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随机矩阵乘积的李雅普诺夫指数的连续性

ARTUR AVILA、亚历克斯·埃斯金和 MARCELO VIANA

Abstract. We prove that the Lyapunov exponents of random products in a (real or complex) matrix group depends continuously on the matrix coefficients and probability weights. More generally, the Lyapunov exponents of the random product defined by any compactly supported probability distribution on $\mathrm{GL}(d)$ vary continuously with the distribution, in a natural topology corresponding to weak*-closeness of the distributions and Hausdorff-closeness of their supports.

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第一部分：李雅普诺夫指数与随机游走

1. Introduction

Lyapunov exponents. The notion of Lyapunov exponents is rooted in the stability theory of differential equations created by Lyapunov [52] at the end of the 19th century. Consider a differential equation

$$x' = L(t)x + R(t, x), \tag{1.1}$$

where $L(t) : \mathbb{R}^d \to \mathbb{R}^d$ is linear and $R(t, \cdot)$ is a perturbation of order greater than 1. The *Lyapunov exponent function* $v \mapsto \lambda(v)$ is defined by

$$\lambda(v) = \limsup_{t\to\infty} \frac{1}{t} \log \|x_v(t)\| \tag{1.2}$$

where x_v is the solution of the linear equation $x' = L(t)x$ with initial condition v. When $\lambda < 0$ the constant solution $x_0(t) \equiv 0$ is exponentially stable for this linear equation. The stability theorem of Lyapunov asserts that, under a technical *Lyapunov regularity* condition, it remains exponentially stable for (1.1).

In 1960, Furstenberg, Kesten [41] proved that the limit in (1.2) exists for almost every x, relative to any probability measure invariant under the flow. A few years later, Oseledets [56] showed that Lyapunov regularity also holds for almost every point. Such results brought the subject of Lyapunov exponents to the realm of ergodic theory, where it has prospered since. Three main problems have a central role in the theory.

The first one is *non-triviality* of the Lyapunov spectrum: *when is it the case that not all Lyapunov exponents are equal?* This was founded by Furstenberg [40] in the 1960s and has been much studied since, especially in the last couple of decades or so. See Viana [65] and references therein. A product of this theory much exploited recently is the Invariance Principle [51, 20, 7, 5], a general statement to the effect that systems with trivial Lyapunov spectra are very rigid.

A related issue is that of *simplicity* of the Lyapunov spectrum: *when are all the Lyapunov exponents distinct, with multiplicity* 1*?* This was initiated by Guivarc'h and Raugi [46] and by Gol'dsheid and Margulis [45], and has also been the object of considerable interest in recent years. See [65] for references and a detailed discussion. An application was the proof of the Zorich–Kontsevich conjecture on the Lyapunov spectrum of the Teichmüller flow on the moduli space of Abelian differentials [6].

Continuity theorem. In the present paper we are mostly concerned with the *dependence* problem: *how do the Lyapunov exponents depend on their underlying system?* Several references to the literature on this problem will be given in a while. Right now, the following special case of our main result illustrates the kind of goals we pursue here.

Let $(A_1, \ldots, A_m)$ be an m-uple of matrices in $G = GL(\mathbb{R}^d)$ and $(p_1, \ldots, p_m)$ be an element of the open simplex Δ_m of dimension $m - 1$, that is, an m-uple of numbers $p_i \in (0, 1)$ such that $\sum_{i=1}^m p_i = 1$. Let ν be the probability measure on G given by

$$\nu = \sum_{i=1}^{m} p_i \delta_{A_i}$$

其中， δ_A 表示位于任意 $A \in G$处的狄拉克质量。设 $\lambda_1 \geqslant \cdots \geqslant \lambda_d$ 为由 ν 诱导的随机矩阵乘积的李雅普诺夫指数（相关定义将在第 2 节中给出）。我们证明:

Theorem A. *For each $1 \leqslant j \leqslant n$, the number λ_j depends continuously on the A_i and the p_i at every point of the domain $G^m \times \Delta_m$.*

The 2-dimensional case of Theorem A has been proved by Bocker and Viana [18]. A different proof of that case that introduces a few of the ideas in this paper appeared in Chapter 10 of the book [65]. Kifer [50] observed the Lyapunov exponents

may actually jump when some weight p_i goes to zero, and that is why Δ_m is taken to be an *open* simplex.

A crucial point in Theorem A is that the conclusion holds even at *reducible* points, that is, when the matrices $A_1, \ldots, A_n$ share one or more invariant proper subspaces. Indeed, continuity of the Lyapunov exponents at irreducible points had already been proved in the 1980s, independently, by Furstenberg and Kifer [42] and by Hennion [47]. As often happens in this field, the reducible case is a lot more subtle, requiring a whole different set of ideas.

我们的方法依赖于对随机游走的定量分析。

$$x \mapsto gx, \quad g \in G \text{ 一个分布为 } \nu \text{ 的随机变量,}$$

defined on the projective space $P = \mathbb{P}(\mathbb{R}^d)$ by the probability measure ν. By Furstenberg and Kifer [42], discontinuity of the Lyapunov exponents can only occur if there is some proper subspace invariant under all the matrices and containing all the "most contracting" directions for the cocycle (see Section 3 for the precise statement).

In a nutshell, we prove that if such an invariant subspace E (the "equator") does exist, typical trajectories of nearby generic random walks spend very little time in its vicinity, rendering the presence of the equator effectively harmless. A bit more precisely, we consider generic distributions ν_k converging to ν as $k \to \infty$, and we show that the stationary measures η_k for the corresponding random walks cannot accumulate on the equator: any limit point η_∞ as $k \to \infty$ (which is automatically a stationary measure for ν) must satisfy $\eta_\infty(E) = 0$. These notions and their basic properties will also be recalled in Section 3.

Margulis functions. The key technical tool to do this is the concept of Margulis function. Such functions have been introduced to the dynamics literature by Margulis in [32]. (In the probability setting, a Margulis function is also called a Foster-Lyapunov (or drift) function, and has been used extensively. See the book [55] for further references.)

In a few words, $\Phi : X \to [0, \infty]$ is a *(multiplicative) Margulis function* for a Markov operator $\mathcal{T}$ on some space X, relative to a set $Y \subset X$, if $\Phi \equiv \infty$ on Y and there exist constants $c < 1$ and $b < \infty$ satisfying

$$\mathcal{T}\Phi(x) \leqslant c\Phi(x) + b \text{ for all } x \in X. \tag{1.3}$$

The distinctive feature implied by this inequality is that $\mathcal{T}\Phi(x)$ is much smaller than $\Phi(x)$ near Y, even if it may be somewhat bigger on other parts of X. Such functions have been used, for example, in [34, 31, 33, 1, 2, 24, 35, 14, 30, 3, 49, 53, 12, 43, 44] and [65, Chapter 10]. For a fairly recent survey on this topic see [38].

Nevertheless, our application of Margulis functions in the present setting comes with a number of novelties. To begin with, we need a different kind of Margulis function, which we introduce here: given a partition (A, B) of the space X, Φ is an *(additive) Margulis function* if there exist positive constants κ_A and κ_B such that

$$\begin{aligned} \mathcal{T}\Psi(x) &\leqslant \Psi(x) - \kappa_A \text{ for every } x \in A \\ \mathcal{T}\Psi(x) &\leqslant \Psi(x) + \kappa_B \text{ for every } x \in B. \end{aligned} \tag{1.4}$$

If Φ is a multiplicative Margulis function then $\log \Phi$ is an additive Margulis function relative to a suitable partition (A, B) (see Remark 6.22). On the other hand, it is not true that if Ψ is an additive Margulis function then $\exp \Psi$ is a multiplicative one. That is because the inequality (1.3) is very sensitive to the

"worst case" behavior of Ψ, whereas (1.4) depend more on the "average case" behavior. For this reason, it is often much easier to construct an additive Margulis function than a multiplicative one. In fact, we do not know how to construct a useful multiplicative Margulis function in our setting beyond the case $d = 2$ (see [65, Chapter 10] and [53]).

Another point worth emphasizing is that in all the previous constructions (apart from [53, 65, 12]), the dynamical system is fixed. Instead, in the present paper we try to make the same Margulis function work for a whole family of dynamical systems, which have very different behaviors near the equator. This introduces quite a lot of new issues. Furthermore, the dynamical behavior near the equator is totally different from the behavior in other parts of phase-space. Thus, we need to carry out a localized analysis of the random walk, which is another important source of difficulties.

进一步的研究视角。矩阵的随机乘积可以表示为一种特殊类型的线性余圈。

$$F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x, v) = (f(x), A(x)v)$$

where the base dynamics $f : M \to M$ is a shift map $f((x_n)_n) = (x_{n+1})_n$ endowed with a Bernoulli measure, and the cocycle function A depends only on the coordinate x_0. The dependence problem extends naturally to this general setting of linear cocyles: usually one takes the base dynamics f and the corresponding invariant probability measure μ to be fixed, and one is interested in understanding how the Lyapunov exponents depend on A.

A natural step is to try and allow for much more general cocycle functions A. For reasons that we will soon discuss, it is convenient to assume some regularity, like Hölder continuity. Moreover, essentially all known results assume the cocycle to satisfy a kind of quasi-conformality condition called *fiber-bunching* (see [20, 7]) which also involves Hölder continuity.

Another natural way to broaden the scope of the theory is to weaken the assumptions on the base dynamics, to consider general dynamical systems more general than shift maps, as well as invariant measures satisfying much milder independence conditions. In this latter direction, Theorem A has been extended to Markov products of 2-dimensional matrices by Malheiro and Viana [53].

For Hölder cocycles, Backes, Brown and Butler [12] extended the 2-dimensional case of Theorem A to general fiber-bunched cocycles whose base transformation f is a hyperbolic homeomorphism on a compact metric space (in the sense of [64]) and whose invariant probability measure has *local product structure* (see [20, 64]), a mild requirement meaning, roughly speaking, that the future depends only weakly on the past. In fact, their statement extends to the class of linear cocycles with invariant holonomies. Both versions had been conjectured in [65, Section 10.6].

Another interesting path to possibly generate further progress is to consider linear cocycles over partially hyperbolic diffeomorphisms, volume-preserving or not. Groundwork in this direction has been laid in [5] and some continuity results have been derived in [7]. See also Avila, Viana and Wilkinson [8, 9], and Poletti and Viana [59].

Bochi [15]的以下结果凸显了余圈函数需要具备某种正则性：如果系统 (f, μ) 是非周期的， $d = 2$，且线性余圈

is continuous and not uniformly hyperbolic, then it may be C^0-approximated by linear cocycles with trivial Lyapunov spectra (the two Lyapunov exponents are equal). Thus, continuity can only hold at cocycles which either are uniformly hyperbolic or have trivial spectra.

In fact, the same is true restricted to the class of derivative cocycles of area-preserving surface diffemorphisms, a much harder fact which was discovered by Mañé [54] and whose proof was completed by Bochi [15]. These results have been extended to arbitrary dimensions by Bochi and Viana [17, 16]. They are generally not true for cocycles over non-invertible maps, even in the $\mathrm{SL}(\mathbb{R}^2)$ case, according to Viana and Yang [66].

On the other hand, the actual relevance of the fiber-bunching condition in this context is presently not entirely clear, indeed this remains one of the outstanding open questions in this area. Examples of discontinuity of the Lyapunov exponents for Hölder continuous linear cocycles which are not fiber-bunched have been found in [65, Section 9.3] and Butler [25].

We have restricted our attention to matrix groups, for good reason. While the basic concepts discussed here (such as Lyapunov exponents, Oseledets regularity, etc.) extend to the more general setting of all (not necessarily invertible) matrices, there is no hope to obtain any general regularity result for Lyapunov exponents in this more general setting.

To explain why, let us consider the Lyapunov exponents of the random product of two real matrices A_1 and A_2, with probability weights $p_1 = p_2 = 1/2$. The Lyapunov exponents are well defined, but the bottom one is equal to $-\infty$ if one of the matrices is not invertible. Moreover, if some finite matrix product involving A_1 and A_2 is zero then the top Lyapunov exponent is $-\infty$ as well.

例如，令 $L(\theta)$ 表示.....的最大李雅普诺夫指数。

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix},$$

viewed as a function of $\theta \in \mathbb{R}$. For $\theta = p/4q$ with p odd and q a non-zero integer, we have that $A_1 A_2^q A_1 = 0$, and so $L(\theta) = -\infty$. By an upper semi-continuity argument, it follows that $L(\theta) = -\infty$ for Baire-generic θ. In fact, it is not difficult to give an explicit generic quantitative condition ensuring that $L(\theta) = -\infty$, and even a sharp one, using the easily checked formula

$$L(\theta) = \sum_{k=0}^{\infty} 2^{-k-2} \log |\cos 2\pi k\theta|,$$

which also shows that the Lyapunov exponent $L(\theta)$ is finite (and discontinuous) at a full Lebesgue measure set of θ.

What happens in this sort of situation is that, while one can still analyze the Lyapunov exponents of cocycles such as this one using a stationary measure η on the projective space, just as we do in the present paper for the invertible case (but taking care of issues such as indeterminacy), in the present setting the measure η becomes atomic, being the sum of Dirac masses with weights 2^{-k-1} on the lines through $(\cos 2\pi k\theta, \sin 2\pi k\theta)$.

连续性的意义与应用。已知李雅普诺夫指数在某个 f 处连续，这一事实本身就能提供有关 f 动力学的信息，正如 Bochi、 Viana [17]，所指出的，以及关于 ..... 的抽象事实。

existence of many (in the Baire sense) continuity points can be leveraged to a fine understanding of the dynamics from the generic point of view.

For instance, continuity of Lyapunov exponents ensures that the Oseledets decomposition varies continuously in a suitable sense: see Backes, Poletti [13]. Moreover, the convergence in the Oseledets theorem is locally uniform on the cocycle. This sort of uniformity is useful in various situations in dynamical systems. An example is the following relevant question in the ergodic theory of volume-preserving diffeomorphisms $f : M \to M$. See [39] or [60] for background.

Let $K \subset M$ be a Pesin block, that is, a compact (non-invariant) set where the Lyapunov exponents are all bounded away from zero and the estimates in the Oseledets theorem hold uniformly. Over such a set, the Pesin stable and unstable manifolds are well defined and depend continuously on the point. In particular they have a definite size, and so nearby points in the Pesin block must belong to the same ergodic component. One may thus ask about the stability of Pesin blocks: is it the case that a smooth perturbation of g must also possess a Pesin block K_g nearby (in the sense that the symmetric difference $K \Delta K_g$ has small measure) such that its Pesin manifolds are close to the unperturbed ones? This can be shown to follow from a suitable control of the dependence of the Lyapunov exponents, and in particular if the averaged Lyapunov exponents depend continuously on the diffeomorphism at the point f.

One setting where knowledge about the continuity of the Lyapunov exponents has been used as an essential ingredient in the understanding of the dynamics is in the study of quasiperiodic Schrödinger operators. Here $f = f_\alpha : x \mapsto x + \alpha$ is a translation on a finite dimensional torus $\mathbb{T}^d$, and the cocycle function takes the form

$$A(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}).$$

In this case the continuity of Lyapunov exponents with respect to both α and A has been proved for analytic A and totally irrational α (meaning that f_α is minimal), by Bourgain and Jitomirskaya [23] when $d = 1$ and by Bourgain [21] in the general case.

It was used, for instance, in the solution by Avila and Jitomirskaya [4] of the Ten Martini Problem, which asked whether the Almost Mathieu Operator has a Cantor spectrum (this can be rephrased as density of uniform hyperbolicity within certain one-parameter families of cocycles). It also appears prominently in the proof of the quantization of the acceleration for $d = 1$, which is the starting point of the so-called global theory of one-frequency Schrödinger operators (Avila [10]).

Continuity is very subtle in this context: for instance, the aforementioned result of Bourgain and Jitomirskaya does not hold when A is merely C^∞ (a result of Wang-You [67]). Continuity as a function of α also may fail at rational α even when A is analytic, see [23] for a discussion.

Another setting which connects with the ideas discussed in this paper is that of $\mathrm{SL}(2, \mathbb{R})$-actions on moduli spaces of Abelian or quadratic differentials. Let η be an $\mathrm{SL}(2, \mathbb{R})$-invariant probability measure. By Eskin-Mirzakhani [36], η is equivalent to Lebesgue measure on some submanifold. The Kontsevich-Zorich cocycle over the Teichmüller flow, which plays a fundamental role in the ergodic theory of translation surfaces, has a nice behavior with respect to η: it is basically a random matrix product (involving countably many matrices), except that the products are merely

"quasi-independent". See [6] for an application of this idea to the issue of simplicity of the Lyapunov spectrum.

Let $(\nu_k)_k$ be a sequence of $\mathrm{SL}(2,\mathbb{R})$-invariant probability measures converging in the weak* sense to some probability measure ν. It would be tempting to use the techniques of our paper to address the issue of continuity of the Lyapunov exponents in this context. However, it turns out that the difficulties addressed in our paper do not show up, and hence simpler techniques can be applied, as was done in Bonatti, Eskin, and Wilkinson [19]. Indeed, by Theorem 2.3 in Eskin, Mirzakhani, and Mohammadi [37] (see also Theorem 2.6 in [19]), the support of ν_k is contained in $\operatorname{supp}\nu$ for every large k. So, in terms of the random matrix models, if there is an invariant subspace for η then it is also invariant for the η_k for large k. This allows one to quotient out bad invariant spaces, and establish continuity by the usual Furstenberg–Kifer argument [42].

Quantitative regularity. Another natural question is how much can the regularity of Lyapunov be upgraded from mere continuity. An old result of Ruelle [61] asserts that if all the matrix coefficients are positive then the largest Lyapunov exponent is a real-analytic function of those coefficients. For locally constant cocycles over Markov shifts, Peres [58] has shown that if the Lyapunov exponents are simple then they depend real-analytically on the transition data, assuming the cocycle function itself is fixed.

For parametrized random matrix products satisfying strong irreducibility and the contraction property, Le Page [57] has proved that the largest Lyapunov exponent is a Hölder continuous function of the parameter. This function is even C^∞ if the probability distributions are absolutely continuous. In the opposite direction, a construction of Halperin (see Simon and Taylor [62, Appendix 3]) shows that for every $\alpha > 0$ one can find random Schrödinger cocycles near which the Lyapunov exponents fail to be α-Hölder continuous.

These results have been sharpened by Duarte and Klein, who developed a unified approach to proving generic moduli of continuity of the Lyapunov exponents for different classes of linear cocycles, both random and quasi-periodic, especially in the 2-dimensional case. See [27, 28] and also [26] for an account of their approach and many applications.

仍然是在二维情形下，Tal 和 Viana [63] 已经证明，在李雅普诺夫谱为单谱的每一个点上，H¨older 连续性均成立。这是对我们在此所发展的那些方法所得估计结果的一种应用，即一个一致有界性结论。

$$\eta(E(r)) \leqslant Cr^\beta$$

for the weight of the neighborhood $E(r)$ of the equator relative to stationary measures η of nearby random walks. Tal and Viana [63] have also shown that, while Hölder continuity may fail when the two Lyapunov exponents coincide, a weaker log-Hölder modulus of continuity does hold at every point. It would be interesting to extend these results to arbitrary dimension.

Still regarding products of finitely many matrices, one problem that has proved to be very resistent to all techniques so far is whether the dependence of the Lyapunov exponent can be much better than Hölder in some non-trivial region of the parameter space. For instance, let us consider random matrix products of two $\mathrm{SL}(2,\mathbb{R})$-matrices A_1 and A_2, with probability weights $p_1 = p_2 = 1/2$. Over the open set $\mathbb{UH}$ of uniformly hyperbolic pairs (A_1, A_2), the top Lyapunov exponent is

clearly a real analytic function. Fix $1 \leqslant k < \infty$. *Is there a non-empty open subset in the complement of* $\mathbb{UH}$ *over which the top Lyapunov exponent is* C^k *?*

It is tempting to try to answer (affirmatively) this question by establishing a suitable spectral gap. Unfortunately the current approaches to the spectral gap use algebraic properties of the matrix coefficients, and thus do not apply over any open set, see [22].

