_{t\in[0,\tau]}\mathcal{G}_{n,k}\{\widehat{\phi}^{\mathrm{RMST},h}_k(t,a;\widehat{\eta}^{\Lambda,h}_k)-\phi^{\mathrm{RMST},h}_\infty(t,a;\eta^{\Lambda,h}_\infty)\}\right|
\end{aligned}
$$

根据 van der Vaart 和 Wellner（1996）的定理 2.14.2，我们有

$$
\begin{aligned}
&\frac{1}{K}\sum_{k=1}^{K}\left|\sup_{t\in[0,\tau]}\mathcal{G}_{n,k}\{\widehat{\phi}^{\mathrm{RMST},h}_k(t,a;\widehat{\eta}^{\Lambda,h}_k)-\phi^{\mathrm{RMST},h}_\infty(t,a;\eta^{\Lambda,h}_\infty)\}\right|\\
\le& \max_k\left\{C^{\mathrm{RMST},h}_1\widetilde{\mathcal{B}}^{\mathrm{RMST},h}_{1,k}(\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)+\sum_{\iota=2}^{6}C^{\mathrm{RMST},h}_\iota\mathcal{B}^{\mathrm{RMST},h}_{\iota,k}(\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)\right\}
\end{aligned}
$$

根据条件 1 和条件 4，$\max_k\left\{C^{\mathrm{RMST},h}_1\widetilde{\mathcal{B}}^{\mathrm{RMST},h}_{1,k}(\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)+\sum_{\iota=2}^{6}C^{\mathrm{RMST},h}_\iota\mathcal{B}^{\mathrm{RMST},h}_{\iota,k}(\widehat{\eta}^{\Lambda,h}_k,\eta^{\Lambda,h}_\infty)\right\}\xrightarrow{P}0$，使得

$$
\frac{1}{K}\sum_{k=1}^{K}\left|\sup_{t\in[0,\tau]}\mathcal{G}_{n,k}\{\widehat{\phi}^{\mathrm{RMST},h}_k(t,a;\widehat{\eta}^{\Lambda,h}_k)-\phi^{\mathrm{RMST},h}_\infty(t,a;\eta^{\Lambda,h}_\infty)\}\right| = o_P(n^{-1/2})
$$

通过将条件 1 和条件 4 替换为条件 5 和条件 7，可得到类似的论证，适用于 $1/K\sum_{k=1}^{K}\left|\mathcal{G}_{n,k}\{\widehat{\phi}^{\mathrm{RMTL},h}_{j,k}(\tau,a;\widehat{\eta}^{\Lambda,h}_{j,k},)-\phi^{\mathrm{RMTL},h}_{j,\infty}(\tau,a;\eta^{\Lambda,h}_{j,\infty},)\}\right|$ 和 $1/K\sum_{k=1}^{K}\left|\sup_{t\in[0,\tau]}\mathcal{G}_{n,k}\{\widehat{\phi}^{\mathrm{RMTL},h}_{j,k}(t,a;\widehat{\eta}^{\Lambda,h}_{j,k},)-\phi^{\mathrm{RMTL},h}_{j,\infty}(t,a;\eta^{\Lambda,h}_{j,\infty},)\}\right|$。

D.6 定理 2 的证明

条件 1. 存在 $\eta^{\Lambda,\infty,h}$，使得对所有 $a,\ \tau$ 以及 $\iota = 1,\ldots,14$，都有 $\max_k\{\mathcal{B}^{\mathrm{RMST},h}_{\iota,k}(\widehat{\eta}^{\Lambda,k,h},\eta^{\Lambda,\infty,h})\}^2 \xrightarrow{P} 0$，其中 $\mathcal{B}^{\mathrm{RMST},\iota,k,h}(\widehat{\eta}^{\Lambda,k,h},\eta^{\Lambda,\infty,h})$ 定义见引理 3。

条件 2. 存在 $\epsilon \in (1,\infty)$，使得 $\{S_k(t\mid a,X)\ S_\infty(t\mid a,X)\ \widehat{\pi}_k(a\mid X)\ \pi_\infty(a\mid X)\ G_k(t\mid a,X)\ G_\infty(t\mid a,X)\} \geq 1/\epsilon$，$h_{\min} \leq \{h_k(X)\ h_\infty(X)\} \leq h_{\max}$，且 $P_{n,k}\{h_k(X)\} > 0$，$E_0\{h_\infty(X)\} > 0$对所有 a，X 和 k均成立。

条件 3。对于所有 $t \in [0,\tau]$、$a \in \{0,1\}$ 以及几乎所有 x，以下条件之一成立：(1) $h_\infty(X) = h_0(X)$ 且 $S_\infty(t\mid a,X) = S_0(t\mid a,X)$；或 (2) $h_\infty(X) = h_0(X)$ 且 $\pi_\infty(a\mid X) = \pi_0(a\mid X)$，并且 $G_\infty(t\mid a,X) = G_0(t\mid a,X)$。

条件 4。 $\max_k\{\widehat{\mathcal{B}}^{\mathrm{RMST},h}_{1,k}(\widehat{\eta}^{\Lambda,k,h},\eta^{\Lambda,\infty,h})\}^2 \xrightarrow{P} 0$ 对每个 a 和 τ 均成立。

条件 5。 存在 $\eta^{\Lambda,j,\infty,h}$，使得 $\max_k\{\mathcal{B}^{\mathrm{RMTL},h}_{j,\iota,k}(\widehat{\eta}^{\Lambda,j,k,h},\eta^{\Lambda,j,\infty,h})\}^2 \xrightarrow{P} 0$ 对于感兴趣的原因 j 均成立，且对所有 $a,\ \tau,\ \iota = 1,\ldots,36$ 均成立，其中 $\mathcal{B}^{\mathrm{RMTL},j,\iota,k,h}(\widehat{\eta}^{\Lambda,j,k,h},\eta^{\Lambda,j,\infty,h})$ 定义见引理 3。

条件 6。对于所有 $t \in [0,\tau]$，$a \in \{0,1\}$、几乎所有 X，且仅针对感兴趣的原因 j，以下条件之一成立：(1) $h_\infty(X) = h_0(X)$ 且 $S_\infty(t\mid a,X) = S_0(t\mid a,X)$ 且 $F_{j,\infty}(t\mid a,X) = F_{j,0}(t\mid a,X)$；或 (2) $h_\infty(X) = h_0(X)$ 且 $\pi_\infty(a\mid X) = \pi_0(a\mid X)$ 且 $G_\infty(t\mid a,X) = G_0(t\mid a,X)$；或 (3) $h_\infty(X) = h_0(X)$ 且 $F_{j,\infty}(t\mid a,X) = F_{j,0}(t\mid a,X)$ 且 $G_\infty(t\mid a,X) = G_0(t\mid a,X)$。

条件 7. $\max_k\{\widehat{\mathcal{B}}^{\mathrm{RMTL},h}_{j,1,k}(\widehat{\eta}^{\Lambda,j,k,h},\eta^{\Lambda,j,\infty,h})\}^2 \xrightarrow{P} 0$ 对于感兴趣的原因 j 以及每个 $a,\ \tau$。

在条件1–3下，

$$\widehat{\psi}^{\mathrm{RMST},h}(\tau,a) \xrightarrow{P} \psi_0^{\mathrm{RMST},h}(\tau,a), \quad \widehat{\psi}^{\mathrm{RMST},h}(\tau) \xrightarrow{P} \psi_0^{\mathrm{RMST},h}(\tau)$$

