arXiv:2305.04144v1 [math.FA] 6 May 2023

在 L L_p 中以可分核线性积分算子表示多项式协变型交换关系

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摘要 研究了在 L_p 上测度空间中的多项式协变型交换关系的线性积分算子表示，并给出了积分算子满足多项式协变型交换关系的核的充要条件。对于与任意单项式及仿射函数相关的重要多项式协变交换关系类别，这些核条件可用单项式和仿射函数的系数来具体表述。应用这些条件，构造了若干定义在 L_p 空间上的积分算子示例；这些算子具有可分核，表示与单项式相关的协变交换关系，其核涉及有界区间上的多参数三角函数、多项式和 Laurent 多项式。计算了这些算子的交换子，并得到了以参数表示的这些算子可交换的精确条件。

关键词： 协方差交换关系，积分算子，可分核**2020年数学主题分类：** 47A62, 47L80, 47L65, 47G10

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1 引言

形如.....的交换关系

$$AB = BF(A) \tag{1}$$

其中， A， B 是结合代数中的元素，而 F 是该代数元素的函数。这类关系在数学的许多领域及其应用中都十分重要。这种交换关系通常称为协变关系、交叉积关系或半直积关系。代数中满足（1）的元素对（A， B）称为该关系 [43]的一个表示。

由线性算子给出的协变交换关系（1）的表示，对于研究群和半群的作用及诱导表示、交叉积算子代数、动力系统、调和分析、小波和分形分析十分重要，因此在物理学和工程学中也有应用 [4, 5,20–22, 27–29, 35, 36, 43]。

文献 [3, 7–11, 13–17, 30–35, 37–41, 43–56]研究了希尔伯特空间上由有界和无界自伴算子、正规算子、酉算子、偏等距算子，以及满足特殊对合条件的其他线性算子给出的形式为（1）的关系的表示结构。相关研究利用了代数元素函数的重排公式、满足协变交换关系的算子的函数演算和谱表示，以及它们与动力系统或由交换关系中映射的迭代所生成的迭代函数系统之间的相互作用。

本文通过测度空间上的巴拿赫空间 L_p 上的线性积分算子，构造了协变交换关系 (1) 的表示。我们研究由满足不同条件的核所定义的线性积分算子表示。我们推导出此类核函数需满足的条件，使得当两个算子均为线性积分型时，相应算子满足多项式 F 的关系。我们构造了测度空间 L_p 上由线性积分算子给出的多项式协变型交换关系表示。与文献 [35, 43, 44, 56] 通过希尔伯特空间上算子的谱理论，研究希尔伯特空间上算子的协变型关系的对合表示不同，我们旨在不施加任何对合条件、也不使用经典算子谱理论，直接构造巴拿赫空间上（比希尔伯特空间更一般的）若干重要算子类中协变型关系的各类表示。此类表示的条件以相应积分算子的核来描述。我们以核为依据，得到积分算子满足多项式协变型交换关系的充要条件。对于与任意单项式及仿射函数相关的重要多项式协变交换关系类，这些核条件可用单项式和仿射函数的系数来具体表述。应用这些条件，我们构造了 L_p 空间上的积分算子示例，它们具有可分核，并表示与单项式相关的协变交换关系；这些算子的核

涉及有界区间上的多参数三角函数、多项式和 Laurent 多项式。我们计算了这些算子的交换子，并得到了用参数表示这些算子可交换的精确条件。

本文分为四节。引言之后，我们在第 2 节介绍一些预备知识、记号、基本定义和一个有用的引理。在第 3 节中，我们给出当算子 A 和 B 都是作用于 L_p 空间上的可分核线性积分算子时的表示。第 4 节中，我们构造一些例子。

2 预备知识与记号

本节介绍本文所需的预备知识、基本定义和记号，更多内容请参阅 [1, 2, 6, 18, 19, 23–26, 42]。

令 $\mathbb{R}$ 为全体实数的集合，$\mathbb{Z}$ 为全体整数的集合，$\mathbb{N}$ 为全体正整数的集合，X 为一个非空集合。设 (X, Σ, μ) 是一个 σ-有限测度空间，其中 Σ 是一个 σ–代数，其可测子集属于 X，且 X 可被至多可数个互不相交的集合 $E_1, E_2, E_3, \ldots$ 覆盖，使得 $E_i \in \Sigma, \mu(E_i) < \infty$, $i = 1, 2, \ldots$，并且 μ 是一个测度。对于 $1 \le p < \infty$，我们用 $L_p(X, \mu)$ 表示等价类（在零测度集合上不同）的全体可测函数 $f\colon X \to \mathbb{R}$ 的集合，这些函数满足 $\int |f(t)|^p d\mu < \infty$。这是一个巴拿赫空间（当 $p = 2$时为希尔伯特空间），其范数为x

对于 $f\mathrm{k}_p = \left(\int_X |f(t)|^p dt! \right)^{\frac{1}{p}}$。我们用 $L_\infty(X, \mu)$ 表示所有等价可测函数类 f 的集合：$X \to \mathbb{R}$ 使得存在 $\lambda > 0, |f(t)| \le \lambda$ 几乎处处成立。这是一个巴拿赫空间，其范数为 $\mathrm{k}\ f\mathrm{k}_\infty = \operatorname{ess\,sup} |f(t)|$。函数$_{t \in X} f$ 的支集 $X \to \mathbb{R}$ 为 supp $f = \{t \in X\colon\ f(t)\ \neq 0\}$。

设 $(\mathbb{R},\ \Sigma,\ \mu)$ 为标准勒贝格测度空间。我们将使用以下记号

$$Q_G(u, v) = \int_G u(t) v(t) d\mu \tag{2}$$

其中 $G \in \Sigma$ 和 $u,\ v$ 是这样的函数 $u,\ v\colon\ G \to \mathbb{R}$ 使得右侧积分存在且有限。

我们将介绍一个关于积分算子的有用引理，本文将用到它。

引理1. 设 (X, Σ, μ) 为一个 σ-有限测度空间。设 $f \in L_q(G_1, \mu)$、$g \in L_q(G_2, \mu)$ 属于 $1 \le q \le \infty$，并令 $G_1, G_2 \in \Sigma, i = 1, 2$。置 $G = G_1 \cap G_2$。则以下陈述等价：

1. *For all* $x \in L_p(X, \mu), 1 \le p \le \infty$ *such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1,$

$$Q_{G_1}(f,x)=\int_{G_1} f(t)x(t)d\mu=\int_{G_2} g(t)x(t)d\mu=Q_{G_2}(g,x).$$

2. 以下条件成立：

a) 对几乎所有的 $t\in G$, $f(t)=g(t)$,

b) 对几乎所有的 $t\in G_1\setminus G$, $f(t)=0$,

c) 对几乎所有的 $t\in G_2\setminus G$, $g(t)=0$。

证明。 $2\Rightarrow 1$ 根据 μ 的可加性，我们有

$$\begin{aligned}\int_{G_1} f(s)x(s)d\mu_s &= \int_{G_1\setminus G} f(s)x(s)d\mu_s+\int_G f(s)x(s)d\mu_s=\int_G g(s)x(s)d\mu_s\\ &=\int_{G_2\setminus G} g(s)x(s)d\mu_s+\int_G g(s)x(s)d\mu_s=\int_{G_2} g(s)x(s)d\mu_s.\end{aligned}$$

$1\Rightarrow 2$ 假设 1 成立。令 $1\le p<\infty$。由霍尔德不等式可知，下列线性泛函 H_f: $L_p(X,\mu)\to\mathbb{R}$, H_g: $L_p(X,\mu)\to\mathbb{R}$, 定义为

$$H_f(x)=\int_{G_1} f(s)x(s)d\mu,\ H_g(x)=\int_{G_2} g(s)x(s)d\mu,$$

是连续的。事实上，由霍尔德不等式可得，对所有 $x\in L_p(X)$

$$\begin{aligned}|H_f(x)|&=\left|\int_{G_1} I_{G_1}(s)f(s)x(s)d\mu\right|\le\left(\int_X |I_{G_1}(s)f(s)|^q d\mu\right)^{1/q}\left(\int_X |x(s)|^p d\mu\right)^{1/p}\\ &=\left(\int_{G_1} |f(s)|^q d\mu\right)^{1/q}\left(\int_X |x(s)|^p d\mu\right)^{1/p}\le\|f\|_{L_q}\|x\|_{L_p},\end{aligned}$$

其中 $f(\cdot)\in L_q(G_1,\mu)$ 根据假设，$1<q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1$。类似地，可以证明 H_g 是连续的。因此，根据线性泛函的黎斯表示定理，[6, 定理 4.11]（对于 $1<p<\infty$）和 [6, 定理 4.14]（对于 $p=1$），存在 $h\in L_q(X,\mu)$，$1<q\le\infty$ 使得 $\frac{1}{p}+\frac{1}{q}=1$，且对所有 $x\in L_p(X,\mu)$

$$\int_X h(s)x(s)d\mu=\int_{G_1} f(s)x(s)d\mu=\int_{G_2} g(s)x(s)d\mu. \tag{3}$$

由于该表示是唯一的，我们有

$$h(s)=I_G(s)f(s)+I_{G_1\setminus G}(s)f(s)+I_{G_2\setminus G}(s)g(s)$$

$$= I_G(s)g(s) + I_{G_1\setminus G}(s)f(s) + I_{G_2\setminus G}(s)g(s),\ G = G_1 \cap G_2.$$

这意味着 $I_G(s)h(s) = I_G(s)f(s) = I_G(s)g(s)$。函数 h 属于 $L_q(X,\mu)$，因为 $f \in L_q(G_1,\mu)$、$g \in L_q(G_2,\mu)$ 和 $L_q(G_i,\mu)$，$i = 1,2$ 是一个向量空间。由 (3) 可得 $I_{X\setminus G_2}(s)h(s) = I_{X\setminus G_1}(s)h(s) = 0$ 几乎处处成立。因此

$$I_{G_1\setminus G}(s)f(s) = I_{G_1\setminus G}(s)h(s) = 0 \text{ almost everywhere since } G_1 \setminus G \subseteq X \setminus G_2,$$
$$I_{G_2\setminus G}(s)g(s) = I_{G_2\setminus G}(s)h(s) = 0 \text{ almost everywhere since } G_2 \setminus G \subseteq X \setminus G_1.$$

令 $p = \infty$。取集合 $H_1 = G_1 \cup G_2$ 的示性函数 $x(t) = I_{H_1}(t)$，可得

$$\int_{G_1} f(t)x(t)d\mu = \int_{G_2} g(t)x(t)d\mu = \int_{G_1} f(t)d\mu = \int_{G_2} g(t)d\mu = \eta,$$

其中 η 是一个常数。现在取 $x(t) = I_{G_1\setminus G}(t)$，可得

$$\int_{G_1} f(t)x(t)d\mu = \int_{G_2} g(t)x(t)d\mu = \int_{G_1\setminus G} f(t)d\mu = \int_{G_2} g(t)\cdot 0 d\mu = 0.$$

于是 $\int_{G_1\setminus G} f(t)d\mu = 0$。类似地，取 $x(t) = I_{G_2\setminus G}(t)$，可得 $\int_{G_2\setminus G} g(t)d\mu = 0$。我们声称，对于几乎处处的 $t \in G_1 \setminus G$，有 $f(t) = 0$；并且对于几乎处处的 $t \in G_2 \setminus G$，有 $g(t) = 0$。我们取集合 $[\alpha_1,\beta_1]\setminus G$ 的一个划分 $\text{Ð}S_i$，使得对于 $i \neq j$，有 $S_i \cap S_j \neq \emptyset$，且每个集合 S_i 都具有正测度。对于每个 $x(t) = I_{S_i}(t)$，有

$$\int_{G_1} f(t)x(t)d\mu = \int_{G_2} g(t)x(t)d\mu = \int_{S_i} f(t)d\mu = \int_{G_2} g(t)\cdot 0 d\mu = 0.$$

因此，$\int_{S_i} f(t)d\mu = 0$。由于我们可以在其每个元素上选择任意具有正测度的划分，$f(t) = 0$ 对于几乎处处的 $t \in G_1 \setminus G$。类似地，$g(t) = 0$ 对于几乎处处的 $t \in G_2 \setminus G$。因此，

$$\eta = \int_{G_1} f(t)d\mu = \int_{G_2} g(t)d\mu = \int_{G} f(t)d\mu = \int_{G} g(t)d\mu.$$

那么，对于所有函数 $x \in L_p(X,\mu)$，我们有 $\int_G f(t)x(t)d\mu = \int_G g(t)x(t)d\mu \Leftrightarrow \int_G [f(t) - g(t)]x(t)d\mu = 0$。设 $x(t) = \begin{cases} 1, & \text{若 } f(t) - g(t) > 0, \\ -1, & \text{若 } f(t) - g(t) < 0, \end{cases}$ 则对于几乎处处的 $t \in G$，且 $x(t) = 0$ 对几乎处处的 $t \in X\setminus G$，我们得到 $\int_G |f(t) - g(t)|d\mu = 0$。这意味着，对于几乎处处的 $t \in G$，有 $f(t) = g(t)$。□

3 线性积分算子表示

设 (X,Σ,μ) 是一个 σ-有限测度空间。本节中，我们讨论协方差型交换关系 (1) 的表示，其中 A 和 B 都是巴拿赫空间 $L_p(X,\mu)$ 上具有 $1 \le p \le \infty$ 的线性积分算子，且其核为可分核:

$$(Ax)(t) = \int_{G_A} \sum_{i=1}^{l_A} a_i(t)c_i(s)x(s)d\mu_s, \quad (Bx)(t) = \int_{G_B} \sum_{j=1}^{l_B} b_j(t)e_j(s)x(s)d\mu_s, \quad (4)$$

对几乎所有的 t，其中 μ_s 中的指标表示积分变量，$G_A \in \Sigma$ 和 $G_B \in \Sigma$。当 $B=0$ 时，对于任意 A，关系式 (1) 都显然成立。如果 $A=0$，则关系式 (1) 化简为 $F(0)B=0$。这意味着，要么 $(F(0)=0$ 且 B 可以是任意定义良好的算子)，要么 $B=0$。因此，我们着重研究 (1) 的非零表示的构造及其性质。我们采用与 Folland [18] 类似的论证，证明这些算子定义良好且有界的充分条件。若 $a_i, b_j \in L_p(X,\mu)$，$c_i \in L_q(G_A,\mu)$，$e_j \in L_q(G_B,\mu)$，i,j,l_A,l_B 为正整数，$i=1,\ldots,l_A$，$j=1,\ldots,l_B$，$1 \le q \le \infty$，$\frac{1}{p}+\frac{1}{q}=1$，则算子 A 和 B 定义良好且有界。事实上，若令 $(A_ix)(t) = \int a_i(t)c_i(s)x(s)d\mu_s$ 对几乎G_Al_A所有 t 成立，且 $(Ax)(t) = \acute{I}(A_ix)(t)$ 对几乎所有 $t \in X$ 成立，并且若 $1<p<\infty$，则由$i=1$霍尔德不等式可得，对所有 $x \in L_p(X,\mu)$，$c_i(\cdot)x(\cdot) \in L_1(G_A,\mu)$，且

$$\begin{aligned}
\|A_ix\|^p_{L_p(X,\mu)} &= \int_X \left| \int_{G_A} a_i(t)c_i(s)x(s)d\mu_s \right|^p d\mu_t \\
&\le \int_X |a_i(t)|^p d\mu_t \left(\int_X |I_{G_A}(s)c_i(s)|^q d\mu_s \right)^{\frac{p}{q}} \int_X |x(s)|^p d\mu_s \\
&= \int_X |a_i(t)|^p d\mu_t \left(\int_{G_A} |c_i(s)|^q d\mu_s \right)^{\frac{p}{q}} \int_X |x(s)|^p d\mu_s \\
&= \|a_i\|^p_{L_p(X,\mu)} \|c_i\|^p_{L_q(G_A,\mu)} \|x\|^p_{L_p(X,\mu)},
\end{aligned}$$

对于 $1<p<\infty, i=1,\ldots,l_A$。对于 $p=\infty$，根据霍尔德不等式，对于所有 $x \in L_\infty(X,\mu)$,$c_i(\cdot)x(\cdot) \in L_1(G_A,\mu)$，以及

$$\|A_ix\|_{L_\infty(X,\mu)} = \operatorname*{ess\,sup}_{t\in X} \left| \int_{G_A} a_i(t)c_i(s)x(s)d\mu_s \right|$$

$$\leq \left(\operatorname*{ess\,sup}_{t\in X}|a_i(t)|\right)\left(\int_X |I_{G_A}(s)c_i(s)|d\mu_s\right)\left(\operatorname*{ess\,sup}_{s\in X}|x(s)|\right)$$

$$= \left(\operatorname*{ess\,sup}_{t\in X}|a_i(t)|\right)\left(\int_{G_A} |c_i(s)|d\mu_s\right)\left(\operatorname*{ess\,sup}_{s\in X}|x(s)|\right)$$

$$= \|a_i\|_{L_\infty(X,\mu)}\|c_i\|_{L_1(G_A,\mu)}\|x\|_{L_\infty(X,\mu)},$$

对于 $i=1,\ldots,l_A$。类似地，可对情形 $p=1$ 进行论证。由于 $L_p(X,\mu)$, $1\leq p\leq\infty$是线性空间，我们得出算子 A, B在 $L_p(X,\mu)$ 上定义良好且有界。

在下面的定理中，我们将考虑另一种等价的写法来表示式（4）中的算子，以简化本文中将要讨论的特殊情形的记号。

定理 1. 设(X,Σ,μ)是 σ-有限测度空间。设 $A\colon L_p(X,\mu)\to L_p(X,\mu)$,$B\colon L_p(X,\mu)\to L_p(X,\mu)$, $1\leq p\leq\infty$是如下定义的非零算子

$$(Ax)(t)=\int_{G_A}\sum_{k=1}^{l_{A,1}}\sum_{i=1}^{l_{A,2}}\sum_{l=1}^{l_{A,3}}\gamma_A^{k,i,l}a_{k,l}(t)c_{i,l}(s)x(s)d\mu_s, \tag{5}$$

$$(Bx)(t)=\int_{G_B}\sum_{u=1}^{l_{B,1}}\sum_{w=1}^{l_{B,2}}\sum_{r=1}^{l_{B,3}}\gamma_B^{u,w,r}b_{u,r}(t)e_{w,r}(s)x(s)d\mu_s, \tag{6}$$

对于几乎所有的 t，其中 μ_s 中的指标表示积分变量，$G_A\in\Sigma$ 且 $G_B\in\Sigma$, $a_{k,l}, b_{u,r}\in L_p(X,\mu)$、$c_{i,l}\in L_q(G_A,\mu)$、$e_{w,r}\in L_q(G_B,\mu)$、$k,i,l,l_A,1,l_A,2,l_A,3,u,w,r,l_B,1,l_B,2,l_B,3$ 均为正整数，使得$1\leq k\leq l_A,1$,$1\leq i\leq l_A,2$, $1\leq l\leq l_A,3$, $1\leq u\leq l_B,1$, $1\leq w\leq l_B,2$, $1\leq r\leq l_B,3$，且 $1\leq q\leq\infty$ 使得 $\frac{1}{p}+\frac{1}{q}=1$, γ^k, $_{A}{}_{i\,l}$, , γ^u,$_B{}^{w\,r}\in\mathbb{R}$, 。考虑由 $F(z)=\overset{n}{\acute{I}}_{j=0}\delta_j z^j$ 定义的多项式，其中 $\delta_j\in\mathbb{R}$ $j=0,\ldots,n$。令 $G=G_A\cap G_B$，并

$$\tilde{\gamma}_1=1,\quad \tilde{\gamma}_m=\prod_{v=1}^{m-1}Q_{G_A}(a_{k_{v+1},l_{v+1}},c_{i_v,l_v}),\ m\geq 2.$$

其中 $Q_\Lambda(u,v)$, $\Lambda\in\Sigma$由(2)定义。则当且仅当满足以下条件时， $AB=BF(A)$成立：

1. 对于几乎处处的$(t,s)\in X\times G$,

$$-\sum_{u,w,r=1}^{l_{B,1},l_{B,2},l_{B,3}}\delta_0\gamma_B^{u,w,r}b_{u,r}(t)e_{w,r}(s)+\sum_{(k,i,l=1)}^{(l_{A,1},l_{A,2},l_{A_3})}\sum_{(u,w,r=1)}^{(l_{B,1},l_{B_2},l_{B_3})}\gamma_A^{k,i,l}\gamma_B^{u,w,r}$$
$$\cdot a_{k,l}(t)Q_{G_A}(c_{i,l},b_{u,r})e_{w,r}(s)$$

$$= \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \sum_{j=1}^{n} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_j,i_j,l_j=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \delta_j \tilde{\gamma}_j \gamma_B^{u,w,r}$$
$$\cdot \prod_{v=1}^{j} \gamma_A^{k_v,i_v,l_v} Q_{G_B}(e_{w,r}, a_{k_1,l_1}) b_{u,r}(t) c_{i_j,l_j}(s)$$

2. 对于几乎处处的$(t,s) \in X \times (G_A \setminus G)$,

$$\sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \sum_{j=1}^{n} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_j,i_j,l_j=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \delta_j \tilde{\gamma}_j \gamma_B^{u,w,r}$$
$$\cdot \prod_{v=1}^{j} \gamma_A^{k_v,i_v,l_v} Q_{G_B}(e_{w,r}, a_{k_1,l_1}) b_{u,r}(t) c_{i_j,l_j}(s) = 0.$$

3. 几乎处处成立 $(t,s) \in X \times (G_B \setminus G)$,

$$\sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \delta_0 \gamma_B^{u,w,r} b_{u,r}(t) e_{w,r}(s) = \sum_{(k,i,l=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \gamma_A^{k,i,l} \gamma_B^{u,w,r} a_{k,l}(t)$$
$$\cdot Q_{G_A}(c_{i,l}, b_{u,r}) e_{w,r}(s).$$

证明。我们注意到，若对所有 $k,i,l, 1 \le k \le l_A,1, 1 \le i \le l_A,2, 1 \le l \le l_A,3, 1 \le u \le l_B,1,$ $1 \le w \le l_B,2, 1 \le r \le l_B,3, a_{k,l}, b_{u,r} \in L_p(X,\mu),\ 1 \le p \le \infty, c_{i,l} \in L_q(G_A,\mu), e_{w,r} \in L_q(G_B,\mu),$ 其中 $1 \le q \le \infty$，且 $\frac{1}{p} + \frac{1}{q} = 1, L_p(X,\mu),\ L_q(X,\mu)$ 是线性空间，则算子 A和 B是良定义的。直接计算可得

$$(A^2x)(t) = \int_{G_A} \sum_{k_1=1}^{l_{A,1}} \sum_{i_1=1}^{l_{A,2}} \sum_{l_1=1}^{l_{A,3}} \gamma_A^{k,i,l} a_{k_1,l_1}(t) c_{i_1,l_1}(s) (Ax)(s) d\mu_s$$
$$= \int_{G_A} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \sum_{(k_2,i_2,l_2=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \gamma_A^{k_1,i_1,l_1} \gamma_A^{k_2,i_2,l_2} a_{k_1,l_1}(t) c_{i_1,l_1}(s) a_{k_2,l_2}(s) d\mu_s$$
$$\cdot \int_{G_A} c_{i_2,l_2}(\tau) x(\tau) d\mu_\tau$$
$$= \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \sum_{(k_2,i_2,l_2=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \gamma_A^{k_1,i_1,l_1} \gamma_A^{k_2,i_2,l_2} a_{k_1,l_1}(t) Q_{G_A}(a_{k_2,l_2}, c_{i_1,l_1})$$
$$\cdot \int_{G_A} c_{i_2,l_2}(\tau) x(\tau) d\mu_\tau$$
$$= \int_{G_A} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \sum_{(k_2,i_2,l_2=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \prod_{v=1}^{2} \gamma_A^{k_v,i_v,l_v} \tilde{\gamma}_2 a_{k_1,l_1}(t) c_{i_2,l_2}(\tau) x(\tau) d\mu_\tau,$$

对几乎处处的 $t_\circ$ 我们假设

$$(A^m x)(t) = \int_{G_A} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_m,i_m,l_m=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \prod_{v=1}^{m} \gamma_A^{k_v,i_v,l_v} \tilde{\gamma}_m a_{k_1,l_1}(t) c_{i_m,l_m}(\tau) d\mu_\tau,\ m \geq 1 \tag{7}$$

对几乎所有的 $t_\circ$ 于是

$$\begin{aligned}(A^{m+1}x)(t) &= \int_{G_A} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{k_m,i_m,l_m=1}^{l_{A,1},l_{A,2},l_{A,3}} \prod_{v=1}^{m} \gamma_A^{k_v,i_v,l_v} \tilde{\gamma}_m a_{k_1,l_1}(t) c_{i_m,l_m}(\tau) \\ &\quad \cdot \Big(\int_{G_A} \sum_{k_{m+1},i_{m+1},l_{m+1}=1}^{l_{A,1},l_{A,2},l_{A,3}} \gamma_A^{k_{m+1},i_{m+1},l_{m+1}} a_{k_{m+1},l_{m+1}}(\tau) c_{i_{m+1},l_{m+1}}(s) x(s) d\mu_s \Big) d\mu_\tau \\ &= \int_{G_A} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_{m+1},i_{m+1},l_{m+1}=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \prod_{v=1}^{m+1} \gamma_A^{k_v,i_v,l_v} \tilde{\gamma}_m a_{k_1,l_1}(t) \\ &\quad \cdot Q_{G_A}(c_{i_m,l_m}, a_{k_{m+1},l_{m+1}}) c_{i_{m+1},l_{m+1}}(s) x(s) d\mu_s \\ &= \int_{G_A} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_{m+1},i_{m+1},l_{m+1}=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \prod_{v=1}^{m+1} \gamma_A^{k_v,i_v,l_v} \gamma_{m+1} a_{k_1,l_1}(t) \\ &\quad \cdot c_{i_{m+1},l_{m+1}}(s) x(s) d\mu_s\end{aligned}$$