2. Statement of main result

We state our main result, Theorem B below, of which Theorem A is an easy consequence. Initially, we recall the notion of Lyapunov exponents and

Given any compactly supported probability measure ν on $G = \mathrm{GL}(\mathbb{R}^d)$, let $\nu^{\mathbb{N}}$ and $\nu^{\mathbb{Z}}$ denote the corresponding Bernoulli measures on $G^{\mathbb{N}}$ and $G^{\mathbb{Z}}$, respectively. Consider the shift maps $\sigma : G^{\mathbb{N}} \to G^{\mathbb{N}}$ and $\sigma : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ given by

$$\sigma\left((g_n)_n\right) = (g_{n+1})_n.$$

By the Oseledets multiplicative ergodic theorem (see [65, Theorems 4.1 and 4.2]), there exist $k \in \{1, \dots, d\}$ and real numbers

$$\chi_1(\nu) > \cdots > \chi_k(\nu) \tag{2.1}$$

such that for $\nu^{\mathbb{N}}$-almost every $\mathfrak{g} = (g_0, \dots, g_n, \dots) \in G^{\mathbb{N}}$ there exists a decreasing family of vector subspaces

$$\mathbb{R}^d = V^1(\mathfrak{g}) > \cdots > V^k(\mathfrak{g}) > V^{k+1}(\mathfrak{g}) = \{0\} \tag{2.2}$$

and for $\nu^{\mathbb{Z}}$-almost every $\mathfrak{g}_\pm = (\dots, g_{-n}, \dots, g_0, \dots, g_n, \dots) \in G^{\mathbb{Z}}$ there exists a direct sum decomposition

$$\mathbb{R}^d = E^1(\mathfrak{g}_\pm) \oplus \cdots \oplus E^k(\mathfrak{g}_\pm) \tag{2.3}$$

使得，对于每个 $i = 1, \dots, k$,

- $g_0 V^i(\mathfrak{g}) = V^i(\sigma(\mathfrak{g}))$ and $g_0 E^i(\mathfrak{g}_\pm) = E^i(\sigma(\mathfrak{g}_\pm))$ for ν-almost every $g_0 \in G$;
- $V^i(\mathfrak{g}) = E^i(\mathfrak{g}_\pm) \oplus V^{i+1}(\mathfrak{g})$ for $\mathfrak{g} = \pi(\mathfrak{g}_\pm)$, where $\pi : G^{\mathbb{Z}} \to G^{\mathbb{N}}$ denotes the canonical projection.
- for every non-zero $v_i \in V^i(\mathfrak{g}) \backslash V^{i+1}(\mathfrak{g})$ and $\nu^{\mathbb{N}}$-almost every $\mathfrak{g} \in G^{\mathbb{N}}$

$$\lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 v_i\| = \chi_i(\nu); \tag{2.4}$$

- for every non-zero $v_i \in E^i(\mathfrak{g}_\pm)$ and $\nu^{\mathbb{Z}}$-almost every $\mathfrak{g}_\pm \in G^{\mathbb{Z}}$

$$\lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 v_i\| = \chi_i(\nu) = \lim_n \frac{1}{-n} \log \|g_{-n}^{-1} \cdots g_{-1}^{-1} v_i\|. \tag{2.5}$$

The maps $\mathfrak{g} \mapsto V^i(\mathfrak{g})$ and $\mathfrak{g}_\pm \mapsto E^i(\mathfrak{g}_\pm)$ with values in the Grassmannian of $\mathbb{R}^d$ are measurable and the dimensions $\dim V^i(\mathfrak{g})$ and $\dim E^i(\mathfrak{g}_\pm)$ are constant on full measure sets. The number $m_i = \dim V^i - \dim V^{i+1} = \dim E^i$ is called the *multiplicity* of the *Lyapunov exponent* $\chi_i(\nu)$. Denote by $\lambda_1(\nu) \geqslant \cdots \geqslant \lambda_d(\nu)$ the Lyapunov exponents *counted with multiplicity.*

Let $P = \mathbb{P}(\mathbb{R}^d)$. The *random walk* defined by ν is described by the pair $(F, \nu^{\mathbb{N}})$, where

$$F : G^{\mathbb{N}} \times P \to G^{\mathbb{N}} \times P, \quad (\mathfrak{g}, v) \mapsto (\sigma(\mathfrak{g}), g_0 v). \tag{2.6}$$

The Lyapunov exponents and the Oseledets filtration may also be obtained from it, as follows. Define

$$\Phi : G \times P \to \mathbb{R}, \quad \Phi(g, v) = \log \frac{\|gv\|}{\|v\|}. \tag{2.7}$$

(For notational simplicity, we use the same symbol (v, say) to denote both a non-zero vector in $\mathbb{R}^d$ and the corresponding element of P; analogously, we use the same notation (L, say) for a vector subspace of $\mathbb{R}^d$ and the subset of P associated to it.) A result of Ledrappier (see [65, Theorem 6.1]) asserts that:

- Given any F-invariant ergodic probability measure m on $G^{\mathbb{N}} \times P$ that projects to $\nu^{\mathbb{N}}$, there exists $j \in \{1, \dots, k\}$ such that

$$\int_{G^{\mathbb{N}} \times P} \Phi \, dm = \chi_j(\nu) \text{ and } m\left(\{(\mathfrak{g}, v) : v \in V^j(\mathfrak{g}) \backslash V^{j+1}(\mathfrak{g})\}\right) = 1. \tag{2.8}$$

- Given any $j \in \{1, \dots, k\}$ there is an ergodic F-invariant probability measure m projecting to $\nu^{\mathbb{N}}$ and satisfying (2.8).

Let $(A_{1,k}, \dots, A_{m,k})$, $k \in \mathbb{N}$ be a sequence of m-uples of matrices converging to some $(A_{1,\infty}, \dots, A_{m,\infty}) \in G^m$ and $(p_{1,k}, \dots, p_{m,k})$, $k \in \mathbb{N}$ be a sequence of probability m-vectors real numbers converging to some $(p_{1,\infty}, \dots, p_{m,\infty}) \in \Delta_m$. Let ν_k and ν_∞ be the probability measures in G given by

$$\nu_k = \sum_{i=1}^{m} p_{i,k} \delta_{A_{i,k}} \text{ and } \nu_\infty = \sum_{i=1}^{m} p_{i,\infty} \delta_{A_{i,\infty}}. \tag{2.9}$$

We want to prove that $\lambda_j(\nu_k) \to \lambda_j(\nu_\infty)$ when $k \to \infty$, for every $j = 1, \dots, d$.

事实上，我们证明了一个更强的结论，其中涉及的概率测度的支撑不必是有限集。令 $\mathcal{P}_c(G)$ 为 G 上紧支撑概率测度所构成的空间，其拓扑是包含以下两种拓扑的最小拓扑 $\mathcal{T}$：

- $\mathcal{W}$ = 概率测度空间中弱* 拓扑在 G• $\operatorname{supp}^* \mathcal{H}$ = 上的限制，即 $\nu \mapsto \operatorname{supp} \nu$ 对 $\mathcal{H}$ 紧子集空间中豪斯多夫拓扑的拉回 G。

This topology $\mathcal{T}$ is metrizable, because both $\mathcal{W}$ and $\mathcal{H}$ are. A sequence (ν_k) converges to ν_∞ in $\mathcal{P}_c(G)$ if and only if

(i) $(\nu_k)_k \to \nu_\infty$ in the weak* topology and
(ii) $(\operatorname{supp} \nu_k)_k \to \operatorname{supp} \nu_\infty$ in the Hausdorff topology.

对于（2.9）中的测度，若对每个 $i = 1, \dots, m$，$A_{i,k} \to A_{i,\infty}$ 且 $p_{i,k} \to p_{i,\infty}$，则情况正是如此；此处，假设 $p_{i,\infty} > 0$ 对于确保支撑的连续性至关重要。

与此相关，Kifer 的例子 [50] 表明，仅有第 (i) 部分不足以确保李雅普诺夫指数的连续性：要使我们的结果成立，不能省略拓扑定义的第 (ii) 部分。

Remark 2.1. If $(\nu_k)_k \to \nu_\infty$ in the weak* topology then, given any $\varepsilon > 0$, the support of ν_∞ is contained in the ε-neighborhood of $\operatorname{supp} \nu_k$ for all large k. Thus, the condition (i) in the definition contains half of the condition (ii). The other half is that, given any $\varepsilon > 0$, the support of ν_k is contained in the ε-neighborhood of $\operatorname{supp} \nu_\infty$ for all large k. This will be used repeatedly.

Also, let us point out that for proving Theorem A it suffices to consider the case $j = 1$. That is because of the following construction. Let $E = \mathbb{R}^d$ and $1 \leqslant l \leqslant d$. The *exterior l-power* $\Lambda^l E$ of E is the vector space of alternating l-linear forms $\omega : E^* \times \cdots \times E^* \to \mathbb{R}$ on the dual space E^*. The *exterior product* of vectors $v_1, \ldots, v_l \in E$, is the alternating l-linear form $v_1 \wedge \cdots \wedge v_l : E^* \times \cdots \times E^* \to \mathbb{R}$ defined by

$$(v_1 \wedge \cdots \wedge v_l)\,(\phi_1, \ldots, \phi_l) = \sum_\sigma \operatorname{sign}(\sigma)\, \phi_{\sigma(1)}(v_1) \cdots \phi_{\sigma(l)}(v_l),$$

where the sum is over all permutations of $\{1, \ldots, l\}$. If $\{e_j : j = 1, \ldots, d\}$ is a basis of E then $\{e_{j_1} \wedge \cdots \wedge e_{j_l} : 1 \leqslant j_1 < \cdots < j_l \leqslant d\}$ is a basis of $\Lambda^l E$. So,

$$\dim \Lambda^l E = \binom{d}{l}.$$

Every $g \in G$ induces an invertible linear map $\Lambda^l g : \Lambda^l E \to \Lambda^l E$, defined by

$$\Lambda^l g(\omega) : (\phi_1, \ldots, \phi_l) \mapsto \omega(\phi_1 \circ g, \ldots, \phi_l \circ g),$$

for $\omega \in \Lambda^l E$ and $\phi_1, \ldots, \phi_l \in E^*$. Thus, any measure ν in G induces a measure $\Lambda^l \nu$ in $\mathrm{GL}(\Lambda^l E)$, by push-forward under $g \mapsto \Lambda^l g$. Moreover, the maps $\nu \mapsto \Lambda^l \nu$ are continuous. One can check (see [65, Proposition 4.17]) that the Lyapunov exponents of the random walk defined by $\Lambda^l \nu$, counted with multiplicity, are the sums

$$\lambda_{i(1)}(\nu) + \cdots + \lambda_{i(l)}(x) \text{ with } 1 \leqslant i_1 < \cdots < i_l \leqslant d.$$

In particular,

$$\lambda_1(\Lambda^l \nu) = \lambda_1(\nu) + \cdots + \lambda_l(\nu).$$

因此，证明对于任意 $1 \leqslant l \leqslant d$，$\nu \mapsto \lambda_1(\Lambda^l \nu)$ 是连续的，将意味着对于任意 $1 \leqslant j \leqslant d$，$\nu \mapsto \lambda_j(\nu)$ 是连续的。

鉴于这些观察，定理 A 将立即由以下结论推出：

Theorem B. *The function $\lambda_1 : \mathcal{P}_c(G) \to \mathbb{R}$, $\nu \mapsto \lambda_1(\nu)$ is continuous, in any dimension $d \geqslant 2$.*

The rest of the paper is devoted to proving Theorem B. In Sections 3 and 4 we present a useful large deviations principle for Lyapunov exponents (Theorem 4.1). In Sections 5 and 6 we introduce several useful tools. In Section 7 we reduce the proof of Theorem B to a main technical result, Theorem 7.1. The proof of the latter result is by induction on the dimension r of the equator, as outlined in Section 7. The case $r = 1$ is carried out in detail in Sections 8 through 11. The inductive step is dealt with in Sections 12 through 16.

Before stating Theorem 7.1 and outlining its proof, we must introduce several general notions and a number of auxiliary results. On the other hand, the proofs of those results, given in Sections 3 to 6, are in themselves not used for establishing Theorem 7.1. Thus the reader is encouraged to skip them at first reading, proceeding as directly as possible to Section 7.

3. Invariant subspaces

In this section we introduce some background material, due mostly to Furstenberg and Kifer [40, 42]. This also allows us to introduce some notations that will be useful in the following. Proofs and more information can also be found in Chapters 4 through 6 of [65].

3.1. **Stationary measures.** Fix $\nu \in \mathcal{P}_c(G)$. We say that a probability measure η on P is *ν-stationary* if

$$\int_P \psi(x)\, d\eta(x) = \int_{G\times P} \psi(gy)\, d\nu(g)\, d\eta(y)$$

for every bounded measurable function $\psi : P \to \mathbb{R}$. In other words, η is ν-stationary if and only if $\mathcal{P}_\nu^*\eta = \eta$, where $\mathcal{P}_\nu^*$ is the operator defined by

$$\mathcal{P}_\nu^*\eta = \int_G (g_*\eta)\ d\nu(g) \tag{3.1}$$

in the space of probability measures. Moreover (see [65, Proposition 5.5]), η is ν-stationary if and only if the probability measure $\nu^{\mathbb{N}} \times \eta$ is invariant under the projective cocycle $F : G^{\mathbb{N}} \times P \to G^{\mathbb{N}} \times P$ defined in (2.6). Stationary measures always exist (see [65, Proposition 5.6]).

We also consider the operator $\mathcal{P}_\nu : \mathcal{B}(P) \to \mathcal{B}(P)$ defined in the space $\mathcal{B}(P)$ of measurable bounded functions $\psi : P \to \mathbb{R}$ by

$$\mathcal{P}_\nu\psi(v) = \int_G \psi(gv)\, d\nu(g). \tag{3.2}$$

A function ψ is said to be *ν-stationary* if $\mathcal{P}_\nu\psi = \psi$. A ν-stationary measure η is *ergodic* if every ν-stationary function is constant on some full η-measure set. This happens (see [65, Proposition 5.13]), if and only if the F-invariant measure $\nu^{\mathbb{N}} \times \eta$ is ergodic for F. The ergodic decomposition theorem (see [65, Theorem 5.14]) asserts that every ν-stationary measure is a convex combination of ergodic ν-stationary measures.

设 Φ 如 (2.7) 中所示。则（参见 [65, 命题 6.7]），

$$\lambda_1(\nu) = \max\Big\{ \int_{G\times P} \Phi\, d(\nu\times\eta) : \eta \text{ is a } \nu\text{-stationary measure}\Big\}. \tag{3.3}$$

对每个 ν-平稳测度 η，记为 $\alpha(\eta) = \int_{G\times P} \Phi\, d(\nu\times\eta)$。遍历分解定理表明，如果将范围限制为遍历的 ν-平稳测度，最大值不会发生变化。

我们也可以将 Φ 视为定义在 $G^{\mathbb{N}} \times P$ 上的一个函数，它仅依赖于 g_0 和 v:

$$\Phi(\mathfrak{g}, v) = \log \frac{\|g_0 v\|}{\|v\|}.$$

Then

$$\lim_n \frac{1}{n} \log \frac{\|g_{n-1}\cdots g_0 v\|}{\|v\|} = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \log \frac{\|g_j g_{j-1}\cdots g_0 v\|}{\|g_{j-1}\cdots g_0 v\|} = \lim_n \frac{1}{n} \sum_{j=0}^{n-1} \Phi(F^j(\mathfrak{g}, v)).$$

因此，对于任意遍历的 ν-平稳测度 η，我们有

$$\lim_n \frac{1}{n} \log \frac{\|g_{n-1}\cdots g_0 v\|}{\|v\|} = \int_{G\times P} \Phi\, d(\nu\times\eta) \tag{3.4}$$

在 $G^{\mathbb{N}} \times P$ 中，除去一个 $\nu^{\mathbb{N}} \times \eta$ 零测集。

Furstenberg 和 Kifer 已证明（参见 [42, 定理 2.1]），如果对于每个（遍历） ν-平稳测度 η，都有 $\alpha(\eta) = \lambda_1(\nu)$，那么对于每个 $v \in P$

$$\lim_n \frac{1}{n} \log \frac{\|g_{n-1}\cdots g_0 v\|}{\|v\|} = \lambda_1(\nu) \text{ for } \nu^{\mathbb{N}}\text{-almost every } \mathfrak{g}. \tag{3.5}$$

This fact is also contained in Theorem 4.1 below. In the next section we analyze what happens when the hypothesis of (3.5) is not fulfilled.

3.2. 赤道。若向量子空间 L of $\mathbb{R}^d$ 满足 $gL = L$ 对于几乎所有 ν g 成立，或者等价地，对于每个 $g \in \operatorname{supp}\nu$ 都成立，则称其为 ν-不变的。注意，如果 L 是一个 ν-不变子空间，那么

$$\lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 \mid L\|$$

exists for every $\mathfrak{g}$ in a full $\nu^{\mathbb{N}}$-measure subset of $G^{\mathbb{N}}$ and is constant on this subset. This direct consequence of Kingman's subadditive ergodic theorem (see [65, Theorem 3.3]), together with the fact that the Bernoulli shift $(\sigma, \nu^{\mathbb{N}})$ is ergodic, will be used repeatedly. It is part of the proof of the Oseledets theorem (see [65, Proposition 4.11]) that for $L = \mathbb{R}^d$ the limit coincides with the largest Lyapunov $\lambda_1(\nu)$.

For any ergodic ν-stationary measure η, define $L(\eta)$ to be the smallest vector subspace such that $\eta(L(\eta)) = 1$. Equivalently, $L(\eta)$ is the vector subspace spanned by the support of η. Then $L(\eta)$ is ν-invariant:

$$\int_{G\times P} \chi_{L(\eta)}(gv)\, d\nu(g) d\eta(v) = \int_P \chi_{L(\eta)}(v)\, d\eta(v) = \eta(L) = 1$$

and this implies

$$\eta(g^{-1}L(\eta)) = \int_P \chi_{L(\eta)}(gv)\, d\eta(v) = 1 \text{ for } \nu\text{-almost every } g.$$

So, it follows from the definition that $g^{-1}L(\eta) = L(\eta)$ for ν-almost every g.

由于 η 的支撑张成了 $L(\eta)$，由（3.4）可知，我们可以找到 $L(\eta)$ 的一组基v_1、...、v_l，以及 $\mathfrak{g}$ 的一个满 $\nu^{\mathbb{N}}$测度子集，使得

$$\lim_n \frac{1}{n} \log \frac{\|g_{n-1} \cdots g_0 v_i\|}{\|v_i\|} = \alpha(\eta) \text{ for } i = 1, \ldots, l.$$

Then

$$\lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 \mid L(\eta)\| = \alpha(\eta). \tag{3.6}$$

还可得出，如果 η 满足 $\alpha(\eta) < \lambda_1(\nu)$，那么 $L(\eta)$ 是真子空间。我们将 ν 的赤道称为极大 ν-不变子空间 E，使得

$$\lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 \mid E\| < \lambda_1(\nu). \tag{3.7}$$

Such a subspace is necessarily proper, but it may not exist. It follows from the previous paragraph that the equator does exist if $\alpha(\eta) < \lambda_1(\nu)$ for some (ergodic) ν-stationary measure; the converse is also true. Moreover, the equator is unique and contains $L(\eta)$ for every ergodic ν-stationary measure η: both claims follow immediately from the observation that if two subspaces satisfy (3.7) then so does their sum.

If the equator does exist, every g in (the group generated by) the support of ν may be written as

$$g = \begin{pmatrix} g^E & h \\ 0 & g^\perp \end{pmatrix} \text{ with } g^E \in \mathrm{GL}(E) \text{ and } g^\perp \in \mathrm{GL}(E^\perp). \tag{3.8}$$

设 ν^E 和 $\nu^\perp$ 分别为 ν 在映射 $g \mapsto g^E$ 和 $g \mapsto g^\perp$ 下的推前。
Furstenberg 和 Kifer 在引理 3.6] 中指出,

$$\lambda_1(\nu) = \max\{\lambda_1(\nu^E), \lambda_1(\nu^\perp)\}.$$

The property (3.7) means that $\lambda_1(\nu^E) < \lambda_1(\nu)$. Hence, $\lambda_1(\nu^\perp) = \lambda_1(\nu)$.