如果还满足条件 4，则

$$\sup_{t\in[0,\tau]}\left|\widehat{\psi}^{\mathrm{RMST},h}(\tau,a) - \psi_0^{\mathrm{RMST},h}(\tau,a)\right| \xrightarrow{P} 0, \quad \sup_{t\in[0,\tau]}\left|\widehat{\psi}^{\mathrm{RMST},h}(\tau) - \psi_0^{\mathrm{RMST},h}(\tau)\right| \xrightarrow{P} 0$$

在条件2和条件 5–6下，

$$\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau,a) \xrightarrow{P} \psi_{j,0}^{\mathrm{RMTL},h}(\tau,a), \quad \widehat{\psi}_j^{\mathrm{RMTL},h}(\tau) \xrightarrow{P} \psi_{j,0}^{\mathrm{RMTL},h}(\tau)$$

如果还满足条件 7，则

$$\sup_{t\in[0,\tau]}\left|\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau,a) - \psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\right| \xrightarrow{P} 0, \quad \sup_{t\in[0,\tau]}\left|\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau) - \psi_{j,0}^{\mathrm{RMTL},h}(\tau)\right| \xrightarrow{P} 0$$

对于条件 3，若 (1) $h_\infty(X) = h_0(X)$ 和 $S_\infty(t\mid a,X) = S_0(t\mid a,X)$ 成立，则 $\mathrm{RMST}_\infty(t\mid a,X) = \mathrm{RMST}_0(t\mid a,X)$、$\Lambda_\infty(t\mid a,X) = \Lambda_0(t\mid a,X)$，且 $\mathrm{d}M_\infty(t\mid a,X) = \mathrm{d}M_0(t\mid a,X)$。我们考虑~ $\widehat{\psi}^{\mathrm{RMST}}$的事件鞅表示，$^h(\tau,a)$，因为 $E_0\{\int_0^{\tau\wedge T}(\cdot)\mathrm{d}M_0(t\mid a,X) \mid A = a, X\} = 0$,

$$
\begin{aligned}
&\widehat{\psi}^{\mathrm{RMST},h}(\tau,a) \xrightarrow{p} \\
&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMST}_0(\tau \mid a,X)\} + \left[E_0\left\{\frac{h_0(X)I(A=a)}{\pi_\infty(a \mid X)}\right\}\right]^{-1} \\
&\times E_0\left[\left\{\frac{h_0(X)I(A=a)}{\pi_\infty(a \mid X)}\right\}\left\{-\mathrm{RMST}_0(\tau \mid a,X)\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0(t \mid a,X)}{S_0(t \mid a,X)G_\infty(t- \mid a,X)}\right.\right. \\
&\left.\left.+\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t \mid a,X)\mathrm{d}M_0(t \mid a,X)}{S_0(t \mid a,X)G_\infty(t- \mid a,X)}\right\}\right] \\
=&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMST}_0(\tau \mid a,X)\} + \left[E_0\left\{\frac{h_0(X)\pi_0(a \mid X)}{\pi_\infty(a \mid X)}\right\}\right]^{-1} \\
&\times E_0\left(\left\{\frac{h_0(X)\pi_0(a \mid X)}{\pi_\infty(a \mid X)}\right\}\left[-\mathrm{RMST}_0(\tau \mid a,X)E\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{d}M_0(t \mid a,X)}{S_0(t \mid a,X)G_\infty(t- \mid a,X)} \mid A=a,X\right\}\right.\right. \\
&\left.\left.+\ E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\mathrm{RMST}_0(t \mid a,X)\mathrm{d}M_0(t \mid a,X)}{S_0(t \mid a,X)G_\infty(t- \mid a,X)} \mid A=a,X\right\}\right]\right) \\
=&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMST}_0(\tau \mid a,X)\} = \psi_0^{\mathrm{RMST},h}(\tau,a)
\end{aligned}
$$

若 (2) $h_\infty(X) = h_0(X)$、$\pi_\infty(a \mid X) = \pi_0(a \mid X)$ 和 $G_\infty(t \mid a,X) = G_0(t \mid a,X)$ 成立，则$\Lambda_\infty^C(t \mid a,X) = \Lambda_0^C(t \mid a,X)$，且 $\mathrm{d}M_\infty^C(t \mid a,X) = \mathrm{d}M_0^C(t \mid a,X)$。我们考虑 $\widehat{\psi}^{\mathrm{RMST}}$的删失鞅表示，$^{h}(\tau,a)$，因为 $E\{\int_0^{\tau\wedge T}(\cdot)\mathrm{d}M_0^C(t \mid a,X) \mid A=a,X\} = 0$,

$$
\begin{aligned}
&\widehat{\psi}^{\mathrm{RMST},h}(\tau,a) \xrightarrow{P} \\
=&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMST}_\infty(\tau \mid a,X)\} + \left[E_0\left\{\frac{h_0(X)I(A=a)}{\pi_0(a \mid X)}\right\}\right]^{-1} \\
&\times E_0\left[\left\{\frac{h_0(X)I(A=a)}{\pi_0(a \mid X)}\right\}\left\{\frac{\Delta(\tau)\min(\widetilde{T},\tau)}{G_0(\tau\wedge\widetilde{T}- \mid a,X)} - \mathrm{RMST}_\infty(\tau \mid a,X)\right.\right. \\
&\left.\left.+\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}M_0^C(t \mid a,X)}{G_0(t- \mid a,X)} + \int_0^{\tau\wedge\widetilde{T}}\frac{\{\mathrm{RMST}_\infty(\tau \mid a,X) - \mathrm{RMST}_\infty(t \mid a,X)\}\mathrm{d}M_0^C(t \mid a,X)}{S_\infty(t \mid a,X)G_0(t- \mid a,X)}\right\}\right] \\
=&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMST}_\infty(\tau \mid a,X)\} + [E_0\{h_0(X)\}]^{-1} \\
&\times E_0\left(h_0(X)\left[E\left\{\frac{\Delta(\tau)\min(\widetilde{T},\tau)}{G_0(\tau\wedge\widetilde{T}- \mid a,X)} \mid A=a,X\right\} - \mathrm{RMST}_\infty(\tau \mid a,X)\right.\right. \\
&+E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{t\mathrm{d}M_0^C(t \mid a,X)}{G_0(t- \mid a,X)} \mid A=a,X\right\} \\
&\left.\left.+E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\{\mathrm{RMST}_\infty(\tau \mid a,X) - \mathrm{RMST}_\infty(t \mid a,X)\}\mathrm{d}M_0^C(t \mid a,X)}{S_\infty(t \mid a,X)G_0(t- \mid a,X)} \mid A=a,X\right\}\right]\right) \\
=&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMST}_0(\tau \mid a,X)\} = \psi_0^{\mathrm{RMST},h}(\tau,a)
\end{aligned}
$$

我们使用了

$$
\begin{aligned}
&E_0\left\{\frac{\Delta(\tau)\min(\widetilde{T},\tau)}{G_0(\tau\wedge\widetilde{T}- \mid a,X)} \mid A=a,X\right\} \\
=&E_0\left\{\frac{I(C>\tau\wedge\widetilde{T}-)\min(T,\tau)}{G_0(\tau\wedge\widetilde{T}- \mid a,X)} \mid A=a,X\right\} = \mathrm{RMST}_0(\tau \mid a,X)
\end{aligned}
$$