对几乎所有的 $t_\circ$ 于是，由归纳法可知，(7) 对所有正整数 m 都成立。因此，我们有

$$\begin{aligned}(F(A)x)(t) = &\delta_0 x(t) + \int_{G_A} \sum_{j=1}^{n} \delta_j \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_j,i_j,l_j=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \prod_{v=1}^{j} \gamma_A^{k_v,i_v,l_v} \tilde{\gamma}_j \\ &\cdot a_{k_1,l_1}(t) c_{i_j,l_j}(s) x(s) d\mu_s\end{aligned}$$

对几乎所有的 $t_\circ$ 然后我们计算，

$$\begin{aligned}(ABx)(t) = &\int_{G_A} \sum_{(k,i,l=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \gamma_A^{k,i,l} a_{k,l}(t) c_{i,l}(s) \Big(\int_{G_B} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \gamma_B^{u,w,r} b_{u,r}(s) \\ &\cdot e_{w,r}(\tau) x(\tau) d\mu_\tau \Big) d\mu_s \\ &\int_{G_B} \sum_{(k,i,l=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \gamma_A^{k,i,l} \gamma_B^{u,w,r} a_{k,l}(t) Q_{G_A}(c_{i,l}, b_{u,r}) e_{w,r}(\tau) x(\tau) d\mu_\tau,\end{aligned}$$

$$(BF(A)x)(t) = \delta_0 \int_{G_B} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \gamma_B^{u,w,r} b_{u,r}(t) e_{w,r}(\tau) x(\tau) d\mu_\tau$$

$$+ \int_{G_B} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \gamma_B^{u,w,r} b_{u,r}(t) e_{w,r}(s) \Big(\int_{G_A} \sum_{j=1}^{n} \delta_j \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{k_j,i_j,l_j=1}^{l_{A,1},l_{A,2},l_{A,3}}$$

$$\Big(\prod_{v=1}^{j} \gamma_A^{k_v,i_v,l_v} \tilde{\gamma}_j a_{k_1,l_1}(s) c_{i_j,l_j}(\tau) x(\tau) d\mu_\tau \Big) \Big) d\mu_s$$

$$= \int_{G_B} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \delta_0 \gamma_B^{u,w,r} b_{u,r}(t) e_{w,r}(\tau) x(\tau) d\mu_\tau$$

$$+ \int_{G_A} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \sum_{j=1}^{n} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_j,i_j,l_j=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \delta_j \tilde{\gamma}_j \gamma_B^{u,w,r} \prod_{v=1}^{j} \gamma_A^{k_v,i_v,l_v}$$

$$\cdot Q_{G_B}(e_{w,r}, a_{k_1,l_1}) b_{u,r}(t) c_{i_j,l_j}(\tau) x(\tau) d\mu_\tau.$$

因此，当且仅当对所有 $x \in L_p(X, \mu)$，$(ABx)(t) = (BF(A)x)(t)$】【。

$$\int_{G_B} \sum_{(k,i,l=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \gamma_A^{k,i,l} \gamma_B^{u,w,r} a_{k,l}(t) Q_{G_A}(c_{i,l}, b_{u,r}) e_{w,r}(\tau) x(\tau) d\mu_\tau$$

$$= \int_{G_B} \sum_{u,w,r=1}^{l_{B,1},l_{B,2},l_{B,3}} \delta_0 \gamma_B^{u,w,r} b_{u,r}(t) e_{w,r}(\tau) x(\tau) d\mu_\tau$$

$$+ \int_{G_A} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \sum_{j=1}^{n} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_j,i_j,l_j=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \delta_j \tilde{\gamma}_j \gamma_B^{u,w,r} \prod_{v=1}^{j} \gamma_A^{k_v,i_v,l_v}$$

$$\cdot Q_{G_B}(e_{w,r}, a_{k_1,l_1}) b_{u,r}(t) c_{i_j,l_j}(\tau) x(\tau) d\mu_\tau.$$

然后应用引理1，我们得出：$AB = BF(A)$ 当且仅当

1. 对于几乎处处的 $(t,s) \in X \times G$,

$$- \sum_{u,w,r=1}^{l_{B,1},l_{B,2},l_{B,3}} \delta_0 \gamma_B^{u,w,r} b_{u,r}(t) e_{w,r}(s)$$

$$+ \sum_{(k,i,l=1)}^{(l_{A,1},l_{A,2},l_{A_3})} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B_2},l_{B_3})} \gamma_A^{k,i,l} \gamma_B^{u,w,r} a_{k,l}(t) Q_{G_A}(c_{i,l}, b_{u,r}) e_{w,r}(s)$$

$$= \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \sum_{j=1}^{n} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_j,i_j,l_j=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \delta_j \tilde{\gamma}_j \gamma_B^{u,w,r} \prod_{v=1}^{j} \gamma_A^{k_v,i_v,l_v}$$

$$\cdot Q_{G_B}(e_{w,r}, a_{k_1,l_1}) b_{u,r}(t) c_{i_j,l_j}(s)$$

2. 对于几乎处处的 $(t,s) \in X \times (G_A \setminus G)$,

$$\sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \sum_{j=1}^{n} \sum_{(k_1,i_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \cdots \sum_{(k_j,i_j,l_j=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \delta_j \tilde{\gamma}_j \gamma_B^{u,w,r} \prod_{v=1}^{j} \gamma_A^{k_v,i_v,l_v}$$
$$\cdot Q_{G_B}(e_{w,r}, a_{k_1,l_1}) b_{u,r}(t) c_{i_j,l_j}(s) = 0;$$

3. 对几乎处处的 $(t,\ s) \in X \times (G_B \setminus G)$,

$$\sum_{u,w,r=1}^{l_{B,1},l_{B,2},l_{B,3}} \delta_0 \gamma_B^{u,w,r} b_{u,r}(t) e_{w,r}(s)$$
$$= \sum_{(k,i,l=1)}^{(l_{A,1},l_{A,2},l_{A,3})} \sum_{(u,w,r=1)}^{(l_{B,1},l_{B,2},l_{B,3})} \gamma_A^{k,i,l} \gamma_B^{u,w,r} a_{k,l}(t) Q_{G_A}(c_{i,l}, b_{u,r}) e_{w,r}(s).$$

□

注1. 算子 A 的形式：$L_p(X,\mu) \to L_p(X,\mu)$，$1 \le p \le \infty$ 在（4）和（5）中给出的形式是等价的。事实上，在（4）中取 $l_{A,1} = l_{A,2} = l_{A,3} = 1$、$\tilde{a}_{i,1}(\cdot) = a_i(\cdot)$、$1 \le i \le l_A$、$\tilde{c}_{j,1}(\cdot) = c_j(\cdot)$、$1 \le j \le l_A$、$\gamma_A^{(i,j,1)} = \begin{cases} 1, & i = j, \\ 0, & i \ne j, \end{cases}$

即可得到（5）。另一方面，根据表达式（5），对于每个三元组 $(i,j,k) \in \mathbb{N}^3$，令 $1 \le i \le l_A$、1、$1 \le j \le l_A$、2、$1 \le k \le l_A$、3，对应的数为 $l = (i-1) l_{A,2} l_{A,3} + (j-1) l_{A,3} + k$，并令函数 $\tilde{a}_l(\cdot) = \gamma_A^{(i,j,k)} a_{i,k}(\cdot)$、$\tilde{c}_l(\cdot) = c_{j,k}(\cdot)$。在 $\mathbb{N}^3 \times \mathbb{N}^3$ 中定义如下偏序关系 $\preceq$

$$(i_1,j_1,k_1) \preceq (i_2,j_2,k_2) \text{ if } i_1 < i_2 \text{ or } (i_1 = i_2 \text{ and } j_1 < j_2) \text{ or } (i_1 = i_2,\ j_1 = j_2,\ k_1 \le k_2).$$

它具有自反性、反对称性和传递性，读者可以在 [18, 25] 中查看相应的定义。它具有自反性，因为对于每个三元组 $(i,j,k) \in \mathbb{N}^3$ 我们有 $i = i,\ j = j,\ k = k$ 因此 $(i,j,k) \preceq (i,j,k)$。如果两个三元组 (i_1,j_1,k_1)、$(i_2,j_2,k_2) \in \mathbb{N}^3$ 满足 $(i_1,j_1,k_1) \preceq (i_2,j_2,k_2)$ 且 $(i_2,j_2,k_2) \preceq (i_1,j_1,k_1)$，我们有

$$\begin{gathered} (i_1 < i_2 \text{ or } i_1 = i_2) \text{ and } (i_2 < i_1 \text{ or } i_2 = i_1) \\ (j_1 < j_2 \text{ or } j_1 = j_2) \text{ and } (j_2 < j_1 \text{ or } j_2 = j_1) \\ (k_1 \le k_2) \text{ and } (k_2 \le k_1). \end{gathered}$$

这些条件意味着 $i_1 = i_2$、$j_1 = j_2$ 和 $k_1 = k_2$，因此（i_1、j_1、k_1）=（i_2、j_2、k_2）。因此，关系 $\preceq$ 具有反对称性。现在假设（i_1、j_1、k_1）、（i_2、j_2、k_2）、（i_3、j_3、k_3）$\in \mathbb{N}^3$，满足（i_1、j_1、k_1）$\preceq$（i_2、j_2、k_2）以及（i_2、j_2、k_2）$\preceq$（i_3、j_3、k_3）。我们有

$$i_1 < i_2 \text{ or } (i_1 = i_2 \text{ and } j_1 < j_2) \text{ or } (i_1 = i_2,\ j_1 = j_2,\ k_1 \le k_2),$$

$$i_2 < i_3 \text{ or } (i_2 = i_3 \text{ and } j_2 < j_3) \text{ or } (i_2 = i_3,\ j_2 = j_3,\ k_2 \le k_3),$$

这意味着

$$i_1 < i_3 \text{ or } (i_1 = i_3 \text{ and } j_1 < j_3) \text{ or } (i_1 = i_3,\ j_1 = j_3,\ k_1 \le k_3).$$

因此 $(i_1, j_1, k_1) \preceq (i_3, j_3, k_3)$，所以关系 $\preceq$ 具有传递性，因而是偏序关系。此外，关系 $\preceq$ 是全序，因为对于任意两个三元组 $(i_1, j_1, k_1) \in \mathbb{N}^3$ 和 $(i_2, j_2, k_2) \in \mathbb{N}^3$，我们有 $(i_1, j_1, k_1) \preceq (i_2, j_2, k_2)$ 或 $(i_2, j_2, k_2) \preceq (i_1, j_1, k_1)$。因此，当 $(1, 1, 1) \preceq (i, j, k) \preceq (l_A,1, l_A,2, l_A,3)$ 时，$l = (i-1) l_{A,2} l_{A,3} + (j-1) l_{A,3} + k$ 被唯一确定，且 $1 \le l \le l_A,1 \cdot l_A,2 \cdot l_A,3$。因此，表达式 (4) 中的 $l_A = l_A,1 \cdot l_A,2 \cdot l_A,3$。

从现在起，我们在陈述中考虑算子 A 和 B 由式 (4) 给出；该式是具有一般可分核的线性积分算子的较简单形式。

定理 2. 设 (X, Σ, μ) 是一个 σ-有限测度空间。设 $A: L_p(X, \mu) \to L_p(X, \mu), B: L_p(X, \mu) \to L_p(X, \mu)$, $1 \le p \le \infty$ 是如下定义的非零算子

$$(Ax)(t) = \int_{G_A} \sum_{i=1}^{l_A} a_i(t) c_i(s) x(s) d\mu_s, \quad (Bx)(t) = \int_{G_B} \sum_{j=1}^{l_B} b_j(t) e_j(s) x(s) d\mu_s,$$

对几乎处处的 t，其中 μ_s 中的下标表示积分变量，$G_A \in \Sigma$ 且 $G_B \in \Sigma$, a_i, $b_j \in L_p(X, \mu)$, $c_i \in L_q(G_A, \mu)$, $e_j \in L_q(G_B, \mu)$, i, j, l_A, l_B 为正整数，使得 $1 \le i \le l_A, 1 \le j \le l_B$ 且 $1 \le q \le \infty$ 使得 $\frac{1}{p} + \frac{1}{q} = 1.n$ 考虑由 $F(z) = \acute{I} \delta_j z^j$ 定义的多项式，其中 $\delta_j \in \mathbb{R}$ $j = 0, 1, 2, \ldots, n.j{=}0$ 设 $G = G_A \cap G_B$，并且

$$\gamma_{i_1} = 1, \quad \gamma_{i_1,\ldots,i_m} = \prod_{l=1}^{m-1} Q_{G_A}(a_{i_{l+1}}, c_{i_l}),\ m \ge 2.$$

其中 Q_Λ（u，v），$\Lambda \in \Sigma$，由（2）定义。当且仅当满足以下条件时，$AB = BF$（A）：

1. 几乎处处 $(t, s) \in X \times G$,

$$-\sum_{k=1}^{l_B} \delta_0 b_k(t) e_k(s) + \sum_{k=1}^{l_B} \sum_{m=1}^{l_A} a_m(t) Q_{G_A}(b_k, c_m) e_k(s)$$
$$= \sum_{k=1}^{l_B} \sum_{j=1}^{n} \sum_{i_1,\ldots,i_j=1}^{l_A} \delta_j b_k(t) Q_{G_B}(e_k, a_{i_1}) \gamma_{i_1,\ldots,i_j} c_{i_j}(s);$$

2. 对于几乎处处的 $(t, s) \in X \times (G_A \setminus G)$,

$$\sum_{k=1}^{l_B}\sum_{j=1}^{n}\sum_{i_1,\dots,i_j=1}^{l_A}\delta_j b_k(t)Q_{G_B}(e_k,a_{i_1})\gamma_{i_1,\dots,i_j}c_{i_j}(s)=0;$$

3. 对于几乎处处的 $(t,s)\in X\times(G_B\setminus G)$,

$$\sum_{k=1}^{l_B}\delta_0 b_k(t)e_k(s)=\sum_{k=1}^{l_B}\sum_{m=1}^{l_A}Q_{G_A}(b_k,c_m)a_m(t)e_k(s).$$

证明。我们注意到，由于对于每个 i,j, $1\le i\le l_A$, $1\le j\le l_B$, $a_i,b_j\in L_p(X,\mu)$, $1\le p\le\infty$, $c_i\in L_q(G_A,\mu)$, $e_j\in L_q(G_B,\mu)$, 其中 $1\le q\le\infty$, 且 $\frac{1}{p}+\frac{1}{q}=1$, $L_p(X,\mu)$, $L_q(X,\mu)$ 是线性空间，因此算子 A 和 B 定义良好。通过直接计算，我们得到

$$\begin{aligned}
(A^2x)(t)&=\int_{G_A}\sum_{i=1}^{l_A}a_i(t)c_i(s)(Ax)(s)d\mu_s\\
&=\int_{G_A}\sum_{i_1,i_2=1}^{l_A}a_{i_1}(t)c_{i_1}(s)a_{i_2}(s)d\mu_s\int_{G_A}c_{i_2}(\tau)x(\tau)d\mu_\tau\\
&=\sum_{i_1,i_2=1}^{l_A}a_{i_1}(t)Q_{G_A}(a_{i_2},c_{i_1})\int_{G_A}c_{i_2}(\tau)x(\tau)d\mu_\tau\\
&=\int_{G_A}\sum_{i_1,i_2=1}^{l_A}\gamma_{i_1,i_2}a_{i_1}(t)c_{i_2}(\tau)x(\tau)d\mu_\tau,\\
(A^3x)(t)&=\int_{G_A}\sum_{i_1=1}^{l_A}a_{i_1}(t)c_{i_1}(s)\left(\int_{G_A}\sum_{i_2,i_3=1}^{l_A}a_{i_2}(s)Q_{G_A}(a_{i_3},c_{i_3})c_{i_2}(\tau)x(\tau)d\mu_\tau\right)d\mu_s\\
&=\sum_{i_1,i_2,i_3=1}^{l_A}\int_{G_A}a_{i_1}(t)Q_{G_A}(a_{i_3},c_{i_2})c_{i_1}(s)a_{i_2}(s)d\mu_s\int_{G_A}c_{i_3}(\tau)x(\tau)d\mu_\tau\\
&=\int_{G_A}\sum_{i_1,i_2,i_3=1}^{l_A}Q_{G_A}(a_{i_3},c_{i_2})Q_{G_A}(a_{i_2},c_{i_1})a_{i_1}(t)c_{i_3}(\tau)x(\tau)d\mu_\tau\\
&=\int_{G_A}\sum_{i_1,i_2,i_3=1}^{l_A}\gamma_{i_1,i_2,i_3}a_{i_1}(t)c_{i_3}(\tau)x(\tau)d\mu_\tau.
\end{aligned}$$

对于几乎所有的 t。我们假设

$$(A^mx)(t)=\int_{G_A}\sum_{i_1,\dots,i_m=1}^{l_A}\gamma_{i_1,\dots,i_m}a_{i_1}(t)c_{i_m}(\tau)x(\tau)d\mu_\tau,\quad m=1,2,\dots$$

对于几乎所有的 $t_\circ$ 那么

$$(A^{m+1}x)(t) = \int_{G_A} \sum_{i_1,\ldots,i_m=1}^{l_A} \gamma_{i_1,\ldots,i_m} a_{i_1}(t) c_{i_m}(\tau) \left(\int_{G_A} \sum_{i_{m+1}=1}^{l_A} a_{i_{m+1}}(\tau) c_{i_{m+1}}(s) d\mu_s \right) d\mu_\tau$$
$$= \int_{G_A} \gamma_{i_1,\ldots,i_{m+1}} a_{i_1}(t) c_{i_{m+1}}(\tau) x(\tau) d\mu_\tau.$$

对几乎所有的 $t_\circ$ 因此，我们有

$$(F(A)x)(t) = \delta_0 x(t) + \int_{G_A} \sum_{j=1}^{n} \sum_{i_1,\ldots,i_m=1}^{l_A} \gamma_{i_1,\ldots,i_j} \delta_j a_{i_1}(t) c_{i_j}(\tau) x(\tau) d\mu_\tau$$

对几乎所有的 $t_\circ$ 然后我们计算得，

$$\begin{aligned} (ABx)(t) &= \int_{G_A} \sum_{m=1}^{l_A} a_m(t) c_m(s) d\mu_s \int_{G_B} \sum_{k=1}^{l_B} b_k(s) e_k(\tau) x(\tau) d\mu_\tau \\ &= \int_{G_B} \sum_{m=1}^{l_A} \sum_{k=1}^{l_B} Q_{G_A}(b_k, c_m) a_m(t) e_k(\tau) x(\tau) d\mu_\tau, \end{aligned} \tag{8}$$

$$\begin{aligned} (BF(A)x)(t) &= \int_{G_B} \sum_{k=1}^{l_B} \delta_0 b_k(t) e_k(\tau) x(\tau) d\mu_\tau \\ &+ \int_{G_B} \sum_{k=1}^{l_B} b_k(t) e_k(s) \int_{G_A} \sum_{j=1}^{n} \sum_{i_1,\ldots,i_j=1}^{l_A} \delta_j \gamma_{i_1,\ldots,i_j} a_{i_1}(s) c_{i_j}(\tau) x(\tau) d\mu_\tau \\ &= \int_{G_B} \sum_{k=1}^{l_B} \delta_0 b_k(t) e_k(\tau) x(\tau) d\mu_\tau \\ &+ \int_{G_A} \sum_{k=1}^{l_B} \sum_{j=1}^{n} \sum_{i_1,\ldots,i_j=1}^{l_A} \delta_j \gamma_{i_1,\ldots,i_j} Q_{G_B}(e_k, a_{i_1}) b_k(t) c_{i_j}(\tau) x(\tau) d\mu_\tau. \end{aligned} \tag{9}$$

因此，所有 $x \in L_p(X, \mu)$ 均满足 $(ABx)(t) = (BF(A)x)(t)$，当且仅当

$$\int_{G_B} \left(-\sum_{k=1}^{l_B} \delta_0 b_k(t) e_k(s) + \sum_{k=1}^{l_B} \sum_{m=1}^{l_A} Q_{G_A}(b_k, c_m) a_m(t) e_k(s) \right) x(s) d\mu_s$$
$$= \int_{G_A} \left(\sum_{k=1}^{l_B} \sum_{j=1}^{n} \sum_{i_1,\ldots,i_j=1}^{l_A} \delta_j \gamma_{i_1,\ldots,i_j} Q_{G_B}(e_k, a_{i_1}) b_k(t) c_{i_j}(s) \right) x(s) d\mu_s.$$

然后应用引理1，我们得出 $AB = BF(A)$，当且仅当

1. 对几乎处处的 $(t,s) \in X \times G$,

$$-\sum_{k=1}^{l_B} \delta_0 b_k(t) e_k(s) + \sum_{k=1}^{l_B} \sum_{m=1}^{l_A} a_m(t) Q_{G_A}(b_k, c_m) e_k(s)$$
$$= \sum_{k=1}^{l_B} \sum_{j=1}^{n} \sum_{i_1,\ldots,i_j=1}^{l_A} \delta_j b_k(t) Q_{G_B}(e_k, a_{i_1}) \gamma_{i_1,\ldots,i_j} c_{i_j}(s),$$

2. 对于几乎所有 $(t, s) \in X \times (G_A \setminus G)$,

$$\sum_{k=1}^{l_B} \sum_{j=1}^{n} \sum_{i_1,\ldots,i_j=1}^{l_A} \delta_j b_k(t) Q_{G_B}(e_k, a_{i_1}) \gamma_{i_1,\ldots,i_j} c_{i_j}(s) = 0,$$

3. 对于几乎处处的 $(t,s) \in X \times (G_B \setminus G)$,

$$\sum_{k=1}^{l_B} \delta_0 b_k(t) e_k(s) = \sum_{k=1}^{l_B} \sum_{m=1}^{l_A} Q_{G_A}(b_k, c_m) a_m(t) e_k(s). \qquad \square$$

注*2.* 在定理 2 中，当 $G_A = G_B = G$ 时，条件 2 和 3 在零测集上成立，因此可以忽略。这样，我们只需考虑条件 1。当 $G_A \neq G_B$ 时，我们还需要检查交集$G = G_A \cap G_B$之外的条件 2 和 3。

নম্নিলখিতি অনুসদ্ধিান্তটি প্রথম-ঘাতরে বহুপদী F-এর সঙ্গে সংশ্লষ্টি কে·ভরেয়িন্টে বনিমিয় সম্পর্করে গুরুত্বপূর্ণ শ্রণেরি ক্ষত্রেে উপপাদ্য 2-এর একটি বিশিষে রূপ।

推论 1. 设 (X, Σ, μ)为 σ-有限测度空间。设 $A: L_p(X,\mu) \to L_p(X,\mu), B: L_p(X,\mu) \to L_p(X,\mu)$, $1 \le p \le \infty$为如下定义的非零算子

$$(Ax)(t) = \int_{G_A} \sum_{i=1}^{l_A} a_i(t) c_i(s) x(s) d\mu_s, \quad (Bx)(t) = \int_{G_B} \sum_{j=1}^{l_B} b_j(t) e_j(s) x(s) d\mu_s,$$

对几乎处处的 t，其中 μ_s 中的下标表示积分变量，$G_A \in \Sigma$且 $G_B \in \Sigma$, a_i, $b_j \in L_p(X,\mu)$, $c_i \in L_q(G_A,\mu)$, $e_j \in L_q(G_B,\mu)$, i, j, l_A, l_B为正整数，满足 $1 \le i \le l_A$, $1 \le j \le l_B$以及 $1 \le q \le \infty$，其中 $\frac{1}{p} + \frac{1}{q} = 1$。考虑多项式 $F(z) = \delta_0 + \delta_1 z$，其中 $\delta_j \in \mathbb{R}$, $j = 0, 1$。设 $G = G_A \cap G_B$和$Q_\Lambda(u,v)$, $\Lambda \in \Sigma$由 *(2)* 定义。那么，当且仅当满足以下条件时，$AB = \delta_0 B + \delta_1 BA$成立:

1. 几乎对每个 $(t,s) \in X \times G$,

$$-\sum_{k=1}^{l_B} \delta_0 b_k(t) e_k(s) + \sum_{k=1}^{l_B} \sum_{m=1}^{l_A} a_m(t) Q_{G_A}(b_k, c_m) e_k(s) = \sum_{k=1}^{l_B} \sum_{i_1=1}^{l_A} \delta_1 b_k(t) Q_{G_B}(e_k, a_{i_1}) c_{i_1}(s)$$

2. 几乎对每个 $(t,s) \in X\times(G_A \setminus G)$,

$$\sum_{k=1}^{l_B}\sum_{i_1=1}^{l_A}\delta_1 b_k(t)Q_{G_B}(e_k,a_{i_1})c_{i_1}(s)=0.$$

3. 对于几乎处处的 $(t,s) \in X\times(G_B \setminus G)$,

$$\sum_{k=1}^{l_B}\delta_0 b_k(t)e_k(s)=\sum_{k=1}^{l_B}\sum_{m=1}^{l_A}Q_{G_A}(b_k,c_m)a_m(t)e_k(s).$$

证明。当 $n=1$ 时，由定理 2 可得。 □

推论 2。 设(X,Σ,μ)为 σ-有限测度空间。设 $A\colon L_p(X,\mu)\to L_p(X,\mu)$,$B\colon L_p(X,\mu)\to L_p(X,\mu)$, $1\le p\le\infty$ 为如下定义的非零算子

$$(Ax)(t)=\int_{G_A}\sum_{i=1}^{l_A}a_i(t)c_i(s)x(s)d\mu_s,\quad (Bx)(t)=\int_{G_B}\sum_{j=1}^{l_B}b_j(t)e_j(s)x(s)d\mu_s,$$