Remark 3.1. Any $g \in G$ that preserves E may written in the form (3.8). Then $(gu)^\perp = g^\perp u^\perp$ for every u, where $\perp$ denotes the component orthogonal to E. Do not mistake this for $gu^\perp = hu^\perp + g^\perp u^\perp$. It is equally clear that $\|g^\perp\| \leqslant \|g\|$. These simple facts will be used several times.

假设存在某个遍历的 $\nu^\perp$-平稳测度 $\eta^\perp$，使得

$$\alpha^\perp(\eta^\perp) = \int_{G\times P} \Phi \, d(\nu^\perp \times \eta^\perp)$$

严格小于 $\lambda_1(\nu^\perp) = \lambda_1(\nu)$。那么， $L(\eta^\perp)$ 是 $E^\perp$ 的一个真 $\nu^\perp$-不变子空间，并且根据 (3.6),

$$\lim_n \frac{1}{n} \log \|g_{n-1}^\perp \cdots g_0^\perp \mid L(\eta^\perp)\| = \alpha^\perp(\eta^\perp).$$

Then $E' = E \oplus L(\eta^\perp)$ is a proper ν-invariant subspace of $\mathbb{R}^d$ and it satisfies the equator property (3.7), because

$$\begin{aligned}\lim_n \frac{1}{n} &\log \|g_{n-1} \cdots g_0 \mid E \oplus L(\eta^\perp)\| \\ &\leqslant \max\big\{ \lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 \mid E\|, \lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 \mid L(\eta^\perp)\|\big\}\end{aligned}$$

由于 E' 严格包含 E，这与赤道的定义相矛盾。这一矛盾证明，对于每个 $\nu^\perp$-平稳测度 $\eta^\perp$， $\alpha^\perp(\eta^\perp) = \lambda_1(\nu)$。

So, by (3.5), for every $v^\perp \in \mathbb{P}(E^\perp)$ there exists a full $\nu^{\mathbb{N}}$-measure subset of $\mathfrak{g}$ for which

$$\lim_n \frac{1}{n} \log \frac{\|g_{n-1}^\perp \cdots g_0^\perp v^\perp\|}{\|v^\perp\|} = \lambda_1(\nu). \tag{3.9}$$

This also implies (see [65, Proposition 4.14]) that for every $v \in \mathbb{R}^d \backslash E$ there exists a full $\nu^{\mathbb{N}}$-measure subset of $\mathfrak{g}$ for which

$$\lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 v\| = \lim_n \frac{1}{n} \log \frac{\|g_{n-1} \cdots g_0 v\|}{\|v\|} = \lambda_1(\nu). \tag{3.10}$$

The following example shows that the equator need not be any of the subspaces V^i in the Oseledets flag (2.2):

示例 3.2. 设 $d = 3$ 、 $m = 2$ 和 $\nu = p_1\delta_{A_1} + p_2\delta_{A_2}$，其中矩阵 A_1 和 A_2 由下式给出。

$$A_1 = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \text{ with } B = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \text{ and } \sigma > 1$$

$$A_2 = \begin{pmatrix} R_\theta B & 0 \\ 0 & 1 \end{pmatrix} \text{ with } R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \text{ and } \theta \neq 0 \text{ small.}$$

The subspace $E_3 = \{(0,0)\} \times \mathbb{R}$ is ν-invariant and corresponds to a zero Lyapunov exponent. The only other ν-invariant subspace is $E_{12} = \mathbb{R}^2 \times \{0\}$. Given any $\varepsilon > 0$, the cone

$$C^u = \{(x, y, 0) \in \mathbb{R}^3 : |y| \leqslant \varepsilon |x|\} \subset E_{12}$$

is forward invariant under both A_1 and A_2, as long as θ is close enough to zero. This implies that some Lyapunov exponent is close to $\log\sigma$. Analogously, the cone

$$C^s = \{(x, y, 0) \in \mathbb{R}^3 : |x| \leqslant \varepsilon|y|\} \subset E_{12}$$

is backward invariant under both A_1 and A_2, which implies that some Lyapunov exponent is close to $-\log\sigma$. So, the Lyapunov exponents are

$$\lambda_1(\nu) \approx \log\sigma \text{ and } \lambda_2(\nu) = 0 \text{ and } \lambda_3(\nu) \approx -\log\sigma.$$

Thus, E_3 is the equator of ν and corresponds to the middle eigenvalue $\lambda_2(\nu) = 0$. Note that both $A_1^\perp = B$ and $A_2^\perp = R_\theta B$ have determinant 1.

4. Uniform convergence in measure

我们需要证明，(3.9) 中的极限关于 $v \in \mathbb{P}(E^\perp)$ 一致收敛于测度。这直接由 (3.5) 中极限的相应性质推出，具体表述如下：

Theorem 4.1. *Assume that $\alpha(\eta) = \lambda_1(\nu)$ for every ν-stationary measure η. Then for any $\varepsilon > 0$ there exist constants $C = C(\nu, \varepsilon) > 0$ and $c = c(\nu, \varepsilon) > 0$ such that for any $v \in P$ and $N \in \mathbb{N}$, there exists a measurable set $\mathcal{E} = \mathcal{E}(\nu, \varepsilon, v, N) \subset G^{\mathbb{N}}$ satisfying:*

(1) $\nu^{\mathbb{N}}(\mathcal{E}^c) \leqslant Ce^{-cN}$ *and*
(2) $\dfrac{1}{n}\log\dfrac{\|g_{n-1}\cdots g_0 v\|}{\|v\|} \in (\lambda_1(\nu) - \varepsilon, \lambda_1(\nu) + \varepsilon)$ *for every $\mathfrak{g} \in \mathcal{E}$ and $n \geqslant N$.*

接下来我们证明定理 4.1。本文其余部分不会用到这个证明，因此读者初次阅读时可以选择跳过本节剩余内容。

Recall that $S = \operatorname{supp}\nu$ is taken to be compact. Let $C(S \times P)$ be the Banach space of continuous functions $\psi : S \times P \to \mathbb{R}$ with the norm $\|\psi\| = \sup|\psi|$. Consider the operator $\mathcal{Q}_\nu$ defined on $C(S \times P)$ by

$$\mathcal{Q}_\nu\psi(g', v) = \int_S \psi(g, gv)\, d\nu(g).$$

Note that $\mathcal{Q}_\nu\psi(g', v)$ does not depend on g'. The dual operator $\mathcal{Q}_\nu^*$ acts in the space of finite signed measures on $S \times P$ by

$$\int_{S\times P} \psi\, d\mathcal{Q}_\nu^*\lambda = \int_{S\times P} \mathcal{Q}_\nu\psi\, d\lambda \text{ for every } \psi \in C(S \times P).$$

A signed measure λ is said to be *$\mathcal{Q}_\nu$-invariant* if $\mathcal{Q}_\nu^*\lambda = \lambda$. Given $\varphi \in C(S \times P)$, denote

$$\Sigma(\nu, \varphi) = \sup\left\{\int_{S\times P} \varphi\, d\lambda : \lambda \text{ is a } \mathcal{Q}_\nu\text{-invariant probability measure}\right\}.$$

Lemma 4.2. *Given any non-negative $\varphi \in C(S \times P)$ and any $\varepsilon > 0$ there exist $\psi, \xi \in C(S \times P)$ such that $\varphi = \mathcal{Q}_\nu\psi - \psi + \xi$ and $\|\xi\| \leqslant \Sigma(\nu, \varphi) + \varepsilon$.*

Proof. Let $\mathcal{W} \subset C(S \times P)$ be the subspace of functions of the form $\varphi = \mathcal{Q}_\nu\psi - \psi$ for some $\psi \in C(S \times P)$. By Hahn–Banach, given any $\varphi \in C(S \times P)$ there exists a continuous linear functional $L : C(S \times P) \to \mathbb{R}$ such that $\|L\| = 1$ and $L \mid \mathcal{W} \equiv 0$ and $L(\varphi) = d(\varphi, \mathcal{W})$. By Riez–Markov, there exists some signed measure λ on

$S \times P$ such that $\|\lambda\| = 1$ and $L(\psi) = \int_{S\times P} \psi \, d\lambda$ for every $\psi \in C(S \times P)$. The fact that L vanishes on W ensures that $\mathcal{Q}_\nu^* \lambda = \lambda$:

$$\int_{S\times P} (\mathcal{Q}_\nu \psi - \psi) \, d\lambda = L(\mathcal{Q}_\nu \psi - \psi) = 0 \text{ for every } \psi \in C(S \times P).$$

Let $\lambda = \lambda^+ - \lambda^-$ be the Hahn decomposition of λ. Then $\|\lambda^+\| + \|\lambda^-\| = \|\lambda\| = 1$ and $\lambda = \mathcal{Q}_\nu^* \lambda = \mathcal{Q}_\nu^* \lambda^+ - \mathcal{Q}_\nu^* \lambda^-$. Since $\mathcal{Q}_\nu^* \lambda^\pm$ are non-negative measures, the latter implies that $\mathcal{Q}_\nu^* \lambda^\pm \geqslant \lambda^\pm$. Since $\mathcal{Q}_\nu 1 = 1$ we have

$$\|\mathcal{Q}_\nu^* \lambda^\pm\| = \int_{S\times P} 1 \, d\mathcal{Q}_\nu^* \lambda^\pm = \int_{S\times P} \mathcal{Q}_\nu 1 \, d\lambda^\pm = \int_{S\times P} 1 \, d\lambda^\pm = \|\lambda^\pm\|.$$

因此， $\mathcal{Q}_\nu^* \lambda^\pm = \lambda^\pm$。令 $\lambda_0 = \lambda^+ / \|\lambda^+\|$。则 $\mathcal{Q}_\nu^* \lambda_0 = \lambda_0$ 和 $\lambda_0 \geqslant \lambda^+ \geqslant \lambda$。由于假设 φ 非负，因此可得。

$$\int_{S\times P} \varphi \, d\lambda_0 \geqslant \int_{S\times P} \varphi \, d\lambda = L(\varphi) = d(\varphi, \mathcal{W}) \tag{4.1}$$

取 $\psi \in C(S \times P)$，使得 $\|\varphi - (\mathcal{Q}_\nu \psi - \psi)\| \leqslant d(\varphi、\ \mathcal{W}) + \varepsilon$，然后定义 $\xi = \varphi - (\mathcal{Q}_\nu \psi - \psi)$。于是，由 **(4.1)** 得，

$$\|\xi\| \leqslant d(\varphi, W) + \varepsilon \leqslant \int_{S\times P} \varphi \, d\lambda_0 + \varepsilon$$

这意味着 $\|\xi\| \leqslant \Sigma(\nu, \varphi) + \varepsilon$。 □

For $n \geqslant 0$, let $\mathcal{B}_n$ be the σ-algebra of $S^{\mathbb{N}}$ generated by the family of cylinders $[0; A_0, \ldots, A_n]$, where the A_j are measurable subsets of S. For a measurable function $X : S^{\mathbb{N}} \to \mathbb{R}$, let $\mathbb{E}(X) = \int_{G^{\mathbb{N}}} X \, d\nu^{\mathbb{N}}$ and $\mathbb{E}(X \mid \mathcal{B}_n)$ denote the *expectation* of X conditioned to $\mathcal{B}_n$, that is, the (essentially unique) $\mathcal{B}_n$-measurable function such that

$$\int_{G^{\mathbb{N}}} \mathbb{E}(X \mid \mathcal{B}_n) \, d\nu^{\mathbb{N}} = \int_{G^{\mathbb{N}}} X \, d\nu^{\mathbb{N}}.$$

The next lemma is a particular instance of the Azuma–Hoeffding inequality (Azuma [11], Hoeffding [48]) for sums of bounded random variables:

Lemma 4.3. *Let $Y_n : G^{\mathbb{N}} \to \mathbb{R}$, $n \in \mathbb{N}$ be such that $A = \sup_n \|Y_n\|$ is finite and*

$$\mathbb{E}(Y_{i_1} \cdots Y_{i_k}) = 0 \text{ for every } 1 \leqslant i_1 < \cdots < i_k. \tag{4.2}$$

那么，对于任意的 $s > 0$ 和 $n \in \mathbb{N}$,

$$\nu^{\mathbb{N}} \left(\{ \mathfrak{g} : |Y_1 + \cdots + Y_n|(\mathfrak{g}) \geqslant s \} \right) \leqslant 2 \exp\left(-\frac{s^2}{2nA^2} \right).$$

证明。由于指数函数是凸函数,

$$e^{ax} = \exp\left(a \left(\frac{1+x}{2} \right) - a \left(\frac{1-x}{2} \right) \right) \leqslant \frac{1+x}{2} e^a + \frac{1-x}{2} e^{-a} = \cosh a + x \sinh a$$

for every $x \in [-1, 1]$ and $a \in \mathbb{R}$. Taking $x = Y_i / \|Y_i\|$ and $a = t\|Y_i\|$, we get that

$$e^{tY_i} \leqslant \cosh(t\|Y_i\|) + \frac{Y_i}{\|Y_i\|} \sinh(t\|Y_i\|)$$

对于任意 $t \in \mathbb{R}$。假设 **(4.2)** 意味着

$$\mathbb{E}\left(\prod_{i=1}^n (a_i + b_i Y_i) \right) = \prod_{i=1}^n a_i$$

for any real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Hence,

$$\mathbb{E}\left(e^{t\sum_{i=1}^n Y_i}\right) \leqslant \mathbb{E}\left(\prod_{i=1}^n \cosh(t\|Y_i\|) + \frac{Y_i}{\|Y_i\|}\sinh(t\|Y_i\|)\right) = \prod_{i=1}^n \cosh(t\|Y_i\|).$$

利用这样一个事实：对于每个 $x \in \mathbb{R}$,

$$\cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \leqslant \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!} = e^{x^2/2}$$

we conclude that

$$\mathbb{E}\left(e^{t\sum_{i=1}^n Y_i}\right) \leqslant \prod_{i=1}^n \exp\left(\frac{t^2}{2}\|Y_i||^2\right) = \exp\left(\frac{t^2}{2}\sum_{i=1}^n \|Y_i\|^2\right)$$

对任意 $t \in \mathbb{R}$。因此，根据切比雪夫不等式,

$$\begin{aligned}\nu^{\mathbb{N}}\left(\{\mathfrak{g} : \sum_{j=1}^n Y_j(\mathfrak{g}) \geqslant s\}\right) &= \nu^{\mathbb{N}}\left(\{\mathfrak{g} : e^{t\sum_{j=1}^n Y_j(\mathfrak{g})} \geqslant e^{ts}\}\right) \\ &\leqslant e^{-ts}\mathbb{E}\left(e^{t\sum_{i=1}^n Y_j}\right) \leqslant \exp\left(-ts + \frac{t^2}{2}\sum_{i=1}^n \|Y_i\|^2\right)\end{aligned}$$

for any $t \in \mathbb{R}$. Taking $t = s/\sum_{i=1}^n \|Y_i\|^2$, we conclude that

$$\nu^{\mathbb{N}}\left(\{\mathfrak{g} : \sum_{i=1}^n Y_j(\mathfrak{g}) \geqslant s\}\right) \leqslant \exp\left(-\frac{s^2}{2\sum_{i=1}^n \|Y_i\|^2}\right) \leqslant \exp\left(-\frac{s^2}{2nA^2}\right)$$

类似地，将每个 Y_j 替换为 $-Y_j$,

$$\nu^{\mathbb{N}}\left(\{\mathfrak{g} : \sum_{i=1}^n Y_j(\mathfrak{g}) \leqslant -s\}\right) \leqslant \exp\left(-\frac{s^2}{2nA^2}\right).$$

将这两个不等式相加，即可得到引理的结论。 □

Lemma 4.4. *For any $\psi \in C(S \times P)$ and $\varepsilon > 0$ there exist $C_1 = C_1(\psi, \varepsilon) > 0$ and $c_1 = c_1(\psi, \varepsilon) > 0$ such that for any $v \in \mathbb{R}^d$ and $N \in \mathbb{N}$ there exists a measurable set $\mathcal{E}_1 = \mathcal{E}_1(\nu, \psi, \varepsilon, v, N)$ satisfying:*

(1) $\nu^{\mathbb{N}}(\mathcal{E}_1^c) \leqslant C_1 e^{-c_1 N}$ *and*

(2) $\frac{1}{n}\sum_{j=0}^{n-1} \mathcal{Q}_\nu\psi(g_{j-1}, g_{j-1}\cdots g_0 v) - \psi(g_j, g_j\cdots g_0 v) \in (-\varepsilon, \varepsilon)$ *for any $\mathfrak{g} \in \mathcal{E}_1$ and $n \geqslant N$.*

Proof. We may suppose that $\|\psi\| > 0$ for otherwise the statement is trivial. Given $v \in \mathbb{R}^d$ and $n \in \mathbb{N}$, define

$$Y_n(\mathfrak{g}) = \mathcal{Q}_\nu\psi(g_{n-1}, g_{n-1}\cdots g_0 v) - \psi(g_n, g_n\cdots g_0 v).$$

Clearly, $\|Y_n\| \leqslant 2\|\psi\|$. Moreover, Y_n is $\mathcal{B}_n$-measurable and, by the definition of $\mathcal{Q}_\nu$,

$$\mathbb{E}(Y_n \mid \mathcal{B}_{n-1}) = \mathcal{Q}_\nu\psi(g_{n-1}, g_{n-1}\cdots g_0 v) - \int_G \psi(g, gg_{n-1}\cdots g_0 v)\, d\nu(g) \equiv 0.$$

因此，对于任意一个 k 元组 $(i_1, \dots, i_{k-1}, i_k)$，其中每个 $j = 1, \dots, k-1$ 都满足 $i_k > i_j$,

$$\mathbb{E}\left(Y_{i_1}\cdots Y_{i_{k-1}}Y_{i_k}\right) = \mathbb{E}\left(Y_{i_1}\cdots Y_{i_{k-1}}\mathbb{E}(Y_{i_k} \mid B_{i_k-1})\right) = 0.$$

因此， $(Y_n)_n$ 满足引理4.4的假设。因此，

$$\nu^{\mathbb{N}}\left(\{\mathfrak{g}\in G^n : \frac{1}{n}|Y_0(\mathfrak{g})+\cdots+Y_{n-1}(\mathfrak{g})|\geqslant\varepsilon\}\right)\leqslant 2\exp\left(-\frac{n\varepsilon^2}{8\|\psi\|^2}\right)$$

for every $\varepsilon>0$ and every $n\in\mathbb{N}$. The conclusion follows by letting $\mathcal{E}_1$ be the set of all $\mathfrak{g}$ such that the claim in part (2) holds for every $n\geqslant N$ and taking $c_1=\varepsilon^2/(8\|\psi\|^2)$ and $C_1=2\sum_{j=0}^{\infty}e^{-cj}$. □

Lemma 4.5. *For every $\varphi\in C(S\times P)$ and $\varepsilon>0$ there exist $C_2=C_2(\nu,\varphi,\varepsilon)>0$ and $c_2=c_2(\nu,\varphi,\varepsilon)>0$ such that, for every $v\in P$ and $N\in\mathbb{N}$ there exists a measurable set $\mathcal{E}_2=\mathcal{E}_2(\nu,\varphi,\varepsilon,v,N)\subset G^{\mathbb{N}}$ satisfying:*

(1) $\nu^{\mathbb{N}}(\mathcal{E}_2^c)\leqslant C_2e^{-c_2N}$ *and*

(2) $\frac{1}{n}\sum_{j=0}^{n-1}\varphi(g_j,g_j\dots g_0v)<\Sigma(\nu,\varphi)+\varepsilon$ *for every $\mathfrak{g}\in\mathcal{E}_2$ and $n\geqslant N$.*

Proof. Neither the hypothesis nor the conclusion are affected if one replaces φ with $\varphi+\text{const}$ (clearly, $\Sigma_{\varphi+\text{const}}=\Sigma(\nu,\varphi)+\text{const}$). Thus, it is no restriction to suppose that φ is non-negative. Then, by Lemma 4.2, for any $\varepsilon>0$ we may find $\psi,\xi\in C(S\times P)$ such that $\varphi=\mathcal{Q}_\nu\psi-\psi+\xi$ and $\|\xi\|<\Sigma(\nu,\varphi)+\varepsilon/4$. Then

$$\begin{aligned}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(g_j,g_j\dots g_0v)&=\frac{1}{n}\sum_{j=1}^{n}\mathcal{Q}_\nu\psi(g_{j-1},g_{j-1}\dots g_0v)-\psi(g_j,g_j\dots g_0v)\\&\quad+\frac{1}{n}\sum_{j=0}^{n-1}\xi(g_j,g_j\dots g_0v)+\frac{1}{n}\psi(g_n,g_n\cdots g_0v)-\frac{1}{n}\psi(g_0,g_0v).\end{aligned}$$