上述证明还意味着 $E_0\{\phi_0^{\mathrm{RMST}}(\tau,a;\eta_0^\Lambda) \mid A=a,X\} =$限制平均生存时间$_0(\tau \mid a,X)$。

对于条件 6，若 (1) $h_\infty(X) = h_0(X)$ 且 $S_\infty(t \mid a, X) = S_0(t \mid a, X)$，并且 $F_{j,\infty}(t \mid a, X) = F_{j,0}(t \mid a, X)$ 成立，则 $\mathrm{RMTL}_{j,\infty}(t \mid a, X) = \mathrm{RMTL}_{j,0}(t \mid a, X)$，$\Lambda_\infty(t \mid a, X) = \Lambda_0(t \mid a, X)$，$\Lambda_{j,\infty}(t \mid a, X) = \Lambda_{j,0}(t \mid a, X)$，$\mathrm{d}M_\infty(t \mid a, X) = \mathrm{d}M_0(t \mid a, X)$，且 $\mathrm{d}M_{j,\infty}(t \mid a, X) = \mathrm{d}M_{j,0}(t \mid a, X)$。我们考虑 $\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau, a)$ 的事件鞅表示，因为 $E[\int_0^{\tau\wedge T}(\cdot)\mathrm{d}M_0(t \mid a, X) \mid A = a, X] = 0$，且 $E[\int_0^{\tau\wedge T}(\cdot)\mathrm{d}M_{j,0}(t \mid a, X) \mid A = a, X] = 0$。

$$
\begin{aligned}
&\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau, a) \xrightarrow{P} \\
&[E_0\{h_0(X)\}]^{-1} E_0\{h_0(X)\mathrm{RMTL}_{j,0}(\tau \mid a, X)\} + \left[E_0\left\{\frac{h_0(X)I(A=a)}{\pi_\infty(a \mid X)}\right\}\right]^{-1} \\
&\times E_0\left[\left\{\frac{h_0(X)I(A=a)}{\pi_\infty(a \mid X)}\right\}\left\{\int_0^{\tau\wedge\widetilde{T}} \frac{(\tau - t)\mathrm{d}M_{j,0}(t \mid a, X)}{G_\infty(t- \mid a, X)} + \int_0^{\tau\wedge\widetilde{T}} \frac{(\tau - t)F_{j,0}(t \mid a, X)\mathrm{d}M_0(t \mid a, X)}{S_0(t \mid a, X)G_\infty(t- \mid a, X)}\right.\right. \\
&\left.\left. - \int_0^{\tau\wedge\widetilde{T}} \frac{\{\mathrm{RMST}_0(\tau \mid a, X) - \mathrm{RMST}_0(t \mid a, X)\}\mathrm{d}M_0(t \mid a, X)}{S_0(t \mid a, X)G_\infty(t- \mid a, X)}\right\}\right] \\
=&[E_0\{h_0(X)\}]^{-1} E_0\{h_0(X)\mathrm{RMTL}_{j,0}(\tau \mid a, X)\} + \left[E_0\left\{\frac{h_0(X)\pi_0(a \mid X)}{\pi_\infty(a \mid X)}\right\}\right]^{-1} \\
&\times E_0\left(\left\{\frac{h_0(X)\pi_0(a \mid X)}{\pi_\infty(a \mid X)}\right\}\left[E_0\left\{\int_0^{\tau\wedge\widetilde{T}} \frac{(\tau - t)\mathrm{d}M_{j,0}(t \mid a, X)}{G_\infty(t- \mid a, X)} \mid A = a, X\right\}\right.\right. \\
&+ E_0\left\{\int_0^{\tau\wedge\widetilde{T}} \frac{(\tau - t)F_{j,0}(t \mid a, X)\mathrm{d}M_0(t \mid a, X)}{S_0(t \mid a, X)G_\infty(t- \mid a, X)} \mid A = a, X\right\} \\
&\left.\left. - E_0\left\{\int_0^{\tau\wedge\widetilde{T}} \frac{\{\mathrm{RMST}_0(\tau \mid a, X) - \mathrm{RMST}_0(t \mid a, X)\}\mathrm{d}M_0(t \mid a, X)}{S_0(t \mid a, X)G_\infty(t- \mid a, X)} \mid A = a, X\right\}\right]\right) \\
=&[E_0\{h_0(X)\}]^{-1} E_0\{h_0(X)\mathrm{RMTL}_{j,0}(\tau \mid a, X)\} = \psi_{j,0}^{\mathrm{RMTL},h}(\tau, a)
\end{aligned}
$$

若 (2) $h_\infty(X) = h_0(X)$ 且 $\pi_\infty(a \mid X) = \pi_0(a \mid X)$，并且 $G_\infty(t \mid a, X) = G_0(t \mid a, X)$ 成立，则 $\Lambda_\infty^C(t \mid a, X) = \Lambda_0^C(t \mid a, X)$，且 $\mathrm{d}M_\infty^C(t \mid a, X) = \mathrm{d}M_0^C(t \mid a, X)$。我们考虑 $\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau, a)$ 的删失鞅表示，因为 $E\{\int_0^{\tau\wedge T}(\cdot)\mathrm{d}M_0^C(t \mid a, X) \mid A = a, X\} = 0$，

$$
\begin{aligned}
&\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau, a) \xrightarrow{P} \\
&[E_0\{h_0(X)\}]^{-1} E_0\{h_0(X)\mathrm{RMTL}_{j,\infty}(\tau \mid a, X)\} + \left[E_0\left\{\frac{h_0(X)I(A=a)}{\pi_0(a \mid X)}\right\}\right]^{-1} E_0\left[\left\{\frac{h_0(X)I(A=a)}{\pi_0(a \mid X)}\right\}\right. \\
&\times \left\{\frac{(\tau - \min(\widetilde{T}, \tau))I(\widetilde{J} = j)}{G_0(\widetilde{T}- \mid a, X)} - \mathrm{RMTL}_{j,\infty}(\tau \mid a, X) - \int_0^{\tau\wedge\widetilde{T}} \frac{(\tau - t)F_{j,\infty}(t \mid a, X)\mathrm{d}M_0^C(t \mid a, X)}{S_\infty(t \mid a, X)G_0(t- \mid a, X)}\right. \\
&\left.\left. + \int_0^{\tau\wedge\widetilde{T}} \frac{\{\mathrm{RMTL}_{j,\infty}(\tau \mid a, X) - \mathrm{RMTL}_{j,\infty}(t \mid a, X)\}\mathrm{d}M_0^C(t \mid a, X)}{S_\infty(t \mid a, X)G_0(t- \mid a, X)}\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
=&[E_0\{h_0(X)\}]^{-1} E_0\{h_0(X)\mathrm{RMTL}_{j,\infty}(\tau \mid a, X)\} + [E_0\{h_0(X)\}]^{-1} \\
&\times E_0\left(h_0(X)\left[E_0\left\{\frac{(\tau - \min(\widetilde{T}, \tau))I(\widetilde{J} = j)}{G_0(\widetilde{T}- \mid a, X)} \mid A = a, X\right\} - \mathrm{RMTL}_{j,\infty}(\tau \mid a, X)\right.\right. \\
&- E_0\left\{\int_0^{\tau\wedge\widetilde{T}} \frac{(\tau - t)F_{j,\infty}(t \mid a, X)\mathrm{d}M_0^C(t \mid a, X)}{S_\infty(t \mid a, X)G_0(t- \mid a, X)} \mid A = a, X\right\} \\
&\left.\left. + E_0\left\{\int_0^{\tau\wedge\widetilde{T}} \frac{\{\mathrm{RMTL}_{j,\infty}(\tau \mid a, X) - \mathrm{RMTL}_{j,\infty}(t \mid a, X)\}\mathrm{d}M_0^C(t \mid a, X)}{S_\infty(t \mid a, X)G_0(t- \mid a, X)} \mid A = a, X\right\}\right]\right) \\
=&[E_0\{h_0(X)\}]^{-1} E_0\{h_0(X)\mathrm{RMTL}_{j,0}(\tau \mid a, X)\} = \psi_{j,0}^{\mathrm{RMTL},h}(\tau, a)
\end{aligned}
$$