对于几乎所有的 t，其中 μ_s 中的下标表示积分变量，$G_A\in\Sigma$ 且 $G_B\in\Sigma$、a_i、$b_j\in L_p(X$、$\mu)$、$c_i\in L_q(G_A$、$\mu)$、$e_j\in L_q(G_B$、$\mu)$、i、j、l_A、l_B 均为正整数，使得 $1\le i\le l_A$、$1\le j\le l_B$ 且 $1\le q\le\infty$，其中$\frac{1}{p}+\frac{1}{q}=1$。设 $G=G_A\cap G_B$，并且 $Q_\Lambda(u$、$v)$、$\Lambda\in\Sigma$ 由 (2) 定义。那么算子 A 和 B 交换，当且仅当满足以下条件:

1. 对于几乎处处的 $(t,s) \in X\times G$,

$$\sum_{k=1}^{l_B}\sum_{m=1}^{l_A}a_m(t)Q_{G_A}(b_k,c_m)e_k(s)=\sum_{k=1}^{l_B}\sum_{i_1=1}^{l_A}b_k(t)Q_{G_B}(e_k,a_{i_1})c_{i_1}(s),$$

2. 对于几乎处处的 $(t,s) \in X\times(G_A \setminus G)$,

$$\sum_{k=1}^{l_B}\sum_{i_1=1}^{l_A}b_k(t)Q_{G_B}(e_k,a_{i_1})c_{i_1}(s)=0,$$

3. 对于几乎处处的 $(t,s) \in X\times(G_B \setminus G)$,

$$\sum_{k=1}^{l_B}\sum_{m=1}^{l_A}Q_{G_A}(b_k,c_m)a_m(t)e_k(s)=0.$$

证明。 由定理 2 可知，当 $n=1,\delta_0=0$ 和 $\delta_1=1$ 时，结论成立。 □

下面的定理 2 推论涉及另一类重要的、与单项式相关联的协变交换关系，并讨论了由积分算子给出的表示 F。

推论 3. 设(X,Σ,μ)是σ-有限测度空间。设$A\colon L_p(X,\mu)\to L_p(X,\mu)$,$B\colon L_p(X,\mu)\to L_p(X,\mu)$, $1\le p\le\infty$为如下定义的非零算子

$$(Ax)(t)=\int_{G_A}\sum_{i=1}^{l_A}a_i(t)c_i(s)x(s)d\mu_s,\quad (Bx)(t)=\int_{G_B}\sum_{j=1}^{l_B}b_j(t)e_j(s)x(s)d\mu_s,$$

对几乎处处的 t，其中 μ_s 中的指标表示积分变量，$G_A\in\Sigma$以及 $G_B\in\Sigma$、 a_i、$b_j\in L_p$（X、μ）， $c_i\in L_q$（G_A、μ）， $e_j\in L_q$（G_B、μ）， i、 j、l_A、l_B 是满足 $1\le i\le l_A$、 $1\le j\le l_B$ 和 $1\le q\le\infty$ 且$\frac{1}{p}+\frac{1}{q}=1$的正整数。考虑多项式 $F\colon\ \mathbb{R}\to\mathbb{R}$，其定义为 $F(z)\ =\delta z^d$，其中 $\delta\in\mathbb{R}$、 $d\in\mathbb{Z}$、 $d>0$。令 $G=G_A\cap G_B$，并且

$$\gamma_{i_1}=1,\quad \gamma_{i_1,\dots,i_m}=\prod_{l=1}^{m-1}Q_{G_A}(a_{i_{l+1}},c_{i_l}),\ m\ge 2.$$

其中 Q_Λ（u、v）、 $\Lambda\in\Sigma$ 由 （2） 定义。则当且仅当满足以下条件时，$AB=BF$（A）：

1. 几乎处处成立于$(t,s)\in X\times G$,

$$\sum_{k=1}^{l_B}\sum_{m=1}^{l_A}a_m(t)Q_{G_A}(b_k,c_m)e_k(s)=\sum_{k=1}^{l_B}\sum_{i_1,\dots,i_d=1}^{l_A}\delta b_k(t)Q_{G_B}(e_k,a_{i_1})\gamma_{i_1,\dots,i_d}c_{i_d}(s)$$

2. 几乎处处成立于$(t,s)\in X\times(G_A\setminus G)$,

$$\sum_{k=1}^{l_B}\sum_{i_1,\dots,i_d=1}^{l_A}\delta b_k(t)Q_{G_B}(e_k,a_{i_1})\gamma_{i_1,\dots,i_d}c_{i_d}(s)=0.$$

3. 几乎处处成立于$(t,s)\in X\times(G_B\setminus G)$,

$$\sum_{k=1}^{l_B}\sum_{m=1}^{l_A}Q_{G_A}(b_k,c_m)a_m(t)e_k(s)=0.$$

证明。 当 $n=d$, $\delta_j=0$ 时，由定理 2 可得，对于 $j\in\{0,\dots,n\}\setminus\{d\}$ 以及$\delta_d=\delta$。 □

推论 4。 设(X,Σ,μ)为 σ-有限测度空间。设$A\colon L_p(X,\mu)\to L_p(X,\mu)$,$B\colon L_p(X,\mu)\to L_p(X,\mu)$, $1\le p\le\infty$为如下定义的非零算子

$$(Ax)(t)=\int_{G}\sum_{i=1}^{l_A}a_i(t)c_i(s)x(s)d\mu_s,\quad (Bx)(t)=\int_{G}\sum_{j=1}^{l_B}b_j(t)e_j(s)x(s)d\mu_s,$$

对于几乎所有的 t，其中 μ_s 中的下标表示积分变量，$G\in\Sigma$、 a_i、 $b_j\in L_p$ $(X,\ \mu)$、 c_i、 $e_j\in L_q(G,\ \mu)$、 i、 j、 l_A、 l_B 均为正整数，使得

$1 \le i \le l_A$、 $1 \le j \le l_B$ 和 $1 \le q \le \infty$，其中$\frac{1}{p} + \frac{1}{q} = 1$。如果对于所有满足 $1 \le m \le l_A$、 $1 \le k \le l_B$ 的正整数 m、 k，以下条件均成立。

$$Q_G(b_k, c_m) = 0, \quad Q_G(e_k, a_m) = 0, \tag{10}$$

则 $AB = \delta BA^d = 0$ 对于某个 $\delta \in \mathbb{R}$ 以及某个 $d \in \mathbb{Z}, d > 0$，其中 $Q_\Lambda(u, v)$, $\Lambda \in \Sigma$, 由 (2) 定义。

证明。 应用推论 3，当 $G_A = G_B = G$ 时，我们有 $AB = \delta BA^d$ 当且仅当

$$\sum_{k=1}^{l_B} \sum_{m=1}^{l_A} a_m(t) Q_G(b_k, c_m) e_k(s) = \sum_{k=1}^{l_B} \sum_{i_1, \ldots, i_d = 1}^{l_A} \delta b_k(t) Q_G(e_k, a_{i_1}) \gamma_{i_1, \ldots, i_d} c_{i_d}(s)$$

对于几乎处处的 $(t, s) \in X \times G$。通过直接计算可得，若条件 (10) 成立，则交换关系 $AB = \delta BA^d$ 成立，且两边均等于零。□

推论 5 ([12])。 设 (X, Σ, μ) 为 σ-有限测度空间。设 $A: L_p(X, \mu) \to L_p(X, \mu)$, $B: L_p(X, \mu) \to L_p(X, \mu)$, $1 \le p \le \infty$ 为如下定义的非零算子

$$(Ax)(t) = \int_{G_A} a(t) c(s) x(s) d\mu_s, \quad (Bx)(t) = \int_{G_B} b(t) e(s) x(s) d\mu_s,$$

对于几乎所有的 t，其中 μ_s 中的下标表示积分变量， $G_A \in \Sigma$以及 $G_B \in \Sigma$, $a, b \in L_p(X, \mu)$、 $c \in L_q(G_A, \mu)$、 $e \in L_q(G_B, \mu)$、 $1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$。$n$考虑由 $F(z) = \sum_{j=0} \delta_j z^j$ 定义的多项式，其中 $\delta_j \in \mathbb{R}$, $j = 0, \ldots, n$。令$G = G_A \cap G_B$, 并且

$$k_1 = \delta_1 Q_{G_B}(a, e) + \sum_{j=2}^{n} \delta_j Q_{G_A}(a, c)^{j-1} Q_{G_B}(a, e), \quad k_2 = Q_{G_A}(b, c),$$

其中 $Q_\Lambda(u, v)$, $\Lambda \in \Sigma$ 由 (2) 定义。则当且仅当满足以下条件时， $AB = BF(A)$:

1. a) 如果几乎处处的 $(t, s) \in supp\ b \times [(supp\, e) \cap G]$, 那么 *(i)* 如果 $k_2 \ne 0$，则 $c(s)k_1 = \lambda e(s)$ 且 $a(t) = (\delta_0 + \lambda) b(t)^{k_2}$，~~其中某个~~实标量为 λ; *(ii)* 如果 $k_2 = 0$，则 $k_1 c(s) = -\delta_0 e(s)$。*b)* 如果 $t \notin supp\ b$，则要么 $k_2 = 0$，要么对于几乎所有 $t \notin supp\, b$, $a(t) = 0$。*c)* 如果 $s \in G \setminus supp\, e$，则要么 $k_1 = 0$，要么对于几乎所有$s \in G \setminus supp\, e$, $c(s) = 0$。

2. $k_2a(t)-\delta_0b(t)=0$ 几乎处处成立于 $t\in X$ 或 $e(s)=0$ 几乎处处成立于$s\in G_B\setminus G$.3. $k_1=0$ 或 $c(s)=0$ 几乎处处成立于 $s\in G_A\setminus G$.

证明。这是定理 2 在 $l_A=l_B=1$ 时的一个特例。为了说明推论 5 中第 1、2 和 3 条更详细条件的由来，我们按照 [12] 给出独立的详细证明。我们注意到，由于$a,\ b\in L_p(X,\ \mu),\ \ 1\le p\le\infty, c\in L_q(G_A,\ \mu),\ e\in L_q(G_B,\ \mu)$，其中 $1\le q\le\infty$，且$\frac{1}{p}+\frac{1}{q}=1$，那么应用霍尔德不等式可知，算子 A和 B是定义良好的。直接计算可得

$$\begin{aligned}(A^2x)(t)&=\int_{G_A}a(t)c(s)(Ax)(s)d\mu_s=\int_{G_A}a(t)c(s)a(s)d\mu_s\int_{G_A}c(\tau_1)x(\tau_1)d\mu_{\tau_1}\\&=Q_{G_A}(a,c)(Ax)(t),\\(A^3x)(t)&=A(A^2x)(t)=Q_{G_A}(a,c)(A^2x)(t)=Q_{G_A}(a,c)^2(Ax)(t)\end{aligned}$$

对几乎所有 t。我们假设

$$(A^mx)(t)=Q_{G_A}(a,c)^{m-1}(Ax)(t),\quad m=1,2,\dots$$

对几乎所有 t。那么

$$(A^{m+1}x)(t)=A(A^mx)(t)=Q_{G_A}(a,c)^{m-1}(A^2x)(t)=Q_{G_A}(a,c)^m(Ax)(t)$$

对几乎所有 t。接着，我们计算

$$\begin{aligned}(ABx)(t)&=\int_{G_A}a(t)c(s)b(s)d\mu_s\int_{G_B}e(\tau_1)x(\tau_1)d\mu_{\tau_1}\\&=k_2\int_{G_B}a(t)e(\tau_1)x(\tau_1)d\mu_{\tau_1},\end{aligned}$$

$$(F(A)x)(t)=\delta_0x(t)+a(t)\sum_{j=1}^{n}\delta_j\left(Q_{G_A}(a,c)\right)^{j-1}\int_{G_A}c(\tau)x(\tau)d\mu_\tau,$$

$$\begin{aligned}(BF(A)x)(t)&=\delta_0b(t)\int_{G_B}e(\tau_1)x(\tau_1)d\mu_{\tau_1}\\&+b(t)\sum_{j=1}^{n}\delta_j\left(Q_{G_A}(a,c)\right)^{j-1}\int_{G_B}e(\tau)a(\tau)d\mu_\tau\int_{G_A}c(\tau_1)x(\tau_1)d\mu_{\tau_1}\\&=\delta_0b(t)\int_{G_B}e(\tau_1)x(\tau_1)d\mu_{\tau_1}+b(t)k_1\int_{G_A}c(\tau_1)x(\tau_1)d\mu_{\tau_1},\end{aligned}$$

对于几乎所有 t。因此，$(AB)x=(BF(A))x$对于所有 $x\in L_p(X,\mu)$ 当且仅当

$$\int_{G_B}[k_2a(t)-\delta_0b(t)]e(s)x(s)d\mu_s=\int_{G_A}k_1b(t)c(s)x(s)d\mu_s.$$

然后根据引理1，我们得出， $AB=BF(A)$ 当且仅当

1. 对几乎所有 $(t,s)\in X\times G$,

$$[k_2a(t)-\delta_0b(t)]e(s)=k_1b(t)c(s);$$

2. $k_2a(t)-\delta_0b(t)=0$ 对几乎所有 $t\in X$，或 $e(s)=0$ 对几乎所有 $s\in G_B\setminus G$;
3. $k_1=0$ 或 $b(t)=0$ 对几乎所有 $t\in X$，或 $c(s)=0$ 对几乎所有 $s\in G_A\setminus G$。

我们可以将第一个条件改写如下:

a) 假设 $(t,s)\in \operatorname{supp} b\times[(\operatorname{supp} e)\cap G]$。

(i) If $k_2\neq 0$, then $k_1\frac{c(s)}{e(s)}=k_2\frac{a(t)}{b(t)}-\delta_0=\lambda$ for some real scalar λ. From this, it follows that $k_1c(s)=e(s)\lambda$ and $a(t)=\frac{\delta_0+\lambda}{k_2}b(t)$.

(ii) If $k_2=0$ then $-\delta_0b(t)e(s)=k_1b(t)c(s)$, which yields $-k_1c(s)=-\delta_0e(s)$.

b) 若 $t\notin\operatorname{supp} b$，则 $k_2a(t)e(s)=0$，由此可得，要么 $k_2=0$，要么 $a(t)=0$几乎处处于 $t\notin\operatorname{supp} b$，要么 $e(s)=0$ 几乎处处成立（这意味着 $B=0$）。

c) 若 $s\in G\setminus\operatorname{supp} e$，则 $k_1b(t)c(s)=0$这意味着，对于几乎所有 $s\in G\setminus\operatorname{supp} e$，要么 $k_1=0$，要么 $c(s)=0$；或者对于几乎所有 t， $b(t)=0$（这意味着 $B=0$）。 □

命题1。 设 (X,Σ,μ) 为 σ-有限测度空间。设

$$A:L_p(X,\mu)\to L_p(X,\mu),\ B:L_p(X,\mu)\to L_p(X,\mu),\ 1\le p\le\infty$$

为按如下方式定义的非零算子

$$(Ax)(t)=\int_G\sum_{i=1}^{l_A}a_i(t)c_i(s)x(s)d\mu_s,\quad (Bx)(t)=\int_G\sum_{j=1}^{l_B}b_j(t)e_j(s)x(s)d\mu_s,$$

对于几乎所有的 t，其中 μ_s 中的下标表示积分变量，$G_A\in\Sigma$以及 $G_B\in\Sigma$、 a_i、 $b_j\in L_p$（X， μ）、 $c_i\in L_q$（G， μ）、 $e_j\in L_q$（G， μ）、 i、 j、 l_A、 l_B 均为正整数，且满足 $1\le i\le l_A$、 $1\le j\le l_B$ 和 $1\le q\le\infty$，其中 $\frac{1}{p}+\frac{1}{q}=1$。令 $G=G_A\cap G_B$，并且 Q_Λ（u， v）、 $\Lambda\in\Sigma$ 由(2)定义。那么，对于几乎所有的 $t\in X$,

1. $(AB)x(t)=\int_G\sum_{k=1}^{l_B}\sum_{m=1}^{l_A}a_m(t)Q_{G_A}(b_k,c_m)e_k(s)x(s)d\mu_s$;
2. $(BA)x(t)=\int_G\sum_{k=1}^{l_B}\sum_{i_1=1}^{l_A}b_k(t)Q_{G_B}(e_k,a_{i_1})c_{i_1}(s)x(s)d\mu_s$;

3. $(AB-BA)x(t)=\int_G\Big(\sum_{k=1}^{l_B}\sum_{m=1}^{l_A}a_m(t)Q_{G_A}(b_k,c_m)e_k(s)$
$-\sum_{k=1}^{l_B}\sum_{i_1=1}^{l_A}b_k(t)Q_{G_B}(e_k,a_{i_1})c_{i_1}(s)\Big)x(s)d\mu_s.$

证明。当 $F(z)=z$ 时，由 (8) 和 (9) 可得。 □

命题 2。 设$(\mathbb{R},\Sigma,\mu)$为实直线上的标准勒贝格测度空间。设 $A\colon L_p(\mathbb{R},\mu)\to L_p(\mathbb{R},\mu)$、 $B\colon L_p(\mathbb{R},\mu)\to L_p(\mathbb{R},\mu)$、 $1\le p\le\infty$为按如下方式定义的算子。

$$\begin{aligned}(Ax)(t)&=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s)+\theta_{A,2}\cos(\omega t)\cos(\omega s)\\&\quad+\theta_{A,3}\sin(\omega t)\sin(\omega s)+\theta_{A,4}\cos(\omega t)\sin(\omega s)]x(s)d\mu_s,\\(Bx)(t)&=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin(\omega t)\cos(\omega s)+\theta_{B,2}\cos(\omega t)\cos(\omega s)\right.\\&\quad\left.+\theta_{B,3}\sin(\omega t)\sin(\omega s)+\theta_{B,4}\cos(\omega t)\sin(\omega s)\right)x(s)d\mu_s,\end{aligned}$$

对于几乎所有的 t，其中 $\theta_{A,\,i},\theta_{B,\,i},\omega\in\mathbb{R}$, $i=1,2,3,4$, $\delta\in\mathbb{R}\setminus\{0\}$， $\alpha,\beta,\alpha_1,\beta_1\in\mathbb{R}$, $\alpha_1<\beta_1$, $\alpha\le\alpha_1$, $\beta\ge\beta_1$, $I_E(\cdot)$ 是集合 E 的示性函数，数$\frac{\omega}{\pi}(\beta_1-\alpha_1)\in\mathbb{Z}$或 $\frac{\omega}{\pi}(\beta_1+\alpha_1)\in\mathbb{Z}$，并且 $\sigma_1,\sigma_2\in\mathbb{R}$ 满足

$$\begin{aligned}\sigma_1&=\int_{\alpha_1}^{\beta_1}(\sin(\omega s))^2d\mu_s=\begin{cases}0, & if\ \omega=0\\ \frac{\beta_1-\alpha_1}{2}-\frac{\cos(\omega(\alpha_1+\beta_1))\sin(\omega(\beta_1-\alpha_1))}{2\omega}, & if\ \omega\ne0,\end{cases}\\\sigma_2&=\int_{\alpha_1}^{\beta_1}(\cos(\omega s))^2d\mu_s=\beta_1-\alpha_1-\sigma_1\\&=\begin{cases}\beta_1-\alpha_1, & if\ \omega=0\\ \frac{\beta_1-\alpha_1}{2}+\frac{\cos(\omega(\alpha_1+\beta_1))\sin(\omega(\beta_1-\alpha_1))}{2\omega}, & if\ \omega\ne0.\end{cases}\end{aligned}$$

那么，对于几乎所有的 t,

$$\begin{aligned}&(AB-BA)x(t)=\qquad(11)\\&\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\big((\theta_{A,3}\theta_{B,1}\sigma_1-\theta_{B,3}\theta_{A,1}\sigma_1+\theta_{A,1}\theta_{B,2}\sigma_2-\theta_{B,1}\theta_{A,2}\sigma_2)\sin(\omega t)\cos(\omega s)\\&+(\theta_{B,1}\theta_{A,4}-\theta_{A,1}\theta_{B,4})\sigma_1\cos(\omega t)\cos(\omega s)+(\theta_{A,1}\theta_{B,4}-\theta_{B,1}\theta_{A,4})\sigma_2\sin(\omega t)\sin(\omega s)\\&+(\theta_{A,4}\theta_{B,3}\sigma_1-\theta_{B,4}\theta_{A,3}\sigma_1+\theta_{A,2}\theta_{B,4}\sigma_2-\theta_{B,2}\theta_{A,4}\sigma_2)\cos(\omega t)\sin(\omega s)\big)x(s)d\mu_s\end{aligned}$$

此外，若 $\sigma_1\ne0,\sigma_2\ne0$，则 $AB=BA$ 当且仅当

$$\theta_{A,3}\theta_{B,1}\sigma_1-\theta_{B,3}\theta_{A,1}\sigma_1=\theta_{B,1}\theta_{A,2}\sigma_2-\theta_{A,1}\theta_{B,2}\sigma_2,\quad\theta_{A,4}\theta_{B,1}=\theta_{B,4}\theta_{A,1}$$

$$\theta_{A,4}\theta_{B,3}\sigma_1 - \theta_{B,4}\theta_{A,3}\sigma_1 = \theta_{B,2}\theta_{A,4}\sigma_2 - \theta_{A,2}\theta_{B,4}\sigma_2.$$

证明。算子 A, B 定义良好且有界。当 $a_i(t) = I_{[\alpha,\beta]}(t)$ 正弦函数(ωt), $i = 1, 3, a_i(t) = I_{[\alpha,\beta]}(t)$ 余弦函数 $(\omega t), i = 2, 4, t \in \mathbb{R}, c_i(s)$ = 余弦函数 (ωs), $i = 1, 2, c_i(s)$ = 正弦函数 (ωs), $i = 3, 4, s \in [\alpha_1, \beta_1]$ 且 $\beta_1\alpha_1, \beta_1, \omega$ 满足 $\int$ 正弦函数(ωs) 余弦函数(ωs) $d\mu_s = 0$ 时，由命题1可得等式 (11)。然后，应用引理1求解$\alpha_1 AB - BA = 0$，证明得证。 □

我们利用命题 2，证明例 1 的情形 2 中某些算子的可交换性。

4 个例子

本节中，我们利用第 3 节推导出的条件，在 L_p空间上构造积分算子的例子。这些算子具有可分核，表示与单项式相关的协方差型交换关系；其核涉及有界区间上的多参数三角函数、多项式和 Laurent 多项式。

例 *1.* 设 $(\mathbb{R}, \Sigma, \mu)$ 为标准勒贝格测度空间。设算子

$$A : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu),\ B : L_p(\mathbb{R}, \mu) \to L_p(\mathbb{R}, \mu),\ 1 < p < \infty$$

定义如下

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} \sum_{m=1}^{4} a_m(t)c_m(s)x(s)d\mu_s, \quad (Bx)(t) = \int_{\alpha_1}^{\beta_1} \sum_{k=1}^{4} b_k(t)e_k(s)x(s)d\mu_s,$$

对于几乎所有的 t，其中 μ 中的下标表示积分变量，

$$\sum_{m=1}^{4} a_m(t)c_m(s) = I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin\left(\frac{2\pi m_1 t}{\lambda_t}\right)\cos\left(\frac{2\pi k_1 s}{\lambda_s}\right) + \theta_{A_2}\cos\left(\frac{2\pi m_2 t}{\lambda_t}\right)\right.$$
$$\left. + \cos\left(\frac{2\pi k_2 s}{\lambda_s}\right) + \theta_{A,3}\sin\left(\frac{2\pi m_3 t}{\lambda_t}\right)\sin\left(\frac{2\pi k_3 s}{\lambda_s}\right) + \theta_{A,4}\cos\left(\frac{2\pi m_4 t}{\lambda_t}\right)\sin\left(\frac{2\pi k_4 s}{\lambda_s}\right)\right),$$
$$\sum_{k=1}^{4} b_k(t)e_k(s) = I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin\left(\frac{2\pi m_1 t}{\lambda_t}\right)\cos\left(\frac{2\pi k_1 s}{\lambda_s}\right) + \theta_{B,2}\cos\left(\frac{2\pi m_2 t}{\lambda_t}\right)\right.$$
$$\left. + \cos\left(\frac{2\pi k_2 s}{\lambda_s}\right) + \theta_{B,3}\sin\left(\frac{2\pi m_3 t}{\lambda_t}\right)\sin\left(\frac{2\pi k_3 s}{\lambda_s}\right) + \theta_{B4}\cos\left(\frac{2\pi m_4 t}{\lambda_t}\right)\sin\left(\frac{2\pi k_4 s}{\lambda_s}\right)\right),$$

对于几乎处处的 $(t,\ s),\ \in \mathbb{R}\times[\alpha_1, \beta_1], \alpha,\ \alpha_1,\ \beta,\ \beta_1$ 是实常数，使得 $\alpha_1 < \beta_1, \alpha \le \alpha_1, \beta \ge \beta_1$，且 $I_E(t)$ 是集合 $E, \lambda_t,\ \lambda_s \in \mathbb{R}\setminus\{0\}, m_1,\ m_2,\ m_3,\ m_4,\ k_1,\ k_2,\ k_3,\ k_4 \in \mathbb{R}$ 的示性函数。因此我们有

$$a_1(t)=\theta_{A,1}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_1 t}{\lambda_t}\right),\quad a_2(t)=\theta_{A,2}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_2 t}{\lambda_t}\right),$$
$$a_3(t)=\theta_{A,3}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_3 t}{\lambda_t}\right),\quad a_4(t)=\theta_{A,4}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_4 t}{\lambda_t}\right),$$
$$c_1(s)=\cos\left(\frac{2\pi k_1 s}{\lambda_s}\right)\qquad c_2(s)=\cos\left(\frac{2\pi k_2 s}{\lambda_s}\right),$$
$$c_3(s)=\sin\left(\frac{2\pi k_3 s}{\lambda_s}\right)\qquad c_4(s)=\sin\left(\frac{2\pi k_4 s}{\lambda_s}\right),$$
$$b_1(t)=\theta_{B,1}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_1 t}{\lambda_t}\right),\quad b_2(t)=\theta_{B,2}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_2 t}{\lambda_t}\right),$$
$$b_3(t)=\theta_{B,3}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_3 t}{\lambda_t}\right),\quad b_4(t)=\theta_{B,4}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_4 t}{\lambda_t}\right),$$
$$e_1(s)=\cos\left(\frac{2\pi k_1 s}{\lambda_s}\right)\qquad e_2(s)=\cos\left(\frac{2\pi k_2 s}{\lambda_s}\right),$$
$$e_3(s)=\sin\left(\frac{2\pi k_3 s}{\lambda_s}\right)\qquad e_4(s)=\sin\left(\frac{2\pi k_4 s}{\lambda_s}\right),$$