Let C_1, c_1 and $\mathcal{E}_1$ be as in Lemma 4.3, with ε replaced with $\varepsilon/4$. The first term in the previous sum is less than $\varepsilon/4$ for every $\mathfrak{g}\in\mathcal{E}_1$. The second term is bounded above by $\|\xi\|<\Sigma(\nu,\varphi)+\varepsilon/4$. Let $L=[\|\psi\|/(4\varepsilon)]$. The third and fourth terms are less than $\varepsilon/4$ if $n>L$. Thus, for every $\mathfrak{g}\in\mathcal{E}_1$ and $n\geqslant N>L$,

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(g_j,g_j\dots g_0v)<\Sigma(\nu,\varphi)+\varepsilon.$$

取 $c_2=c_1$ 和 $C_2=\max\{C_1,e^{c_2L}\}$。定义 $\mathcal{E}_2=\mathcal{E}_1$，如果 $N>L$，否则定义为 $\mathcal{E}_2=G^{\mathbb{N}}$。
□

Corollary 4.6. *Let $\varphi\in C(S\times P)$ be such that $\int_{S\times P}\varphi\,d\lambda=\Sigma(\nu,\varphi)$ for every $\mathcal{Q}_\nu$-invariant probability measure λ. For $\varepsilon>0$ there exist $C_3=C_3(\nu,\varphi,\varepsilon)>0$ and $c_3=c_3(\nu,\varphi,\varepsilon)>0$ such that, for every $v\in P$ and $N\in\mathbb{N}$ there exists a measurable set $\mathcal{E}_3=\mathcal{E}_3(\nu,\varphi,\varepsilon,v,N)\subset G^{\mathbb{N}}$ satisfying:*

(1) $\nu^{\mathbb{N}}(\mathcal{E}_3^c)\leqslant C_3e^{-c_2N}$ *and*

(2) $\frac{1}{n}\sum_{j=0}^{n-1}\varphi(g_j,g_j\dots g_0v)\in(\Sigma(\nu,\varphi)-\varepsilon,\Sigma(\nu,\varphi)+\varepsilon)$ *for $\mathfrak{g}\in\mathcal{E}_3$ and $n\geqslant N$.*

Proof. The assumption implies that $\Sigma(\nu,-\varphi)=-\Sigma(\nu,\varphi)$. So, applying Lemma 4.5 also to the function $-\varphi$ we get $C_2(-\varphi,\varepsilon)$ and $c_2(-\varphi,\varepsilon)$ and $\mathcal{E}_2(-\varphi,\varepsilon,v,N)$ such that

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(g_j,g_j\dots g_0v)>\Sigma(\nu,\varphi)-\varepsilon.$$

for any $\mathfrak{g} \in \mathcal{E}_2(-\varphi, \varepsilon, v, N)$ and $n \geqslant N$. To conclude, take C_3 to be the sum of $C_2(\nu, \pm\varphi, \varepsilon)$ and c_3 to be the minimum of $c_2(\nu, \pm\varphi, \varepsilon)$ and $\mathcal{E}_3$ to be the intersection of $\mathcal{E}_2(\nu, \pm\varphi, \varepsilon, v, N)$. □

We also need to interpret $\Sigma(\nu, \varphi)$ in terms of the ν-stationary measures. That is the purpose of the next lemma:

Lemma 4.7. *Let $\iota : S \times P \to S \times P$, $\iota(g, v) = (g, gv)$. Then $\eta \mapsto \iota_*(\nu \times \eta)$ maps the set of ν-stationary measures bijectively to the set of $\mathcal{Q}_\nu$-invariant probability measures. Its inverse is the push-forward π_* of the canonical projection $\pi : S \times P \to P$.*

Proof. Given any probability measure λ with $\mathcal{Q}_\nu^*\lambda = \lambda$, let η be its projection to P. For any bounded measurable function $\varphi : S \times P \to \mathbb{R}$,

$$\begin{aligned}\int_{S\times P} \varphi(g', v)\, d\lambda(g', v) &= \int_{S\times P} \mathcal{Q}_\nu\varphi(g', v)\, d\lambda(g', v)\\ &= \int_{S\times S\times P} \varphi(g, gv)\, d\nu(g)\, d\lambda(g', v) = \int_{S\times P} \varphi(g, gv)\, d\nu(g)\, d\eta(v)\end{aligned}$$

(because the integrand does not depend on g'). Moreover, the special case when φ does not depend on g means that

$$\begin{aligned}\int_P \phi(v)\, d\eta(v) = \int_{S\times P} \phi(v)\, d\lambda(g', v) &= \int_{S\times P} \mathcal{Q}_\nu\phi(g', v)\, d\lambda(g', v)\\ &= \int_{S\times S\times P} \phi(gv)\, d\nu(g)\, d\lambda(g', v) = \int_{S\times P} \phi(gv)\, d\nu(g)\, d\eta(v)\end{aligned}$$

对于任意有界可测函数 ϕ: $P \to \mathbb{R}$。换句话说， η 是 ν-平稳的。

Conversely, given any ν-平稳测度 η，在 $S \times P$ 上定义 λ，通过

$$\int_{S\times P} \varphi(g', v)\, d\lambda(g', v) = \int_{S\times P} \varphi(g, gv)\, d\nu(g)\, d\eta(v).$$

对于任意有界可测函数 φ： $S \times P \to \mathbb{R}$。接着。

$$\begin{aligned}\int_{S\times P} \mathcal{Q}_\nu\varphi(g', v)\, d\lambda(g', v) &= \int_{S\times P} \mathcal{Q}_\nu\varphi(g', g'v)\, d\nu(g')\, d\eta(v)\\ &= \int_{S\times S\times P} \varphi(g, gg'v)\, d\nu(g)\, d\nu(g')\, d\eta(v).\end{aligned}$$

由于 η 是 ν-平稳的，右侧可以改写为

$$\int_{S\times P} \varphi(g, gv)\, d\nu(g)\, d\eta(v) = \int_{S\times P} \varphi(g, v)\, d\lambda(g, v).$$

结合这两个恒等式，可以看出 $\mathcal{Q}_\nu^*\lambda = \lambda$。 □

Proof of Theorem 4.1. Consider $\varphi = \Phi \circ \iota^{-1}$, that is,

$$\varphi : S \times P \to \mathbb{R}, \quad \varphi(g, v) = \log \frac{\|v\|}{\|g^{-1}v\|}.$$

引理 4.7 表明,

$$\begin{aligned}\Sigma(\nu, \varphi) &= \sup\Big\{ \int_{S\times P} \varphi\, d\lambda : \mathcal{Q}_\nu^*\lambda = \lambda\Big\}\\ &= \sup\Big\{ \int_{S\times P} \Phi\, d(\nu \times \eta) :\ \eta \text{ is a } \nu\text{-stationary measure}\Big\} = \lambda_1(\nu)\end{aligned}$$

and the hypothesis of Corollary 4.6 corresponds precisely to the hypothesis of Theorem 4.1. Take $C = C_3(\nu, \varphi, \varepsilon)$ and $c = c_3(\nu, \varphi, \varepsilon)$ and $\mathcal{E} = \mathcal{E}_3(\nu, \varphi, \varepsilon, v, N)$. Then $\nu^{\mathbb{N}}(\mathcal{E}^c) \leqslant Ce^{-cn}$ and for any $\mathfrak{g} \in \mathcal{E}$,

$$\frac{1}{n} \log \frac{\|g_{n-1} \cdots g_0 v\|}{\|v\|} = \frac{1}{n} \sum_{j=0}^{n-1} \Phi\left(g_j, g_{j-1} \cdots g_0 v\right) = \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(g_j, g_j g_{j-1} \cdots g_0 v\right)$$
$$\in (\Sigma(\nu, \varphi) - \varepsilon, \Sigma(\nu, \varphi) + \varepsilon) = (\lambda_1(\nu) - \varepsilon, \lambda_1(\nu) + \varepsilon).$$

定理的证明至此完成。 □

第二部分：预备知识与证明概要

5. The equator is a repeller

设 $(\nu_k)_k$ 是一个收敛到空间 $\mathcal{P}_c(G)$ 中某个 ν_∞ 的序列。根据 (3.3)，对于每个 k，都存在一个 ν_k-平稳测度 η_k，定义在 P 上，使得

$$\lambda_1(\nu_k) = \int_{G \times P} \Phi \, d(\nu_k \times \eta_k).$$

由于 P 上的概率测度空间是弱*-紧的，为了证明$(\lambda_1(\nu_k))$ 收敛到 $\lambda_1(\nu_\infty)$，不妨假设序列 $(\eta_k)_k$收敛到某个概率测度 η_∞。然后

$$\lambda_1(\nu_k) \to \int_{G \times P} \Phi \, d(\nu_\infty \times \eta_\infty) \text{ when } k \to \infty.$$

测度 η_∞ 必然是 ν_∞-平稳的（参见 [65, 命题 5.9]）。

Now there are two alternatives. If $\int_{G \times P} \Phi \, d(\nu_\infty \times \eta_\infty) = \lambda_1(\nu_\infty)$ then

$$\lambda_1(\nu_k) \to \lambda_1(\nu_\infty),$$

as we wanted to prove. Otherwise, we are in the setting of Section 4: there is a proper subspace E of $\mathbb{R}^d$ such that

(i) E 是 ν_∞不变的，且

$$\lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 \mid E\| < \lambda_1(\nu_\infty).$$

(ii) For any $\varepsilon > 0$ and $\delta > 0$ there exists $N = N(\nu_\infty, \varepsilon, \delta) \in \mathbb{N}$ and for every $v^\perp \in \mathbb{P}(E^\perp)$ there exists $\mathcal{E} = \mathcal{E}(\nu_\infty, \varepsilon, \delta, v^\perp)$ such that $\nu_\infty^{\mathbb{N}}(\mathcal{E}^c) < \delta$ and

$$\frac{1}{n} \log \frac{\|g_{n-1}^\perp \cdots g_0^\perp v^\perp\|}{\|v^\perp\|} \in (\lambda_1(\nu_\infty) - \varepsilon, \lambda_1(\nu_\infty) + \varepsilon)$$

对任意 $\mathfrak{g} \in \mathcal{E}$ 和 $n \geqslant N$。

(iii) $\eta_\infty(E) > 0$.

We are going to see that the properties (i) - (iii) are incompatible with the fact that η_∞ is the limit of stationary measures for nearby random walks. Indeed, if E satisfies (i) and (ii) then it is a kind of repeller for the random walk on P associated to ν_∞ (the precise statements are in Section 5.2). As we are going to see, that implies that the η_k-measure of any neighborhood of E is small when k is large, so that the limit η_∞ cannot satisfy (iii); the proof will use some general tools that we introduce in Sections 6.1 through 6.4. Hence the second alternative above cannot actually occur, and thus Theorem B will follow.

5.1. **Generic measures.** Actually, it suffices to carry these arguments in a special case, as we are going to explain. For each $1 \leqslant r \leqslant d$, let $\mathrm{Gr}(r,d)$ be the Grassmannian manifold of r-dimensional subspaces of $\mathbb{R}^d$. Moreover, let $\mathcal{F}(r,d)$ be the space of flags

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r \subset \mathbb{R}^d,$$

where each F_i has dimension i. Note that $\mathrm{Gr}(1,d) = \mathcal{F}(1,d)$ coincides with the projective space P.

The natural action of G on the projective space extends to group actions on every $\mathrm{Gr}(r,d)$ and $\mathcal{F}(r,d)$. Thus, in particular, to each probability measure ν on G we may associate operators $\mathcal{P}_\nu$ and $\mathcal{P}_\nu^*$ acting, respectively, on bounded measurable functions and on measures of $\mathrm{Gr}(r,d)$ or $\mathcal{F}(r,d)$, just as we did for P in (3.1) and (3.2):

$$\mathcal{P}_\nu\psi(v) = \int_G \psi(gv)\, d\nu(g) \text{ and } \mathcal{P}_\nu^*\eta = \int_G (g_*\eta)\, d\nu(g). \tag{5.1}$$

We continue to say that a function ψ is *ν-stationary* if $\mathcal{P}_\nu\psi = \psi$ and a probability measure ζ is *ν-stationary* if $\mathcal{P}_\nu^*\zeta = \zeta$.

A subset of an algebraic variety X is *Zariski-dense* if it is not contained in any proper algebraic subvariety of X. The cases we are interested in are $X = \mathrm{Gr}(r,d)$, $\mathcal{F}(r,d)$, or G. We call a measure η on the algebraic variety *generic* if $\eta(M) = 0$ for any proper algebraic subvariety M. Then, in particular, η is *non-atomic*, meaning that $\eta(\{p\}) = 0$ for every point p in the domain.

Remark 5.1. The restrictions $\eta \mid U$ of generic measures are generic, and so are the products $\eta_1 \times \eta_2$ of generic measures. Moreover, if $\{\eta_t : t \in T\}$ is a family of generic measures, and ξ is a measure on T then $\eta = \int_T \eta_t\, d\xi(t)$ is a generic measure. In particular, the generic measures form a vector subspace.

命题 5.2。设 $X = \mathrm{Gr}(r,d)$ or $X = \mathcal{F}(r,d)$。如果 ν 是一个概率测度，其支撑在 G 中扎里斯基稠密，那么 ν 在 X上存在唯一的平稳测度 η，且该测度是典型的。

Proof. Lemmas 4.2 and 4.5 in Gol'dsheid, Margulis [45]. □

Proposition 5.3. *Every $\nu \in \mathcal{P}_c(G)$ is approximated by generic probability measures whose supports are Zariski-dense.*

Proof. For each $j \in \mathbb{N}$, let $\xi_j \in \mathcal{P}_c(G)$ be the normalized restriction of the Haar measure of G to the ball of radius $1/j$ around the identity I relative to some left-invariant distance on G. It is clear that $\operatorname{supp}\xi_j$ coincides with that ball, and ξ_j vanishes on any proper subvariety of G. Moreover, $(\xi_j)_j$ converges to the Dirac mass δ_I in the topology of $\mathcal{P}_c(G)$. For each $j \in \mathbb{N}$, let $\nu_j = \nu * \xi_j$ be the probability measure on G such that, for any limited function $\phi : G \to \mathbb{R}$,

$$\int_G \phi\, d\nu_j = \int_G \phi(gh)\, d\nu(g)\, d\xi_j(h).$$

The assumption on ξ_j ensures that ν_j is generic. Moreover, $(\nu_j)_j$ converges to ν in the weak* topology. Furthermore, $\operatorname{supp}\nu_j$ coincides with the $(1/j)$-neighborhood of the support of ν, and so $(\operatorname{supp}\nu_j)_j \to \operatorname{supp}\nu$ in the Hausdorff topology. This proves that $(\nu_j)_j \to \nu$ in $\mathcal{P}_c(G)$. The fact that $\operatorname{supp}\nu_j$ has non-empty interior implies that it is Zariski-dense, for every $j \in \mathbb{N}$. □

Corollary 5.4. *Suppose that* $(\lambda_1(\nu'_k))_k \to \lambda_1(\nu_\infty)$ *for any sequence* $(\nu'_k)_k$ *converging to* ν_∞ *in* $\mathcal{P}_c(G)$ *such that* ν'_k *is generic and* $\operatorname{supp}\nu'_k$ *is Zariski-dense in* G *for every* k*. Then* $(\lambda_1(\nu_k))_k \to \lambda_1(\nu_\infty)$ *for every sequence* $(\nu_k)_k$ *converging to* ν_∞ *in* $\mathcal{P}_c(G)$*.*

Proof. Let $(\nu_k)_k$ be any sequence converging to ν_∞ in $\mathcal{P}_c(G)$, and d be any distance generating the topology of $\mathcal{P}_c(G)$. By Proposition 5.3, for each k we may find a generic probability measure ν'_k arbitrarily close to ν_k and whose support is Zariski-dense. Take ν'_k such that $d(\nu_k, \nu'_k) < 1/k$. Using the well-known fact that the function $\zeta \mapsto \lambda_1(\zeta)$ is upper semi-continuous (this is a consequence of (3.3) below), we may also suppose that $\lambda_1(\nu'_k) \leqslant \lambda_1(\nu_k) + 1/k$. Then $(\nu'_k)_k \to \nu_\infty$ and so $\lim_k \lambda_1(\nu'_k) = \lambda_1(\nu_\infty)$. Moreover,

$$\liminf_k \lambda_1(\nu_k) \geqslant \lim_k \lambda_1(\nu'_k) = \lambda_1(\nu_\infty).$$

Using semi-continuity once more, this implies $\lim_k \lambda_1(\nu_k) = \lambda_1(\nu_\infty)$. □

Thus, to prove Theorem B it suffices to consider sequences $(\nu_k)_k \to \nu_\infty$ of generic measures such that every $\operatorname{supp}\nu_k$ is Zariski-dense in G. We do so in all that follows. Then, by Proposition 5.2, the ν_k-stationary measure η_k is unique and generic.

5.2. 排斥行为。给定任意子空间 $W \subset \mathbb{R}^d$，令 $\Pi_W : \mathbb{R}^d \to W^\perp$be 为沿 W 的正交投影。当 W 是一维时，我们也用 $\Pi_W = \Pi_w$ 表示任意非零向量 $w \in W$。

对于任意的 $g \in G$ 和 $v \in P$，令 $Dg_v : T_vP \to T_{gv}P$ 表示$g : G \to G$ 在点 $v \in P$处的导数。切空间 T_vP 自然地与正交空间 $\{v\}^\perp$相识别。然后

$$Dg_v\dot{v} = \Pi_{gv} g\dot{v} \frac{\|v\|}{\|gv\|}. \tag{5.2}$$

For $v \in E$, let

$$Dg_v^\perp = \Pi_E \circ Dg_v \mid E^\perp : E^\perp \to E^\perp, \quad Dg_v^\perp v^\perp = g^\perp v^\perp \frac{\|v\|}{\|gv\|}. \tag{5.3}$$

对于 $n \in \mathbb{N}$ 以及定义在 G 上的任意概率测度 ν，令 $\nu^{(n)}$ 表示 n-卷积，即 $\nu^{\mathbb{N}}$ 在映射 $G^{\mathbb{N}} \to G$ 下的推前，其中 $G^{\mathbb{N}} \to G$ 由 $\mathfrak{g} \mapsto g_{n-1} \cdots g_0$ 定义：

$$\nu^{(n)} = \nu^{\mathbb{N}}(\{\mathfrak{g} \in G^{\mathbb{N}} : g_{n-1} \cdots g_0 \in B\}) \tag{5.4}$$

for any measurable set $B \subset G$. Note that the map $\nu \mapsto \nu^{(n)}$ is continuous relative to the weak* topology.