我们代入了

$$
\begin{aligned}
&E_0\left\{\frac{(\tau-\min(\widetilde{T},\tau))I(\widetilde{J}=j)}{G_0(\widetilde{T}-\mid a,X)}\mid A=a,X\right\}\\
=&E_0\left\{\frac{\Delta(\tau-\min(T,\tau))I(J=j)}{G_0(\widetilde{T}-\mid a,X)}\mid A=a,X\right\}=\mathrm{RMTL}_{j,0}(\tau\mid a,X)
\end{aligned}
$$

若 (3) $h_\infty(X)=h_0(X)$ 且 $F_{j,\infty}(t\mid a,X)=F_{j,0}(t\mid a,X)$ 和 $G_\infty(t\mid a,X)=G_0(t\mid a,X)$ 成立，则 $\mathrm{RMTL}_{j,\infty}(t\mid a,X)=\mathrm{RMTL}_{j,0}(t\mid a,X)$，$\Lambda_\infty^C(t\mid a,X)=\Lambda_0^C(t\mid a,X)$，且 $\mathrm{d}M_\infty^C(t\mid a,X)=\mathrm{d}M_0^C(t\mid a,X)$。我们考虑 $\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau,a)$ 与 $E[\int_0^{\tau\wedge T}\cdot\,\mathrm{d}M_0^C(t\mid a,X)]$ 的删失鞅表示，因为 $\int\cdot\,\mathrm{d}(M_0^C(t\mid a,X))=0$

$$
\begin{aligned}
&\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau,a)\xrightarrow{P}\\
&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}+\left[E_0\left\{\frac{h_0(X)I(A=a)}{\pi_\infty(a\mid X)}\right\}\right]^{-1}E_0\left[\left\{\frac{h_0(X)I(A=a)}{\pi_\infty(a\mid X)}\right\}\right.\\
&\times\left\{\frac{(\tau-\min(\widetilde{T},\tau))I(\widetilde{J}=j)}{G_0(\widetilde{T}-\mid a,X)}-\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\int_0^{\tau\wedge\widetilde{T}}\frac{(\tau-t)F_{j,0}(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_\infty(t\mid a,X)G_0(t-\mid a,X)}\right.\\
&\left.\left.+\int_0^{\tau\wedge\widetilde{T}}\frac{\{\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0^C(t\mid a,X)}{S_\infty(t\mid a,X)G_0(t-\mid a,X)}\right\}\right]\\
=&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}+\left[E_0\left\{\frac{h_0(X)\pi_0(a\mid X)}{\pi_\infty(a\mid X)}\right\}\right]^{-1}\\
&\times E_0\left(\left[E_0\left\{\frac{h_0(X)\pi_0(a\mid X)}{\pi_\infty(a\mid X)}\right\}\left\{\frac{(\tau-\min(\widetilde{T},\tau))I(\widetilde{J}=j)}{G_0(\widetilde{T}-\mid a,X)}\mid A=a,X\right\}-\mathrm{RMTL}_{j,0}(\tau\mid a,X)\right.\right.\\
&-E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{(\tau-t)F_{j,0}(t\mid a,X)\mathrm{d}M_0^C(t\mid a,X)}{S_\infty(t\mid a,X)G_0(t-\mid a,X)}\mid A=a,X\right\}\\
&\left.\left.+E_0\left\{\int_0^{\tau\wedge\widetilde{T}}\frac{\{\mathrm{RMTL}_{j,0}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(t\mid a,X)\}\mathrm{d}M_0^C(t\mid a,X)}{S_\infty(t\mid a,X)G_0(t-\mid a,X)}\mid A=a,X\right\}\right]\right)\\
=&[E_0\{h_0(X)\}]^{-1}E_0\{h_0(X)\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}=\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)
\end{aligned}
$$

上述证明还意味着 $E_0\{\phi_{j,0}^{\mathrm{RMTL}}(\tau,a;\eta_{j,0}^{\Lambda})\mid A=a,X\}=\mathrm{RMTL}_{j,0}(\tau\mid a,X)$。

根据 Westling 等人（2021年）的引理 2,

$$
\begin{aligned}
&|\widehat{\psi}^{\mathrm{RMST},h}(\tau,a)-\psi_0^{\mathrm{RMST},h}(\tau,a)|\\
\le&|P_n\{\varphi_\infty^{\mathrm{RMST},h}(\tau,a;\eta_\infty^{\Lambda,h})\}|+\left|\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(\tau,a;\eta_\infty^{\Lambda,h})\}\right|\\
&+\left|\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k}{n}P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(\tau,a;\eta_\infty^{\Lambda,h})\}\right|
\end{aligned}
$$

在条件 3 和弱大数定律下，$P_n\{\varphi_\infty^{\mathrm{RMST},h}(t,a;\eta_\infty^{\Lambda,h})\}\xrightarrow{P}0$。根据条件 1、2 和引理 4，第二项为 $o_P(n^{-1/2})$。第三项由 $2[\max_k P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(\tau,a;\eta_\infty^{\Lambda,h})\}^2]^{1/2}$ 界定，进而可根据引理进一步界定，并依概率收敛至零。因此，$|\widehat{\psi}^{\mathrm{RMST},h}(\tau,a)-\psi_0^{\mathrm{RMST},h}(\tau,a)|=o_P(1)$。

就一致一致性而言，

$$\sup_{t\in[0,\tau]}|\widehat{\psi}^{\mathrm{RMST},h}(t,a)-\psi_0^{\mathrm{RMST},h}(t,a)|$$

$$\leq \sup_{t\in[0,\tau]}|P_n\{\varphi_\infty^{\mathrm{RMST},h}(t,a;\eta_\infty^{\Lambda,h})\}|+\sup_{t\in[0,\tau]}\left|\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(t,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(t,a;\eta_\infty^{\Lambda,h})\}\right|$$

$$+\sup_{t\in[0,\tau]}\left|\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k}{n}P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(t,a;\widehat{\eta}_k^{\Lambda,h})-\phi_\infty^{\mathrm{RMST},h}(t,a;\eta_\infty^{\Lambda,h})\}\right|$$

在 Donsker $\phi^{\mathrm{RMST}}{}_{,\infty}{}^{h}(t,a;\eta^{\Lambda}{}_{,\infty}{}^{h})$ 条件下，右端的第一项为 $O_P(n^{-1/2})$。当条件2和4成立时，第二项为$o_P(n^{-1/2})$。根据引理 3，第三项依概率收敛于零。最终，$\sup_{t\in[0,\tau]}|\widehat{\psi}^{\mathrm{RMST},h}(t,a)-\psi^{\mathrm{RMST}}{}_{,0}{}^{h}(t,a)|=o_P(1)$。类似的推理也适用于$|\widehat{\psi}^{\mathrm{RMTL}}{}_{,j}{}^{h}(\tau,a)-\psi_0^{\mathrm{RMTL}}{}_{,j}{}^{h},(\tau,a)|=o_P(1)$ 和 $\sup_{t\in[0,\tau]}|\widehat{\psi}^{\mathrm{RMTL}}{}_{,j}{}^{h}(t,a)-\psi_0^{\mathrm{RMTL}}{}_{,j}{}^{h},(t,a)|=o_P(1)$。