这些算子是良定义的；事实上，对于任意固定的 $p, 1<p<\infty$，函数a_i，$b_j\in L_p(\mathbb{R},\mu)$，$c_i$，$e_j\in L_q([\alpha_1,\beta_1],\mu)$，$i=1,2,3,4, j=1,2,3,4, 1<q<\infty$，$\frac{1}{p}+\frac{1}{q}=1$。事实上，

$$\int_{\mathbb{R}}|a_i(s)|^p d\mu=\int_{\alpha}^{\beta}|\tilde{a}_i(s)|^p d\mu<\infty,\ i=1,2,3,4,$$

其中 $\tilde{a}_j(s)=\theta_{A,j}$正弦函数 $\left(\frac{2\pi m_j s}{\lambda_s}\right), j=1,3, \tilde{a}_j(s)=\theta_{A,j}$余弦函数 $\left(\frac{2\pi m_j s}{\lambda_s}\right)$，$j=2,4$ 是连续函数。类似地，

$$\int_{\mathbb{R}}|b_j(s)|^p d\mu=\int_{\alpha}^{\beta}|\tilde{b}_j(s)|^p d\mu<\infty,\ j=1,2,3,4.$$

其中，$\tilde{b}(s)$ =余弦函数 $\left(\frac{2\pi m_j s}{\lambda_s}\right), j=1,2, \tilde{b}_j(s)$ =正弦函数 $\left(\frac{2\pi m_j t}{\lambda_t}\right), j=3,4$ 均为连续函数。类似地，我们有

$$\int_{\alpha_1}^{\beta_1}|c_i(s)|^q d\mu<\infty,\ i=1,2,3,4,\quad \int_{\alpha_1}^{\beta_1}|e_j(s)|^q d\mu<\infty,\ j=1,2,3,4.$$

请注意，在这种情况下，推论 3 的条件 1、2 和 3 简化为仅需满足条件 1，因为集合 $G_A=G_B=[\alpha_1,\beta_1]$，因而 $G=[\alpha_1,\beta_1]$、$G_A\setminus G=G_B\setminus G=\emptyset$。因此，根据注2，条件 2 和 3 在零测集上成立，故它们得到满足。

考虑多项式 $F(z)=\delta z^2, \delta\in\mathbb{R}$。我们将寻求使算子满足 $AB=BF(A)$ 的条件。通过在 $n=2$ 时应用推论 3，我们得到

$$\sum_{k=1}^{4}\sum_{m=1}^{4} a_m(t)Q_{G_A}(b_k,c_m)e_k(s)=\sum_{i=1}^{4}\sum_{i_1,i_2=1}^{4}[\delta b_i(t)a_m(t)Q_{G_B}(e_i,a_{i_1})\gamma_{i_1,i_2}c_{i_2}(s) \tag{12}$$

对于几乎所有的 $(t,\ s)\in\mathbb{R}\times[\alpha_1,\beta_1]$。因此，我们有

$$a_i(t)e_j(s)=\begin{cases}\theta_{A,i}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i t}{\lambda_t}\right)\cos\left(\frac{2\pi k_j s}{\lambda_s}\right),\ i=1,3,\ j=1,2\\ \theta_{A,i}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i t}{\lambda_t}\right)\sin\left(\frac{2\pi k_j s}{\lambda_s}\right),\ i=1,3,\ j=3,4\\ \theta_{A,i}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i t}{\lambda_t}\right)\cos\left(\frac{2\pi k_j s}{\lambda_s}\right),\ i=2,4,\ j=1,2\\ \theta_{A,i}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i t}{\lambda_t}\right)\sin\left(\frac{2\pi k_j s}{\lambda_s}\right),\ i=2,4,\ j=3,4,\end{cases}$$

$$b_i(t)c_j(s)=\begin{cases}\theta_{B,i}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i t}{\lambda_t}\right)\cos\left(\frac{2\pi k_j s}{\lambda_s}\right),\ i=1,3,\ j=1,2\\ \theta_{B,i}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i t}{\lambda_t}\right)\sin\left(\frac{2\pi k_j s}{\lambda_s}\right),\ i=1,3,\ j=3,4\\ \theta_{B,i}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i t}{\lambda_t}\right)\cos\left(\frac{2\pi k_j s}{\lambda_s}\right),\ i=2,4,\ j=1,2\\ \theta_{B,i}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i t}{\lambda_t}\right)\sin\left(\frac{2\pi k_j s}{\lambda_s}\right),\ i=2,4,\ j=3,4.\end{cases}$$

情况有很多种，但我们将考虑其中两种情况。

情况 ***1***:

若 $(\frac{m_i}{\lambda_t},\frac{k_j}{\lambda_s})\neq(\frac{m_u}{\lambda_t},\frac{k_w}{\lambda_s})$，每当（$(i,j)\neq(u,w)$），$i,j,u,w=1,2,3,4$，并令$a_j$（$t$）$=\theta_{A,j}\tilde{a}_j$（$t$），$b_j$（$t$）$=\theta_{B,j}\tilde{b}_j$（$t$），由（12）得到如下结果

$$a_i(t)e_j(t)Q_{G_A}(b_j,c_i)=\delta b_i(t)c_j(s)\sum_{k=1}^{4}Q_{G_A}(e_i,a_k)\gamma_{k,j},\ i,j=1,2,3,4,$$

对几乎所有的 $(t,s)\in\mathbb{R}\times[\alpha_1,\beta_1]$，其中 $\gamma_{k,i}=Q_{G_A}(a_i,c_k)$。因此，我们有

$$Q_{G_A}(\tilde{b}_j,c_i)\theta_{A,i}\theta_{B,j}=\delta\theta_{B,i}\theta_{A,j}\sum_{k=1}^{4}Q_{G_B}(e_i,a_k)Q_{G_A}(\tilde{a}_j,c_k),\ i,j=1,2,3,4.$$

利用 $Q_\Lambda(\cdot,\cdot)$ 的双线性和对称性，我们可以将前面的方程展开为以下方程组:

$$\begin{aligned}Q_{G_A}(\tilde{b}_1,c_1)\theta_{A,1}\theta_{B,1}&=\delta\theta_{B,1}\theta_{A,1}\left(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_1)Q_{G_A}(\tilde{a}_1,c_1)\right.\\&\quad+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_1)Q_{G_A}(\tilde{a}_1,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_1)Q_{G_A}(\tilde{a}_1,c_3)\\&\quad\left.+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_1)Q_{G_A}(\tilde{a}_1,c_4)\right),\\Q_{G_A}(\tilde{b}_2,c_1)\theta_{A,2}\theta_{B,1}&=\delta\theta_{B,1}\theta_{A,2}\left(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_1)Q_{G_A}(\tilde{a}_2,c_1)\right.\\&\quad\left.+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_1)Q_{G_A}(\tilde{a}_2,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_1)Q_{G_A}(\tilde{a}_2,c_3)\right.\end{aligned}$$

$$+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_1)Q_{G_A}(\tilde{a}_2,c_4)\big),Q_{G_A}(\tilde{b}_3,c_1)\theta_{A,1}\theta_{B,3}=\delta\theta_{B,1}\theta_{A,3}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_1)Q_{G_A}(\tilde{a}_3,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_1)Q_{G_A}(\tilde{a}_3,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_1)Q_{G_A}(\tilde{a}_3,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_1)Q_{G_A}(\tilde{a}_3,c_4)\big),Q_{G_A}(\tilde{b}_4,c_1)\theta_{A,1}\theta_{B,4}=\delta\theta_{B,1}\theta_{A,4}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_1)Q_{G_A}(\tilde{a}_4,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_1)Q_{G_A}(\tilde{a}_4,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_1)Q_{G_A}(\tilde{a}_4,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_1)Q_{G_A}(\tilde{a}_4,c_4)\big),Q_{G_A}(\tilde{b}_1,c_2)\theta_{A,2}\theta_{B,1}=\delta\theta_{B,2}\theta_{A,1}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_2)Q_{G_A}(\tilde{a}_1,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_2)Q_{G_A}(\tilde{a}_1,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_2)Q_{G_A}(\tilde{a}_1,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_2)Q_{G_A}(\tilde{a}_1,c_4)\big),Q_{G_A}(\tilde{b}_2,c_2)\theta_{A,2}\theta_{B,2}=\delta\theta_{B,2}\theta_{A,2}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_2)Q_{G_A}(\tilde{a}_2,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_2)Q_{G_A}(\tilde{a}_2,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_2)Q_{G_A}(\tilde{a}_2,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_2)Q_{G_A}(\tilde{a}_2,c_4)\big),Q_{G_A}(\tilde{b}_3,c_2)\theta_{A,2}\theta_{B,3}=\delta\theta_{B,2}\theta_{A,3}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_2)Q_{G_A}(\tilde{a}_3,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_2)Q_{G_A}(\tilde{a}_3,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_2)Q_{G_A}(\tilde{a}_3,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_2)Q_{G_A}(\tilde{a}_3,c_4)\big),Q_{G_A}(\tilde{b}_4,c_2)\theta_{A,2}\theta_{B,4}=\delta\theta_{B,2}\theta_{A,4}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_2)Q_{G_A}(\tilde{a}_2,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_2)Q_{G_A}(\tilde{a}_4,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_2)Q_{G_A}(\tilde{a}_4,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_2)Q_{G_A}(\tilde{a}_4,c_4)\big),Q_{G_A}(\tilde{b}_1,c_3)\theta_{A,3}\theta_{B,1}=\delta\theta_{B,3}\theta_{A,1}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_3)Q_{G_A}(\tilde{a}_1,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_3)Q_{G_A}(\tilde{a}_1,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_3)Q_{G_A}(\tilde{a}_1,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_3)Q_{G_A}(\tilde{a}_1,c_4)\big),Q_{G_A}(\tilde{b}_2,c_3)\theta_{A,2}\theta_{B,3}=\delta\theta_{B,2}\theta_{A,3}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_3)Q_{G_A}(\tilde{a}_2,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_3)Q_{G_A}(\tilde{a}_2,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_3)Q_{G_A}(\tilde{a}_2,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_3)Q_{G_A}(\tilde{a}_2,c_4)\big),Q_{G_A}(\tilde{b}_3,c_3)\theta_{A,3}\theta_{B,3}=\delta\theta_{B,3}\theta_{A,3}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_3)Q_{G_A}(\tilde{a}_3,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_3)Q_{G_A}(\tilde{a}_3,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_3)Q_{G_A}(\tilde{a}_3,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_3)Q_{G_A}(\tilde{a}_3,c_4)\big),Q_{G_A}(\tilde{b}_4,c_3)\theta_{A,3}\theta_{B,4}=\delta\theta_{B,3}\theta_{A,4}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_3)Q_{G_A}(\tilde{a}_4,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_3)Q_{G_A}(\tilde{a}_4,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_3)Q_{G_A}(a_4,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_3)Q_{G_A}(\tilde{a}_4,c_4)\big),Q_{G_A}(\tilde{b}_1,c_4)\theta_{A,4}\theta_{B,1}=\delta\theta_{B,4}\theta_{A,1}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_4)Q_{G_A}(\tilde{a}_1,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_4)Q_{G_A}(\tilde{a}_1,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_4)Q_{G_A}(a_1,c_3)+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_4)Q_{G_A}(\tilde{a}_1,c_4)\big),Q_{G_A}(\tilde{b}_2,c_4)\theta_{A,4}\theta_{B,2}=\delta\theta_{B,4}\theta_{A,2}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_4)Q_{G_A}(\tilde{a}_2,c_1)+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_4)Q_{G_A}(\tilde{a}_2,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_4)Q_{G_A}(a_2,c_3)$$

$$+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_4)Q_{G_A}(\tilde{a}_2,c_4)\big),$$
$$Q_{G_A}(\tilde{b}_3,c_4)\theta_{A,4}\theta_{B,3}=\delta\theta_{B,4}\theta_{A,3}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_4)Q_{G_A}(\tilde{a}_3,c_1)$$
$$+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_4)Q_{G_A}(\tilde{a}_3,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_4)Q_{G_A}(\tilde{a}_3,c_3)$$
$$+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_4)Q_{G_A}(\tilde{a}_3,c_4)\big),$$
$$Q_{G_A}(\tilde{b}_4,c_4)\theta_{A,4}\theta_{B,4}=\delta\theta_{B,4}\theta_{A,4}\big(\theta_{A,1}Q_{G_B}(\tilde{a}_1,e_4)Q_{G_A}(\tilde{a}_4,c_1)$$
$$+\theta_{A,2}Q_{G_B}(\tilde{a}_2,e_4)Q_{G_A}(\tilde{a}_4,c_2)+\theta_{A,3}Q_{G_B}(\tilde{a}_3,e_4)Q_{G_A}(\tilde{a}_4,c_3)$$
$$+\theta_{A,4}Q_{G_B}(\tilde{a}_4,e_4)Q_{G_A}(\tilde{a}_4,c_4)\big),$$

~我们计算 $Q_{G_A}(b_i,c_j)$、$Q_{G_A}(\tilde{a}_i,e_j)$、$Q_{G_B}(e_j,\tilde{a}_i)$ 和 $Q_{G_A}(\tilde{a}_i,c_j)$:

$$Q_{G_A}(\tilde{b}_i,c_j)=\int_{\alpha_1}^{\beta_1}\tilde{b}_i(s)c_j d\mu_s$$
$$=\frac{\lambda_t\lambda_s}{2\pi(m_i\lambda_s+k_j\lambda_t)}\sin\pi\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1+\alpha_1)\sin\pi\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1)$$
$$+\frac{\lambda_t\lambda_s}{2\pi(m_i\lambda_s-k_j\lambda_t)}\sin\pi\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1+\alpha_1)\sin\pi\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1),$$
$$\text{if } \frac{m_i}{\lambda_t}\neq\frac{k_j}{\lambda_s},\ \frac{m_i}{\lambda_t}\neq-\frac{k_j}{\lambda_s},\ i=1,3,\ j=1,2,$$
$$Q_{G_A}(\tilde{b}_i,c_j)=\frac{\lambda_t}{2\pi m_i}\left(\left(\sin\left(\frac{2\pi m_i}{\lambda_t}\beta_1\right)\right)^2-\left(\sin\left(\frac{2\pi m_i}{\lambda_t}\alpha_1\right)\right)^2\right),$$
$$\text{if } \frac{m_i}{\lambda_t}=\frac{k_j}{\lambda_s},\ i=1,3,\ j=1,2,$$
$$Q_{G_A}(\tilde{b}_i,c_j)=\frac{\lambda_t\lambda_s}{2\pi(m_i\lambda_s-k_j\lambda_t)}\sin\pi\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1)\cos\pi\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1+\alpha_1)$$
$$-\frac{\lambda_t\lambda_s}{2\pi(m_i\lambda_s+k_j\lambda_t)}\sin\pi\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1)\cos\pi\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1+\alpha_1),$$
$$\text{if } \frac{m_i}{\lambda_t}\neq\frac{k_j}{\lambda_s},\ \frac{m_i}{\lambda_t}\neq-\frac{k_j}{\lambda_s},\ i=1,3,\ j=3,4,$$
$$Q_{G_A}(\tilde{b}_i,c_j)=\frac{1}{2}(\beta_1-\alpha_1)-\frac{\lambda_t}{4\pi m_i}\sin 2\pi\frac{m_i}{\lambda_t}(\beta_1-\alpha_1)\cos 2\pi\frac{m_i}{\lambda_t}(\beta_1+\alpha_1),$$
$$\frac{m_i}{\lambda_t}=\frac{k_j}{\lambda_s},\ i=1,3,\ j=3,4.$$
$$Q_{G_A}(\tilde{b}_i,c_j)=\frac{\lambda_t\lambda_s}{2\pi(m_i\lambda_s+k_j\lambda_t)}\sin\pi\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1)\cos\pi\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1+\alpha_1)$$
$$+\frac{\lambda_t\lambda_s}{2\pi(m_i\lambda_s-k_j\lambda_t)}\sin\pi\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1)\cos\pi\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1+\alpha_1),$$
$$\text{if } \frac{m_i}{\lambda_t}\neq\frac{k_j}{\lambda_s},\ \frac{m_i}{\lambda_t}\neq-\frac{k_j}{\lambda_s},\ i=2,4,\ j=1,2,$$

$$Q_{G_A}(\tilde{b}_i,c_j)=\frac{1}{2}(\beta_1-\alpha_1)+\frac{\lambda_t}{4\pi m_i}\sin 2\pi\frac{m_i}{\lambda_t}(\beta_1-\alpha_1)\cos 2\pi\frac{m_i}{\lambda_t}(\beta_1+\alpha_1),$$

if $\frac{m_i}{\lambda_t}=\frac{k_j}{\lambda_s}$, $i=2,4, j=1,2.$

$$Q_{G_A}(\tilde{b}_i,c_j)=\frac{\lambda_t\lambda_s}{2\pi(m_i\lambda_s+k_j\lambda_t)}\sin\pi\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1+\alpha_1)\sin\pi\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1)$$

$$\cdot\frac{\lambda_t\lambda_s}{2\pi(m_i\lambda_s-k_j\lambda_t)}\sin\pi\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1+\alpha_1)\sin\pi\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1),$$

if $\frac{m_i}{\lambda_t}\neq\frac{k_j}{\lambda_s}$, $\frac{m_i}{\lambda_t}\neq-\frac{k_j}{\lambda_s}$, $i=2,4$, $j=3,4$,

$$Q_{G_A}(\tilde{b}_i,c_j)=\frac{\lambda_t}{2\pi m_i}\left(\left(\sin\left(\frac{2\pi m_i}{\lambda_t}\beta_1\right)\right)^2-\left(\sin\left(\frac{2\pi m_i}{\lambda_t}\alpha_1\right)\right)^2\right),$$

if $\frac{m_i}{\lambda_t}=\frac{k_j}{\lambda_s}$, $i=2,4, j=3,4.$

如果对于所有 i，$j=1$，2，3，4，除非 $i=j=2$、$i=j=3$ 以及数值 $\left(\frac{m_i}{\lambda_t}-\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1)$ 和 $\left(\frac{m_i}{\lambda_t}+\frac{k_j}{\lambda_s}\right)(\beta_1-\alpha_1)$、$i$，$j=1$，2，3，4 均为整数，那么该系统

$$\left\{\sin\left(\frac{2\pi m_i}{\lambda_t}t\right),\cos\left(\frac{2\pi k_j}{\lambda_s}t\right),\cos\left(\frac{2\pi m_i}{\lambda_t}t\right),\sin\left(\frac{2\pi k_j}{\lambda_s}t\right)\right\}_{i,j=1,2,3,4}$$

是正交的，即 $Q_{G_A}(\tilde{a}_i,c_j)=0$ 当 $\tilde{a}_i\neq c_j$ 时。在这种情况下，若 $G_A=G_B=[\alpha_1,\beta_1]$，$\frac{m_2}{\lambda_t}=\frac{k_2}{\lambda_s}$ 和 $\frac{m_3}{\lambda_t}=\frac{k_3}{\lambda_s}$，则关系式 (12) 化简为

$$\begin{cases}Q_{G_A}(\tilde{b}_2,c_2)\theta_{A,2}\theta_{B,2}=\delta\theta_{B,2}\theta_{A,2}^2Q_{G_A}(\tilde{b}_2,c_2)Q_{G_B}(\tilde{a}_2,e_2)\\ Q_{G_A}(\tilde{b}_3,c_3)\theta_{A,3}\theta_{B,3}=\delta\theta_{B,3}\theta_{A,3}^2Q_{G_A}(\tilde{b}_3,c_3)Q_{G_B}(\tilde{a}_3,e_3).\end{cases}\tag{13}$$

˜由于 $G_A=G_B=[\alpha_1,\beta_1]$, $\tilde{a}_i=b_i$, $c_j=e_j$, $i,j=2,3$，我们令 $\sigma_1=Q_{G_A}$(正弦函数($\omega_3 s$),正弦函数($\omega_3 s$))，$\sigma_2=Q_{G_A}$(余弦函数($\omega_2 s$),余弦函数($\omega_2 s$))，其中 $\omega_2=\frac{2\pi m_2}{\lambda_t}$,$\omega_3=\frac{2\pi m_3}{\lambda_t}$。因此，若 $\sigma_1\neq 0$, $\sigma_2\neq 0$, 则方程组 (13) 有以下解:

1. $\{\theta_{A,2}=0,\theta_{A,3}=0,\theta_{A,1},\theta_{A,4},\theta_{B,1},\theta_{B,2},\theta_{B,3},\theta_{B,4}\in\mathbb{R}\}$; 2. $\{\theta_{A,2}=0,\theta_{B,3}=0,\theta_{A,1},\theta_{A,3},\theta_{A,4},\theta_{B,1},\theta_{B,2},\theta_{B,4}\in\mathbb{R}\}$; 3. n θ_A, $_2=0,\theta_{A,3}=\frac{1}{\delta\sigma_1},\delta\neq 0,\theta_A$,1,$\theta_A$,4,$\theta_B$,1,$\theta_B$,2,$\theta_B$,3,$\theta_{B,4}\in\mathbb{R}$o ; 4. $\{\theta_{B,2}=0,\theta_{A,3}=0,\theta_{A,1},\theta_{A,2},\theta_{A,4},\theta_{B,1},\theta_{B,3},\theta_{B,4}\in\mathbb{R}\}$; 5. $\{\theta_{B,2}=0,\theta_{B,3}=0,\theta_{A,1},\theta_{A,2},\theta_{A,3},\theta_{A,4},\theta_{B,1},\theta_{B,4}\in\mathbb{R}\}$; 6. n θ_B, $_2=0,\theta_{A,3}=\frac{1}{\delta\sigma_1},\delta\neq 0,\theta_A$,1,$\theta_A$,2,$\theta_A$,4,$\theta_B$,1,$\theta_B$,3,$\theta_{B,4}\in\mathbb{R}$o ;

7. $\left\{\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{A,3}=0,\delta\neq 0,\theta_{A,1},\theta_{A,4},\theta_{B,1},\theta_{B,2},\theta_{B,3},\theta_{B,4}\in\mathbb{R}\right\}$;
8. $\left\{\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{B,3}=0,\delta\neq 0,\theta_{A,1},\theta_{A,3},\theta_{A,4},\theta_{B,1},\theta_{B,2},\theta_{B,4}\in\mathbb{R}\right\}$;
9. $\left\{\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{A,3}=\frac{1}{\delta\sigma_1},\delta\neq 0,\theta_{A,1},\theta_{A,4},\theta_{B,1},\theta_{B,2},\theta_{B,3},\theta_{B,4}\in\mathbb{R}\right\}$.