Proposition 5.5. *There exists* $\kappa_0 = \kappa_0(\nu_\infty) > 0$ *and for each* $\delta > 0$ *there exist* $N_0 = N_0(\nu_\infty, \delta) \in \mathbb{N}$ *and* $\tau_0 = \tau_0(\nu_\infty, \delta) > 0$ *such that for every* $n \geqslant N$ *and* $v^\perp \in \mathbb{P}(E^\perp)$ *there exists* $\mathcal{E}_0 = \mathcal{E}_0(\nu_\infty, \delta, n, v^\perp) \subset \operatorname{supp}\nu_\infty^{(n)}$ *with* $\nu_\infty^{(n)}(\mathcal{E}_0^c) < \delta$ *and*

(1) $\log \dfrac{\|Dg_v^\perp v^\perp\|}{\|v^\perp\|} > \kappa_0 n$ *for any* $g \in \mathcal{E}_0$ *and* $v \in E$*;*

(2) $\dfrac{\|g^\perp v^\perp\|}{\|gv^\perp\|} > \tau_0$ *for any* $g \in \mathcal{E}_0$*.*

证明。固定数字 $\alpha = \alpha(\nu_\infty)$ 和 $\beta = \beta(\nu_\infty)$，使得

$$\lim_n \frac{1}{n} \log \|g_{n-1} \cdots g_0 \mid E\| < \alpha < \beta < \lambda_1(\nu_\infty)$$

然后选择 $0 < \kappa_0 < \beta - \alpha_\circ$ 设 $\delta > 0_\circ$ 根据性质 (i)，存在 $N' \in \mathbb{N}$ 和 $\mathcal{E}' \subset G^{\mathbb{N}}$，使得 $\nu_\infty^{\mathbb{N}}((\mathcal{E}')^c) < \delta/4$，并且

$$\log \frac{\|g_{n-1}\cdots g_0 v\|}{\|v\|} \leqslant \alpha n \text{ for any } \mathfrak{g} \in \mathcal{E}' \text{ and } v \in E \text{ and } n \geqslant N'. \tag{5.5}$$

根据性质 (ii)，存在 $N'' \in \mathbb{N}$，并且对于每个 $v^\perp \in E^\perp$，存在 $\mathcal{E}'' \subset G^{\mathbb{N}}$，使得 $\nu_\infty^{\mathbb{N}}((\mathcal{E}'')^c) < \delta/4$，并且

$$\log \frac{\|g_{n-1}^\perp \cdots g_0^\perp v^\perp\|}{\|v^\perp\|} \geqslant \beta n \text{ for any } \mathfrak{g} \in \mathcal{E}'' \text{ and } n \geqslant N''. \tag{5.6}$$

Let $N''' = \max\{N', N''\}$ and for each $n \geqslant N'''$ and $v^\perp \in \mathbb{P}(E^\perp)$ define

$$\mathcal{E}''' = \{g_{n-1}\cdots g_0 : \mathfrak{g} \in \mathcal{E}' \cap \mathcal{E}''\}.$$

定义（5.4）给出 $\nu_\infty^{(n)}((\mathcal{E}''')^c) \leqslant \nu_\infty^{\mathbb{N}}((\mathcal{E}')^c \cup (\mathcal{E}'')^c) < \delta/2_\circ$ 此外，如果$g \in \mathcal{E}'''$，那么由（5.5）和（5.6）可得。

$$\log \frac{\|Dg_v^\perp v^\perp\|}{\|v^\perp\|} = \log \frac{\|g^\perp v^\perp\|}{\|v^\perp\|} - \log \frac{\|gv\|}{\|v\|} \geqslant (\beta - \alpha)n > \kappa_0 n \tag{5.7}$$

对于任意的 $v \in E$ 和 $n \geqslant N'''_\circ$ 只要我们选择 $N_0 \geqslant N'''$ 和 $\mathcal{E}_0 \subset \mathcal{E}'''$，就可以得到结论 (1)，而这正是我们将在下一段中所做的。

Now let us explain how to obtain claim (2). The following elementary inequality will be used a couple of times:

$$|\sin\angle(v_1, v_2)| \leqslant \frac{\|v_1 \pm v_2\|}{\|v_2\|} \text{ for any non-zero } v_1, v_2 \in \mathbb{R}^d. \tag{5.8}$$

Let $a = (\chi_1 - \chi_2)/8$ where $\chi_1 > \chi_2$ are the two largest Lyapunov exponents in (2.1). Clearly, we may assume that β has been chosen greater than $\chi_1 - a$ (keep in mind that $\chi_1 = \lambda_1$). Consider the Oseledets splitting in (2.3) and denote $E^* = E^2 \oplus \cdots \oplus E^k$. By the Oseledets theorem (see [65, Theorem 4.2]), there exists a measurable function $\mathfrak{g}_\pm \mapsto c(\mathfrak{g}_\pm)$ with values in $(0, 1)$ such that

$$c(\mathfrak{g}_\pm)\, e^{n(\chi_1 - a)}||v_1\| \leqslant \|g_{n-1}\cdots g_0 v_1\| \leqslant c(\mathfrak{g}_\pm)^{-1} e^{n(\chi_1 + a)}\|v_1\| \tag{5.9}$$

$$\|g_{n-1}\cdots g_0 v_*\| \leqslant c(\mathfrak{g}_\pm)^{-1} e^{n(\chi_2 + a)}\|v_*\| \tag{5.10}$$

for any $v_1 \in E^1(\mathfrak{g}_\pm)$, $v_* \in E^*(\mathfrak{g}_\pm)$ and $n \in \mathbb{N}$, and for $\nu_\infty^{\mathbb{Z}}$-almost every $\mathfrak{g}_\pm$. In particular, $E^1(\mathfrak{g}_\pm) \cap E^*(\mathfrak{g}_\pm) = \{0\}$. Since the growth rate of every vector $v \in E$ is strictly less than $\lambda_1 = \chi_1$ (property (i) above), we also have that $E^1(\mathfrak{g}_\pm) \cap E = \{0\}$. Thus, up to reducing the function $c(\mathfrak{g}_\pm)$, we may suppose that

$$|\sin\angle(E^1(\mathfrak{g}_\pm), E^*(\mathfrak{g}_\pm))| \geqslant c(\mathfrak{g}_\pm) \text{ and } |\sin\angle(E^1(\mathfrak{g}_\pm), E)| \geqslant c(\mathfrak{g}_\pm) \tag{5.11}$$

for $\nu_\infty^{\mathbb{Z}}$-almost every $\mathfrak{g}_\pm$. Fix $b = b(\nu_\infty, \delta) > 0$ small enough that the set

$$\mathcal{A} = \{\mathfrak{g}_\pm \in G^{\mathbb{Z}} : c(\mathfrak{g}_\pm) \geqslant b\}$$

has $\nu_\infty^{\mathbb{Z}}(\mathcal{A}^c) < \delta/4$. Then fix $M = M(\nu_\infty, \delta) \in \mathbb{N}$ such that

$$e^{-an} < \frac{b^2}{2} \text{ for every } n \geqslant M. \tag{5.12}$$

Let $N_0 = \max\{N', N'', M\}$ and for each $n \geqslant N_0$ and $v^\perp \in \mathbb{P}(E^\perp)$ define

$$\mathcal{E}_0 = \{g_{n-1}\cdots g_0 : \mathfrak{g}_\pm \in \pi^{-1}\mathcal{E}' \cap \pi^{-1}\mathcal{E}'' \cap \mathcal{A} \cap \sigma^{-n}\mathcal{A}\} \cap \operatorname{supp} \nu_\infty^{(n)}$$

($\pi : G^{\mathbb{Z}} \to G^{\mathbb{N}}$ is the canonical projection). The choices of $\mathcal{E}'$, $\mathcal{E}''$ and $\mathcal{A}$ ensure that $\nu_\infty^{(n)}(\mathcal{E}_0^c) < \delta$. For each $g \in \mathcal{E}_0$, take $\mathfrak{g}_\pm \in \mathcal{E}' \cap \mathcal{E}'' \cap \mathcal{A} \cap \sigma^{-n}\mathcal{A}$ such that $g = g_{n-1} \cdots g_0$ and then let $v^\perp = v_1 + v_*$ be the decomposition of $v^\perp$ with respect to the splitting $\mathbb{R}^d = E^1(\mathfrak{g}_\pm) \oplus E^*(\mathfrak{g}_\pm)$. From property (5.11) and the fact that $\mathfrak{g}_\pm \in \mathcal{A}$ we get

$$\|v_*\| \leqslant \frac{\|v^\perp\|}{|\sin \angle(E^1(\mathfrak{g}_\pm), E^*(\mathfrak{g}_\pm))|} \leqslant c(\mathfrak{g}_\pm)^{-1}\|v^\perp\| \leqslant b^{-1}\|v^\perp\|. \tag{5.13}$$

由于 $\beta > \chi_1 - a$，不等式 (5.6) 给出

$$\|gv^\perp\| \geqslant \|g^\perp v^\perp\| \geqslant e^{(\chi_1 - a)n}\|v^\perp\|. \tag{5.14}$$

性质（5.9）和（5.10）给出

$$\|gv^\perp\| \leqslant b^{-1}e^{(\chi_1+a)n}\|v_1\| + b^{-1}e^{(\chi_2+a)n}\|v_*\|. \tag{5.15}$$

Putting the relations (5.13) through (5.15) together, and using (5.12), we obtain

$$\begin{aligned}\|v^\perp\| &\leqslant b^{-1}e^{2an}\|v_1\| + b^{-1}e^{(\chi_2-\chi_1+2a)n}\|v_*\| \\ &\leqslant b^{-1}e^{2an}\|v_1\| + b^{-2}e^{-6an}\|v^\perp\| \leqslant b^{-1}e^{2an}\|v_1\| + \frac{1}{2}\|v^\perp\|\end{aligned}$$

这证明了 $\|v_1\| \geqslant (b/2)e^{-2an}\|v^\perp\|$。将该不等式与 (5.13) 以及性质 (5.9) 和 (5.10) 结合起来，

$$\begin{aligned}\|gv_1\| &\geqslant be^{(\chi_1-a)n}\|v_1\| \geqslant (b^2/2)e^{(\chi_1-3a)n}\|v^\perp\| \\ \|gv_*\| &\leqslant b^{-1}e^{(\chi_2+a)n}\|v_*\| \leqslant b^{-2}e^{(\chi_2+a)n}\|v^\perp\|\end{aligned}$$

鉴于我们对 a 的选择以及关系式 (5.13)，这意味着

$$\frac{\|gv_*\|}{\|gv_1\|} \leqslant 2b^{-4}e^{(\chi_2-\chi_1+4a)n} \leqslant 2b^{-4}e^{-4an} < \frac{b}{2}e^{-2an} \leqslant \frac{b}{2}.$$

Then it follows that

$$|\sin \angle(gv^\perp, E^1(\sigma^n\mathfrak{g}_\pm))| \leqslant |\sin \angle(gv^\perp, gv_1)| \leqslant \frac{\|gv_*\|}{\|gv_1\|} < b/2.$$

Now, property (5.11) implies that $|\sin \angle(E^1(\sigma^n\mathfrak{g}_\pm), E)| \geqslant b$, because we have taken $\mathfrak{g}_\pm$ such that $\sigma^n\mathfrak{g}_\pm \in \mathcal{A}$. Since $|\sin|$ is a subadditive function, these two inequalities imply that $|\sin \angle(gv^\perp, E))| > b/2$, which means that $\|g^\perp v^\perp\| > (b/2)\|gv^\perp\|$. This proves (2) with $\tau_0 = b/2$. □

设 $d(\cdot,\cdot)$ 为定义在射影空间 P 上的距离，由

$$d(u, v) = \left\|\Pi_u \frac{v}{\|v\|}\right\| = |\sin \angle(u, v)|. \tag{5.16}$$

注意，对于每个 u，都有 $0 \leqslant d(u,v) \leqslant 1$ $v \in P$。接下来，我们将命题 5.5 第 (1) 部分中的无穷小估计用到赤道的距离来表述：

Corollary 5.6. *For each $n \geqslant N$ there exists $\rho_0 = \rho_0(\nu_\infty, n) > 0$ and for each $x \in P$ with $d(x, E) \leqslant \rho_0$ there exists $\mathcal{D}_\infty(x) = \mathcal{D}_\infty(\nu_\infty, \delta, n, x) \subset \operatorname{supp} \nu_\infty^{(n)}$ with $\nu_\infty^{(n)}(\mathcal{D}_\infty(x)^c) < \delta$ and*

$$-\log d(gx, E) \leqslant -\log d(x, E) - \frac{3\kappa_0}{4}n \quad \text{for any } g \in \mathcal{D}_\infty(x). \tag{5.17}$$

Proof. Let exp denote the exponential map of the Riemannian manifold P. For each $x \in P$ close to the equator we may write $x = \exp_v v^\perp$ for a (unique) $v^\perp \in E^\perp$ with $\|v^\perp\| = d(x,v) = d(x,E)$. Then

$$\lim_{x\to E} \frac{d(gx,E)}{\|Dg_v^\perp v^\perp\|} = \lim_{x\to E} \frac{d(\exp_{gv} Dg_v v^\perp, E)}{\|Dg_v^\perp v^\perp\|} = 1$$

and the limits are uniform in $g \in \operatorname{supp}\nu_\infty^{(n)}$. In particular, there exists $\rho_0 = \rho_0(\nu_\infty, n) > 0$ such that

$$d(x,E) \leqslant \rho_0 \quad \Rightarrow \quad \Big|\log \frac{d(gx,E)}{d(x,E)} - \log \frac{\|Dg_v^\perp v^\perp\|}{\|v^\perp\|}\Big| < \frac{\kappa_0}{4} n \tag{5.18}$$

for every $g \in \operatorname{supp}\nu_\infty^{(n)}$. Define $\mathcal{D}_\infty(x)$ to be the set $\mathcal{E}_0(\nu_\infty, \delta, n, v^\perp)$ given by Proposition 5.5. Then (5.17) follows from (5.18) and part (1) of Proposition 5.5. □

我们还需要将这些估计从 $k = \infty$ 扩展到任意大的 $k \in \mathbb{N}$:

Corollary 5.7. *For any $n \geqslant N$ and $\rho \in (0, \rho_0)$ there is $k_0 = k_0(\nu_\infty, \delta, n, \rho) \in \mathbb{N}$ and for any $x \in P$ with $\rho \leqslant d(x,E) \leqslant \rho_0$ there is $\mathcal{D}_k(x) = \mathcal{D}_k(\nu_\infty, \delta, n, \rho, x) \subset \operatorname{supp}\nu_k^{(n)}$ such that $\nu_k^{(n)}(\mathcal{D}_k(x)^c) < \delta$ and*

$$-\log d(gx,E) \leqslant -\log d(x,E) - \frac{\kappa_0}{2} n \quad \text{for any } g \in \mathcal{D}_k(x) \text{ and } k \geqslant k_0. \tag{5.19}$$

Proof. Let $n \geqslant N$ and $\rho \in (0, \rho_0)$ be fixed. The set $K = \{x \in P : \rho \leqslant d(x,E) \leqslant \rho_0\}$ is compact. By continuity, there exists $\theta = \theta(\nu_\infty, n, \rho) > 0$ (keep in mind that $\kappa_0 = \kappa_0(\nu_\infty)$ and $\rho_0 = \rho_0(\nu_\infty, n)$), such that

$$\Big|\log \frac{d(gx,E)}{d(x,E)} - \log \frac{d(hy,E)}{d(y,E)}\Big| < \frac{\kappa_0}{4} n \tag{5.20}$$

for any $g \in B(h,\theta)$ and $x \in B(y,\theta)$ and $h \in \operatorname{supp}\nu_\infty^{(n)}$ and $y \in K$. Choose a finite set $\{x_j : j = 1, \dots, l\} \subset K$ such that the balls $B(x_j, \theta)$, $j = 1, \dots, l$ cover K. For each $x \in K$ choose $j \in \{1, \dots, l\}$ such that $x \in B(x_j, \theta)$ and define

$$\mathcal{D}_k(x) = \big[\ \theta\text{-neighborhood of } \mathcal{D}_\infty(x_j)\big] \cap \operatorname{supp}\nu_k^{(n)}. \tag{5.21}$$

设 $x \in K$ 和 $g \in \mathcal{D}_k(x)$。根据定义，存在 $x_j \in K$ 和 $h \in \mathcal{D}_\infty(x_j)$，使得 $d(x、x_j) < \theta$ 以及 $d(g、h) < \theta$。于是，由 (5.20) 可得

$$-\log d(gx,E) \leqslant -\log d(x,E) - \log d(hx_j,E) + \log d(x_j,E) + \frac{\kappa_0}{4} n,$$

whereas (5.17) gives that $-\log d(hx_j,E) \leqslant -\log d(x_j,E) - (3/4)\kappa_0 n$. Substituting the latter in the former one obtains (5.19). Since $\nu_k^{(n)}$ converges to $\nu_\infty^{(n)}$ in the weak* topology, the limit inferior of the $\nu_k^{(n)}$-measures of (5.21) is greater than or equal to

$$\nu_\infty^{(n)}\left(\mathcal{D}_\infty(x_j)\right) > 1 - \delta$$

for any $j = 1, \dots, l$. In particular, there is $k_0 = k_0(\nu_\infty, \delta, n, \rho) \in \mathbb{N}$ such that

$$\nu_k^{(n)}(\mathcal{D}_k(x)) > 1 - \delta \text{ for every } k \geqslant k_0 \text{ and } x \in K.$$

This completes the argument. □

6. A TOOLBOX

Here we collect several fairly general ideas and facts that are required for the continuation of our arguments. The proofs themselves will not be used in what follows, so the reader is advised to skip them at this stage, and to return to this section for reference along the way, as needed. In our applications, the metric spaces X and X' will be suitable subsets of Grassmannian manifolds or flag varieties.

6.1. **Couplings.** Let η and η' be measures on measurable spaces X and X', respectively, with $\eta(X) = \eta'(X')$. A measure $\tilde{\eta}$ on $X \times X'$ is a *coupling* of η and η' if it projects to η on the first factor and to η' on the second factor, that is, if

$$\tilde{\eta}(A \times X') = \eta(A) \text{ and } \tilde{\eta}(X \times A') = \eta'(A')$$

for any measurable sets $A \subset X$ and $A' \subset X'$. When $\eta = \eta'$ (and $X = X'$) we call $\tilde{\eta}$ a *self-coupling* of η. A self-coupling is *symmetric* if it is invariant under the involution $\iota : (x, x') \mapsto (x', x)$.

For example, the *diagonal embedding* of a measure η on X is the symmetric self-coupling $\tilde{\eta}$ of η defined by

$$\tilde{\eta}(B) = \eta(\{x \in X : (x, x) \in B\}). \tag{6.1}$$

耦合的另一个简单例子是缩放乘积

$$\tilde{\eta} = \frac{1}{c}(\eta \times \eta'), \text{ where } c = \eta(X) = \eta'(X'). \tag{6.2}$$

Couplings are far from being unique, which turns out to be a very convenient feature in our context. Especially, we will use the following elementary construction:

Lemma 6.1. *Suppose that $A \subset X$ and $A' \subset X'$ are such that $\eta(A) < \eta(X \backslash A)$ and $\eta'(A') < \eta'(X' \backslash A')$. Then there exists a coupling $\tilde{\eta}$ of η and η' such that $\tilde{\eta}(A \times A') = 0$.*

证明。令 $B = X \backslash A$ 和 $B' = X' \backslash A'$，然后取

$$\begin{aligned} \tilde{\eta} = {} & \frac{1}{\eta'(B')}(\eta \mid A) \times (\eta' \mid B') + \frac{1}{\eta(B)}(\eta \mid B) \times (\eta' \mid A') \\ & + \left(\frac{1}{\eta(B)} + \frac{1}{\eta'(B')} - \frac{c}{\eta(B)\eta'(B')} \right) (\eta \mid B) \times (\eta' \mid B'), \end{aligned} \tag{6.3}$$

其中 $c = \eta(X) = \eta'(X')$。 □

引理 6.2. 设 $A_j \subset C_j \subset X$ 和 $A'_j \subset C'_j \subset X'$、$j = 1$、.....、$k$ 满足以下条件。

(1) $\eta(C_j) < \eta(X \backslash C_j)$ *and* $\eta'(C'_j) < \eta'(X' \backslash C'_j)$ *for* $1 \leqslant j \leqslant k$;
(2) $A_j \times (X' \backslash C'_j)$ *and* $(X \backslash C_j) \times A'_j$ *are disjoint from* $A_i \times A'_i$ *for* $1 \leqslant i < j \leqslant k$.

Then there exists a coupling $\tilde{\eta}$ of η and η' such that $\tilde{\eta}(A_j \times A'_j) = 0$ for $j = 1, \ldots, k$.