条件 1 和 5 要求估计的干扰函数依概率收敛到其极限对应函数，这些函数用于控制 $\widehat{\psi}^{\mathrm{RMST},h}(\tau,a)-\psi^{\mathrm{RMST}}{}_{,0}{}^{h}(\tau,a)$ 和 $\widehat{\psi}^{\mathrm{RMTL}}{}_{,j}{}^{h}(\tau,a)-\psi_0^{\mathrm{RMTL}}{}_{,j}{}^{h},(\tau,a)$ 的冯·米塞斯展开中的经验过程项。条件 2 对条件分布进行界定；这些分布在分解 $P\{\widehat{\phi}^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}^{\Lambda,h})-\phi^{\mathrm{RMST}}{}_{,0}{}^{h}(\tau,a;\eta^{\Lambda}{}_{,0}{}^{h})\}^2$ 和 $P\{\widehat{\phi}_j^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}^{\Lambda}{}_{,j}{}^{h})-\phi_0^{\mathrm{RMTL}}{}_{,j}{}^{h},(\tau,a;\eta^{\Lambda}{}_{,j}{}_0{}^{h},)\}^2$ 后出现在分母中，并要求它们在所有层中远离零。通过对极端值进行缩尾处理，可在经验上确保这一条件。条件 3 和 6 通常称为模型双重稳健性，即所得估计量的一致性仅依赖于对部分干扰函数的一致估计。当 $h_0(x)$ 是 $\pi_0(a\mid x)$ 的函数时，条件 3 简化为 (1)$\pi_\infty(a\mid X)=\pi_0(a\mid X)$ 且 $S_\infty(t\mid a,X)=S_0(t\mid a,X)$，或 (2) $\pi_\infty(a\mid X)=\pi_0(a\mid X)$ 且 $G_\infty(t\mid a,X)=G_0(t\mid a,X)$；而条件 6 则简化为 (1) $\pi_\infty(a\mid X)=\pi_0(a\mid X)$ 且 $S_\infty(t\mid a,X)=S_0(t\mid a,X)$ 且 $F_{j,\infty}(t\mid a,X)=F_{j,0}(t\mid a,X)$，或 (2) $\pi_\infty(a\mid X)=\pi_0(a\mid X)$ 且 $G_\infty(t\mid a,X)=G_0(t\mid a,X)$。当 $h_0(x)=1$ 时，条件 3 简化为以下任一情形：(1) $S_\infty(t\mid a,X)=S_0(t\mid a,X)$，或 (2) $\pi_\infty(a\mid X)=\pi_0(a\mid X)$ 且 $G_\infty(t\mid a,X)=G_0(t\mid a,X)$。这一条件类似于 ATE 中生存函数双重稳健估计量的一致性条件（ Zhang 和 Schaubel，2012 ； Bai 等人，2013 ）等。当 $h_0(x)=1$ 时，条件 6 放宽为 (1) $S_\infty(t\mid a,X)=S_0(t\mid a,X)$ 且 $F_{j,\infty}(t\mid a,X)=F_{j,0}(t\mid a,X)$，或 (2)$\pi_\infty(a\mid X)=\pi_0(a\mid X)$ 且 $G_\infty(t\mid a,X)=G_0(t\mid a,X)$，或 (3) $F_{j,\infty}(t\mid a,X)=F_{j,0}(t\mid a,X)$ 且 $G_\infty(t\mid a,X)=G_0(t\mid a,X)$；这与 Ozenne 等人（2020）中关于原因特异性累积发生函数的定理 1 相对应。通过交叉拟合可以避免要求干扰模型具有有界复杂度（通常意味着熵积分有界）的 Donsker 条件（Chernozhukov 等人，2018）。

定理 3 的证明

点态和一致渐近线性还需要满足额外条件。

条件 8. 满足 $\max_k|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1}|=o_P(n^{-1/2})$, $\max_k E_0|\widehat{h}_k(X)-h_0(X)|=o_P(n^{-1/2})$。

条件 9. 满足 $\widehat{r}_1(\tau,a)=o_P(n^{-1/2})$。

条件 10. 满足 $\widehat{r}_2^{\mathrm{RMST}}(\tau,a)=o_P(n^{-1/2})$。

条件 11。此外，$\sup_{t\in[0,\tau]}\widehat{r}_1(t,a)=o_P(n^{-1/2})$ 以及 $\sup_{t\in[0,\tau]}\widehat{r}_2^{\mathrm{RMST}}(t,a)=o_P(n^{-1/2})$。

条件 12。满足 $\widehat{r}_2^{\mathrm{RMTL}}{}_{,j}(\tau,a)=o_P(n^{-1/2})$ 以及 $\widehat{r}_3^{\mathrm{RMTL}}{}_{,j}(\tau,a)=o_P(n^{-1/2})$.

条件 13。此外，$\sup_{t\in[0,\tau]}\widehat{r}_1(t,a)=o_P(n^{-1/2})$，$\sup_{t\in[0,\tau]}\widehat{r}_{2\,j}^{\mathrm{RMTL}}{}_{,}(t,a)=o_P(n^{-1/2})$，且$\sup_{t\in[0,\tau]}\widehat{r}_{3\,j}^{\mathrm{RMTL}}{}_{,}(t,a)=o_P(n^{-1/2})$。

在条件 1、2、3、8、9、10 下，

$$\widehat{\psi}^{\mathrm{RMST},h}(\tau,a)=\psi_0^{\mathrm{RMST},h}(\tau,a)+P_n\{\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta_0^{\Lambda,h})\}+o_P(n^{-1/2})$$
$$n^{1/2}\{\widehat{\psi}^{\mathrm{RMST},h}(\tau,a)-\psi_0^{\mathrm{RMST},h}(\tau,a)\}\xrightarrow{D}\mathcal{N}(0,P_0[\{\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta_0^{\Lambda,h})\}^2])$$

此外，若条件 4 和条件 11 成立，

$$\sup_{t\in[0,\tau]}\left|\widehat{\psi}^{\mathrm{RMST},h}(t,a)-\psi_0^{\mathrm{RMST},h}(t,a)-P_n\{\varphi_0^{\mathrm{RMST},h}(t,a;\eta_0^{\Lambda,h})\}\right|=o_P(n^{-1/2})$$

使得 $n^{1/2}\{\widehat{\psi}^{\mathrm{RMST},h}(t,a)-\psi^{\mathrm{RMST}}{}_{,0}{}^{h}(t,a)\}\,(t\in[0,\tau])$ 弱收敛到一个紧致的零均值高斯过程，其协方差为 $\sigma^{\mathrm{RMST}}{}_{,0}{}^{h}(u,t,a)=P\{\varphi_0^{\mathrm{RMST},h}(u,a;\,\eta^{\Lambda}{}_{,0}{}^{h})\varphi^{\mathrm{RMST}}{}_{,0}{}^{h}(t,a;\,\eta^{\Lambda}{}_{,0}{}^{h})\}$.