相应的算子对 $(A,\ B)$ 为

1. 对于 $\{\theta_{A,2}=0,\theta_{A,3}=0,\theta_{A,1},\theta_{A,4},\theta_{B,1},\theta_{B,2},\theta_{B,3},\theta_{B,4}\in\mathbb{R},\sigma_1\neq 0,\sigma_2\neq 0\}$,

$$
\begin{aligned}
(Ax)(t)=&\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&\left.+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,\\
(Bx)(t)=&\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&+\theta_{B,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{B,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)\\
&\left.+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,
\end{aligned}
$$

对于几乎处处的 t。这些算子满足 $AB=\delta BA^2=BA=0$。

2. 对于 $\{\theta_{A,2}=0,\theta_{B,3}=0,\theta_{A,1},\theta_{A,3},\theta_{A,4},\theta_{B,1},\theta_{B,2},\theta_{B,4}\in\mathbb{R},\sigma_1\neq 0,\sigma_2\neq 0\}$,

$$
\begin{aligned}
(Ax)(t)=&\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&\left.+\theta_{A,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,\\
(Bx)(t)=&\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&\left.+\theta_{B,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,
\end{aligned}
$$

对于几乎处处的 t。这些算子满足 $AB=\delta BA^2=BA=0$。

3. 对于 $\theta_{A,\ 2}=0,\ \ \theta_{A,\ 3}=\overline{\delta\sigma_1},\ \ {}^1\delta\neq 0,\ \ \sigma_1\neq 0,\ \ \sigma_2\neq 0,\ \ \theta_{A,}\ 1,\ \ \theta_{A,}\ 4,\ \ \theta_{B,}\ 1,\ \ \theta_{B,}\ 2,\ \ \theta_{B,}\ 3,\ \ \theta_{B,\ 4}\in\mathbb{R}$,

$$
(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.
$$

$$+\frac{1}{\delta\sigma_1}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\Bigg)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.$$

$$+\theta_{B,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{B,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)$$

$$\left.+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB=\delta BA^2=BA=0$。4. 对于 $\{\theta_{B,2}=0,\theta_{A,3}=0,\theta_{A,1},\theta_{A,2},\theta_{A,4},\sigma_1\neq0,\sigma_2\neq0,\theta_{B,1},\theta_{B,3},\theta_{B,4}\in\mathbb{R}\}$,

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.$$

$$+\theta_{A,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\Bigg)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.$$

$$+\theta_{B,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\Bigg)x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB=\delta BA^2=BA=0$。
5. 对于 $\{\theta_{B,2}=0,\theta_{B,3}=0,\theta_{A,1},\theta_{A,2},\theta_{A,3},\theta_{A,4},\sigma_1\neq0,\sigma_2\neq0,\theta_{B,1},\theta_{B,4}\in\mathbb{R}\}$,

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.$$

$$+\theta_{A,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{A,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)$$

$$\left.+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.$$

$$\left.+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,$$

对于几乎所有的 t，这些算子满足 $AB=\delta BA^2=BA=0$。

6. 对于 $\theta_{B,2}=0$, $\theta_{A,3}=\frac{1}{\delta\sigma_1}$, $\delta\neq 0$, $\sigma_1\neq 0$, $\sigma_2\neq 0$, $\theta_{A,1}$, $\theta_{A,2}$, $\theta_{A,4}$, $\theta_{B,1}$, $\theta_{B,3}$, $\theta_{B,4}\in\mathbb{R}$,

$$
\begin{aligned}
(Ax)(t) &= \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&+\theta_{A,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\frac{1}{\delta\sigma_1}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)\\
&\left.+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,\\
(Bx)(t) &= \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&\left.+\theta_{B,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,
\end{aligned}
$$

对于几乎处处的 t，这些算子满足 $AB=\delta BA^2=BA$。

7. 若 $\theta_{A,2}=\frac{1}{\delta\sigma_2}$, $\theta_{A,3}=0$, $\delta\neq 0$, $\sigma_1\neq 0$, $\sigma_2\neq 0$, $\theta_{A,1}$, $\theta_{A,4}$, $\theta_{B,1}$, $\theta_{B,2}$, $\theta_{B,3}$, $\theta_{B,4}\in\mathbb{R}$,

$$
\begin{aligned}
(Ax)(t) &= I_{[\alpha,\beta]}(t)\int_{\alpha_1}^{\beta_1}\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&\left.+\frac{1}{\delta\sigma_2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,\\
(Bx)(t) &= I_{[\alpha,\beta]}(t)\int_{\alpha_1}^{\beta_1}\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&+\theta_{B,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{B,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)\\
&\left.+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,
\end{aligned}
$$

对于几乎处处的 t，这些算子满足 $AB=\delta BA^2=BA$。

8. 对于 $\theta_{A,2}=\frac{1}{\delta\sigma_2}, \theta_{B,3}=0, \delta\neq 0, \sigma_1\neq 0, \sigma_2\neq 0, \theta_{A,1}, \theta_{A,3}, \theta_{A,4}, 1, \theta_{B,2}, \theta_{B,4}\in\mathbb{R}$,

$$
\begin{aligned}
(Ax)(t) &= I_{[\alpha,\beta]}(t)\int_{\alpha_1}^{\beta_1}\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.\\
&\left.+\frac{1}{\delta\sigma_2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{A,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)\right.
\end{aligned}
$$

$$+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\Bigg)x(s)d\mu_s,$$

$$(Bx)(t)=I_{[\alpha,\beta]}(t)\int_{\alpha_1}^{\beta_1}\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.$$

$$\left.+\theta_{B,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,$$

对于几乎处处的 t。这些算子满足 $AB=\delta BA^2=BA$。

9. 对于 $\theta_{A,2}=\frac{1}{\delta\sigma_2}$，$\theta_{A,3}=\frac{1}{\delta\sigma_1}$，$\delta\neq 0$，$\sigma_1\neq 0$，$\sigma_2\neq 0$，$\theta_{A,1}$，$\theta_{A,4}$，$\theta_{B,1}$，$\theta_{B,2}$，$\theta_{B,3}$，$\theta_{B,4}\in\mathbb{R}$，

$$(Ax)(t)=I_{[\alpha,\beta]}(t)\int_{\alpha_1}^{\beta_1}\left(\theta_{A,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.$$

$$+\frac{1}{\delta\sigma_2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\frac{1}{\delta\sigma_1}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)$$

$$\left.+\theta_{A,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,$$

$$(Bx)(t)=I_{[\alpha,\beta]}(t)\int_{\alpha_1}^{\beta_1}\left(\theta_{B,1}\sin\left(\frac{2\pi m_1}{\lambda_t}t\right)\cos\left(\frac{2\pi k_1}{\lambda_s}s\right)\right.$$

$$+\theta_{B,2}\cos\left(\frac{2\pi m_2}{\lambda_t}t\right)\cos\left(\frac{2\pi k_2}{\lambda_s}s\right)+\theta_{B,3}\sin\left(\frac{2\pi m_3}{\lambda_t}t\right)\sin\left(\frac{2\pi k_3}{\lambda_s}s\right)$$

$$\left.+\theta_{B,4}\cos\left(\frac{2\pi m_4}{\lambda_t}t\right)\sin\left(\frac{2\pi k_4}{\lambda_s}s\right)\right)x(s)d\mu_s,$$

几乎对所有 t 都成立。这些算子满足 $AB=\delta BA^2=BA$。

情形 **2**:

情形 2 中，我们假设函数正弦函数(·)、余弦函数(·)的自变量中 t 和 s 的系数相同（ω），且正弦函数(·)与余弦函数(·)在 $[\alpha_1,\beta_1]$ 中正交，即 $\int\sin(\omega s)\cos(\omega s)d\mu_s=0$。在所得算子中，$\alpha_1$ 有些满足交换关系 $AB=\delta BA^2$，但彼此不可交换；而对于那些可交换的算子，其交换条件可通过直接计算或命题 2 得出。若对所有 $i,j=1,2,3,4$ 都有 $\frac{m_i}{\lambda_t}=\frac{k_j}{\lambda_s}$，则对所有 $i,j=1,2,3,4$，令 $\omega=\frac{2\pi m_i}{\lambda_t}=\frac{2\pi k_j}{\lambda_s}$。系统 $\{\sin(\omega s),\cos(\omega s)\}$ 是正

交的，当且仅当 $Q_{G_A}(\sin(\omega s),\cos(\omega s))=0$，且 $\frac{\omega}{\pi}(\beta_1-\alpha_1)\in\mathbb{Z}$ 或 $\frac{\omega}{\pi}(\beta_1+\alpha_1)\in\mathbb{Z}$。在这些条件及 $G_A=G_B=[\alpha_1,\beta_1]$ 下，有 $Q_{G_A}(\sin(\omega s),\cos(\omega s))=0$。因此，由（12）可得以下方程组：

$$\begin{cases}\theta_{A,3}\theta_{B,1}\sigma_1+\theta_{A,1}\theta_{B,2}\sigma_2=\delta\theta_{A,3}\theta_{A,1}\theta_{B,3}\sigma_1^2+\delta\theta_{A,4}\theta_{A,1}\theta_{B,1}\sigma_1\sigma_2\\ \qquad\qquad +\delta\theta_{A,1}\theta_{A,2}\theta_{B,3}\sigma_1\sigma_2+\delta\theta_{A,2}^2\theta_{B,1}\sigma_2^2\\ \theta_{A,4}\theta_{B,1}\sigma_1+\theta_{A,2}\theta_{B,2}\sigma_2=\delta\theta_{A,3}\theta_{A,1}\theta_{B,4}\sigma_1^2+\delta\theta_{A,4}\theta_{A,1}\theta_{B,2}\sigma_1\sigma_2\\ \qquad\qquad +\delta\theta_{A,1}\theta_{A,2}\theta_{B,4}\sigma_1\sigma_2+\delta\theta_{A,2}^2\theta_{B,2}\sigma_2^2\\ \theta_{A,3}\theta_{B,3}\sigma_1+\theta_{A,1}\theta_{B,4}\sigma_2=\delta\theta_{A,3}^2\theta_{B,3}\sigma_1^2+\delta\theta_{A,4}\theta_{A,3}\theta_{B,1}\sigma_1\sigma_2\\ \qquad\qquad +\delta\theta_{A,4}\theta_{A,1}\theta_{B,3}\sigma_1\sigma_2+\delta\theta_{A,2}\theta_{A,4}\theta_{B,1}\sigma_2^2\\ \theta_{A,4}\theta_{B,3}\sigma_1+\theta_{A,2}\theta_{B,4}\sigma_2=\delta\theta_{A,3}^2\theta_{B,4}\sigma_1^2+\delta\theta_{A,4}\theta_{A,3}\theta_{B,2}\sigma_1\sigma_2\\ \qquad\qquad +\delta\theta_{A,4}\theta_{A,1}\theta_{B,4}\sigma_1\sigma_2+\delta\theta_{A,2}\theta_{A,4}\theta_{B,2}\sigma_2^2\end{cases}\tag{14}$$

其中 $\sigma_1=Q_{G_A}(\sin(\omega s),\sin(\omega s))$ 和 $\sigma_2=Q_{G_A}(\cos(\omega s),\cos(\omega s))$，即，

$$\sigma_1=\int_{\alpha_1}^{\beta_1}(\sin(\omega s))^2d\mu_s=\begin{cases}0, & \text{if } \omega=0\\ \frac{\beta_1-\alpha_1}{2}-\frac{\cos(\omega(\alpha_1+\beta_1))\sin(\omega(\beta_1-\alpha_1))}{2\omega}, & \text{if } \omega\neq 0\end{cases},\tag{15}$$

$$\begin{aligned}\sigma_2&=\int_{\alpha_1}^{\beta_1}(\cos(\omega s))^2d\mu_s=\beta_1-\alpha_1-\sigma_1\\ &=\begin{cases}\beta_1-\alpha_1, & \text{if } \omega=0\\ \frac{\beta_1-\alpha_1}{2}+\frac{\cos(\omega(\alpha_1+\beta_1))\sin(\omega(\beta_1-\alpha_1))}{2\omega}, & \text{if } \omega\neq 0\end{cases}.\end{aligned}\tag{16}$$

方程组 (14) 关于 $\theta_{B,i},\ i=1,2,3,4$ 是线性的，可以写成如下形式：

$$\begin{cases}0=(\theta_{A,3}\sigma_1-\delta\theta_{A,4}\theta_{A,1}\sigma_1\sigma_2-\delta\theta_{A,2}^2\sigma_2^2)\theta_{B,1}+\theta_{A,1}\sigma_2\theta_{B,2}\\ \quad -(\delta\theta_{A,3}\theta_{A,1}\sigma_1^2+\delta\theta_{A,1}\theta_{A,2}\sigma_1\sigma_2)\theta_{B,3}\\ 0=\theta_{A,4}\sigma_1\theta_{B,1}+(\theta_{A,2}\sigma_2-\delta\theta_{A,4}\theta_{A,1}\sigma_1\sigma_2-\delta\theta_{A,2}^2\sigma_2^2)\theta_{B,2}\\ \quad -(\delta\theta_{A,3}\theta_{A,1}\sigma_1^2+\delta\theta_{A,1}\theta_{A,2}\sigma_1\sigma_2)\theta_{B,4}\\ 0=(-\delta\theta_{A,2}\theta_{A,4}\sigma_2^2-\delta\theta_{A,4}\theta_{A,3}\sigma_1\sigma_2)\theta_{B,1}\\ \quad +(\theta_{A,3}\sigma_1-\delta\theta_{A,3}^2\sigma_1^2-\delta\theta_{A,4}\theta_{A,1}\sigma_1\sigma_2)\theta_{B,3}+\theta_{A,1}\sigma_2\theta_{B,4}\\ 0=(-\delta\theta_{A,4}\theta_{A,3}\sigma_1\sigma_2-\delta\theta_{A,2}\theta_{A,4}\sigma_2^2)\theta_{B,2}+\theta_{A,4}\sigma_1\theta_{B,3}\\ \quad +(\theta_{A,2}\sigma_2-\delta\theta_{A,3}^2\sigma_1^2-\delta\theta_{A,4}\theta_{A,1}\sigma_1\sigma_2)\theta_{B,4}.\end{cases}$$

情形 ***2A***：

我们认为算子 A 已给定，并求算子 B。考虑到$\theta_{A,\ i},\ i=1,\ 2,\ 3,\ 4$ 已给定，与 (14) 对应的矩阵方程为 $V\Theta_B=0$，其中 $V=[v_{ij}]_{4\times 4}$，

$v_{11}=\theta_{A,3}\sigma_1-\delta\theta_{A,4}\theta_{A,1}\sigma_1\sigma_2-\delta\theta_{A,2}^2\sigma_2^2$，　$v_{12}=\theta_{A,}\sigma_2$

$v_{13}=-\ \delta\theta_{A\ 3}\theta_{A\ 1}\sigma_1^2+\delta\theta_{A\ 1}\theta_{A\ 2}\sigma_1\sigma_2$　　(、、、)，

$v_{14}=0$

$v_{21}=\theta_{A\ 4}\sigma_1$　　$v_{22}=\theta_{A,}{}_{,2}\sigma_2-\delta\theta_{A,4}\theta_{A,1}\sigma_1\sigma_2-\delta\theta_{A,2}^2\sigma_2^2$

$v_{23}=0,$　　$v_{24}=-\ (\delta\theta_{A,\ 3}\theta_{A,\ 1}\sigma_1^2+\delta\theta_{A,\ 1}\theta_{A,\ 2}\sigma_1\sigma_2)$

$v_{31}=-\delta\theta_{A\ 2}\theta_{A\ 4}\sigma_2^2-\delta\theta_{A\ 4}\theta_{A\ 3}\sigma_1\sigma_2$　　$v_{32}=0$ ，，，，，，，，，，，

$v_{33}=\theta_{A\ 3}\sigma_1-\delta\theta_{A\ 3}^2\sigma_1^2-\delta\theta_{A\ 4}\theta_{A\ 1}\sigma_1\sigma_2$　　$v_{34}=\theta_{A\ 1}\sigma_2$

$v_{41}=0$　　$v_{42}=-\delta\theta_{A}$ ，，

$_4\theta_{A\ 3}\sigma_1\sigma_2-\delta\theta_{A\ 2}\theta_{A\ 4}\sigma_2^2$，，，，，，，，$_1\sigma_1\sigma_2$

$v_{43}=\theta_{A\ 4}\sigma_1$　　$v_{44}=\theta_{A\ 2}\sigma_2-\delta\theta_{A\ 3}^2\sigma_1^2-\delta\theta_{A\ 4}\theta_A$

以及 $\Theta_B=[\theta_{B,1j}]_{1\times4}$、$\theta_{B,\ 1j}=\theta_{B,\ j}$、$j=1,\ 2,\ 3,\ 4$。相应的行列式为

$$det(V)=\sigma_1\sigma_2\Big(\theta_{A,1}^4\theta_{A,4}^4\delta^4\sigma_1^3\sigma_2^3+\theta_{A,1}^3\theta_{A,2}^2\theta_{A,4}^3\delta^4\sigma_1^2\sigma_2^4-2\theta_{A,1}^3\theta_{A,2}\theta_{A,3}\theta_{A,}^3$$
$$_4\delta^4\sigma_1^3\sigma_2^3-\theta_{A,1}^3\theta_{A,2}\theta_{A,4}^4\delta^4\sigma_1^2\sigma_2^4+\theta_{A,1}^3\theta_{A,2}^2\theta_{A,4}^3\delta^4\sigma_1^4\sigma_2^2+\theta_{A,1}^3\theta_{A,3}\theta_{A,4}^4\delta^4\sigma_1^3\sigma_2^3-2\theta_{A,}^2$$
$$_1\theta_{A,2}^3\theta_{A,3}\theta_{A,4}^2\delta^4\sigma_1^2\sigma_2^4+2\theta_{A,1}^2\theta_{A,2}^2\theta_{A,3}^2\theta_{A,4}^3\delta^4\sigma_1^3\sigma_2^3+2\theta_{A,1}^2\theta_{A,2}^2\theta_{A,3}\theta_{A,4}^3\delta^4\sigma_1^2\sigma_2^4-2\theta_{A,}^2$$
$$_1\theta_{A,2}\theta_{A,3}^3\theta_{A,4}^2\delta^4\sigma_1^4\sigma_2^2-\theta_{A,1}^2\theta_{A,3}^3\theta_{A,4}^3\delta^4\sigma_1^4\sigma_2^2+\theta_{A,1}^2\theta_{A,2}^4\theta_{A,3}^2\theta_{A,4}\delta^4\sigma_1^2\sigma_2^4-2\theta_{A,1}\theta_{A,}^3$$
$$_2\theta_{A,3}^3\theta_{A,4}\delta^4\sigma_1^3\sigma_2^3-\theta_{A,1}\theta_{A,2}^3\theta_{A,3}^2\theta_{A,4}^2\delta^4\sigma_1^3\sigma_2^3+\theta_{A,1}\theta_{A,2}^2\theta_{A,3}^4\theta_{A,4}\delta^4\sigma_1^4\sigma_2^2+\theta_{A,1}\theta_{A,}^2$$
$$_2\theta_{A,3}^3\theta_{A,4}^2\delta^4\sigma_1^3\sigma_2^3+2\theta_{A,1}\theta_{A,2}\theta_{A,3}^4\theta_{A,4}^2\delta^4\sigma_1^4\sigma_2^2+\theta_{A,2}^4\theta_{A,3}^4\delta^4\sigma_1^3\sigma_2^3-\theta_{A,2}^3\theta_{A,3}^4\theta_{A,}$$
$$_4\delta^4\sigma_1^3\sigma_2^3-\theta_{A,2}^2\theta_{A,3}^5\theta_{A,4}\delta^4\sigma_1^4\sigma_2^2-\theta_{A,1}^3\theta_{A,2}\theta_{A,3}^3\delta^3\sigma_1^2\sigma_2^3-2\theta_{A,1}^3\theta_{A,3}\theta_{A,4}^3\delta^3\sigma_1^3\sigma_2^2-\theta_{A,}^3$$
$$_1\theta_{A,4}^4\delta^3\sigma_1^2\sigma_2^3-\theta_{A,1}^2\theta_{A,2}^2\theta_{A,3}\theta_{A,}^2{}_{,4}\delta^3\sigma_1^2\sigma_2^3-\theta_{A,1}^2\theta_{A,2}\theta_{A,4}^3\delta^3\sigma_1\sigma_2^4-\theta_{A,1}^2\theta_{A,2}\theta_{A,3}^2\theta_{A,}^2$$
$$_4\delta^3\sigma_1^3\sigma_2^2+4\theta_{A,1}^2\theta_{A,2}\theta_{A,3}\theta_{A,}^3{}_{,4}\delta^3\sigma_1^2\sigma_2^3-3\theta_{A,1}^2\theta_{A,3}^2\theta_{A,4}^2\delta^3\sigma_1^3\sigma_2^2+2\theta_{A,1}^2\theta_{A,3}^3\theta_{A,}^3$$
$$_4\delta^3\sigma_1^3\sigma_2^2-\theta_{A,1}\theta_{A,2}^4\theta_{A,3}\theta_{A,}^2{}_{,4}\delta^3\sigma_1\sigma_2^4-\theta_{A,1}\theta_{A,2}^3\theta_{A,3}^2\theta_{A,4}\delta^3\sigma_1^2\sigma_2^3+3\theta_{A,1}\theta_{A,2}^3\theta_{A,3}\theta_{A,}^2$$
$$_4\delta^3\sigma_1\sigma_2^4-2\theta_{A,1}\theta_{A,2}\theta_{A,3}^3\theta_{A,4}\delta^3\sigma_1^3\sigma_2^2+3\theta_{A,1}\theta_{A,2}\theta_{A,3}^3\theta_{A,4}^2\delta^3\sigma_1^3\sigma_2^2-\theta_{A,1}\theta_{A,3}^5\theta_{A,}$$
$$_4\delta^3\sigma_1^5+3\theta_{A,1}\theta_{A,3}^4\theta_{A,4}^2\delta^3\sigma_1^4\sigma_2-\theta_{A,2}^4\theta_{A,3}^3\delta^3\sigma_1^2\sigma_2^3-\theta_{A,2}^3\theta_{A,3}^4\delta^3\sigma_1^3\sigma_2^2+\theta_{A,2}^3\theta_{A,3}^3\theta_{A,}$$
$$_4\delta^3\sigma_1^2\sigma_2^3-\theta_{A,2}^2\theta_{A,3}^5\delta^3\sigma_1^4\sigma_2+\theta_{A,2}^2\theta_{A,3}^4\theta_{A,4}\delta^3\sigma_1^3\sigma_2^2+\theta_{A,2}\theta_{A,3}^5\theta_{A,4}\delta^3\sigma_1^4\sigma_2+\theta_{A,3}^6\theta_{A,}$$
$$_4\delta^3\sigma_1^5-2\theta_{A,1}^3\theta_{A,4}^3\delta^2\sigma_1^2\sigma_2^2-3\theta_{A,1}^2\theta_{A,2}^2\theta_{A,4}^2\delta^2\sigma_1\sigma_2^3-2\theta_{A,1}^2\theta_{A,3}^2\theta_{A,4}^2\delta^2\sigma_1^3\sigma_2+\theta_{A,1}^2\theta_{A,}$$
$$_3\theta_{A,4}^3\delta^2\sigma_1^2\sigma_2^2-\theta_{A,1}\theta_{A,2}^4\theta_{A,4}\delta^2\sigma_2^4+\theta_{A,1}\theta_{A,2}^3\theta_{A,3}\theta_{A,4}\delta^2\sigma_1\sigma_2^3+\theta_{A,1}\theta_{A,2}^2\theta_{A,3}^2\theta_{A,}$$
$$_4\delta^2\sigma_1^2\sigma_2^2+\theta_{A,1}\theta_{A,2}^2\theta_{A,3}\theta_{A,}^2{}_{,4}\delta^2\sigma_1\sigma_2^3+3\theta_{A,1}\theta_{A,2}\theta_{A,3}^3\theta_{A,4}\delta^2\sigma_1^3\sigma_2-2\theta_{A,1}\theta_{A,2}\theta_{A,3}^3\theta_{A,}$$
$$_4\delta^2\sigma_1^3\sigma_2-\theta_{A,1}\theta_{A,3}^3\theta_{A,}^2{}_{,4}\delta^2\sigma_1^3\sigma_2+\theta_{A,2}^4\theta_{A,3}\theta_{A,4}\delta^2\sigma_2^4+\theta_{A,2}^3\theta_{A,3}^3\delta^2\sigma_1^2\sigma_2^2+\theta_{A,2}^3\theta_{A,3}^2\theta_{A,}$$
$$_4\delta^2\sigma_1\sigma_2^3+\theta_{A,2}^2\theta_{A,3}^4\delta^2\sigma_1^3\sigma_2$$

$$-\theta_{A,1}^4\theta_{A,3}^2\theta_{A,4}\delta^3\sigma_1\sigma_2^4$$

$$+\theta_{A,2}^2\theta_{A,3}^3\theta_{A,4}\delta^2\sigma_1^2\sigma_2^2+\theta_{A,2}\theta_{A,3}^5\delta^2\sigma_1^4-\theta_{A,2}\theta_{A,3}^4\theta_{A,4}\delta^2\sigma_1^3\sigma_2-\theta_{A,3}^5\theta_{A,4}\delta^2\sigma_1^4$$
$$+\theta_{A,1}^2\theta_{A,2}\theta_{A,4}^2\delta\sigma_1\sigma_2^2+2\theta_{A,1}^2\theta_{A,3}\theta_{A,4}^2\delta\sigma_1^2\sigma_2+\theta_{A,1}^2\theta_{A,4}^3\delta\sigma_1\sigma_2^2+\theta_{A,1}\theta_{A,2}^3\theta_{A,4}\delta\sigma_2^3$$
$$+\theta_{A,1}\theta_{A,2}^2\theta_{A,3}\theta_{A,4}\delta\sigma_1\sigma_2^2-\theta_{A,1}\theta_{A,2}\theta_{A,3}^2\theta_{A,4}\delta\sigma_1^2\sigma_2-2\theta_{A,1}\theta_{A,2}\theta_{A,3}\theta_{A,4}^2\delta\sigma_1\sigma_2^2$$
$$+\theta_{A,1}\theta_{A,3}^3\theta_{A,4}\delta\sigma_2^3-\theta_{A,2}^3\theta_{A,3}\theta_{A,4}\delta\sigma_2^3-\theta_{A,2}^2\theta_{A,3}^2\theta_{A,4}\delta\sigma_1\sigma_2^2-\theta_{A,2}\theta_{A,3}^4\delta\sigma_1^3$$
$$-\theta_{A,2}\theta_{A,3}^3\theta_{A,4}\delta\sigma_1^2\sigma_2+\theta_{A,1}^2\theta_{A,4}^2\sigma_1\sigma_2-\theta_{A,1}\theta_{A,2}\theta_{A,3}\theta_{A,4}\sigma_1\sigma_2$$
$$-\theta_{A,1}\theta_{A,3}\theta_{A,4}^2\sigma_1\sigma_2+\theta_{A,2}\theta_{A,3}^2\theta_{A,4}\sigma_1\sigma_2\Big).$$

线性方程系统 (14) 有非零解，当且仅当 V 的行列式为零。对于情形 $\theta_{A,\ 4}=0$, 行列式为

$$\det(V)=\delta\theta_{A,3}^3\theta_{A,2}\sigma_1^3\sigma_2(\theta_{A,2}^2\delta\sigma_2^2-\theta_{A,3}\sigma_1)(\delta\theta_{A,2}\sigma_2-1)(\delta\theta_{A,3}\sigma_1-1).$$

因此，当$\theta_{A,\ 4}=0$时， V 的行列式等于零，当且仅当以下情况之一成立：

1. $\{\theta_{A,1},\theta_{A,2},\theta_{A,3},\delta\in\mathbb{R},\sigma_2\neq0,\sigma_1=0\}$;
2. $\{\theta_{A,1},\theta_{A,2},\theta_{A,3},\sigma_1,\delta\in\mathbb{R},\sigma_2=0\}$;
3. $\{\theta_{A,1},\theta_{A,2},\theta_{A,3}\in\mathbb{R},\sigma_1\neq0,\sigma_2\neq0,\delta=0\}$;
4. $\{\theta_{A,1}\in\mathbb{R},\theta_{A,3}\neq\frac{1}{\delta\sigma_1},\theta_{A,3}\neq0,\sigma_1\neq0,\sigma_2\neq0,\delta\neq0,\theta_{A,2}=0\}$;
5. $\{\theta_{A,1}\in\mathbb{R},\theta_{A,2}\neq0,\delta\neq0,\sigma_1\neq0,\sigma_2\neq0,\theta_{A,3}=0\}$;
6. $\{\theta_{A,1},\theta_{A,2}\in\mathbb{R},\delta\neq0,\sigma_1\neq0,\sigma_2\neq0,\theta_{A,3}=\frac{1}{\delta\sigma_1}\}$;
7. $\{\theta_{A,1}\in\mathbb{R},\theta_{A,3}\neq\frac{1}{\delta\sigma_1},\theta_{A,3}\neq0,\delta\neq0,\sigma_1\neq0,\sigma_2\neq0,\theta_{A,2}=\frac{1}{\delta\sigma_2}\}$;
8. $\left\{\theta_{A,1}\in\mathbb{R},\theta_{A,2}\neq0,\theta_{A,2}^2\neq\frac{1}{\delta^2\sigma_2^2},\delta\neq0,\sigma_1\neq0,\sigma_2\neq0,\theta_{A,3}=\frac{\delta\theta_{A,2}^2\sigma_2^2}{\sigma_1}\right\}$.