Proof. We are going to construct couplings $\tilde{\eta}_j$, $j = 1, \ldots, k$ of η and η' such that

$$\tilde{\eta}_j(A_i \times A'_i) = 0 \text{ for any } 1 \leqslant i \leqslant j. \tag{6.4}$$

那么只需取 $\tilde{\eta} = \tilde{\eta}_k$。

The case $j = 1$ of (6.4) is contained in Lemma 6.1. We proceed by induction. Let $j \in \{2, \ldots, k\}$ and assume that we have constructed a coupling $\tilde{\eta}_{j-1}$ of η and η' such that $A_i \times A_i$ has zero measure for every $i = 1, \ldots, j-1$. We claim that

$$\tilde{\eta}_{j-1}(C_j \times C'_j) < \tilde{\eta}_{j-1}(C_j^c \times (C'_j)^c), \tag{6.5}$$

其中 $C_j^c = X\backslash C_j$ 和 $(C_j')^c = X'\backslash C_j'$。事实上，假设 $\eta'(C_j') \leqslant \eta(C_j)$。回顾一下，根据假设， $\eta(C_j) < \eta(C_j^c)$。此外，

- $\tilde\eta_{j-1}(C_j \times C_j') + \tilde\eta_{j-1}(C_j^c \times C_j') = \eta'(C_j')$,
- $\tilde\eta_{j-1}(C_j^c \times C_j') + \tilde\eta_{j-1}(C_j^c \times (C_j')^c) = \eta(C_j^c)$.

结合这些关系，我们得到（6.5）中的不等式。 $\eta(C_j) \leqslant \eta'(C_j')$ 的情况类似，因此命题得证。特别地，以下数值小于 1：

$$\theta_j = \frac{\tilde\eta_{j-1}(A_j \times A_j')}{\tilde\eta_{j-1}(C_j^c \times (C_j')^c)}.$$

Let $\pi : X \times X' \to X$ and $\pi' : X \times X' \to X'$ be the canonical projections. Let ζ_j be the scaled product of $\pi_*\left(\tilde\eta_{j-1} \mid A_j \times A_j'\right)$ and $\theta_j\pi'_*\left(\tilde\eta_{j-1} \mid C_j^c \times (C_j')^c\right)$, and ζ_j' be the scaled product of $\theta_j\pi_*\left(\tilde\eta_{j-1} \mid C_j^c \times (C_j')^c\right)$ and $\pi'_*\left(\tilde\eta_{j-1} \mid A_j \times A_j'\right)$. Then define

$$\tilde\eta_j = \tilde\eta_{j-1} - \left(\tilde\eta_{j-1} \mid A_j \times A_j'\right) - \theta_j\left(\tilde\eta_{j-1} \mid C_j^c \times (C_j')^c\right) + \zeta_j + \zeta_j'. \tag{6.6}$$

显然， $\tilde\eta_j$ 是一个（正）测度，因为 $\theta_j < 1$。同样显然， $\tilde\eta_j$ 是 η 和 η'的一个耦合：

$$\begin{aligned}\pi_*\tilde\eta_j = \pi_*\tilde\eta_{j-1} - \pi_*\left(\tilde\eta_{j-1} \mid A_j \times A_j'\right) - \theta_j\pi_*\left(\tilde\eta_{j-1} \mid C_j^c \times (C_j')^c\right)\\ + \pi_*\left(\tilde\eta_{j-1} \mid A_j \times A_j'\right) + \theta_j\pi_*\left(\eta_{j-1} \mid C_j^c \times (C_j')^c\right) = \pi_*\tilde\eta_{j-1} = \eta\end{aligned}$$

and, analogously, $\pi'_*\tilde\eta_j = \eta'$. Note also that $\tilde\eta_j(A_j \times A_j') = 0$. Moreover, ζ_j is concentrated in $A_j \times (C_j')^c$ and ζ_j' is concentrated in $C_j^c \times A_j'$. Since both sets are assumed to be disjoint from $A_i \times A_i'$, we get that

$$\tilde\eta_j\left(A_i \times A_i'\right) \leqslant \tilde\eta_{j-1}\left(A_i \times A_i'\right) = 0 \text{ for } i = 1, \ldots, j-1.$$

This completes the induction. □

Remark 6.3. By construction, the coupling $\tilde\eta$ varies continuously with η and η' in the weak*-topology at all points such that the boundaries of all A_j and C_j have zero η-measure and the boundaries of all A_j' and C_j' have zero η'-measure.

Remark 6.4. The constructions in Lemmas 6.1 and 6.2 preserve the class of non-atomic measures and, when X is an algebraic variety, also the class of generic measures: if η and η' are generic then so is the coupling $\tilde\eta$. That follows directly from Remark 5.1 and the expressions (6.3) and (6.6).

Remark 6.5. When $X = X'$, these constructions are *involution-invariant* in the following sense. First, in Lemma 6.1 the coupling $\tilde\eta$ given by (6.3) is replaced with $\iota_*\tilde\eta$ when one exchanges the roles of η, A and η', A'. In particular, if $\eta = \eta'$ and $A = A'$ then the self-coupling $\tilde\eta$ is symmetric. In Lemma 6.2 we may take $\zeta_j' = \iota_*\zeta_j$ and then the coupling $\tilde\eta_j$ is replaced with $\iota_*\tilde\eta_j$ when the roles of η, A_i, C_i and η', A_i', C_i' are exchanged. In particular, if $\eta = \eta'$, $A_i = A_i'$, and $C_i = C_i'$ then the self-coupling $\tilde\eta = \tilde\eta_k$ is symmetric.

Proposition 6.6. *Let K be a compact subset of the product $X \times X'$ such that*

$$K(x') = \{y \in X : (y, x') \in K\} \text{ and } K'(x) = \{y' \in X' : (x, y') \in K\}.$$

satisfy

$$\eta\left(K(x')\right) < \eta\left(X\backslash K(x')\right) \text{ and } \eta'\left(K'(x)\right) < \eta'\left(X'\backslash K'(x)\right) \tag{6.7}$$

for every $(x, x') \in X \times X'$. Then there exists a coupling $\tilde{\eta}$ of η and η' that vanishes on a neighborhood of K.

证明。我们将找到开集 $A_j \subset C_j \subset X$ 和 $A'_j \subset C'_j \subset X'$、 $j = 1$、...、 k, 使得

(a) $\eta(C_j) < \eta(X \backslash C_j)$ and $\eta(C'_j) < \eta(X \backslash C'_j)$;
(b) $A_j \times A'_j$ is disjoint from $(X \backslash C_i) \times A'_i$ and $A_i \times (X \backslash C'_i)$;
(c) and the union of the products $A_j \times A'_j$ contains K.

Then, by Lemma 6.2, there exists a coupling $\tilde{\eta}$ of η and η' vanishing on the union of $A_j \times A'_j$, $j = 1, \dots, k$, which gives the claim of the present lemma. So let us explain how to construct such sets.

By compactness, there exists $\varepsilon > 0$ such that (6.7) remains valid when $K(x')$ and $K'(x)$ are replaced with their 10ε-neighborhoods. Let ε be fixed. Also by compactness, the maps $x \mapsto K'(x)$ and $x' \mapsto K(x')$ are upper semicontinuous. In particular, for any $x \in X$ and $x' \in X'$ there exist $\delta(x) > 0$ and $\delta'(x') > 0$ such that

$$\begin{aligned} &K'(y) \subset B_\varepsilon\big(K'(x)\big) \text{ if } d(x, y) < 4\delta(x) \text{ and} \\ &K(y') \subset B_\varepsilon\big(K(x')\big) \text{ if } d(x', y') < 4\delta(x'). \end{aligned} \tag{6.8}$$

It is no restriction to assume that $\delta(x)$ and $\delta'(x')$ are bounded by ε, and we do so. Let $(x_1, x'_1), \dots, (x_k, x'_k) \in K$ be such that $B(x_j, \delta(x_j)) \times B(x'_j, \delta'(x'_j))$, $j = 1, \dots, k$ cover K. Initially, define

$$\begin{aligned} A_j = B(x_j, 3\delta(x_j)), \quad A'_j = B(x'_j, 3\delta'(x'_j)) \\ C_j = B\left(K(x'_j), 10\varepsilon\right), \quad C'_j = B\left(K'(x_j), 10\varepsilon\right). \end{aligned} \tag{6.9}$$

Note that $A_j \subset C_j$ and $A'_j \subset C'_j$ since $(x_j, x'_j) \in K$ and $\delta, \delta' \leqslant \varepsilon$. Conditions (a) and (c) are clearly satisfied, but not necessarily (b). In the following we replace the sets A_j and A'_j with suitable subsets, in such a way as to achieve (b) while preserving (c). The condition (a) is clearly not affected.

PSfrag replacements

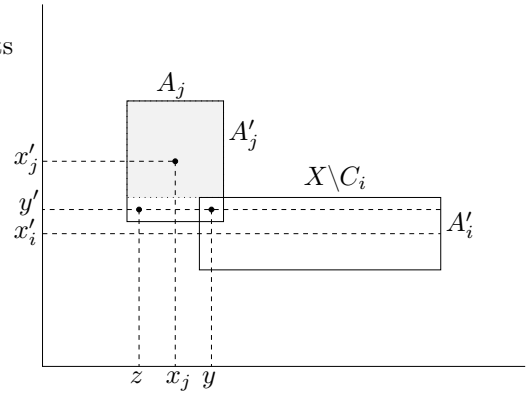

FIGURE 1. Trimming the sets A_j and A'_j: to achieve the disjointness condition (c), in the situation described in the figure $A_j \times A'_j$ is replaced with $A_j \times (A'_j \backslash \bar{A}'_i)$, which corresponds to the shaded region. A dual operation is applied to A_j, and the whole procedure is repeated for every i and j.

Consider any $i, j = 1, \dots, k$. If $A_j \times A'_j$ is disjoint from $A_i \times (X' \backslash C'_i)$ and $(X \backslash C_i) \times A'_i$ there is nothing to do. Next, let us consider the case when there exists (y, y') in $\big(A_j \times A'_j\big) \cap \big((X \backslash C_i) \times A'_i\big)$. See Figure 1. We claim that $A_j \times B(x'_i, 4\delta(x'_i))$ is disjoint from K. Indeed, suppose there existed $z \in A_j$ and $y'' \in B(x'_i, 4\delta(x'_i))$ such that $(z, y') \in K$. Since $d(z, y) < 6\delta(x_j) \leqslant 6\varepsilon$, it would follow that $y \in B(K(y''), 6\varepsilon)$. On the other hand, $K(y'') \subset B(K(x'_i), \varepsilon)$ because $d(y'', x'_i) < 4\delta(x'_i)$. Hence, we would have $y \in B(K(x'_i), 7\varepsilon)$, which would contradict the fact that $y \in X \backslash C_i$. This contradiction proves our claim. Now, this ensures that $A_j \times \bar{A}'_i$ is disjoint and at a definite distance from K. Thus, we may replace A'_j with $A'_j \backslash \bar{A}'_i$ in our construction without affecting condition (d) and, by doing it, we get that $A_j \times A'_j$ becomes disjoint from $(X \backslash C_i) \times A'_i$. The case when there exists (y, y') in $(A_j \times A'_j) \cap (A_i \times (X' \backslash C'_i))$ is treated in the same way, trimming A'_j instead. Repeating this procedure for each i and j, we get all three conditions (a) to (c). □

注 6.7. 对所有 $j = 1, \dots, k$ 的边界 ∂A_j 取并集，在经过修剪操作后，其并集不会增大；对于 $j = 1, \dots, k$ 上的边界 $\partial A'_j$ 的并集，同样如此。

Next we want to state and prove a parametrized version of Proposition 6.6. The following elementary fact will be useful at other places as well:

Lemma 6.8. *Let $g : Z \times T \to \mathbb{R}$ be a bounded measurable function, where Z is a metric space and (T, μ) is a probability space. Let $z_0 \in Z$ be such that the set $D(z_0)$ of values of $t \in T$ such that $z \mapsto g(z, t)$ is discontinuous at z_0 has zero μ-measure. Then $z \mapsto \int_T g(z, t)\, d\mu(t)$ is continuous at $z = z_0$.*

Proof. Fix any $\varepsilon > 0$. For each $k \in \mathbb{N}$, denote by T_k the set of values of $t \in T$ such that $|g(z_0, t) - g(z, t)| \leqslant \varepsilon$ for any z in the $(1/k)$-neighborhood of z_0. The sequence T_k is non-decreasing and the assumption ensures that $\cup_k T_k$ has full μ-measure. Fix k such that $\mu(T_k^c) < \varepsilon$. Then for any z in the $(1/k)$-neighborhood of z_0,

$$\left| \int_T g(z_0, \cdot)\, d\mu - \int_T g(z, \cdot)\, d\mu \right| \leqslant \varepsilon + \left| \int_{T_k^c} g(z_0, \cdot)\, d\mu \right| + \left| \int_{T_k^c} g(z, \cdot)\, d\mu \right| \leqslant (1 + 2\|g\|)\varepsilon.$$

由于 $\varepsilon > 0$ 任意，这就证明了 z_0 是一个连续点。 □

Proposition 6.9. *Let X, X', Y, and Y' be compact metric spaces, K be a compact subset of $X \times X'$, and $\eta = \{\eta_y : y \in Y\}$ and $\eta' = \{\eta'_{y'} : y' \in Y'\}$ be continuous families of probability measures on X and X', respectively, such that*

$$\eta_y\big(K(x')\big) < \eta_y\big(X \backslash K(x')\big) \text{ and } \eta'_{y'}\big(K'(x)\big) < \eta'_{y'}\big(X' \backslash K'(x)\big). \tag{6.10}$$

for every $(x, x') \in X \times X'$ and $(y, y') \in Y \times Y'$. Then there exists a continuous family $\tilde{\eta} = \{\tilde{\eta}_{y,y'} : (y, y') \in Y \times Y'\}$ of probability measures on $X \times X'$ such that each $\tilde{\eta}_{y,y'}$ is a coupling of η_y and $\eta'_{y'}$ vanishing on a uniform neighborhood of K.

Proof. By compactness, the assumption (6.10) implies that there exists $\varepsilon > 0$ such that

$$\eta_y(B(K(x'), 10\varepsilon)) < \eta_y(X \backslash B(K(x'), 10\varepsilon)) \text{ and}$$
$$\eta'_{y'}(B(K'(x), 10\varepsilon)) < \eta'_{y'}(X' \backslash B(K'(x), 10\varepsilon))$$

for every $(x, x') \in X \times X'$ and $(y, y') \in Y \times Y'$. Fix ε and let $\delta(x), \delta'(x') \in (0, \varepsilon)$ be as in (6.8). Let $(x_1, x'_1), \dots, (x_k, x'_k) \in K$ be such that $B(x_j, \delta(x_j)) \times B(x'_j, \delta'(x'_j))$,

$j = 1, \dots, k$ cover K. For each $s \in [0,1]$ and $j = 1, \dots, k$, define

$$\begin{aligned} A_{j,s} &= B(x_j, (3-s)\delta(x_j)), \quad A'_{j,s} = B(x'_j, (3-s)\delta'(x'_j)) \\ C_{j,s} &= B\left(K(x'_j), (10-s)\varepsilon\right), \quad C'_{j,s} = B\left(K'(x_j), (10-s)\varepsilon\right). \end{aligned} \tag{6.11}$$

Applying to the measures η_y and $\eta'_{y'}$, and the sets $A_{j,s}$, $C_{j,s}$, $A'_{j,s}$, and $C'_{j,s}$ the same construction in Lemma 6.2, we find for each $s \in [0,1]$ and $(y, y') \in Y \times Y'$ a coupling $\tilde\eta_{y,y',s}$ of η_y and $\eta'_{y'}$ which vanishes on a neighborhood of K independent of both s and (y, y'). (recall that the trimming is always done at a definite distance from K).

By Remark 6.3, for each fixed $s \in [0,1]$ the map $(y, y') \mapsto \tilde\eta_{y,y',s}$ is discontinuous at a given point $(z, z') \in Y \times Y'$ only if the union of the boundaries of $A_{j,s}$ and $C_{j,s}$ has positive measure for η_z or the union of the boundaries of $A'_{j,s}$ and $C'_{j,s}$ has positive measure for $\eta'_{z'}$. In the setting of (6.11), the boundaries of the $A_{j,s}$ are pairwise disjoint for each fixed j, and the same is true for the $C_{j,s}$, $A'_{j,s}$, and $C'_{j,s}$. Thus, positive measure may occur only for a countable subset of values of s. Remark 6.7 ensures that the latter conclusion remains valid after the trimming. In conclusion, every $(z, z') \in Y \times Y'$ is a continuity point of $(y, y') \mapsto \tilde\eta_{y,y',s}$ for all but countably many values of $s \in [0,1]$. Then, using Lemma 6.8,

$$\tilde\eta_{y,y'} = \int_0^1 \tilde\eta_{y,y',s}\, ds$$

is a coupling of η_y and $\eta'_{y'}$ depending continuously on y and y' and vanishing on a uniform neighborhood of K. □

Remark 6.10. When $X = X'$ and the set $K \subset X \times X$ is symmetric, one may exchange A_j, C_j with A'_j, C'_j in (6.9), and $A_{j,s}, C_{j,s}$ with $A'_{j,s}, C'_{j,s}$ in (6.11). Thus (recall Remark 6.5) when the roles of η and η' are exchanged the coupling $\tilde\eta$ is replaced with $\iota_*\tilde\eta$ in Proposition 6.6 and each $\tilde\eta_{y',y}$ is replaced with $\iota_*\tilde\eta_{y,y'}$ in Proposition 6.9. In particular, if $\eta = \eta'$ and $Y = Y'$ then the self-coupling $\tilde\eta$ in Proposition 6.6 is symmetric, and the family $\tilde\eta$ in Proposition 6.9 is involution-invariant, meaning that $\tilde\eta_{y',y} = \iota_*\tilde\eta_{y,y'}$ for every $(y, y') \in Y \times Y$.

命题 6.6 的以下特殊情形将会很有用：

Corollary 6.11. *Let $X = X'$ and assume that η and η' satisfy $\eta(\{x\}) < \eta(X\backslash\{x\})$ and $\eta'(\{x\}) < \eta'(X\backslash\{x\})$ for every $x \in X$. Then there exists a coupling $\tilde\eta$ of η and η' that vanishes on a neighborhood of the diagonal of $X \times X$. If $\eta = \eta'$ then the self-coupling $\tilde\eta$ may be chosen to be symmetric.*

6.2. **Markov operators.** Let X be a metric space. We denote by $\mathcal{B}(X)$ the Banach space of bounded measurable functions on X, with the norm

$$\|\psi\| = \sup\{|\psi(x)| : x \in X\}.$$

马尔可夫算子是一种线性算子 $\mathcal{T}$: $\mathcal{B}(X) \to \mathcal{B}(X)$，其形式为

$$\mathcal{T}\psi(x) = \int_X \psi(y)\, d\sigma_x(y) \tag{6.12}$$

where $\{\sigma_x : x \in X\}$ is a measurable family of probability measures on X. It is clear that $\mathcal{T}$ is a bounded operator, with $\|\mathcal{T}\| = 1$. We call it *continuous* if the map $x \mapsto \sigma_x$ is continuous relative to the weak* topology. Then $\mathcal{T}$ preserves the subspace $\mathcal{C}(X)$ of bounded continuous functions.

The *dual operator* $\mathcal{T}^*$ is defined on the space of bounded finitely additive signed measures η on X with the total variation norm (see [29, IV.4.5]) by

$$\mathcal{T}^*\eta = \int_X \sigma_x \, d\eta(x). \tag{6.13}$$

The two are related by

$$\int_X \psi \, d\,(\mathcal{T}^*\eta) = \int_X (\mathcal{T}\psi)\; d\eta \text{ for every } \psi \text{ and } \eta. \tag{6.14}$$

A measure η is $\mathcal{T}$*-invariant* if $\mathcal{T}^*\eta = \eta$. If $\mathcal{T}$ is continuous then $\mathcal{T}^*$ preserves the subspace of bounded (countably additive) signed measures on X.

Remark 6.12. If $\{\sigma_x : x \in X\}$ is such that every σ_x is a generic measure then, cf. Remark 5.1, the range of the associated dual Markov operator $\mathcal{T}^*\eta = \int_X \sigma_x \, d\eta(x)$ is contained in the space of generic measures. In particular, any $\mathcal{T}$-invariant measure is a generic measure.