在条件2、5、6、8、9、12下，

$$\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau,a)=\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)+P_n\{\varphi_{j,0}^{\mathrm{RMTL},h}(\tau,a;\eta_0^{\Lambda,h})\}+o_P(n^{-1/2})$$
$$n^{1/2}\{\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau,a)-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\}\xrightarrow{D}\mathcal{N}(0,P_0[\{\varphi_{j,0}^{\mathrm{RMTL},h}(\tau,a;\eta_0^{\Lambda,h})\}^2])$$

此外，若条件 7 和 13 成立，

$$\sup_{t\in[0,\tau]}\left|\widehat{\psi}_j^{\mathrm{RMTL},h}(t,a)-\psi_{j,0}^{\mathrm{RMTL},h}(t,a)-P_n\{\varphi_{j,0}^{\mathrm{RMTL},h}(t,a;\eta_{j,0}^{\Lambda,h})\}\right|=o_P(n^{-1/2})$$

使得 $n^{1/2}\{\widehat{\psi}_j^{\mathrm{RMTL},h}(t,a)-\psi_0^{\mathrm{RMTL}}{}_{,j}{}^{h}{}_{,}(t,a)\}\,(t\in[0,\tau])$ 弱收敛到一个均值为零、协方差为 $\sigma_{j\,0}^{\mathrm{RMTL}\,h}{}_{,,}(u,t,a)=P\{\varphi_{j,0}^{\mathrm{RMTL},h}(u,a;\,\eta^{\Lambda}{}_{,j\,0}{}^{h}{}_{,})\varphi_0^{\mathrm{RMTL}}{}_{,j}{}^{h}{}_{,}(t,a;\,\eta^{\Lambda}{}_{,j\,0}{}^{h}{}_{,})\}$ 的紧高斯过程。

沿用定理 2 的证明，当 $\eta^{\Lambda}{}_{,\infty}{}^{h}=\eta^{\Lambda}{}_{,0}{}^{h}$ 时，我们有

$$\begin{aligned}&\widehat{\psi}^{\mathrm{RMST},h}(\tau,a)-\psi_0^{\mathrm{RMST},h}(\tau,a)\\=&P_n\{\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta_0^{\Lambda,h})\}+\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_0^{\mathrm{RMST},h}(\tau,a;\eta_0^{\Lambda,h})\}\\&+\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k}{n}[P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})\}-\psi_0^{\mathrm{RMST},h}(\tau,a)]\end{aligned}$$

在条件 1、2、4 和引理 3 下，

$$\frac{1}{K}\sum_{k=1}^{K}\frac{Kn_k^{1/2}}{n}\sup_{t\in[0,\tau]}\left|\mathcal{G}_{n,k}\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})-\phi_0^{\mathrm{RMST},h}(\tau,a;\eta_0^{\Lambda,h})\}\right|=o_P(n^{-1/2})$$

对于右端的第三项，我们将杜阿梅尔方程代入

$$\frac{S_0(t-\mid a,X)}{\widehat{S}(t\mid a,X)}\{\mathrm{d}\widehat{\Lambda}(t\mid a,X)-\mathrm{d}\Lambda_0(t\mid a,X)\}=\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}(t\mid a,X)}-1\right\}$$

引理 2 中,

$$
\begin{aligned}
&P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})\}-\psi_0^{\mathrm{RMST},h}(\tau,a)\\
=&[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)\widehat{\mathrm{RMST}}_k(\tau\mid a,X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right]\\
&+E_0\left[\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\widehat{\mathrm{RMST}}_k(\tau\mid a,X)\left\{\frac{S_0(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-1\right\}\right]\\
&-E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau\widehat{\mathrm{RMST}}_k(t\mid a,X)\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right]\\
&-E_0\left(\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\left[\widehat{\mathrm{RMST}}_k(\tau\mid a,X)\left\{\frac{S_0(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}\right\}-\mathrm{RMST}_0(\tau\mid a,X)\right]\right)\\
&+([P_{n,k}\{\widehat{h}_k(X)\}]^{-1}-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1})E_0\{\widehat{h}_k(X)\widehat{\mathrm{RMST}}_k(\tau\mid a,X)\}\\
&+([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1})E_0\{\widehat{h}_k(X)\mathrm{RMST}_0(\tau\mid a,X)\}\\
&+[E_0\{h_0(X)\}]^{-1}E_0[\{\widehat{h}_k(X)-h_0(X)\}\mathrm{RMST}_0(\tau\mid a,X)]
\end{aligned}
$$

使得

$$
\begin{aligned}
&\left|P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})\}-\psi_0^{\mathrm{RMST},h}(\tau,a)\right|\\
\le&2h_{\min}^{-1}h_{\max}\epsilon\tau E_0\left|\int_0^\tau\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right|\\
&+h_{\min}^{-1}h_{\max}\epsilon E_0\left|\{\widehat{\pi}_k(a\mid X)-\pi_0(a\mid X)\}\{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)-\mathrm{RMST}_0(\tau\mid a,X)\}\right|\\
&+2h_{\min}^{-1}h_{\max}\tau h_{\max}\tau\left|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1}\right|+h_{\min}^{-1}\tau E_0|\{\widehat{h}_k(X)-h_0(X)|
\end{aligned}
$$

其中我们定义

$$
\begin{aligned}
\widehat{r}_1(\tau,a)&=\max_k E_0\left|\int_0^\tau\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right|\\
\widehat{r}_2^{\mathrm{RMST}}(\tau,a)&=\max_k E_0\left|\{\widehat{\pi}_k(a\mid X)-\pi_0(a\mid X)\}\{\widehat{\mathrm{RMST}}_k(\tau\mid a,X)-\mathrm{RMST}_0(\tau\mid a,X)\}\right|
\end{aligned}
$$

注意，$1/K\sum_{k=1}^K Kn_k/n\le 2$，使得

$$
\begin{aligned}
&\left|\frac{1}{K}\sum_{k=1}^K\frac{Kn_k}{n}[P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_k^{\Lambda,h})\}-\psi_0^{\mathrm{RMST},h}(\tau,a)]\right|\\
\le&4h_{\min}^{-1}h_{\max}\epsilon\tau\widehat{r}_1(\tau,a)+2h_{\min}^{-1}h_{\max}\epsilon\widehat{r}_2^{\mathrm{RMST}}(\tau,a)\\
&+4h_{\min}^{-1}h_{\max}\tau h_{\max}\tau\left|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1}\right|+2h_{\min}^{-1}\tau E_0|\{\widehat{h}_k(X)-h_0(X)|
\end{aligned}
$$

根据条件 8、9、10，右端为 $o_P(n^{-1/2})$。因此 $\widehat{\psi}^{\mathrm{RMST},h}(\tau,a)=\psi_0^{\mathrm{RMST},h}(\tau,a)+P_n\{\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta^{\Lambda},{}_0{}^h)\}+o_P(n^{-1/2})$。由于 $P_0\{\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta^{\Lambda},{}_0{}^h)\}=0$, $P_0[\{\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta^{\Lambda},{}_0{}^h)\}]^2<\infty$，且其一致有界，$n^{1/2}P_n\{\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta^{\Lambda},{}_0{}^h)\}\xrightarrow{D}\mathcal{N}(0,P_0[\{\varphi_0^{\mathrm{RMST},h}(\tau,a;\eta^{\Lambda},{}_0{}^h)\}^2])$。