我们在这些情况下寻找对应的解：$\theta_{B,\ i}$、 $i=1,\ 2,\ 3,\ 4$。

1. 若$\{\theta_{A,\ 1},\ \theta_{A,\ 2},\ \theta_{A,\ 3},\ \delta\in\mathbb{R},\ \sigma_2\neq0,\ \sigma_1=0\}$，则我们进一步分为以下子情况。

• 若$\theta_{A,\ 1}=\theta_{A,\ 2}=0$，则$\theta_{B,\ 1},\ \theta_{B,\ 2},\ \theta_{B,\ 3},\ \theta_{B,\ 4}\in\mathbb{R}$可以是任意值。对应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\theta_{A,3}\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$
$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)[\theta_{B,1}\sin(\omega t)\cos(\omega s)+\theta_{B,2}\cos(\omega t)\cos(\omega s)$$
$$+\theta_{B,3}\sin(\omega t)\sin(\omega s)+\theta_{B,4}\cos(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB=\delta BA^2=BA=0$。

•如果 $\delta \neq 0$， $\theta_{A,\ 2} = \frac{1}{\delta\sigma_2}$那么$\theta_{B,\ 4} = 0$， $\theta_{B,\ 1} = \delta\theta_{A,\ 1}\theta_{B,\ 2}\sigma_2$， $\theta_{B,}$ 2， $\theta_{B,\ 3} \in \mathbb{R}$。相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s) + \theta_{A,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\delta\theta_{A,1}\theta_{B,2}\sigma_2\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

对于几乎处处的 t。这些算子满足 $AB = \delta BA^2 = BA$。

•如果 $\delta = 0$， $\theta_{A,\ 1} = \theta_{A,\ 2} = 0$，那么 $\theta_{B,}$ 1， $\theta_{B,}$ 2， $\theta_{B,}$ 3， $\theta_{B,\ 4} \in \mathbb{R}$ 可以是任意值。相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{A,3}\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{B,1}\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s) + \theta_{B,4}\cos(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

对于几乎处处的 t。这些算子满足 $AB = 0$。

•如果 $\delta = 0$， $\theta_{A,\ 1} \neq 0$ 或 $\theta_{A,\ 2} \neq 0$，那么 $\theta_{B,\ 2} = \theta_{B,\ 4} = 0$， 以及 $\theta_{B,}$ 1， $\theta_{B,\ 3} \in \mathbb{R}$。相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta A,2\cos(\omega t)\cos(\omega s) + \theta_{A,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{B,1}\sin(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

对于几乎处处的 t。这些算子满足 $AB = 0$。

2. 若 $\{\theta_A,1, \theta_A,2, \theta_A,3, \sigma_1, \delta \in \mathbb{R}, \sigma_2 = 0\}$，则我们细分为以下几个子情形。

• 如果 $\sigma_1 = 0$，那么 $\alpha_1 = \beta_1$，这意味着算子 $A = 0$ 和 $B = 0$;

•如果 $\sigma_1 \neq 0$ 且 $\theta_{A,\,3} = 0$，那么 $\theta_{B,\,1}$，$\theta_{B,\,2}$，$\theta_{B,\,3}$，$\theta_{B,\,4} \in \mathbb{R}$ 可以是任意的。相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,2}\cos(\omega t)\cos(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{B,1}\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s) + \theta_{B,4}\cos(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

对几乎所有的 t。这些算子满足 $AB = \delta BA^2 = 0$。此外，对于所有 $x \in L_p(\mathbb{R},\ \mu)$，$1 < p < \infty$ 我们有，对几乎所有的 t，

$$(AB - BA)x(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\sigma_1\theta_{A,1}[\theta_{B,3}\sin(\omega t)\cos(\omega s) + \theta_{B,4}\cos(\omega t)\cos(\omega s)]x(s)d\mu_s.$$

根据命题 2，由于 $\alpha_1 < \beta_1$，$[\alpha,\beta] \supseteq [\alpha_1,\beta_1]$ 和 $\sigma_1 \neq 0$，算子 A 与 B 交换，当且仅当以下条件之一成立：$\theta_{A,\,1} = 0$ 或 $(\theta_{B,\,3} = 0$ 且 $\theta_{B,\,4} = 0)$。

•如果 $\sigma_1 \neq 0$，$\delta = 0$ 和 $\theta_{A,\,3} \neq 0$，那么 $\theta_{B,\,1} = \theta_{B,\,3} = 0$，以及 $\theta_{B,\,2}$，$\theta_{B,\,4} \in \mathbb{R}$。相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,2}\cos(\omega t)\cos(\omega s) + \theta_{A,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{B,2}\cos(\omega t)\cos(\omega s) + \theta_{B,4}\cos(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

对几乎所有的 t。这些算子满足 $AB = 0$。

•如果 $\sigma_1 \neq 0$，$\delta \neq 0$，$\theta_{A,1} = 0$，$\theta_{A,3} \neq 0$ 以及 $\theta_{A,3} \neq \frac{1}{\delta\sigma_1}$，那么 $\theta_{B,1} = \theta_{B,3} = \theta_{B,\,4} = 0$，以及 $\theta_{B,2} \in \mathbb{R}$。相应的算子是

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,2}\cos(\omega t)\cos(\omega s) + \theta_{A,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,2}\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB = \delta BA^2 = BA = 0$。

• 如果 $\sigma_1 \neq 0$、$\delta \neq 0$ 和 $\theta_{A,3} = \frac{1}{\delta\sigma_1}$，那么 $\theta_{B,4} = 0$，$\theta_{B,1} = \delta\theta_{A,1}\theta_{B,3}\sigma_1$、$\theta_{B,3}$，$\theta_{B,2} \in \mathbb{R}$。相应的算子为

$$\begin{aligned}(Ax)(t) &= \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,2}\cos(\omega t)\cos(\omega s) \\ &\quad + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s, \\ (Bx)(t) &= \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\delta\theta_{A,1}\theta_{B,3}\sigma_1\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s) \\ &\quad + \theta_{B,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,\end{aligned}$$

对几乎所有 t 都成立。这些算子满足 $AB = \delta BA^2 = BA$。

3. 如果 $\{\theta_{A,1}, \theta_{A,2}, \theta_{A,3} \in \mathbb{R}, \sigma_1 \neq 0, \sigma_2 \neq 0, \delta = \theta_{A,4} = 0\}$，那么我们细分为以下子情况。

• 如果 $\theta_{A,2} = 0$、$\theta_{A,3} \neq 0$，那么得到以下解

$$\left\{\theta_{B,1} = -\frac{\theta_{A,1}\sigma_2\theta_{B,2}}{\theta_{A,3}\sigma_1}, \theta_{B,3} = -\frac{\theta_{A,1}\sigma_2\theta_{B,4}}{\theta_{A,3}\sigma_1}, \theta_{B,2}, \theta_{B,4} \in \mathbb{R}\right\}.$$

这对应于以下算子：

$$\begin{aligned}(Ax)(t) &= \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s, \\ (Bx)(t) &= \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(-\frac{\theta_{A,1}\sigma_2\theta_{B,2}}{\theta_{A,3}\sigma_1}\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s)\right. \\ &\quad \left. -\frac{\theta_{A,1}\sigma_2\theta_{B,4}}{\theta_{A,3}\sigma_1}\sin(\omega t)\sin(\omega s) + \theta_{B,4}\cos(\omega t)\sin(\omega s)\right)x(s)d\mu_s,\end{aligned}$$

对于几乎所有的 t。这些算子满足 $AB = 0$。

• 如果 $\theta_{A,3} = 0$ 并且 ($\theta_{A,1} \neq 0$ 或者 $\theta_{A,2} \neq 0$)，那么我们得到以下解

$$\{\theta_{B,2} = \theta_{B,4} = 0, \theta_{B,1}, \theta_{B,3} \in \mathbb{R}\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,2}\cos(\omega t)\cos(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB = 0$。

4. 若 $\{\theta_{A,1} \in \mathbb{R}, \theta_{A,3} \neq \frac{1}{\delta\sigma_1}, \delta \neq 0, \sigma_1 \neq 0, \sigma_2 \neq 0, \theta_{A,2} = \theta_{A,4} = 0\}$，则我们细分为以下子情况。

• 若 $\theta_{A,3} \neq 0$，则得到以下解：

$$\left\{\theta_{B,1} = -\frac{\theta_{A,1}\sigma_2\theta_{B,2}}{\theta_{A,3}\sigma_1}, \theta_{B,2} \in \mathbb{R}, \theta_{B,3} = \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,3}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(-\frac{\theta_{A,1}\sigma_2\theta_{B,2}}{\theta_{A,3}\sigma_1}\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s)\right) \cdot x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB = \delta BA^2 = BA = 0$。

• 若 $\theta_{A,3} = 0$ 且 $\theta_{A,1} \neq 0$，则得到以下解

$$\left\{\theta_{B,1}, \theta_{B,3} \in \mathbb{R},\ \theta_{B,2} = \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{A,1}\sin(\omega t)\cos(\omega s)x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对于几乎所有的 t，它们满足 $AB = \delta BA^2 = 0$。此外，对于所有 $x \in L_p(\mathbb{R}, \mu)$，$1 < p < \infty$ 我们有

$$(AB - BA)x(t) = -(BA)x(t) = -\theta_{B,3}\theta_{A,1}\sigma_1 \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\sin(\omega t)\cos(\omega s)x(s)d\mu_s$$

几乎处处对 t 成立。应用命题 2，由于 $\alpha_1 < \beta_1$、$[\alpha,\beta] \supseteq [\alpha_1,\beta_1]$ 和 $\sigma_1 \neq 0$，算子 A 和 B 交换，当且仅当以下两种情况之一成立：$\theta_{A,1} = 0$ $(A = 0)$ 或 $\theta_{B,3} = 0$。

5. 如果 $\{\theta_{A,1} \in \mathbb{R},\ \theta_{A,2} \neq 0,\ \delta \neq 0,\ \sigma_1 \neq 0,\ \sigma_2 \neq 0,\ \theta_{A,3} = \theta_{A,4} = 0\}$，则我们细分为以下子情况。

• 如果 $\theta_{A,2} \neq \frac{1}{\delta\sigma_2}$，则得到以下解：

$$\left\{\theta_{B,1} = -\frac{\theta_{A,1}\sigma_1\theta_{B,3}}{\theta_{A,2}\sigma_2}, \theta_{B,3} \in \mathbb{R}, \theta_{B,2} = \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,2}\cos(\omega t)\cos(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(-\frac{\theta_{A,1}\sigma_1}{\theta_{A,2}\sigma_2}\theta_{B,3}\sin(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对于几乎处处的 t。这些算子满足 $AB = \delta BA^2 = BA = 0$。

• 如果 $\theta_{A,2} = \frac{1}{\delta\sigma_2}$，则得到以下解：

$$\left\{\theta_{B,1} = \delta\theta_{A,1}(\sigma_2\theta_{B,2} - \sigma_1\theta_{B,3}),\ \theta_{B,2}, \theta_{B,3} \in \mathbb{R}, \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s)\right)x(s)d\mu_s,$$

$$\begin{aligned}(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\ &\big(\delta\theta_{A,1}(\sigma_2\theta_{B,2} - \sigma_1\theta_{B,3})\sin(\omega t)\cos(\omega s)\\ &+\theta_{B,2}\cos(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)\big)x(s)d\mu_s,\end{aligned}$$

对于几乎所有的 t。这些算子满足 $AB = \delta BA^2 = BA$。

6. 如果 $\{\theta_{A,}1, \theta_{A,2} \in \mathbb{R}, \sigma_1 \neq 0, \sigma_2 \neq 0,\ \delta \neq 0, \theta_{A,3} = \frac{1}{\delta\sigma_1}, \theta_{A,4} = 0\}$，那么我们分为以下子情况。

•如果 $\theta_{A,2}\sigma_2\delta \neq 1$、$\theta_{A,2}\sigma_2\delta \neq -1$、$\theta_{A,1} \neq 0$ 和 $\theta_{A,2} \neq 0$，那么我们得到以下解：

$$\left\{\theta_{B,3} = -\frac{\theta_{B,1}(\delta\sigma_2\theta_{A,2} - 1)}{\delta\theta_{A,1}\sigma_1}, \theta_{B,1} \in \mathbb{R}, \theta_{B,2} = \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,2}\cos(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin(\omega t)\cos(\omega s) - \frac{\theta_{B,1}(\delta\sigma_2\theta_{A,2}-1)}{\delta\theta_{A,1}\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对于几乎所有的 t，它们满足 $AB = \delta BA^2 = BA$。

- 如果 $\theta_{A,\ 2} = -\frac{1}{\delta\sigma_2}$、$\theta_{A,\ 1} \neq 0$ 和 $\theta_{A,\ 2} \neq 0$，那么我们得到以下解：

$$\left\{\theta_{B,1}, \theta_{B,3} \in \mathbb{R}, \theta_{B,2} = \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin(\omega t)\cos(\omega s) - \frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

几乎对所有 t 成立。它们满足 $AB = \delta BA^2$。此外，对于所有 $x \in L_p(\mathbb{R},\ \mu)$，$1 < p < \infty$ 我们有

$$(AB - BA)x(t) = \left(\frac{2\theta_{B,1}}{\delta} - \sigma_1\theta_{A,1}\theta_{B,3}\right)\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\sin(\omega t)\cos(\omega s)x(s)d\mu_s$$

对于几乎所有的 t。应用命题 2，由于 $\alpha_1 < \beta_1$、$[\alpha,\beta] \supseteq [\alpha_1,\beta_1]$ 和 $\delta \neq 0$，算子 A 和 B 交换，当且仅当 $\theta_{B,\ 1} = \frac{\delta}{2}\sigma_1\theta_{A,\ 1}\theta_{B,\ 3}$。

- 如果 $\theta_{A,\ 2}\sigma_2\delta \neq 1$、$\theta_{A,\ 1} = 0$ 和 $\theta_{A,\ 2} \neq 0$，那么我们得到以下解：

$$\left\{\theta_{B,3} \in \mathbb{R}, \theta_{B,1} = \theta_{B,2} = \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)[\theta_{A,2}\cos(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,3}\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$

对几乎所有的 t，这些算子满足 $AB = \delta BA^2 = BA$。

- 若 $\theta_{A,2} = 0$ 且 $\theta_{A,1} = 0$，则得到以下解：

$$\{\theta_{B,2}, \theta_{B,3} \in \mathbb{R}, \theta_{B,1} = \theta_{B,4} = 0\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,2}\cos(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB = \delta BA^2 = BA$。

- 若 $\theta_{A,2} = 0$ 且 $\theta_{A,1} \neq 0$，则得到以下解：

$$\{\theta_{B,1} = \delta\theta_{A,1}(\sigma_1\theta_{B,3} - \sigma_2\theta_{B,2}),\ \theta_{B,2}, \theta_{B,3} \in \mathbb{R},\ \theta_{B,4} = 0\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} [\theta_{A,1}\sin(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} \big(\delta\theta_{A,1}(\sigma_1\theta_{B,3} - \sigma_2\theta_{B,2})\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)\big)x(s)d\mu_s,$$

对几乎处处的 t。这些算子满足 $AB = \delta BA^2 = BA$。

- 若 $\theta_{A,\ 2} = \frac{1}{\delta\sigma_2}$ 且 $\theta_{A,\ 1} \neq 0$，则得到以下解：

$$\left\{\theta_{B,2} = \frac{2\sigma_1}{\sigma_2}\theta_{B,3},\ \theta_{B,1}, \theta_{B,3} \in \mathbb{R},\ \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin(\omega t)\cos(\omega s) + \frac{2\sigma_1}{\sigma_2}\theta_{B,3}\cos(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对几乎所有的 t，它们都满足 $AB = \delta BA^2$。此外，对于所有 $x \in L_p(\mathbb{R}, \mu)$，$1 < p < \infty$ 我们有，对几乎所有的 t，

$$(AB - BA)x(t) = \sigma_1\theta_{A,1}\theta_{B,3}\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\sin(\omega t)\cos(\omega s)x(s)d\mu_s.$$

根据命题 2，由于 $\alpha_1 < \beta_1$、$[\alpha,\beta] \supseteq [\alpha_1,\beta_1]$ 和 $\sigma_1 \neq 0$，算子A 和 B 交换，当且仅当 $\theta_{A,1} = 0$ 或 $\theta_{B,3} = 0$。

- 如果 $\theta_{A,2} = \frac{1}{\delta\sigma_2}$ 且 $\theta_{A,1} = 0$，那么我们得到以下解：

$$\{\theta_{B,1}, \theta_{B,2}, \theta_{B,3}, \theta_{B,4} \in \mathbb{R}\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,1}\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s) + \theta_{B,3}\sin(\omega t)\sin(\omega s) + \theta_{B,4}\cos(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对几乎所有 t。这些算子满足 $AB = \delta BA^2 = BA$。

7. 如果 $\{\theta_{A,1} \in \mathbb{R}, \theta_{A,3} \neq 0, \theta_{A,3} \neq \frac{1}{\delta\sigma_1}, \sigma_1 \neq 0, \sigma_2 \neq 0,\ \delta \neq 0, \theta_{A,2} = \frac{1}{\delta\sigma_2}, \theta_{A,4} = 0\}$，那么我们得到以下解：

$$\left\{\theta_{B,1} = \frac{\delta\theta_{A,1}\sigma_2}{1 - \delta\theta_{A,3}\sigma_1}\theta_{B,2}, \theta_{B,2} \in \mathbb{R}, \theta_{B,3} = \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s) + \theta_{A,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\frac{\delta\theta_{A,1}\sigma_2}{1-\delta\theta_{A,3}\sigma_1}\theta_{B,2}\sin(\omega t)\cos(\omega s) + \theta_{B,2}\cos(\omega t)\cos(\omega s)\right)x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB = \delta BA^2 = BA$。

8. 若 $\left\{\theta_{A,1}\in\mathbb{R}, \theta_{A,2}\neq 0, \theta_{A,2}^2 \neq \frac{1}{\delta^2\sigma_2^2}, \delta\neq 0, \sigma_1\neq 0, \sigma_2\neq 0, \theta_{A,3} = \frac{\delta\theta_{A,2}^2\sigma_2^2}{\sigma_1}\right\}$，则得到如下解：

$$\left\{\theta_{B,1}\in\mathbb{R}, \theta_{B,2} = \theta_{B,3} = \theta_{B,4} = 0\right\}.$$

这对应于以下算子：

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,1}\sin(\omega t)\cos(\omega s) + \theta_{A,2}\cos(\omega t)\cos(\omega s) + \frac{\delta\theta_{A,2}^2\sigma_2^2}{\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,1}\sin(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对几乎处处的 t 成立。它们满足 $AB=\delta BA^2$。此外，对于所有 $x\in L_p(\mathbb{R},\mu)$，$1<p<\infty$ 对几乎处处的 t，我们有：

$$(AB-BA)x(t) = \theta_{B,1}\theta_{A,2}\sigma_2(\delta\sigma_2\theta_{A,2}-1)\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\sin(\omega t)\cos(\omega s)x(s)d\mu_s.$$

应用命题 2，由于 $\alpha_1<\beta_1$，$[\alpha,\beta]\supseteq[\alpha_1,\beta_1]$，$\sigma_2\neq 0$，$\delta\neq 0$ 和 $\theta_{A,2}\neq\frac{1}{\delta\sigma_2}$，算子 A 和 B 交换，当且仅当 $\theta_{A,2}=0$ 或 $\theta_{B,1}=0$ $(B=0)$。

情形 ***2B***：

设算子 B 已给定，我们要寻找算子 A。可以通过求解 θ_A 中的方程组来寻找算子 A，$_i$，$i=1$，2，3，4。在这种情况下

方程组（14）是非线性的，通常更难求解。因此，我们考虑一些特殊情形。如果算子 B 使得系数满足 $\theta_{B,1}=\theta_{B,3}$ 且 $\theta_{B,2}=\theta_{B,4}$，那么，例如，对于 $\{\theta_{A,i}\}$，我们有以下非零解：

$$\left\{\theta_{A,1}=\theta_{A,3}=\frac{\theta_{B,1}}{\delta(\theta_{B,1}\sigma_1+\theta_{B,2}\sigma_2)},\ \theta_{A,2}=\theta_{A,4}=\frac{\theta_{B,2}}{\delta(\theta_{B,1}\sigma_1+\theta_{B,2}\sigma_2)},\right.$$
$$\left.\delta(\theta_{B,1}\sigma_1+\theta_{B,2}\sigma_2)\neq 0\right\}, \tag{17}$$

其中，σ_1 由（15）给出，σ_2 由（16）给出。当算子 A 的系数由（17）给出时，相应的算子对（A、B）为

$$\begin{aligned}(Ax)(t)=&\frac{1}{\delta(\theta_{B,1}\sigma_1+\theta_{B,2}\sigma_2)}\int_{\alpha_1}^{\beta_1}\big(\theta_{B,1}\sin(\omega t)\cos(\omega s)+\theta_{B,2}\cos(\omega t)\cos(\omega s)\\&+\theta_{B,1}\sin(\omega t)\sin(\omega s)+\theta_{B,2}\cos(\omega t)\sin(\omega s)\big)x(s)d\mu_s\\(Bx)(t)=&\int_{\alpha_1}^{\beta_1}[\theta_{B,1}\sin(\omega t)\cos(\omega s)+\theta_{B,2}\cos(\omega t)\cos(\omega s)\\&+\theta_{B,1}\sin(\omega t)\sin(\omega s)+\theta_{B,2}\cos(\omega t)\sin(\omega s)]x(s)d\mu_s,\end{aligned}$$

对于几乎处处的 t均成立。此外，它们满足 $AB=\delta BA^2=BA$。

如果算子 B 的系数满足 $\theta_{B,3}=\frac{2\sigma_2}{\sigma_1}\theta_{B,2}$，$\theta_{B,4}=0$，$\theta_{B,1}$、$\theta_{B,2}\in\mathbb{R}$，$\sigma_1\neq 0$ 由（15）给出，$\sigma_2\neq 0$ 由（16）给出，则 $AB=\delta BA^2$ 当且仅当以下条件之一成立

1. $\{\theta_{A,1}=\theta_{A,2}=\theta_{A,3}=\theta_{A,4}=0\}$
2. $\{\theta_{B,1}=\theta_{B,2}=\theta_{B,3}=\theta_{B,4}=0\}$
3. $\{\theta_{A,3}=\frac{\delta\sigma_2^2}{\sigma_1}\theta_{A,2}^2,\theta_{A,1},\theta_{A,2},\delta\in\mathbb{R},\theta_{A,4}=\theta_{B,2}=\theta_{B,3}=\theta_{B,4}=0,\sigma_1,\sigma_2\in\mathbb{R}\setminus\{0\},$ $\theta_{B,1}\in\mathbb{R}\setminus\{0\}\}$
4. $\left\{\theta_{A,1}=\frac{\theta_{B,1}}{\theta_{B,2}\delta\sigma_2},\theta_{A,3}=\frac{1}{\delta\sigma_1},\ \theta_{A,2}=\theta_{A,4}=0,\delta\neq 0,\sigma_1\neq 0,\sigma_2\neq 0,\theta_{B,2}\neq 0\right\}$
5. $\left\{\theta_{A,1}=-\frac{\theta_{B,1}}{\theta_{B,2}\delta\sigma_2},\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{A,3}=\theta_{A,4}=0,\delta\neq 0,\sigma_1\neq 0,\sigma_2\neq 0,\theta_{B,2}\neq 0\right\}$
6. $\left\{\theta_{A,1}=0,\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{A,3}=\frac{1}{\delta\sigma_1},\theta_{A,4}=0,\delta\neq 0,\ \sigma_1\neq 0,\sigma_2\neq 0,\theta_{B,1}\neq 0\right.$ $\left.\theta_{B,2}\neq 0\right\}$
7. $\left\{\theta_{A,1}\in\mathbb{R}\setminus\{0\},\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{A,3}=\frac{1}{\delta\sigma_1},\theta_{A,4}=0,\delta\neq 0,\sigma_1\neq 0,\sigma_2\neq 0,\theta_{B,1}\neq 0,\right.$ $\left.\theta_{B,2}=\theta_{B,3}=\theta_{B,4}=0\right\}$
8. $\left\{\theta_{A,1}=0,\ \theta_{A,2}=\frac{1}{\delta\sigma_2},\ \theta_{A,3}=\frac{1}{\delta\sigma_1},\ \theta_{A,4}\in\mathbb{R},\delta\neq 0,\sigma_1\neq 0,\sigma_2\neq 0,\theta_{B,1}=0,\right.$ $\left.\theta_{B,2}\neq 0\right\}$.