Suppose that X comes with a transitive G–action $(g, x) \mapsto gx$. Grassmannian manifolds $\mathrm{Gr}(r, d)$ and flag varieties $\mathcal{F}(r, d)$ are the examples we have in mind. Then, to any probability measure ν on G we may associate the Markov operators $\mathcal{P}_\nu$ and $\mathcal{P}_\nu^*$ in (3.1), (3.2), and (5.1)

$$\mathcal{P}_\nu\psi(x) = \int_G \psi(gx)\, d\nu(g) \text{ and } \mathcal{P}_\nu^*\eta = \int_G (g_*\eta)\; d\nu(g). \tag{6.15}$$

This corresponds to (6.12) with $\sigma_x =$ the push-forward of ν under the map $g \mapsto gx$. Note that $\mathcal{P}_\nu$ is continuous if the G-action is continuous, and a measure η on Z is $\mathcal{P}_\nu$-invariant precisely if it is ν-stationary. These are the fundamental examples of Markov operators in our context, but we will have to deal with other types as well.

One reason is that the kind of conclusion we are seeking, namely, that stationary measures give small weights to a neighborhood of the equator is local in nature. That is consistent with the fact that the information on the dynamics we can extract from Proposition 5.5 is clearly local. In contrast, the property of being a stationary measure is *not* local: the restriction of a $\mathcal{P}_\nu$-invariant measure η to some set $U \subset X$ is usually not a $\mathcal{P}_\nu$-invariant measure. The way we handle this is by finding a "localized" Markov operator, related to the original one and to the domain U in an explicit manner, with respect to which the restriction $\eta \mid U$ is indeed an invariant measure.

Remark 6.13. The assumption that the G-action on X is transitive means that $G \to X$, $g \mapsto gx$ is surjective for any fixed $x \in X$. Then every probability measure σ on X lifts (non-uniquely) to a probability measure μ on G: use the Hahn–Banach and Riesz–Markov theorems. In particular, (6.12) may be written in the form

$$\mathcal{T}\psi(x) = \int_G \psi(gx)\, d\mu_x(g)$$

对于某个由 $\{\mu_x : x \in X\}$ 上的概率测度构成的族而言。本文并未使用这一一般性结论，但其在一个特殊情形下的显式构造将在引理 16.1 中给出。

Proposition 6.14. *Let $\mathcal{T} : \mathcal{B}(X) \to \mathcal{B}(X)$ be a Markov operator, η be a $\mathcal{T}$-invariant measure, and $U \subset X$ be such that $\eta(U) > 0$. Then there exists a Markov operator $\mathcal{T}_U : \mathcal{B}(U) \to \mathcal{B}(U)$ that leaves invariant the normalized restriction η_U of η to the subset U.*

证明。我们将找到 $\{\sigma_{U,x}\colon\ x \in U\}$ 使得 $\mathcal{T}_U\psi(x) = \int_U \psi(y)\, d\sigma_{U,x}(y)$保持 η_U。令 χ_U表示 U的特征函数。由于 η是 $\mathcal{T}$-不变的，

$$\begin{aligned} 0 &= \int_X (\mathcal{T}\chi_U - \chi_U)\ d\eta = \int_X \left[\int_X \chi_U(y)\, d\sigma_x(y) - \chi_U(x)\right] d\eta(x) \\ &= \int_U [\sigma_x(U) - 1]\ d\eta(x) + \int_{U^c} \sigma_x(U)\, d\eta(x), \end{aligned}$$

that is,

$$\int_U \sigma_x(U^c)\, d\eta(x) = \int_{U^c} \sigma_x(U)\, d\eta(x). \tag{6.16}$$

令该数为 J。如果 $J = 0$，则无需做太多处理： η_U 结果证明是$\mathcal{T}$-不变的，因此只需取 $\sigma_{U,x} = \sigma_x$，其中 $x \in U$。如果 $J > 0$，则定义

$$\sigma_{U,x} = (\sigma_x \mid U) + \sigma_x(U^c)\frac{1}{J}\int_{U^c} (\sigma_z \mid U)\, d\eta(z) \tag{6.17}$$

对每个 $x \in U$。换言之，

$$\begin{aligned} \mathcal{T}_U\psi(x) &= \int_U \psi(y)\, d\sigma_x(y) + \sigma_x(U^c)\frac{1}{J}\int_{U^c}\int_U \psi(y)\, d\sigma_z(y)\, d\eta(z) \\ \text{and } \mathcal{T}_U^*\xi &= \int_U (\sigma_x \mid U)\, d\xi(x) + \int_U \sigma_x(U^c)\, d\xi(x)\frac{1}{J}\int_{U^c} (\sigma_z \mid U)\, d\eta(z) \end{aligned}$$

注意，$\sigma_{U,\ x}$ 是 U上的一个概率：

$$\sigma_{U,x}(U) = \sigma_x(U) + \sigma_x(U^c)\frac{1}{I}\int_{U^c} \sigma_x(U)\, d\xi(x) = \sigma_x(U) + \sigma_x(U^c) = 1.$$

此外，根据 J 的定义，

$$\begin{aligned} \mathcal{T}^*\eta_U &= \int_U (\sigma_x \mid U)\, d\eta_U(x) + \int_U \sigma_x(U^c)\, d\eta_U(x)\frac{1}{J}\int_{U^c} (\sigma_z \mid U)\, d\eta(z) \\ &= \frac{1}{\eta(U)}\left(\int_U (\sigma_x \mid U)\, d\eta(x) + \int_{U^c} (\sigma_z \mid U)\, d\eta(z)\right) = \frac{1}{\eta(U)}\int_X (\sigma_x \mid U)\, d\eta(x). \end{aligned}$$

Each $\psi \in \mathcal{B}(U)$ may be viewed as an element of $\mathcal{B}(X)$ that vanishes outside U. Then

$$\begin{aligned} \int_U \psi\, d\,(\mathcal{T}_U^*\eta_U) &= \frac{1}{\eta(U)}\int_X\int_U \psi(y)\, d\sigma_x(y)\, d\eta(x) = \frac{1}{\eta(U)}\int_X\int_X \psi(y)\, d\sigma_x(y)\, d\eta(x) \\ &= \frac{1}{\eta(U)}\int_X \mathcal{T}\psi(x)\, d\eta(x) = \frac{1}{\eta(U)}\int_X \psi(x)\, d\eta(x) = \int_U \psi\, d\eta_U. \end{aligned}$$

因此， $\mathcal{T}^*\eta_U = \eta_U$，这正是我们想要证明的。 □

The operator $\mathcal{T}_U$ in Proposition 6.14 need not be continuous, in general. In the next proposition we fix that problem.

Proposition 6.15. *Let $\mathcal{T} : \mathcal{B}(X) \to \mathcal{B}(X)$ be a Markov operator, η be a $\mathcal{T}$-invariant measure, and $U \subset X$ be such that $\eta(U) > 0$. Assume that $x \mapsto \sigma_x$ is continuous on U. Then there is a continuous Markov operator $\mathcal{T}_U : \mathcal{B}(U) \to \mathcal{B}(U)$ such that the normalized restriction η_U is $\mathcal{T}_U$-invariant.*

Proof. The strategy is to consider a monotone family $\{U_t : t \in [0,1]\}$ of subsets of U with pairwise disjoint boundaries, and to associate to each U_t a Markov operator $\mathcal{T}_t$ such that $\mathcal{T}_t^* \eta_U = \eta_U$. These operators will still be discontinuous, but we can get rid of the discontinuities by integrating with respect to t. The details follow.

As before, let J be the number in (6.16). When $J = 0$ there is nothing to do, because in that case the construction in Proposition 6.14 does yield a continuous Markov operator $\mathcal{T}_U$. From now on, assume that $J > 0$. Let $a > 0$ be a small number. For each $t \in [0,1]$, define $U_t = \{x \in U : d(x, U^c) \geqslant at\}$ and

$$J(t) = \int_U \sigma_x(U_t^c)\, d\eta(x).$$

Note that $J(t) \geqslant J > 0$. Then let ξ_t be the probability measure defined on U by

$$\int_U (\sigma_x \mid U_t)\, d\eta(x) + J(t)\xi_t = \eta \mid U. \tag{6.18}$$

注意， ξ_t 是定义良好的（每个 $\sigma_x \mid U_t$ 都可以视为 U 上的一个测度，因为 $U_t \subset U$，因此该恒等式中的所有项都是 U 上的测度），并且它确实是一个概率测度。

Now let $\mathcal{T}_t : \mathcal{B}(U) \to \mathcal{B}(U)$ be the Markov operator associated to the family

$$\sigma_{x,t} = (\sigma_x \mid U_t) + \sigma_x(U_t^c)\xi_t \tag{6.19}$$

U上的概率测度。条件 (6.18) 意味着 η_U 是 $\mathcal{T}_t$-不变的:

$$\begin{aligned} \mathcal{T}_t^* \eta_U &= \int_U (\sigma_x \mid U_t)\, d\eta_U(x) + \int_U \sigma_x(U_t^c)\xi_t\, d\eta_U(x) \\ &= \int_U (\sigma_x \mid U_t)\, d\eta_U(x) + \frac{J(t)}{\eta(U)}\xi_t = \eta_U. \end{aligned} \tag{6.20}$$

接下来，将 $\mathcal{T}_U$ 定义为与该概率测度族相关联的马尔可夫算子。

$$\sigma_{U,x} = \int_0^1 \sigma_{x,t}\, dt. \tag{6.21}$$

由 (6.20) 可知， η_U 是 $\mathcal{T}_U$-不变的。

We are left to show that the map $x \mapsto \sigma_{U,x}$ is continuous with respect to the weak* topology, that is, that

$$x \mapsto \int \varphi\, d\sigma_{U,x} = \int_0^1 \left(\int_{U_t} \varphi(y)\, d\sigma_x(y) + \sigma_x(U_t^c) \int_U \varphi(z)\, d\xi_t(z) \right) dt \tag{6.22}$$

is continuous for any bounded continuous function $\varphi : U \to \mathbb{R}$. This will be a consequence of the following fact:

Lemma 6.16. *Let Z be a metric space, σ_0 be a probability measure on Z and $g : Z \to \mathbb{R}$ be a measurable function such that the closure of the set of discontinuity points has zero σ_0-measure. Then σ_0 is a continuity point of the map $\sigma \mapsto \int_Z g\, d\sigma$ in the space of probability measures on Z with the weak* topology.*

Proof. Denote by R the closure of the set of discontinuity points. Given $\varepsilon > 0$, let V be an open neighborhood of R whose closure $\overline{V}$ satisfies $\sigma_0(\overline{V}) < \varepsilon$. Then $\sigma(\overline{V}) < \varepsilon$ for any σ in a weak* neighborhood of σ_0. By the Tietze extension theorem, there

exists a continuous function $h : Z \to \mathbb{R}$ coinciding with g on the complement of V and satisfying $\|h\| \leqslant \|g\|$. Then

$$\left|\int_Z g\,d\sigma - \int_Z g\,d\sigma_0\right| \leqslant \left|\int_Z h\,d\sigma - \int_Z h\,d\sigma_0\right| + 4\|g\|\varepsilon \leqslant (1 + 4\|g\|)\,\varepsilon$$

对于 σ 中的任意元素，只要其属于 σ_0 的一个弱邻域。因此， σ_0 是一个连续点。 □

Going back to proving the proposition, fix any $z \in U$ and consider $Z = X$ and $\sigma_0 = \sigma_z$. Keep in mind that $x \mapsto \sigma_x$ is assumed to be continuous on U. Thus, applying Lemma 6.16 twice, to $g = \varphi\chi_{U_t}$ and to $g = \chi_{U_t^c}$, we see that z is a point of discontinuity of

$$x \mapsto \int_{U_t} \varphi(y)\,d\sigma_x(y) + \sigma_x(U_t^c)\int_U \varphi(z)\,d\xi_t(z) \tag{6.23}$$

only if the boundary $\partial U_t = \partial U_t^c$ has positive measure for σ_z. Since these boundaries are pairwise disjoint when t varies, the latter can only happen for countably many values of t. Thus, we may apply Lemma 6.8 to $Z = X$ and the function $g(x, t)$ given by the right-hand side of (6.23), to conclude that (6.22) is continuous. □

Remark 6.17. The localization procedure in Propositions 6.14 and 6.15 preserves the class of non-atomic measures and, when X is an algebraic variety, also the class of generic measures. That is a direct consequence of (6.17), (6.18), (6.19), and Remark 5.1.

6.3. **Invariant couplings.** Let X be a metric space X and $\mathcal{T} : \mathcal{B}(X) \to \mathcal{B}(X)$ be a Markov operator, given by

$$\mathcal{T}\psi(x) = \int_X \psi(y)\,d\sigma_x(y).$$

A *self-coupling* of $\mathcal{T}$ is a Markov operator $\tilde{\mathcal{T}} : \mathcal{B}(X \times X) \to \mathcal{B}(X \times X)$ of the form

$$\tilde{\mathcal{T}}\tilde{\psi}(x, x') = \int_{X\times X'} \tilde{\psi}(y, y')\,d\tilde{\sigma}_{x,x'}(y, y') \tag{6.24}$$

~其中每个 $\tilde{\sigma}_{x,x'}$ 都是 σ_x 和 $\sigma'_{x'}$ 的一种耦合。如果映射 $(x,x') \mapsto \sigma_{x,x'}$ 在 $X \times X$ 上是连续的，则自耦合 $\mathcal{T}$ 是连续的。

~引理 6.18. 如果 $\tilde{\eta}$ 是 η 和 η' 的一个耦合，并且 $\mathcal{T}$ 是 $\mathcal{T}$ 的一个自耦合，那么~$\mathcal{T}^*\tilde{\eta}$ 是 $\mathcal{T}^*\eta$ 和 $\mathcal{T}^*\eta'$的一个耦合。

~证明。设 ψ： $X \times X \to \mathbb{R}$ 为任意一个仅依赖于第一个变量的有界可测函数: $\psi(x, x') = \psi(x)$ 其中 $\psi \in \mathcal{B}(X)$。根据定义，

$$\begin{aligned}\int_{X\times X} \tilde{\psi}\,d(\tilde{\mathcal{T}}^*\tilde{\eta}) &= \int_{X\times X}\int_{X\times X} \tilde{\psi}(y, y')\,d\tilde{\sigma}_{x,x'}(y, y')\,d\tilde{\eta}(x, x') \\ &= \int_{X\times X}\int_{X\times X} \psi(y)\,d\tilde{\sigma}_{x,x'}(y, y')\,d\tilde{\eta}(x, x').\end{aligned}$$

Since $\tilde{\sigma}_{x,x'}$ projects to σ_x and $\tilde{\eta}$ projects to η on the first factor, this last expression may be written as

$$\int_{X\times X}\int_X \psi(y)\,d\sigma_x(y)\,d\tilde{\eta}(x, x') = \int_X\int_X \psi(y)\,d\sigma_x(y)\,d\eta(x) = \int_X \psi\,d(\mathcal{T}^*\eta).$$

~这证明 $\mathcal{T}^*\tilde{\eta}$ 在第一个因子上的投影为 $\mathcal{T}^*\eta$。类似地，它在第二个因子上的投影为 $\mathcal{T}^*\eta'$。□

~引理 6.19。假设 X 是紧致的，且 $\mathcal{T}$ 是连续的，并设 η 和 η' 是 $\mathcal{T}$ 上的 X 不变概率测度。给定 η 和 η' 的任意一个耦合 $\tilde{\eta}_0$，序列的每一个聚点 $\tilde{\eta}$

$$\tilde{\eta}_n = \frac{1}{n}\sum_{j=0}^{n-1} \widetilde{\mathcal{T}}^{n*}\tilde{\eta}_0$$

is a $\widetilde{\mathcal{T}}$-invariant coupling of η and η'. In particular, $\widetilde{\mathcal{T}}$-invariant couplings do exist.

Proof. By Lemma 6.18, every $\tilde{\eta}_n$ is a coupling of η and η'. By compactness, there exists $(n_i)_i \to \infty$ such that $(\tilde{\eta}_{n_i})_i$ converges to some $\tilde{\eta}$ in the weak* topology. Clearly, $\tilde{\eta}$ is still a coupling of η and η'. Let $\tilde{\psi} : X \times X \to \mathbb{R}$ be any bounded continuous function. The assumption ensures that $\widetilde{\mathcal{T}}\tilde{\psi}$ is also continuous. Thus,

$$\begin{aligned}\int_{X\times X}\left(\widetilde{\mathcal{T}}\tilde{\psi}-\tilde{\psi}\right)d\tilde{\eta} &= \lim_i \int_{X\times X}\left(\widetilde{\mathcal{T}}\tilde{\psi}-\tilde{\psi}\right)d\tilde{\eta}_{n_i}\\ &= \lim_i \int_{X\times X}\frac{1}{n_i}\left(\widetilde{\mathcal{T}}^{n_i}\tilde{\psi}-\tilde{\psi}\right)d\tilde{\eta}_0 = 0\end{aligned}$$

~ ~ ~ ~（回顾一下，对于每个 n，都有 $\|\mathcal{T}^n\psi\| \leqslant \|\psi\|$）。因此，$\mathcal{T}^*\tilde{\eta} = \tilde{\eta}$，这正是我们想要证明的。 □

Remark 6.20. A self-coupling $\widetilde{\mathcal{T}}$ is *symmetric* if $\sigma_{x',x} = \iota_*\sigma_{x,x'}$ for all $x, x' \in X$. If $\widetilde{\mathcal{T}}$ is symmetric and $\tilde{\eta}$ is a symmetric self-coupling of η' then $\widetilde{\mathcal{T}}^*\tilde{\eta}$ is a symmetric self-coupling of $\widetilde{\mathcal{T}}^*\eta$. Moreover, the $\widetilde{\mathcal{T}}$-invariant self-coupling $\tilde{\eta}$ in Lemma 6.19 may be taken to be symmetric.

6.4. 马古利斯函数。与之前一样，设

$$\mathcal{T} : \mathcal{B}(X) \to \mathcal{B}(X), \quad \mathcal{T}\psi(x) = \int_X \psi(y)\, d\sigma_x(y)$$

be a Markov operator on a metric space X. Let $X = A \cup B$ be a partition of X into disjoint sets A and B. An *(additive) Margulis function* for $\mathcal{T}$ relative to (A, B) is a measurable function $\Psi : X \to [0, \infty]$ such that there exist $\kappa_A > 0$ and $\kappa_B > 0$ such that

$$\mathcal{T}\Psi(x) \leqslant \Psi(x) - \kappa_A \text{ for every } x \in A \tag{6.25}$$

$$\mathcal{T}\Psi(x) \leqslant \Psi(x) + \kappa_B \text{ for every } x \in B. \tag{6.26}$$

(Ψ is usually not bounded, but its image under $\mathcal{T}$ is easily defined using monotone convergence: let $\mathcal{T}\Psi = \lim_n \mathcal{T}(\min\{\Psi, n\})$. Then $\int_X \mathcal{T}\Psi\, d\zeta = \int \Psi\, d(\mathcal{T}^*\zeta)$ for any probability measure ζ on X.) We make following technical assumption, which is used in the context of (6.30): there exists $L > 0$ such that Ψ is lower semi-continuous on $\Psi^{-1}([L, \infty])$.

Margulis functions are a very effective tool for estimating the spatial distribution of $\mathcal{T}$-invariant measures. The simple lemma that follows illustrates this idea:

Lemma 6.21. *Let $\Psi : X \to [0, \infty]$ be a Margulis function for a Markov operator $\mathcal{T}$ relative to a partition (A, B). Let ζ be any measure on X such that $\int \Psi\, d\zeta < \infty$ and $\int_X \mathcal{T}\Psi\, d\zeta \geqslant \int_X \Psi\, d\zeta$. Then*

$$\zeta(B) \geqslant \frac{\kappa_A}{\kappa_A + \kappa_B}\zeta(X). \tag{6.27}$$

In particular, this holds if ζ is $\mathcal{T}$-invariant and satisfies $\int \Psi\, d\zeta < \infty$.