当条件 11 成立

$$
\begin{aligned}
&\sup_{t\in[0,\tau]}\left|\frac{1}{K}\sum_{k=1}^K\frac{Kn_k}{n}[P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(t,a;\widehat{\eta}_k^{\Lambda,h})\}-\psi_0^{\mathrm{RMST},h}(t,a)]\right|\\
\le&4h^{-1}{}_{\text{最小化}}h_{\max}\epsilon\tau\text{上确界}_{t\in[0,\tau]}\ \widehat{r}_1(t,a)+2h^{-1}{}_{\text{最小化}}h_{\max}\epsilon\text{上确界}_{(t,a)^{t\in[0,\tau]}}\ \widehat{r}_2^{\mathrm{RMST}}\\
&+4h^{-1}{}_{\text{最小值}}h_{\max}\tau h_{\max}\tau\left|[P_{n,k}\{\widehat{w}_k^h(a,\ X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1}\right|+2h^{-1}{}_{\text{最小值}}\tau E_0|\{\widehat{h}_k(X)-h_0(X)|=o_P(n^{-1/2})
\end{aligned}
$$

因此,

$$\sup_{t\in[0,\tau]}\left|\widehat{\psi}^{\mathrm{RMST},h}(t,a)-\psi_0^{\mathrm{RMST},h}(t,a)-P_n\{\varphi_0^{\mathrm{RMST},h}(t,a;\eta_0^{\Lambda,h})\}\right|=o_P(n^{-1/2})$$

由于 $\varphi^{\mathrm{RMST}}{}_{,0}{}^{h}(t,a;\,\eta^{\Lambda}{}_{,0}{}^{h})$ 对 $t\in[0,\tau]$ 一致有界，因此 $n^{1/2}\{P_n\varphi_0^{\mathrm{RMST},h}(t,a;\,\eta^{\Lambda}{}_{,0}{}^{h})\}$ 弱收敛至一个均值为零、协方差为 $\sigma^{\mathrm{RMST}}{}_{,0}{}^{h}(u,t,a)=P_0\{\varphi_0^{\mathrm{RMST},h}(u,a;\,\eta^{\Lambda}{}_{,0}{}^{h})\varphi^{\mathrm{RMST}}{}_{,0}{}^{h}(t,a;\,\eta$ 的紧高斯过程

我们将杜阿梅尔方程代入引理 2 的竞争风险情境,

$$\begin{aligned}
&P_0\{\widehat{\phi}_{j,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h})\}-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\\
=&[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\left(-E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)\tau}{\widehat{\pi}_k(a\mid X)}\int_0^\tau S_0(t\mid a,X)\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\right.\right.\\
&\left.\times\{\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\right]-E_0\left[\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\right.\\
&\left.\times\tau\int_0^\tau S_0(t\mid a,X)\{\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\right]\\
&+E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau tS_0(t\mid a,X)\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\{\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\right]\\
&+E_0\left[\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\int_0^\tau tS_0(t\mid a,X)\{\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\right]\\
&+E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right]\\
&+E_0\left[\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\left\{\frac{S_0(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-1\right\}\right]\\
&-E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right]\\
&-E_0\left(\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\left[\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\left\{\frac{S_0(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}\right\}-\int_0^\tau\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}\widehat{F}_{j,k}(t\mid a,X)\mathrm{d}t\right]\right)\\
&-E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)\tau}{\widehat{\pi}_k(a\mid X)}\int_0^\tau\widehat{F}_{j,k}(t\mid a,X)\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right]\\
&-E_0\left[\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\tau\left\{\frac{\widehat{F}_{j,k}(\tau\mid a,X)S_0(\tau\mid a,X)}{\widehat{S}_k(\tau\mid a,X)}-\int_0^\tau S_0(\tau\mid a,X)\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)\right\}\right]\\
&+E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau t\widehat{F}_{j,k}(t\mid a,X)\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right]
\end{aligned}$$

$$\begin{aligned}
&+E_0\left[\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\left\{\frac{\widehat{F}_{j,k}(\tau\mid a,X)S_0(\tau\mid a,X)\tau}{\widehat{S}_k(\tau\mid a,X)}-\int_0^\tau\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}\widehat{F}_{j,k}(t\mid a,X)\mathrm{d}t\right.\right.\\
&\left.\left.-\int_0^\tau tS_0(\tau\mid a,X)\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)\right\}\right]\\
&+([P_{n,k}\{\widehat{h}_k(X)\}]^{-1}-[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1})E_0\{\widehat{h}_k(X)\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\}\\
&+([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1})E_0\{\widehat{h}_k(X)\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}\\
&+[E_0\{h_0(X)\}]^{-1}E_0[\{\widehat{h}_k(X)-h_0(X)\}\mathrm{RMTL}_{j,0}(\tau\mid a,X)]
\end{aligned}$$

消去共同项后，我们得到

$$
\begin{aligned}
&P_0\{\widehat{\phi}_{j,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h})\}-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\\
=&[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}\left(-E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau(\tau-t)S_0(t\mid a,X)\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\right.\right.\\
&\left.\times\{\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\right]+E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau\{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)-\widehat{\mathrm{RMTL}}_{j,k}(t\mid a,X)\}\right.\\
&\times\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\Bigg]-E_0\left[\frac{\widehat{h}_k(X)\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}\int_0^\tau(\tau-t)\widehat{F}_{j,k}(t\mid a,X)\right.\\
&\left.\times\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right]+E_0\left[\widehat{h}_k(X)\left\{\frac{\pi_0(a\mid X)}{\widehat{\pi}_k(a\mid X)}-1\right\}\right.\\
&\left.\times\{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}\right]\\
&+([P_n\{\widehat{h}_k(X)\}]^{-1}-[P_n\{\widehat{w}_k^h(a,X)\}]^{-1})E_0\{\widehat{h}_k(X)\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)\}\\
&+([P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1})E_0\{\widehat{h}_k(X)\mathrm{RMTL}_j(\tau\mid a,X)\}\\
&+[E_0\{h_0(X)\}]^{-1}E_0[\{\widehat{h}_k(X)-h_0(X)\}\mathrm{RMTL}_{j,0}(\tau\mid a,X)]
\end{aligned}
$$

最后，

$$
\begin{aligned}
&\left|P_0\{\widehat{\phi}_{j,k}^{\mathrm{RMTL},h}(\tau,a;\widehat{\eta}_{j,k}^{\Lambda,h})\}-\psi_{j,0}^{\mathrm{RMTL},h}(\tau,a)\right|\\
=&2h_{\min}^{-1}h_{\max}\epsilon\tau E_0\left|\int_0^\tau\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\{\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\right|\\
&+4h_{\min}^{-1}h_{\max}\epsilon\tau E_0\left|\int_0^\tau\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\mathrm{d}\left\{\frac{S_0(t\mid a,X)}{\widehat{S}_k(t\mid a,X)}-1\right\}\right|\\
&+h_{\min}^{-1}h_{\max}\epsilon E_0\left|\{\widehat{\pi}_k(a\mid X)-\pi_0(a\mid X)\}\{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}\right|\\
&+2h_{\min}^{-1}h_{\max}\tau h_{\max}\tau\left|[P_{n,k}\{\widehat{w}_k^h(a,X)\}]^{-1}-[E_0\{h_0(X)\}]^{-1}\right|+h_{\min}^{-1}\tau E_0|\{\widehat{h}_k(X)-h_0(X)|
\end{aligned}
$$