现在，我们来寻找这些不同情形下对应的算子。

1. 如果 $\{\theta_{A,1}=\theta_{A,2}=\theta_{A,3}=\theta_{A,4}=0\}$，那么 $A=0$。2. 如果 $\{\theta_{B,1}=\theta_{B,2}=\theta_{B,3}=\theta_{B,4}=0\}$，那么 $B=0$。3 如果 $\{\theta_{A,3}=\frac{\delta\sigma_2^2}{\sigma_1}\theta_{A,2}^2,\theta_{A,1},\theta_{A,2},\delta\in\mathbb{R},\theta_{A,4}=\theta_{B,2}=\theta_{B,3}=\theta_{B,4}=0,\sigma_1,\sigma_2,\in\mathbb{R}\backslash\{0\},\theta_{B,1}\in\mathbb{R}\backslash\{0\}\}$，那么对应的非零算子对 (A,B) 为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\Big(\theta_{A,1}\sin(\omega t)\cos(\omega s)+\theta_{A,2}\cos(\omega t)\cos(\omega s)+\frac{\delta\sigma_2^2\theta_{A,2}^2}{\sigma_1}\sin(\omega t)\sin(\omega s)\Big)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\theta_{B,1}\sin(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对几乎处处的 t。它们满足 $AB=\delta BA^2$。这对算子是在情形 2A 的第 8 项中找到的。

4. 如果 n $\theta_{A,1}=\frac{\theta_{B,1}}{\delta\theta_{B,2}\sigma_2}$，$\theta_{A,3}=\frac{1}{\delta\sigma_1}$，$\theta_{A,2}=\theta_{A,4}=0$，$\delta\neq0$，$\sigma_1\neq0$，$\sigma_2\neq0$，$\theta_{B,2}\neq0$，$\theta_{B,1}\in\mathbb{R}\}$，则相应的非零算子对（$A$，$B$）为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\frac{\theta_{B,1}}{\delta\theta_{B,2}\sigma_2}\sin(\omega t)\cos(\omega s)+\frac{1}{\sigma_1\delta}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\big(\theta_{B,1}\sin(\omega t)\cos(\omega s)+\theta_{B,2}\cos(\omega t)\cos(\omega s)+\frac{2\sigma_2}{\sigma_1}\theta_{B,2}\sin(\omega t)\sin(\omega s)\big)x(s)d\mu_s,$$

对几乎处处的 t。它们满足 $AB=\delta BA^2=BA$。

5. 如果 n $\theta_{A,1}=-\frac{\theta_{B,1}}{\delta\theta_{B,2}\sigma_2}$，$\theta_{A,2}=\frac{1}{\delta\sigma_2}$，$\theta_{A,3}=\theta_{A,4}=0$，$\delta\neq0$，$\sigma_1\neq0$，$\sigma_2\neq0$，$\theta_{B,2}\neq0$，$\theta_{B,1}\in\mathbb{R}\}$，则相应的非零算子对（$A$，$B$）为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(-\frac{\theta_{B,1}}{\delta\theta_{B,2}\sigma_2}\sin(\omega t)\cos(\omega s)+\frac{1}{\sigma_2\delta}\cos(\omega t)\cos(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\big(\theta_{B,1}\sin(\omega t)\cos(\omega s)+\theta_{B,2}\cos(\omega t)\cos(\omega s)+\frac{2\sigma_2}{\sigma_1}\theta_{B,2}\sin(\omega t)\sin(\omega s)\big)x(s)d\mu_s,$$

对于几乎所有的 t。它们满足 $AB=\delta BA^2=BA$。

6. 如果 $\theta_{A,1}=0,\ \theta_{A,2}=\frac{1}{\delta\sigma_2},\ \theta_{A,3}=\frac{1}{\delta\sigma_1},\ \theta_{A,4}=0,\delta\neq 0,\ \sigma_1\neq 0,\sigma_2\neq 0,\theta_{B,1}\neq 0,\theta_{B,2}\neq 0\}$，那么相应的非零算子对 (A,B) 是

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\frac{1}{\sigma_2\delta}\cos(\omega t)\cos(\omega s)+\frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\big(\theta_{B,1}\sin(\omega t)\cos(\omega s)+\theta_{B,2}\cos(\omega t)\cos(\omega s)$$
$$+\frac{2\sigma_2}{\sigma_1}\theta_{B,2}\sin(\omega t)\sin(\omega s)\big)x(s)d\mu_s,$$

对于几乎所有的 t。它们满足 $AB=\delta BA^2=BA$。

7. 如果 $\theta_{A,1}\in\mathbb{R}\setminus\{0\},\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{A,3}=\frac{1}{\delta\sigma_1},\theta_{A,4}=0,\ \delta\neq 0,\ \sigma_1\neq 0,\sigma_2\neq 0,\theta_{B,1}\neq 0,$ $\theta_{B\,2}=\theta_{B\,3}=\theta_{B\,4}=0$ $A\ B$ ...，那么相应的非零算子对（,）为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\big(\theta_{A,1}\sin(\omega t)\cos(\omega s)+\frac{1}{\sigma_2\delta}\cos(\omega t)\cos(\omega s)$$
$$+\frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\big)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,1}\sin(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对于几乎所有的 t。它们满足 $AB=\delta BA^2=BA$。

8. 如果 $\theta_{A,1}=0,\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{A,3}=\frac{1}{\delta\sigma_1},\theta_{A,4}\in\mathbb{R}\setminus\{0\},\ \delta\neq 0,\sigma_1\neq 0,\sigma_2\neq 0,\theta_{B,1}=0,\theta_{B,2}\neq 0\}$,，那么相应的非零算子对（$A,B$）为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\big(\frac{1}{\sigma_2\delta}\cos(\omega t)\cos(\omega s)+\frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)$$
$$+\theta_{A,4}\cos(\omega t)\sin(\omega s)\big)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\big(\theta_{B,2}\cos(\omega t)\cos(\omega s)+\frac{2\sigma_2}{\sigma_1}\theta_{B,2}\sin(\omega t)\sin(\omega s)\big)x(s)d\mu_s,$$

几乎处处满足 t。它们满足 $AB=\delta BA^2$。此外，对于 $x\in L_p(\mathbb{R},\ \mu),\ 1<p<\infty,$

$$(AB-BA)x(t)=\theta_{A,4}\sigma_2\theta_{B,2}\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\cos(\omega t)\sin(\omega s)x(s)d\mu_s,$$

几乎处处满足 t_0。根据命题 2，由于 $\alpha_1 < \beta_1$, $[\alpha,\beta] \supseteq [\alpha_1,\beta_1]$ 和$\sigma_2 \neq 0$，算子 A 和 B 交换，当且仅当 $\theta_{A,\,4} = 0$ 或 $\theta_{B,\,2} = 0$ $(B = 0)$。

若算子 B 满足系数关系 $\theta_{B,\,1} = \theta_{B,\,4} = 0$，且算子 A 的第一个系数为零，即 $\theta_{A,\,1} = 0$，那么根据（12），交换关系$AB = \delta BA^2$ 成立当且仅当

$$\begin{cases} \theta_{A,2}\theta_{B,2}\sigma_2 = \delta\theta_{A,2}^2\theta_{B,2}\sigma_2^2, \\ \theta_{A,3}\theta_{B,3}\sigma_1 = \delta\theta_{A,3}^2\theta_{B,3}\sigma_1^2, \\ \theta_{A,4}\theta_{B,3}\sigma_1 = \delta\theta_{A,4}\theta_{A,3}\theta_{B,2}\sigma_1\sigma_2 + \delta\theta_{A,2}\theta_{A,4}\theta_{B,2}\sigma_2^2. \end{cases} \tag{18}$$

若 $\sigma_1 \neq 0$ 由（15）给出，$\sigma_2 \neq 0$ 由（16）给出，则方程组（18）成立当且仅当以下情况之一成立

1. $\{\theta_{B,1} = \theta_{B,4} = 0, \theta_{B,2}, \theta_{B,3}, \delta \in \mathbb{R}, \theta_{A,1} = \theta_{A,2} = \theta_{A,3} = \theta_{A,4} = 0\}$;2. $\{\theta_{B,1} = \theta_{B,2} = \theta_{B,4} = 0, \theta_{B,3} \in \mathbb{R}\setminus\{0\}, \theta_{A,1} = \theta_{A,3} = \theta_{A,4}, \theta_{A,2} \in \mathbb{R}\setminus\{0\}, \delta = 0\}$;

3. $\{\theta_{B,1} = \theta_{B,3} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{A,1} = \theta_{A,2} = 0, \theta_{A,3}, \theta_{A,4} \in \mathbb{R}, \delta = 0, |\theta_{A,3}| + |\theta_{A,4}| > 0\}$; *4.* $\{\theta_{B,1} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{B,3} \in \mathbb{R} \mid \{\theta_{B,2}\frac{\sigma_2}{\sigma_1}\}, \theta_{A,1} = \theta_{A,2} = \theta_{A,4} = 0, \theta_{A,3} = \frac{1}{\delta\sigma_1}, \delta \neq 0\}$; *5.* $\{\theta_{B,1} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{B,3} = \theta_{B,2}\frac{\sigma_2}{\sigma_1}, \theta_{A,1} = \theta_{A,2} = 0, \theta_{A,3} = \frac{1}{\delta\sigma_1}, \theta_{A,4} \in \mathbb{R}, \delta \neq 0\}$; *6.* $\{\theta_{B,1} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{B,3} \in \mathbb{R} \mid \{\theta_{B,2}\frac{\sigma_2}{\sigma_1}\}, \theta_{A,1} = \theta_{A,3} = \theta_{A,4} = 0, \theta_{A,2} = \frac{1}{\delta\sigma_2}, \delta \neq 0\}$; *7.* $\{\theta_{B,1} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{B,3} = \theta_{B,2}\frac{\sigma_2}{\sigma_1}, \theta_{A,1} = \theta_{A,3} = 0, \theta_{A,2} = \frac{1}{\delta\sigma_2}, \theta_{A,4} \in \mathbb{R}, \delta \neq 0\}$; *8.* $\{\theta_{B,1} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{B,3} \in \mathbb{R} \mid \{\theta_{B,2}\frac{2\sigma_2}{\sigma_1}\}, \theta_{A,1} = \theta_{A,4} = 0, \delta \neq 0, \theta_{A,2} = \frac{1}{\delta\sigma_2}, \theta_{A,3} = \frac{1}{\delta\sigma_1}\}$; *9.* $\{\theta_{B,1} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{B,3} = \theta_{B,2}\frac{2\sigma_2}{\sigma_1}, \theta_{A,1} = 0, \theta_{A,2} = \frac{1}{\delta\sigma_2}, \delta \neq 0, \theta_{A,3} = \frac{1}{\delta\sigma_1}, \theta_{A,4} \in \mathbb{R}\}$; *10.* $\{\theta_{B,1} = \theta_{B,2} = \theta_{B,4} = 0, \theta_{B,3} \in \mathbb{R} \mid \{0\}, \theta_{A,1} = \theta_{A,3} = \theta_{A,4} = 0, \theta_{A,2} \in \mathbb{R} \mid \{0\}, \delta \neq 0\}$; *11.* $\{\theta_{B,1} = \theta_{B,2} = \theta_{B,4} = 0, \theta_{B,3} \in \mathbb{R} \mid \{0\}, \theta_{A,1} = \theta_{A,4} = 0, \theta_{A,3} = \frac{1}{\delta\sigma_1}\theta_{A,2} \in \mathbb{R}, \delta \neq 0\}$;

12. $\{\theta_{B,1} = \theta_{B,3} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{A,1} = \theta_{A,2} = \theta_{A,4} = 0, \theta_{A,3} \in \mathbb{R} \mid \{0\}, \delta \neq 0\}$; *13.* $\{\theta_{B,1} = \theta_{B,3} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{A,1} = \theta_{A,2} = \theta_{A,3} = 0, \theta_{A,4} \in \mathbb{R}, \delta \neq 0\}$; *14.* $\{\theta_{B,1} = \theta_{B,3} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{A,1} = \theta_{A,4} = 0, \theta_{A,2} = \frac{1}{\delta\sigma_1}, \theta_{A,3} \in \mathbb{R}, \delta \neq 0\}$;

15. $\{\theta_{B,1} = \theta_{B,3} = \theta_{B,4} = 0, \theta_{B,2} \in \mathbb{R} \mid \{0\}, \theta_{A,1} = 0, \theta_{A,2} = \frac{1}{\delta\sigma_2}, \theta_{A,3} = -\frac{1}{\delta\sigma_1}, \delta \neq 0, \theta_{A\,4} \in \mathbb{R}\}$.

现在我们考虑这些不同情况下的相应算子。1. 如果 $\{\theta_{B,1}=\theta_{B,4}=0,\theta_{B,2},\theta_{B,3},\delta\in\mathbb{R},\theta_{A,1}=\theta_{A,2}=\theta_{A,3}=\theta_{A,4}=0,\sigma_1\neq 0,\sigma_2\neq 0\}$，那么 $A=0$；2. 如果 $\{\theta_{B,1}=\theta_{B,2}=\theta_{B,4}=0,\theta_{B,3}\in\mathbb{R}\setminus\{0\},\theta_{A,1}=\theta_{A,3}=\theta_{A,4}=0,\theta_{A,2}\in\mathbb{R}\setminus\{0\},\delta=0,\sigma_1\neq 0,\sigma_2\neq 0\}$，那么相应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}\theta_{A,2}I_{[\alpha,\beta]}(t)\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}\theta_{B,3}I_{[\alpha,\beta]}(t)\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$

对几乎所有 t 均成立。此外，这些算子满足 $AB=0$。

3. 若 $\{\theta_{B,1}=\theta_{B,3}=\theta_{B,4}=0,\theta_{B,2}\in\mathbb{R}\setminus\{0\},\theta_{A,1}=\theta_{A,2}=0,\theta_{A,3},\theta_{A,4}\in\mathbb{R},\delta=0,|\theta_{A,3}|+|\theta_{A,4}|>0,\sigma_1\neq 0,\sigma_2\neq 0\}$，则相应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)[\theta_{A,3}\sin(\omega t)\sin(\omega s)+\theta_{A,4}\cos(\omega t)\sin(\omega s)]x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}\theta_{B,2}I_{[\alpha,\beta]}(t)\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对几乎所有 t 均成立。这些算子满足 $AB=0$。

4. 若 $\{\theta_{B,1}=\theta_{B,4}=0,\theta_{B,2}\in\mathbb{R}\setminus\{0\},\theta_{B,3}\in\mathbb{R}\setminus\theta_{B,2}\frac{\sigma_2}{\sigma_1},\theta_{A,1}=\theta_{A,2}=0,\theta_{A,4}=0,\theta_{A,3}=\frac{1}{\delta\sigma_1},\delta\neq 0,\sigma_1\neq 0,\sigma_2\neq 0\}$，则相应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\theta_{B,2}\cos(\omega t)\cos(\omega s)+\theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对几乎所有的 t，这些算子都满足 $AB=\delta BA^2=BA$。

5. 如果 $\{\theta_{B,1}=\theta_{B,4}=0,\theta_{B,2}\in\mathbb{R}\setminus\{0\},\theta_{B,3}=\theta_{B,2}\frac{\sigma_2}{\sigma_1},\theta_{A,1}=\theta_{A,2}=0,\theta_{A,3}=\frac{1}{\delta\sigma_1},\theta_{A,4}\in\mathbb{R},\delta\neq 0,\sigma_1\neq 0,\sigma_2\neq 0\}$，则相应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1}I_{[\alpha,\beta]}(t)\left(\frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)+\theta_{A,4}\cos(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,2}\cos(\omega t)\cos(\omega s)+\frac{\sigma_2\theta_{B,2}}{\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对于几乎所有的 t，这些算子满足 $AB=\delta BA^2=BA$。6. 若 $\{\theta_{B,\ 1}=\theta_{B,\ 4}=0,\ \theta_{B,\ 2}\in\mathbb{R}\setminus\{0\},\ \theta_{B,\ 3}\in\mathbb{R}\setminus\{\theta_{B,\ 2}\frac{\sigma_2}{\sigma_1}\},\ \theta_{A,\ 1}=\theta_{A,\ 3}=\theta_{A,\ 4}=0,\ \theta_{A,\ 2}=\frac{1}{\delta\sigma_2},\ \delta\neq 0,\ \sigma_1\neq 0,\ \sigma_2\neq 0\}$，则相应的算子是

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,2}\cos(\omega t)\cos(\omega s)+\theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对几乎处处的 t。这些算子满足 $AB=\delta BA^2=BA$。7. 若 $\{\theta_{B,\ 1}=\theta_{B,\ 4}=0,\ \theta_{B,\ 2}\in\mathbb{R}\setminus\{0\},\ \theta_{B,\ 3}=\theta_{B,\ 2}\frac{\sigma_2}{\sigma_1},\ \theta_{A,\ 1}=\theta_{A,\ 3}=0,\ \theta_{A,\ 2}=\frac{1}{\delta\sigma_2},\ \delta\neq 0,\ \theta_{A,\ 4}\in\mathbb{R},\ \sigma_1\neq 0,\ \sigma_2\neq 0\}$，则相应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s)+\theta_{A,4}\cos(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,2}\cos(\omega t)\cos(\omega s)+\frac{\sigma_2\theta_{B,2}}{\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对几乎处处的 t。这些算子满足 $AB=\delta BA^2=BA$。

8. 若 $\{\theta_{B,\ 1}=\theta_{B,\ 4}=0,\ \theta_{B,\ 2}\in\mathbb{R}\setminus\{0\},\ \theta_{B,\ 3}\in\mathbb{R}\setminus\{\theta_{B,\ 2}\frac{2\sigma_2}{\sigma_1}\},\ \theta_{A,\ 1}=\theta_{A,\ 4}=0,\ \theta_{A,\ 2}=1\frac{}{\delta\sigma_2},\ \theta_{A,\ 3}=\frac{1}{\delta\sigma_1},\ \delta\neq 0,\ \sigma_1\neq 0,\ \sigma_2\neq 0\}$，则相应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s)+\frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,2}\cos(\omega t)\cos(\omega s)+\theta_{B,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对于几乎所有的 t，这些算子满足 $AB=\delta BA^2=BA$。

9. 如果 $\{\theta_{B,1}=\theta_{B,4}=0,\theta_{B,2}\in\mathbb{R}\setminus\{0\},\theta_{B,3}=\theta_{B,}{}_2\frac{\sigma_2}{\sigma_1},\ \theta_{A,1}=0,\theta_{A,2}=\frac{1}{\delta\sigma_2},\ \theta_{A,\ 3}=\frac{1}{\delta\sigma_1},\ \theta_{A,\ 4}\in\mathbb{R},\ \delta\neq 0,\ \sigma_1\neq 0,\ \sigma_2\neq 0\}$，则相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\right.$$
$$\left.+\theta_{A,4}\cos(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$
$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{B,2}\cos(\omega t)\cos(\omega s) + \frac{2\sigma_2\theta_{B,2}}{\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

对于几乎所有的 t。它们满足 $AB = \delta BA^2$。算子 $AB - BA$ 是在情形 2B 的第 8 项中计算得出的。

10. 如果 $\{\theta_{B,1} = \theta_{B,2} = \theta_{B,4} = 0, \theta_{B,3} \in \mathbb{R}\setminus\{0\}, \theta_{A,1} = \theta_{A,3} = \theta_{A,4} = 0, \theta_{A,2} \in \mathbb{R}\setminus\{0\}, \delta \neq 0, \sigma_1 \neq 0, \sigma_2 \neq 0\}$, 则相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{A,2}\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$
$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,3}\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$

对于几乎所有的 t，这些算子满足 $AB = \delta BA^2 = BA = 0$。11. 如果 $\{\theta_{B,1} = \theta_{B,2} = \theta_{B,4} = 0, \theta_{B,3} \in \mathbb{R}\setminus\{0\}, \theta_{A,1} = \theta_{A,4} = 0, \theta_{A,3} = \frac{1}{\delta\sigma_1}\theta_{A,2} \in \mathbb{R},$
$\delta \neq 0\ \sigma_1 \neq 0\ \sigma_2 \neq 0$,}, 则相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\theta_{A,2}\cos(\omega t)\cos(\omega s) + \frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$
$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,3}\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$

对于几乎所有的 t，这些算子满足 $AB = \delta BA^2 = BA$。

12. 如果 $\{\theta_{B,1} = \theta_{B,3} = \theta_{B,4} = 0,\ \theta_{B,2} \in \mathbb{R}\setminus\{0\},\ \theta_{A,1} = \theta_{A,2} = \theta_{A,4} = 0,\ \theta_{A,3} \in \mathbb{R}\setminus\{0\},\ \delta \neq 0,\ \sigma_1 \neq 0,\ \sigma_2 \neq 0\}$，则相应的算子为

$$(Ax)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{A,3}\sin(\omega t)\sin(\omega s)x(s)d\mu_s,$$
$$(Bx)(t) = \int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,2}\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB = \delta BA^2 = BA = 0$。

13. 如果 $\{\theta_{B,1}=\theta_{B,3}=\theta_{B,4}=0,\theta_{B,2}\in\mathbb{R}\setminus\{0\},\theta_{A,1}=\theta_{A,2}=\theta_{A,3}=0,\theta_{A,4}\in\mathbb{R},\delta\neq 0,\sigma_1\neq 0,\sigma_2\neq 0\}$, 则相应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{A,4}\cos(\omega t)\sin(\omega s)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,2}\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对于几乎处处的 t，它们满足 $AB=\delta BA^2=0$。此外，对于所有 $x\in L_p(\mathbb{R},\mu)$，$1<p<\infty$ 我们有，对于几乎处处的 t,

$$(AB-BA)x(t)=-(BA)x(t)=-\theta_{A,4}\sigma_2\theta_{B,2}\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\cos(\omega t)\sin(\omega s)x(s)d\mu_s.$$

应用命题 2，由于 $\alpha_1<\beta_1$、$[\alpha,\beta]\supseteq[\alpha_1,\beta_1]$ 和 $\sigma_2\neq 0$，算子 A和 B 交换，当且仅当以下两种情况之一成立：$\theta_{A,4}=0$ $(A=0)$ 或 $\theta_{B,2}=0$ $(B=0)$。

14. 如果 $\{\theta_{B,1}=\theta_{B,3}=\theta_{B,4}=0,\ \theta_{B,2}\in\mathbb{R}\setminus\{0\},\ \theta_{A,1}=\theta_{A,4}=0,\ \theta_{A,2}=\frac{1}{\delta\sigma_1},\ \theta_{A,3}\in\mathbb{R},\ \delta\neq 0,\ \sigma_1\neq 0,\ \sigma_2\neq 0\}$，则相应的算子为

$$(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\left(\frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s)+\theta_{A,3}\sin(\omega t)\sin(\omega s)\right)x(s)d\mu_s,$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,2}\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对几乎所有的 t，这些算子满足 $AB=\delta BA^2=BA$。15. 若 $\{\theta_{B,1}=\theta_{B,3}=\theta_{B,4}=0,\theta_{B,2}\in\mathbb{R}\setminus\{0\},\theta_{A,1}=0,\theta_{A,2}=\frac{1}{\delta\sigma_2},\theta_{A,3}=-\frac{1}{\delta\sigma_1},\delta\neq 0,\theta_{A,4}\in\mathbb{R},\sigma_1\neq 0,\sigma_2\neq 0\}$，则相应的算子为

$$\begin{aligned}(Ax)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\Big(&\frac{1}{\delta\sigma_2}\cos(\omega t)\cos(\omega s)-\frac{1}{\delta\sigma_1}\sin(\omega t)\sin(\omega s)\\&+\theta_{A,4}\cos(\omega t)\sin(\omega s)\Big)x(s)d\mu_s,\end{aligned}$$

$$(Bx)(t)=\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\theta_{B,2}\cos(\omega t)\cos(\omega s)x(s)d\mu_s,$$

对于几乎所有的 t。这些算子满足 $AB=\delta BA^2$。此外，对于所有 $x\in L_p(\mathbb{R},\mu)$，$1<p<\infty$ 有

$$(AB-BA)x(t)=\theta_{A,4}\sigma_2\theta_{B,2}\int_{\alpha_1}^{\beta_1} I_{[\alpha,\beta]}(t)\cos(\omega t)\sin(\omega s)x(s)d\mu_s,$$

对于几乎所有的 t。应用命题 2，由于 $\alpha_1<\beta_1$、$[\alpha,\beta]\supseteq[\alpha_1,\beta_1]$ 和 $\sigma_2\neq 0$，算子 A 和 B 交换，当且仅当 $\theta_{A,\ 4}=0$ 或 $\theta_{B,\ 2}=0$ $(B=0)$。

例 2。设 $(\mathbb{R},\ \Sigma,\ \mu)$ 为标准勒贝格测度空间。设算子

$$A: L_p(\mathbb{R},\mu)\to L_p(\mathbb{R},\mu),\quad B: L_p(\mathbb{R},\mu)\to L_p(\mathbb{R},\mu),\ 1<p<\infty$$

定义如下

$$\begin{aligned}(Ax)(t)=\int_{\alpha_1}^{\beta_1}\Bigg(&\sum_{i=1}^{l_{A,1}}\sum_{j=1}^{l_{A,2}}\theta_{i,j}^{A}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i}{\lambda_t}t\right)\cos\left(\frac{2\pi k_j}{\lambda_s}s\right)\\&+\sum_{i=1}^{l_{A,1}}\sum_{j=1}^{l_{A,2}}\pi_{i,j}^{A}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i}{\lambda_t}t\right)\cos\left(\frac{2\pi k_j}{\lambda_s}s\right)\\&+\sum_{i=1}^{l_{A,1}}\sum_{j=1}^{l_{A,2}}\nu_{i,j}^{A}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i}{\lambda_t}t\right)\sin\left(\frac{2\pi k_j}{\lambda_s}s\right)\\&+\sum_{i=1}^{l_{A,1}}\sum_{j=1}^{l_{A,2}}\varsigma_{i,j}^{A}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i}{\lambda_t}t\right)\sin\left(\frac{2\pi k_j}{\lambda_s}s\right)\Bigg)x(s)d\mu_s,\\(Bx)(t)=\int_{\alpha_1}^{\beta_1}\Bigg(&\sum_{i=1}^{l_{A,1}}\sum_{j=1}^{l_{A,2}}\theta_{i,j}^{B}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i}{\lambda_t}t\right)\cos\left(\frac{2\pi k_j}{\lambda_s}s\right)\\&+\sum_{i=1}^{l_{A,1}}\sum_{j=1}^{l_{A,2}}\pi_{i,j}^{B}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i}{\lambda_t}t\right)\cos\left(\frac{2\pi k_j}{\lambda_s}s\right)\\&+\sum_{i=1}^{l_{A,1}}\sum_{j=1}^{l_{A,2}}\nu_{i,j}^{B}I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i}{\lambda_t}t\right)\sin\left(\frac{2\pi k_j}{\lambda_s}s\right)\\&+\sum_{i=1}^{l_{A,1}}\sum_{j=1}^{l_{A,2}}\varsigma_{i,j}^{B}I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i}{\lambda_t}t\right)\sin\left(\frac{2\pi k_j}{\lambda_s}s\right)\Bigg)x(s)d\mu_s,\end{aligned}$$