其中，我们定义了

$$
\begin{aligned}
\widehat{r}_{2,j}^{\mathrm{RMTL}}(\tau,a)=&\max_k E_0\left|\{\widehat{\pi}_k(a\mid X)-\pi_0(a\mid X)\}\{\widehat{\mathrm{RMTL}}_{j,k}(\tau\mid a,X)-\mathrm{RMTL}_{j,0}(\tau\mid a,X)\}\right|\\
\widehat{r}_{3,j}^{\mathrm{RMTL}}(\tau,a)=&\max_k E_0\left|\int_0^\tau\left\{\frac{G_0(t-\mid a,X)}{\widehat{G}_k(t-\mid a,X)}-1\right\}\{\mathrm{d}\widehat{\Lambda}_{j,k}(t\mid a,X)-\mathrm{d}\Lambda_{j,0}(t\mid a,X)\}\right|
\end{aligned}
$$

其余证明与 $P_0\{\widehat{\phi}_k^{\mathrm{RMST},h}(\tau,a;\widehat{\eta}_{,k}^{\Lambda,h})\}-\psi_{,0}^{\mathrm{RMST},h}(\tau,a)$ 的证明类似。

关于所提估计量的效率界，$\widehat{\psi}^{\mathrm{RMST},h}(\tau)$ 的条件方差为

$$
\begin{aligned}
&\operatorname{var}\{\widehat{\psi}^{\mathrm{RMST},h}(\tau) \mid O_n\} \\
=&\frac{1}{n}[P_n\{\widehat{h}(X)\}]^{-2}P_n[\widehat{h}^2(X)\operatorname{var}\{\widehat{\mathrm{RMST}}(\tau \mid 1, X) \mid O_n\}] + \frac{1}{n}[P_n\{\widehat{w}^h(1,X)\}]^{-2} \\
&\times P_n\left[\frac{\widehat{h}^2(X)I(A=1)}{\widehat{\pi}^2(1 \mid X)}\operatorname{var}\left\{-\int_0^{\tau\wedge\widetilde{T}}\frac{\{\widehat{\mathrm{RMST}}(\tau \mid 1,X) - \widehat{\mathrm{RMST}}(t \mid 1,X)\}\mathrm{d}\widehat{M}(t \mid 1,X)}{\widehat{S}(t \mid 1,X)\widehat{G}(t- \mid 1,X)} \mid O_n\right\}\right] \\
&+ \frac{1}{n}[P_n\{\widehat{h}(X)\}]^{-2}P_n[\widehat{h}^2(X)\operatorname{var}\{\widehat{\mathrm{RMST}}(\tau \mid 0, X) \mid O_n\}] + \frac{1}{n}[P_n\{\widehat{w}^h(0,X)\}]^{-2} \\
&\times P_n\left[\frac{\widehat{h}^2(X)I(A=0)}{\widehat{\pi}^2(0 \mid X)}\operatorname{var}\left\{-\int_0^{\tau\wedge\widetilde{T}}\frac{\{\widehat{\mathrm{RMST}}(\tau \mid 0,X) - \widehat{\mathrm{RMST}}(t \mid 0,X)\}\mathrm{d}\widehat{M}(t \mid 0,X)}{\widehat{S}(t \mid 0,X)\widehat{G}(t- \mid 0,X)} \mid O_n\right\}\right]
\end{aligned}
$$

根据 Slutsky定理,

$$
\begin{aligned}
&n \times \operatorname{var}\{\widehat{\psi}^{\mathrm{RMST},h}(\tau) \mid O_n\} \\
\to&\left\{\int h_0(x)p_0(x)\mathcal{F}(dx)\right\}^{-2}\int\left[\frac{\operatorname{var}\{\phi_0^{\mathrm{RMST}}(\tau,1;\eta_0^{\Lambda}) \mid x\}}{\pi_0(1 \mid x)} + \frac{\operatorname{var}\{\phi_0^{\mathrm{RMST}}(\tau,0;\eta_0^{\Lambda}) \mid x\}}{\pi_0(0 \mid x)}\right]h_0^2(x)p_0(x)\mathcal{F}(dx)
\end{aligned}
$$

其中， $p(x)$是相对于测度 $\mathcal{F}(\cdot)$的 X 概率密度函数。假设未倾斜、未中心化的真实 EIF 满足条件同方差，即 $\operatorname{var}\{\phi_0^{\mathrm{RMST}}(\tau,1;\eta_0^{\Lambda}) \mid x\} = \operatorname{var}\{\phi_0^{\mathrm{RMST}}(\tau,0;\eta_0^{\Lambda}) \mid x\} = C_{\mathrm{var}}^{\mathrm{RMST}}$，则有

$$
n \times \operatorname{var}\{\widehat{\psi}^{\mathrm{RMST},h}(\tau) \mid O_n\} \to C_{\mathrm{var}}^{\mathrm{RMST}}\left\{\int h_0(x)p_0(x)\mathcal{F}(dx)\right\}^{-2}\int\left[\frac{h_0^2(x)p_0(x)\mathcal{F}(dx)}{\pi_0(1 \mid x)\pi_0(0 \mid x)}\right]
$$

根据柯西-施瓦茨不等式,

$$
\begin{aligned}
&\left\{\int h_0(x)p_0(x)\mathcal{F}(dx)\right\}^2 = \left\{\int \frac{h_0(x)}{\sqrt{\pi_0(1 \mid x)\pi_0(0 \mid x)}}\sqrt{\pi_0(1 \mid x)\pi_0(0 \mid x)}p_0(x)\mathcal{F}(dx)\right\}^2 \\
\le&\int \frac{h_0^2(x)}{\pi_0(1 \mid x)\pi_0(0 \mid x)}p_0(x)\mathcal{F}(dx) \times \int \pi_0(1 \mid x)\pi_0(0 \mid x)p_0(x)\mathcal{F}(dx)
\end{aligned}
$$

等号成立当且仅当 $h_0(x)/\{\sqrt{\pi_0(1 \mid x)\pi_0(0 \mid x)}\} \propto \sqrt{\pi_0(1 \mid x)\pi_0(0 \mid x)}$，恒等地$h_0(x) \propto \pi_0(1 \mid x)\pi_0(0 \mid x) \propto \operatorname{var}(a \mid x)$。此时，在未倾斜、未中心化的 EIF 满足条件同方差的情况下，所提估计量达到 WATE 的半参数效率界。对于 $\operatorname{var}\{\widehat{\psi}_j^{\mathrm{RMTL},h}(\tau) \mid O_n\}$ 的证明类似。

注意，当 $h_0(x) = \pi(1 \mid x)\pi(0 \mid x)$ 时，我们有

$$
\psi_0^{\mathrm{RMST,ATO}}(\tau) = \frac{E_0[\operatorname{cov}\{A, \min(T,\tau) \mid X\}]}{E_0\{\operatorname{var}(A \mid X)\}}
$$

and

$$
\psi_{j,0}^{\mathrm{RMTL,ATO}}(\tau) = \frac{E_0(\operatorname{cov}[A, \{\tau - \min(T,\tau)\}I(J=j) \mid X])}{E_0\{\operatorname{var}(A \mid X)\}}
$$

Vansteelandt 和 Dukes（2022年）讨论了连续和二元结局下部分线性模型的自适应估计。然而，由于生存时间存在删失，结构嵌套模型需要对限制平均生存时间作出额外的半参数假设（Hagiwara 等，2020），因此本文不予探讨。