对几乎所有的 t，其中 μ 中的下标表示积分变量，$\alpha,\alpha_1,\beta,\beta_1$ 是实常数，满足 $\alpha_1<\beta_1,\ \alpha\le\alpha_1,\ \beta\ge\beta_1$，而 $I_E(t)$ 是集合 E，$\lambda_t,\lambda_s\in\mathbb{R}\setminus\{0\}$ 的示性函数；$l_A,1$，$l_A,2$ 是正整数，$m_i,k_j,\theta_{i}^{A},_j,\theta_{i}^{B},_j,\pi_{i}^{A},_j,\pi_{i}^{B},_j,\nu_{i}^{A},_j,\nu_{i}^{B},_j,\varsigma_{i}^{A},_j,\varsigma_{i}^{B},_j\in\mathbb{R}$，$1\le i\le l_A,1$，$1\le j\le l_A,2$。令

$$a_{i,l}(t)=b_{i,l}(t)=\begin{cases} I_{[\alpha,\beta]}(t)\sin\left(\frac{2\pi m_i t}{\lambda_t}\right), & l=1 \text{ or } l=3 \\ I_{[\alpha,\beta]}(t)\cos\left(\frac{2\pi m_i t}{\lambda_t}\right), & l=2 \text{ or } l=4 \end{cases}$$

$$c_{i,l}(s)=e_{i,l}(s)=\begin{cases} \cos\left(\frac{2\pi k_i s}{\lambda_s}\right), & l=1 \text{ or } l=2 \\ \sin\left(\frac{2\pi k_i s}{\lambda_s}\right), & l=3 \text{ or } l=4 \end{cases}$$

$a_{i,\ l},\ \ b_{i,\ l}\in L_p(\mathbb{R},\ \mu),\ \ 1<p<\infty, 1\le i\le l_{A,\ 1}, 1\le l\le 4, c_{i,\ l},\ \ e_{i,\ l}\in L_q[\alpha_1,\beta_1]$，其中 $1<q<\infty$ 满足 $\frac{1}{p}+\frac{1}{q}=1, 1\le i\le l_{A,\ 1}, 1\le l\le 4$。

与例 1 中类似的论证以及 $L_p(\mathbb{R},\ \mu)$ 是一个向量空间这一事实表明，这些算子是良定义的。根据定理 1 $AB=\delta BA^2$，对于某些$\delta\in\mathbb{R}$、$G_A=G_B=[\alpha_1,\beta_1]$，当且仅当几乎处处对 $(t,\ s)$ 有 $\in\mathbb{R}\times[\alpha_1,\beta_1]$

$$\begin{aligned} &\sum_{(i,j,l=1)}^{(l_{A,1},l_{A,2},l_{A,3})}\ \sum_{(u,w,r=1)}^{(l_{A,1},l_{A,2},l_{A,3})}\gamma_A^{i,j,l}\gamma_B^{u,w,r}a_{i,l}(t)Q_{G_A}(c_{j,l};b_{u,r})e_{w,r}(s) \\ =&\sum_{(u,w,r=1)}^{(l_{A,1},l_{A,2},l_{A,3})}\ \sum_{(i_1,j_1,l_1=1)}^{(l_{A,1},l_{A,2},l_{A,3})}\ \sum_{(i_2,j_2,l_2=1)}^{(l_{A,1},l_{A,2},l_{A,3})}\delta Q_{G_A}(a_{i_2,l_2};c_{j_1,l_1})\gamma_B^{u,w,r}\gamma_A^{i_1,j_1,l_1}\gamma_A^{i_2,j_2,l_2} \\ &\cdot Q_{G_A}(e_{w,r};a_{i_1,l_1})b_{u,r}(t)c_{i_2,l_2}(s), \end{aligned}$$

其中 $l_{A,\ 3}=4, Q_\Lambda(\cdot,\ \cdot)$ 在 (2) 中定义，

$$\gamma_A^{i,j,l}=\begin{cases} \theta_{i,j}^A, & \text{if } l=1 \\ \pi_{i,j}^A, & \text{if } l=2 \\ \nu_{i,j}^A, & \text{if } l=3 \\ \varsigma_{i,j}^A, & \text{if } l=4 \end{cases} \text{ and } \gamma_B^{i,j,l}=\begin{cases} \theta_{i,j}^B, & \text{if } l=1 \\ \pi_{i,j}^B, & \text{if } l=2 \\ \nu_{i,j}^B, & \text{if } l=3 \\ \varsigma_{i,j}^B, & \text{if } l=4. \end{cases}$$

例 *3.* 设 $(\mathbb{R},\ \Sigma,\ \mu)$ 是标准勒贝格测度空间。设

$$A: L_p(\mathbb{R},\mu)\to L_p(\mathbb{R},\mu),\quad B: L_p(\mathbb{R},\mu)\to L_p(\mathbb{R},\mu),\ 1<p<\infty$$

定义如下

$$(Ax)(t)=\int_0^1\sum_{m=1}^2 a_m(t)c_m(s)x(s)d\mu_s,\quad (Bx)(t)=\int_0^1\sum_{k=1}^2 b_k(t)e_k(s)x(s)d\mu_s,$$

对几乎所有 t，其中 μ 中的指标表示积分变量，$a_m,\ \ b_k\in L_p(\mathbb{R},\ \mu),\ e_m,\ c_k\in L_q([0,1],\ \mu),\ k,\ \ m=1,\ \ 2,$ 使得

$$\begin{gathered} Q_{[0,1]}(e_i,a_j)=0,\ i\ne j,\ Q_{[0,1]}(c_i,a_j)=0,\ i\ne j, Q_{[0,1]}(b_i,c_j)=0,\ i\ne j, \\ a_m(t)=\alpha_m b_m(t),\ c_k(s)=\pi_k e_k(s),\ k,m=1,2. \end{gathered}$$

其中 $\delta\in\mathbb{R}\setminus\{0\},\ \ \alpha_m,\pi_m\in\mathbb{R}, m=1,2$。如果 $\alpha_m=0$或 $\pi_m=0$或 $Q_{[0,1]}(b_m,e_m)=0$ 或 $\delta\alpha_m\pi_m Q_{[0,1]}(b_m,e_m)=1, m=1,2,$，那么算子 A和 B满足 $AB=\delta BA^2$。

事实上，通过应用推论 3，并利用 $Q_\Lambda(\cdot,\cdot)$ 的双线性，

$$\begin{aligned}
&\sum_{k=1}^{2}\sum_{m=1}^{2} a_m(t)Q_{[0,1]}(b_k,c_m)e_k(s)\\
&\quad=\sum_{m=1}^{2}[a_m(t)Q_{[0,1]}(b_1,c_m)e_1(s)+a_m(t)Q_{[0,1]}(b_2,c_m)e_2(s)]\\
&\quad=Q_{[0,1]}(b_1,c_1)a_1(t)e_1(s)+Q_{[0,1]}(b_2,c_2)a_2(t)e_2(s)\\
&\quad=Q_{[0,1]}(b_1,\pi_1e_1)a_1(t)e_1(s)+Q_{[0,1]}(b_2,\pi_2e_2)a_2(t)e_2(s)\\
&\quad=\pi_1Q_{[0,1]}(b_1,e_1)a_1(t)e_1(s)+\pi_2Q_{[0,1]}(b_2,e_2)a_2(t)e_2(s)\\
&\quad=\pi_1\alpha_1Q_{[0,1]}(b_1,e_1)b_1(t)e_1(s)+\pi_2\alpha_2Q_{[0,1]}(b_2,e_2)b_2(t)e_2(s)
\end{aligned}$$

对几乎所有的（t，s）$\in\mathbb{R}\times[0,1]$。另一方面，

$$\begin{aligned}
&\sum_{k=1}^{2}\sum_{i_1,i_2=1}^{2}\delta b_k(t)Q_{[0,1]}(e_1,a_{i_1})\gamma_{i_1,i_2}c_{i_2}(s)\\
&=\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\delta[b_1(t)Q_{[0,1]}(e_1,a_{i_1})\gamma_{i_1,i_2}c_{i_2}(s)+b_2(t)Q_{[0,1]}(e_2,a_{i_1})\gamma_{i_1,i_2}c_{i_2}(s)]\\
&=\sum_{i_2=1}^{2}\delta[b_1(t)Q_{[0,1]}(e_1,a_1)\gamma_{1,i_2}c_{i_2}(s)+b_2(t)Q_{[0,1]}(e_2,a_1)\gamma_{1,i_2}c_{i_2}(s)]\\
&\quad+\sum_{i_2=1}^{2}\delta[b_1(t)Q_{[0,1]}(e_1,a_2)\gamma_{2,i_2}c_{i_2}(s)+b_2(t)Q_{[0,1]}(e_2,a_2)\gamma_{2,i_2}c_{i_2}(s)]\\
&=\delta b_1(t)Q_{[0,1]}(e_1,a_1)Q_{[0,1]}(a_1,c_1)c_1(s)\\
&\qquad\qquad+\delta b_2(t)Q_{[0,1]}(e_2,a_2)Q_{[0,1]}(a_2,c_2)c_2(s)\\
&=\delta\alpha_1^2\pi_1^2b_1(t)Q_{[0,1]}(e_1,b_1)^2e_1(s)+\delta\alpha_2^2\pi_2^2b_2(t)Q_{[0,1]}(e_2,b_2)^2e_2(s)
\end{aligned}$$

对几乎所有的（t，s）$\in\mathbb{R}\times[0,1]$。若 $\alpha_m=0$ 或 $\pi_m=0$ 或 $Q_{[0,1]}$（b_m，e_m）$=0$ 或 $\delta\alpha_m\pi_mQ_{[0,1]}$（$b_m$，$e_m$）$=1$，$m=1$，2，则有

$$\begin{aligned}
&\pi_1\alpha_1Q_{[0,1]}(b_1,e_1)b_1(t)e_1(s)+\pi_2\alpha_2Q_{[0,1]}(b_2,e_2)b_2(t)e_2(s)\\
&=\delta\alpha_1^2\pi_1^2b_1(t)Q_{[0,1]}(e_1,b_1)^2e_1(s)+\delta\alpha_2^2\pi_2^2b_2(t)Q_{[0,1]}(e_2,b_2)^2e_2(s).
\end{aligned}\tag{19}$$

事实上，若 $\alpha_1=0$ 或 $\pi_1=0$ 或 $Q_{[0,1]}$（e_1，b_1）$=0$，则方程（19）变为

$$\pi_2\alpha_2Q_{[0,1]}(b_2,e_2)b_2(t)e_2(s)=\delta\alpha_2^2\pi_2^2b_2(t)Q_{[0,1]}(e_2,b_2)^2e_2(s).$$

这等价于 $\alpha_2=0$ 或 $\pi_2=0$ 或 $Q_{[0,1]}$（e_2，b_2）$=0$ 或 $\delta\alpha_2\pi_2Q_{[0,1]}(b_2,e_2)=1$。类似地，如果 $\alpha_2=0$ 或 $\pi_2=0$ 或 $Q_{[0,1]}(e_2,b_2)=0$，那么方程 (19) 等价于 $\alpha_1=0$ 或 $\pi_1=0$ 或 $Q_{[0,1]}(e_1,b_1)=0$ 或 $\delta\alpha_1\pi_1Q_{[0,1]}(b_1,e_1)=1$。如果 $\delta\alpha_m\pi_mQ_{[0,1]}(b_m,e_m)=1$ 对于 $m=1,2$ 成立，那么通过直接计算可知方程 (19) 成立。因此，应用推论 3 可得 $AB=\delta BA^2$。特别地，考虑以下算子

$A\colon L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu)$, $B\colon L_p(\mathbb{R},\mu) \to L_p(\mathbb{R},\mu)$, $1 < p < \infty$定义如下

$$(Ax)(t) = \int_0^1 \sum_{m=1}^{2} a_m(t)c_m(s)x(s)d\mu_s\,, \quad (Bx)(t) = \int_0^1 \sum_{k=1}^{2} b_k(t)e_k(s)x(s)d\mu_s,$$

对于几乎所有的 t，其中 μ 中的指标表示积分变量，

$$\sum_{m=1}^{2} a_m(t)c_m(s) = I_{[\alpha,\beta]}(t)(-6t(4s-3)+12t^2(3s-2)),$$

$$\sum_{k=1}^{2} b_k(t)e_k(s) = I_{[\alpha,\beta]}(t)(6t(4s-3)-12t^2(3s-2)),$$

对于几乎所有的 $(t,\ s) \in \mathbb{R}\times[0,1]$，$\alpha$，$\beta$ 是满足 $\alpha \le 0$，$\beta \ge 1$ 的实常数，并且 $I_E(t)$ 是集合 E 的示性函数。因此我们有

$$\begin{aligned} &a_1(t) = I_{[\alpha,\beta]}(t)(-6t)\,, && a_2(t) = I_{[\alpha,\beta]}(t)12t^2\\ &c_1(s) = 4s-3\,, && c_2(s) = 3s-2\\ &b_1(t) = I_{[\alpha,\beta]}(t)6t\,, && b_2(t) = I_{[\alpha,\beta]}(t)(-12)t^2\\ &e_1(s) = 4s-3\,, && e_2(s) = 3s-2. \end{aligned}$$

这些算子定义良好，并满足 $AB = BA^2$。事实上，当 $\delta = 1$、$n = 2$、$G_A = G_B = [0,1]$ 时，应用推论 3 可得

$$\begin{aligned} &Q_{[0,1]}(e_1,a_1) = \int_0^1 -6s(4s-3)d\mu = 1\,, && Q_{[0,1]}(e_1,a_2) = \int_0^1 (4s-3)12s^2d\mu = 0,\\ &Q_{[0,1]}(e_2,a_1) = \int_0^1 -6s(3s-2)d\mu = 0\,, && Q_{[0,1]}(e_2,a_2) = \int_0^1 12s^2(3s-2)d\mu = 1,\\ &\gamma_{1,1} = Q_{[0,1]}(a_1,c_1) = \int_0^1 -6s(4s-3)d\mu = 1\,, && \gamma_{2,1} = Q_{[0,1]}(a_1,c_2) = \int_0^1 -6s(3s-2)d\mu = 0,\\ &\gamma_{1,2} = Q_{[0,1]}(a_2,c_1) = \int_0^1 12s^2(4s-3)d\mu = 0\,, && \gamma_{2,2} = Q_{[0,1]}(a_2,c_2) = \int_0^1 -12s^2(3s-2)d\mu = 1. \end{aligned}$$

因此，我们有

$$\begin{aligned} &\sum_{k=1}^{2}\sum_{m=1}^{2} a_m(t)Q_{G_A}(b_k,c_m)e_k(s)\\ &= \sum_{m=1}^{2}[a_m(t)Q_{G_A}(b_1,c_m)e_1(s) + a_m(t)Q_{G_A}(b_2,c_m)e_2(s)]\\ &= a_1(t)e_1(s) + a_2(t)e_2(s) = I_{[\alpha,\beta]}(t)(6t(4s-3)-12t^2(3s-2)) \end{aligned}$$

几乎处处（t,s）$\in \mathbb{R}\times[0,1]$。另一方面，

$$\begin{aligned}&\sum_{k=1}^{2}\sum_{i_1,i_2=1}^{2} b_k(t)Q_{G_B}(e_1,a_{i_1})\gamma_{i_1,i_2}c_{i_2}(s)\\&\quad= b_1(t)Q_{G_B}(e_1,a_1)\gamma_{1,1}c_1(s)+b_2(t)Q_{G_B}(e_2,a_2)\gamma_{2,2}c_2(s)\\&\quad= I_{[\alpha,\beta]}(t)(6t(4s-3)-12t^2(3s-2))\end{aligned}$$

对于几乎所有的 $(t,s)\in\mathbb{R}\times[0,1]$。因此，推论 3 的条件得到满足，所以 $AB=BA^2$。此外，$AB=BA^2=-A$ 和 $A^2=A$，也就是说，A 和 B 交换。

例*4*（*Laurent* 多项式）。设 $(\mathbb{R},\Sigma,\mu)$ 为标准勒贝格测度空间。设 $A\colon L_p(\mathbb{R},\mu)\to L_p(\mathbb{R},\mu)$，$B\colon L_p(\mathbb{R},\mu)\to L_p(\mathbb{R},\mu)$，$1<p<\infty$ 定义如下

$$(Ax)(t)=\int_1^2\sum_{m=1}^{4}a_m(t)c_m(s)x(s)d\mu_s,\quad (Bx)(t)=\int_1^2\sum_{k=1}^{2}b_k(t)e_k(s)x(s)d\mu_s,$$

对几乎处处的 t，其中 μ 中的下标表示积分变量，

$$\begin{aligned}\sum_{m=1}^{4}a_m(t)c_m(s)&=I_{[\alpha,\beta]}(t)(\gamma_{A,0}+\gamma_{A,1}\tfrac{1}{s}+\gamma_{A,2}\tfrac{1}{t}+\gamma_{A,3}\tfrac{1}{ts}),\\\sum_{k=1}^{2}b_k(t)e_k(s)&=I_{[\alpha,\beta]}(t)(\gamma_{B,1}\tfrac{1}{s}+\gamma_{B,2}\tfrac{1}{t}),\end{aligned}$$

对几乎处处的 (t,s)，$\in\mathbb{R}\times[1,2]$，α,β 是实常数，使得 $0<\alpha\le 1,\beta\ge 2$,$I_E(t)$ 是集合 $E,\gamma_{A,m},\gamma_{B,k}\in\mathbb{R}$, $m=1,2,3,4$, $k=1,2$ 的示性函数。因此我们有

$$\begin{aligned}&a_1(t)=I_{[\alpha,\beta]}(t)\gamma_{A,0},&&a_2(t)=I_{[\alpha,\beta]}(t)\gamma_{A,1}\\&a_3(t)=I_{[\alpha,\beta]}(t)\gamma_{A,2}\tfrac{1}{t},&&a_4(t)=I_{[\alpha,\beta]}(t)\gamma_{A,3}\tfrac{1}{t}\\&c_1(s)=c_3(s)=1&&c_2(s)=c_4(s)=\tfrac{1}{s}\\&b_1(t)=I_{[\alpha,\beta]}(t)\gamma_{B,1},&&b_2(t)=I_{[\alpha,\beta]}(t)\gamma_{B,2}\tfrac{1}{t}\\&e_1(s)=\tfrac{1}{s},&&e_2(s)=1.\end{aligned}$$

这些算子定义良好；事实上，对于任意固定的 p, $1<p<\infty$，函数$a_i,b_j\in L_p(\mathbb{R},\mu)$、$c_i,e_j\in L_q([1,2],\mu)$、$i=1,2,3,4$, $j=1,2$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$。事实上，

$$\int_{\mathbb{R}}|a_i(s)|^p d\mu=\int_{\alpha}^{\beta}|a_i(s)|^p d\mu<\infty,\ i=1,2,3,4.$$

Similarly, $\int_{\mathbb{R}}|b_j(s)|^p d\mu=\int_{\alpha}^{\beta}|\tilde{b}_j(s)|^p d\mu<\infty,\ j=1,2$. Analogously, we have

$$\int_1^2|c_i(s)|^q d\mu<\infty,\ i=1,2,3,\ \text{and}\ \int_1^2|e_j(s)|^q d\mu<\infty,\ j=1,2.$$

因此我们考虑单项式 $F(z) = \delta z^2, \delta \in \mathbb{R}$, $G_A = G_B = [1,2]$，根据推论 3，我们有 $AB = \delta BA^2$ 当且仅当

$$\sum_{k=1}^{2}\sum_{m=1}^{4} a_m(t)Q_{G_A}(b_k,c_m)e_k(s) = \sum_{k=1}^{2}\sum_{i_1,i_2=1}^{4} \delta b_k(t)Q_{G_B}(e_k,a_{i_1})c_{i_2}(s)\gamma_{i_1,i_2}.$$

因此，我们得到以下方程组:

$$\begin{cases}
\gamma_{A,0}\gamma_{B,2}\ln 2+\gamma_{A,1}\gamma_{B,2}\frac{1}{2} & =\delta\gamma_{B,1}\gamma_{A,0}^2\ln 2+\delta\gamma_{B,1}\gamma_{A,0}\gamma_{A,2}(\ln 2)^2 \\
& +\delta\gamma_{B,1}\gamma_{A,1}\gamma_{A,0}(\ln 2)^2+\delta\gamma_{B,1}\gamma_{A,1}\gamma_{A,2}(\ln 2)^2 \\
& +\delta\gamma_{B,1}\gamma_{A,2}\gamma_{A,0}\frac{1}{2}+\delta\gamma_{B,1}\gamma_{A,2}^2\frac{\ln 2}{2} \\
& +\delta\gamma_{B,1}\gamma_{A,3}\gamma_{A,0}\frac{\ln 2}{2}+\delta\gamma_{B,1}\gamma_{A,3}\gamma_{A,2}\frac{1}{4} \\
\gamma_{A,0}+\gamma_{A,1}\gamma_{B,1}\ln 2 & =\delta\gamma_{B,1}\gamma_{A,0}\gamma_{A,1}\ln 2+\delta\gamma_{B,1}\gamma_{A,0}\gamma_{A,2}^2(\ln 2)^2 \\
& +\delta\gamma_{B,1}\gamma_{A,1}\gamma_{A,3}\frac{\ln 2}{2}+\delta\gamma_{B,1}\gamma_{A,1}\gamma_{A,3}\frac{\ln 2}{2}+\delta\gamma_{B,1}\gamma_{A,2}\gamma_{A,1}\frac{1}{2} \\
& +\delta\gamma_{B,1}\gamma_{A,2}\gamma_{A,3}\frac{\ln 2}{2}+\delta\gamma_{B,1}\gamma_{A,3}\gamma_{A,1}\frac{\ln 2}{2}+\delta\gamma_{B,1}\gamma_{A,3}^2\frac{1}{4} \\
\gamma_{A,2}\gamma_{B,2}\ln 2+\gamma_{A,3}\gamma_{A,2}\frac{1}{2} & =\delta\gamma_{B,2}\gamma_{A,0}^2+\delta\gamma_{B,2}\gamma_{A,0}\gamma_{A,2}\ln 2+\delta\gamma_{B,2}\gamma_{A,1}\gamma_{A,0}\ln 2 \\
& +\delta\gamma_{B,2}\gamma_{A,1}\gamma_{A,2}\ln 2+\delta\gamma_{B,2}\gamma_{A,2}^2(\ln 2)^2 \\
& +\delta\gamma_{B,2}\gamma_{A,3}\gamma_{A,0}(\ln 2)^2+\delta\gamma_{B,2}\gamma_{A,3}\gamma_{A,2}\frac{\ln 2}{2} \\
\gamma_{A,2}\gamma_{B,1}\ln 2+\gamma_{A,3}\gamma_{B,1}\ln 2 & =\delta\gamma_{B,1}\gamma_{A,0}\gamma_{A,1}\ln 2+\delta\gamma_{B,2}\gamma_{A,0}\gamma_{A,3}\ln 2 \\
& +\delta\gamma_{B,2}\gamma_{A,2}\gamma_{A,1}\ln 2+\delta\gamma_{B,2}\gamma_{A,2}\gamma_{A,3}(\ln 2)^2 \\
& +\delta\gamma_{B,2}\gamma_{A,3}\gamma_{A,1}(\ln 2)^2+\delta\gamma_{B,2}\gamma_{A,3}^2\frac{\ln 2}{2}
\end{cases}$$

该方程组有许多解，例如 $\gamma_{A,1} = \gamma_{B,1} = 0, \gamma_{A,3} = -2\ln 2\gamma_{A,}2$, $\gamma_{A,}2$, $\gamma_{B,2} \in \mathbb{R}$。这对应于以下算子

$$(Ax)(t) = \int_1^2 I_{[\alpha,\beta]}(t)\left(\frac{\gamma_{A,2}}{t} - \frac{2\ln 2\gamma_{A,2}}{ts}\right)x(s)d\mu_s,$$

$$(Bx)(t) = \int_1^2 I_{[\alpha,\beta]}(t)\frac{\gamma_{B,2}}{t}x(s)d\mu_s,$$

对于几乎所有的 t，这些算子都满足 $AB = \delta BA^2 = 0$。此外，对于所有 $x \in L_p(\mathbb{R},\mu)$, $1 < p < \infty$，我们有

$$(AB-BA)x(t) = -(BAx)(t) = -\gamma_{A,2}\gamma_{B,2}\ln 2\int_1^2 I_{[\alpha,\beta]}(t)\frac{1}{t}\left(1-\frac{2\ln 2}{s}\right)x(s)d\mu_s,$$

几乎处处的 t。由于 $\alpha < \beta$，应用引理1可得， A 与 B 交换，当且仅当 $\gamma_{A,2} = 0$ ($A = 0$) 或 $\gamma_{B,2} = 0$ ($B = 0$)。

致谢

本研究得到瑞典国际发展合作署（Sida）支持，资助来源为其在莫桑比克开展的数学研究与高等教育能力建设双边计划。Domingos Djinja 谨向数学与应用数学研究环境 MAM、数学与物理部、教育、文化与传播学院及梅拉达伦大学致谢，感谢其为数学研究提供了优良的环境。

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