Proof. We have

$$\int_X \Psi\, d\zeta \leqslant \int_X \mathcal{T}\Psi\, d\zeta \leqslant \int_X \Psi\, d\zeta - \kappa_A \zeta(A) + \kappa_B \zeta(B).$$

Thus, $\kappa_A \zeta(A) - \kappa_B \zeta(B) \leqslant 0$, which is the same as (6.27). To get the last claim, just note that $\int_X \mathcal{T}\Psi\, d\zeta = \int_X \Psi\, d\zeta$ if ζ is $\mathcal{T}$-invariant. □

Remark 6.22. Given a set $Y \subset X$, we call *multiplicative Margulis function* for $\mathcal{T}$ relative to (X, Y) any measurable function $\Phi : X \to [1, \infty]$ such that $\log \Phi$ is uniformly continuous, $\Phi(x) = \infty$ if and only if $x \in Y$, Φ is a proper function on $X \backslash Y$, and there exist constants $c < 1$ and $b < \infty$ such that

$$\mathcal{T}\Phi(x) \leqslant c\Phi(x) + b \text{ for all } x \in X. \tag{6.28}$$

If Φ is a multiplicative Margulis function then $\log \Phi$ is an additive Margulis function relative to the partition (A, B) defined by

$$A = \{x \in X : \Phi(x) > \alpha\} \text{ and } B = \{x \in X : \Phi(x) \leqslant \alpha\}$$

对于任意 $\alpha > b/(1-c)$。事实上，詹森不等式意味着

$$\mathcal{T} \log \Phi(x) \leqslant \log \mathcal{T}\Phi(x) \leqslant \log(c\Phi(x) + b) \text{ for every } x.$$

Moreover,

$$\log(cy + b) \leqslant \begin{cases} \log y + \log(c + b) & \text{for every } y \geqslant 1 \\ \log y + \log(c + b/\alpha) & \text{if } y > \alpha. \end{cases}$$

Thus, we may take $\kappa_A = -\log(c + b/\alpha) > 0$ and any $\kappa_B \geqslant \log(c + b)$. On the other hand, as was already pointed out in the Introduction, it is not true that if Ψ is an additive Margulis function then $\exp \Psi$ is a multiplicative one.

Proposition 6.23. *Assume that X is compact and let $\widetilde{\mathcal{T}}$ be a continuous self-coupling of $\mathcal{T}$. Let $\Psi : X \times X \to [0, +\infty]$ be a Margulis function for $\widetilde{\mathcal{T}}$ which is bounded outside every neighborhood of the diagonal, and let η be a non-atomic $\mathcal{T}$-invariant measure on X. Then one can find a $\widetilde{\mathcal{T}}$-invariant self-coupling $\tilde{\eta}$ of η and a sequence $(\tilde{\eta}_j)_j$ of self-couplings of η converging to $\tilde{\eta}$ in the weak* topology and such that*

$$\int_{X\times X} \Psi\, d\tilde{\eta}_j < \infty \text{ and } \int_{X\times X} \widetilde{\mathcal{T}}\Psi\, d\tilde{\eta}_j \geqslant \int_{X\times X} \Psi\, d\tilde{\eta}_j \text{ for every } j. \tag{6.29}$$

Proof. By Corollary 6.11, there exists some self-coupling $\hat{\eta}_0$ of η that vanishes on a neighborhood of the diagonal. Then $\int_{X\times X} \Psi\, d\hat{\eta}_0$ is finite. Conditions (6.25) and (6.26) imply that $\widetilde{\mathcal{T}}^j \Psi(x) \leqslant \Psi(x) + j\kappa_B$ for every $x \in X$, and so

$$\int_{X\times X} \widetilde{\mathcal{T}}^j \Psi\, d\hat{\eta}_0 \leqslant \int_{X\times X} \Psi\, d\hat{\eta}_0 + j\kappa_B < \infty$$

for every j. Let $\tilde{\eta} = \lim_i \hat{\eta}_{n_i}$ be any weak* accumulation point of the sequence

$$\hat{\eta}_n = \frac{1}{n} \sum_{j=0}^{n-1} \widetilde{\mathcal{T}}^{n*} \hat{\eta}_0.$$

As noted in Lemma 6.19, every $\hat{\eta}_n$ is a self-coupling of η with $\int_{X\times X} \Psi\, d\hat{\eta}_n < \infty$, and $\tilde{\eta}$ is a $\widetilde{\mathcal{T}}$-invariant self-coupling of η.

If $\int_X \Psi\, d\tilde{\eta}$ is finite then the claim follows by taking $\tilde{\eta}_j = \tilde{\eta}$ for every j. In this case the equality holds on the second part of (6.29). Now suppose that $\int_X \Psi\, d\tilde{\eta}$ is infinite. By the lower semi-continuity assumption on Ψ, this implies that

$$\int_X \Psi\, d\hat{\eta}_{n_i} \to \infty \text{ as } i \to \infty. \tag{6.30}$$

Then $\int_X \Psi\, d\left(\widetilde{\mathcal{T}}^{*n}\hat{\eta}_0\right)$ must be unbounded. In particular, one can find $(m_j)_j \to \infty$ such that

$$\int_{X\times X} \widetilde{\mathcal{T}}\Psi\, d\hat{\eta}_{m_j} - \int_{X\times X} \Psi\, d\hat{\eta}_{m_j} = \frac{1}{m_j}\left(\int_{X\times X} \Psi\, d\left(\widetilde{\mathcal{T}}^{*m_j}\hat{\eta}_0\right) - \int_{X\times X} \Psi\, d\hat{\eta}_0\right) \geqslant 0.$$

Thus, it suffices to take $\tilde{\eta}_j = \hat{\eta}_{m_j}$ for every j. □

6.5. **Adapted operators.** Let Z be an algebraic variety endowed with a continuous G–action $(g, z) \mapsto gz$. Let ν be a compactly supported probability measure on G, and η be a ν-stationary probability measure on Z. For each $z \in Z$, let ν_z denote the push-forward of ν under the map $g \mapsto gz$.

Let $X \subset Z$ and $a_X > 0$ be some small number. Consider the family of subsets $X_t = \{x \in X : d(x, Z\backslash X) \geqslant a_X t\}$, $t \in [0, 2]$. Note that $t \mapsto X_t$ is monotone decreasing. Define the *ν–core* of X to be

$$\mathcal{X}_\nu X = \{x \in X_2 : gx \in X_2 \text{ for all } g \in \operatorname{supp}\nu\}. \tag{6.31}$$

补集 $\partial_\nu X = X\backslash\mathcal{X}_\nu X$ 称为 X 的 ν–边缘。另定义

$$\mathcal{X}_\nu^{\#} X = \left\{x \in \mathcal{X}_\nu X : g^{-1}x \in \mathcal{X}_\nu X \text{ for all } g \in \operatorname{supp}\nu\right\}. \tag{6.32}$$

It is clear that $\mathcal{X}_\nu^{\#} X \subset \mathcal{X}_\nu X \subset X_2 \subset X$. Moreover,

$$\nu_x\left(X\backslash X_2\right) = 0 \text{ for all } x \in \mathcal{X}_\nu X, \text{ and} \tag{6.33}$$

$$\nu_x\left(\mathcal{X}_\nu^{\#} X\right) = 0 \text{ for all } x \in \partial_\nu X. \tag{6.34}$$

Example 6.24. In our initial applications, Z will be a Grassmannian manifold $\mathrm{Gr}(r, d)$ and X will be the closed ε-neighborhood $E_r(\varepsilon)$ of the subset of r-dimensional subspaces of $\mathbb{R}^d$ contained in the equator E. Later we will also take Z to be a flag variety $\mathcal{F}(r, d)$ and X to be the closed subset $E_r^\diamond(\varepsilon)$ of flags whose r-coordinate F_r is in $E_r(\varepsilon)$. We will always take $a_X = \varepsilon/100$, which means that $X_t = E_r((1-t/100)\varepsilon)$ and $X_t = E_r^\diamond((1 - t/100)\varepsilon)$, respectively, for all $t \in [0, 2]$.

A Markov operator $\mathcal{T} : \mathcal{B}(X) \to \mathcal{B}(X)$ is said to be *adapted to* (ν, X) if the associated family of probability measures $\{\sigma_x : x \in X\}$ satisfies

(a) 对于每个 x, $\sigma_x = \nu_x$ 都在 ν-核 $\mathcal{X}_\nu X$ 的一个邻域内; (b) 对于 x 中的每个元素, $\sigma_x(\mathcal{X}_\nu^{\#} X) = 0$ 都在 ν- 边缘 $\partial_\nu X$ 上; (c) 对于每个 $x \in X$, σ_x 都是典型的。

The assumption of the next proposition implies that $x \mapsto \sigma_x$ is continuous on U, and so Proposition 6.15 does hold in this setting. The assumption is satisfied, in particular, if the Markov operator $\mathcal{T}$ is adapted to (ν, X) and U is a subset of the ν-core of X.

Proposition 6.25. *Let $\mathcal{T} : \mathcal{B}(X) \to \mathcal{B}(X)$ be a Markov operator, η be a $\mathcal{T}$-invariant measure, and $U \subset X$ be such that $\eta(U) > 0$. Suppose that*

(a) *$\sigma_x = \nu_x$ for every $x \in U$;*
(b) *$\sigma_x(\mathcal{X}_\nu^{\#} U) = 0$ for every $x \notin U$;*

(c) 对于每个 $x \in X$， σ_x 都是典型测度。

Then the continuous Markov operator $\mathcal{T}_U : \mathcal{B}(U) \to \mathcal{B}(U)$ given by Proposition 6.15 is adapted to (ν, U).

Proof. Let $U^c = X \backslash U$ and $U_t^c = X \backslash U_t$. Recall that $\mathcal{T}_U$ is given by the probability measures $\sigma_{U,x}$ defined in (6.21). Since $\eta = \int_X \sigma_y \, d\eta(y)$, because η is assumed to be $\mathcal{T}$-invariant, the definition (6.18) of ξ_t means that

$$\xi_t = \frac{1}{J(t)} \left(\int_{U^c} (\sigma_y \mid U) \, d\eta(y) + \int_U (\sigma_y \mid U \cap U_t^c) \, d\eta(y) \right),$$

and so

$$\sigma_{x,t} = (\sigma_x \mid U_t) + \frac{\sigma_x(U_t^c)}{J(t)} \left(\int_{U^c} (\sigma_y \mid U) \, d\eta(y) + \int_U (\sigma_y \mid U \cap U_t^c) \, d\eta(y) \right). \tag{6.35}$$

Condition (c) in the definition of an adapted operator follows directly from Remark 6.17. Let us check conditions (a) and (b).

If $x \in \mathcal{X}_\nu U$ then $gx \in U_2$ for every $g \in \operatorname{supp} \nu$. Since U_1 is a neighborhood of U_2, and the support of ν is compact, it follows that there exists a neighborhood V of $\mathcal{X}_\nu U$ such that $gx \in U_1$ for every $x \in V$ and $g \in \operatorname{supp} \nu$. Thus $\sigma_x(U_1^c) = \nu_x(U_1^c) = 0$, and so, for all $t \in [0,1]$, the second term on the right hand side of (6.35) vanishes, which means that $\sigma_{x,t} = \sigma_x = \nu_x$. Integrating with respect to t, we find that $\sigma_{U,x} = \sigma_x = \nu_x$ for all $x \in V$, which proves condition (a).

Now consider $x \in \partial_\nu U$. We claim that all three terms on the right-hand side of (6.35) vanish on $\mathcal{X}_\nu^\# U$. Initially, (6.34) implies that $\sigma_x(\mathcal{X}_\nu^\# U) = \nu_x(\mathcal{X}_\nu^\# U) = 0$, which sets the claim for the first term. The case of the second term is an immediate consequence of the assumption that $\sigma_y(\mathcal{X}_\nu^\# U) = 0$ for every $y \in U^c$. Finally, $(\sigma_y \mid U \cap U_t^c)(\mathcal{X}_\nu^\# U) = 0$ for every $y \in U$ and $t \in [0,1]$, just because $\mathcal{X}_\nu^\# U$ is contained in U_t. Hence, the third term is also zero on $\mathcal{X}_\nu^\# U$. This proves that $\sigma_{x,t}(\mathcal{X}_\nu^\# U) = 0$ for all $t \in [0,1]$, which implies that $\sigma_{U,x}$ satisfies (b). □

Remark 6.26. By definition $\nu_x(\mathcal{X}_\nu^\# U) = 0$ for every $x \notin U$. Compare conditions (a) and (b) in Proposition 6.25.

7. 证明概要

给定 $\mathbb{R}^d$ 的任意子空间 U 和 V，定义

$$d(U,V) = \sup_{u \in U} \inf_{v \in V} d(u,v) = \sup_{u \in U} \inf_{v \in V} |\sin \angle(u,v)|. \tag{7.1}$$

当且仅当 $U \subset V$ 且 $d(U$ 时， $0 \leqslant d(U$、$V) \leqslant 1$，其中 $d(U$、$V) = 0$；当且仅当 $U \cap V^\perp \neq \{0\}$ 时， $V) = 1$。特别地，

$$\dim U > \dim V \quad \Rightarrow \quad d(U,V) = 1. \tag{7.2}$$

It is also clear that $d(U_2, V_1) \geqslant d(U_1, V_2)$ whenever $U_1 \subset U_2$ and $V_1 \subset V_2$.

The function $d(\cdot,\cdot)$ in (7.1) is clearly not symmetric, in general. However, its restriction to each Grassmannian manifold $\mathrm{Gr}(r,d)$ turns out to be a distance, invariant under the action of the orthogonal group $\mathrm{O}(d)$ on the Grassmannian. The case $r = 1$ is just (5.16).

Let $E_r \subset \mathrm{Gr}(r,d)$ be the set of r-dimensional subspaces contained in the equator E. In particular, $\mathrm{Gr}(1,d) = P$ and $E_1 = E$. For $\varepsilon > 0$, let $E_r(\varepsilon) \subset \mathrm{Gr}(r,d)$ be the closed ε-neighborhood of E_r. We also denote $E_r(\varepsilon, \varepsilon') = E_r(\varepsilon) \backslash E_r(\varepsilon')$ for

$0 < \varepsilon' < \varepsilon$。对于每个 $X = E_r(\varepsilon)$，我们在 ν–核 $\mathcal{X}_\nu X$ 和 ν–边缘 $\partial_\nu X$ 的定义 (6.31) 中始终取 $a_X = \varepsilon/100$。

7.1. **Main inductive statement.** We are going to prove Theorem B by contradiction: suppose that there exists a discontinuity point ν_∞ for the largest Lyapunov exponent λ_1. Then, as we saw in Section 5 (Proposition 5.2 and Corollary 5.4), there exists a sequence $(\nu_k)_k$ of generic measures with Zariski-dense supports converging to some ν_∞ on G and such that their (unique) stationary measures $(\eta_k)_k$ on P converge to a ν_∞-stationary measure η_∞ satisfying $\eta_\infty(E) > 0$.

Theorem 7.1. *Let $1 \leqslant r \leqslant d$ and suppose that there exist constants $\varepsilon_r > 0$ and $n_r \in \mathbb{N}$ such that:*

(i) 对于每个 $k \in \mathbb{N}$，都存在一个连续马尔可夫算子

$$\mathcal{T}_{k,r} : \mathcal{B}(E_r(\varepsilon_r)) \to \mathcal{B}(E_r(\varepsilon_r)), \quad \mathcal{T}_{k,r}\varphi(x) = \int_{E_r(\varepsilon_r)} \varphi(y)\, d\sigma_{k,r,x}(y)$$

适用于 $(\nu_k^{(n_r)}, E_r(\varepsilon_r))$。

(ii) 对于每个 $k \in \mathbb{N}$，存在一个 $\mathcal{T}_{k,r}$-不变概率测度 $\eta_{k,r}$ 定义在 $E_r(\varepsilon_r)$ 上，使得 $\eta_{\infty,r} = \lim_k \eta_{k,r}$ 存在且满足 $\eta_{\infty,r}(E_r) > 0$。

如果 $r <$ 维数为 E，则存在常数 ε_{r+1} 和 $K_{r+1} \in \mathbb{N}$，使得当 r 替换为 $r+1$时，(i) 和(ii)成立。如果 $r =$ 维数为 E，则(i)和(ii)不可能同时成立。

根据注 6.12，不变测度 $\eta_{k,r}$ 自动是典型测度。

Remark 7.2. Since the support of ν_k is Zariski-dense, it follows from Proposition 5.2 that for each $1 \leqslant r \leqslant d$ there exists a unique ν_k-stationary probability measure $\hat{\eta}_{k,r}$ in $\mathrm{Gr}(r,d)$. However, even if we assume that there exists a subsequence along which $(\hat{\eta}_{k,1})_k$ converges to a measure $\hat{\eta}_{\infty,1}$ with $\hat{\eta}_{\infty,1}(E) > 0$, it is not clear that for any $r > 1$ the sequence $(\hat{\eta}_{k,r})_k$ admits a subsequence converging to some measure $\hat{\eta}_{\infty,r}$ with $\hat{\eta}_{\infty,r}(E_r) > 0$. Thus, Theorem 7.1 is not immediately obvious. Indeed, our measures $\eta_{k,r}$ on $E_r(\varepsilon_r)$ are *not* the normalized restrictions of the $\hat{\eta}_{k,r}$.

为推导定理 B，注意到对于 $r = 1$，定理 7.1 的假设在 $n_1 = 1$ 和 $\varepsilon_1 = 1$ 下成立，因此 $E_1(\varepsilon_1) = P$，并且

$$\mathcal{T}_{k,1} = \mathcal{P}_{\nu_k} : \mathcal{B}(P) \to \mathcal{B}(P) \text{ and } \eta_{k,1} = \eta_k \text{ for } k \in \mathbb{N}.$$

Indeed, it is clear that $\mathcal{P}_{\nu_k}$ is adapted to (ν_k, P) and leaves η_k invariant. Recall that ν_k is taken to be generic for $k \in \mathbb{N}$. Moreover, the limit $\eta_{\infty,1} = \eta_\infty$ satisfies $\eta_\infty(E) > 0$. Then we can iterate Theorem 7.1 until we arrive at the case $r = \dim E$, which leads to a contradiction, which proves Theorem B.

The proof of Theorem 7.1 occupies what is left of this paper. In the remainder of the present section we outline the main ideas. Initially, we discuss the case $r = 1$, which involves many of the ingredients of the general step, though not all. Then we hint at how these ideas can be extended to $r > 1$.

Throughout, $E_1(\varepsilon)^2 = E_1(\varepsilon) \times E_1(\varepsilon)$ and $E_1(\varepsilon, \varepsilon')^2 = E_1(\varepsilon, \varepsilon') \times E_1(\varepsilon, \varepsilon')$ for any $0 < \varepsilon' < \varepsilon$. The following simple fact will be used a few times:

Remark 7.3. Given any $\beta > 0$ and $n \in \mathbb{N}$, the neighborhood $E_r(\alpha)$ is contained in the $\nu_\infty^{(n)}$-core of $E_r(\beta/2)$ for all $1 \leqslant r \leqslant d$ if α is sufficiently small, depending only on ν_∞, n and β. This is because the equator E is ν_∞-invariant. Since $\operatorname{supp} \nu_k^{(n)}$

converges to $\operatorname{supp}\nu_\infty^{(n)}$, it follows that if $\alpha > 0$ is sufficiently small and $k \in \mathbb{N}$ is large enough, depending only on ν_∞, n and β, then $E_r(\alpha)$ is contained in the $\nu_k^{(n)}$-core of $E_r(\beta)$ for all $1 \leqslant r \leqslant d$.

7.2. **First step.** Let us consider constants $\varepsilon_1 > 0$ and $n_1 \in \mathbb{N}$, continuous Markov operators

$$\mathcal{T}_{k,1} : \mathcal{B}(E_1(\varepsilon_1)) \to \mathcal{B}(E_1(\varepsilon_1)), \quad \mathcal{T}_{k,1}\varphi(x) = \int_G \varphi(y)\, d\sigma_{k,1,x}(y)$$

and $\mathcal{T}_{k,1}$-invariant measures $\eta_{k,1}$ as in Theorem 7.1. We may start from $n_1 = 1$ and $\varepsilon_1 = 1$, as in the previous section, but along the way we replace the operators with convenient iterates, which means that n_1 is increased, and we also localize them to suitable neighborhoods of the equator E, using Propositions 6.15 and 6.25 and Remark 7.3, which entails reducing ε_1.

The way we set this is by replacing n_1 with a variable $n \in \mathbb{N}$, which we think of as a free parameter, and by taking $\mathcal{T}_{k,1}$ and ε_1 as functions of n. The conditions on ε_1 are given in (7.3) and (9.12), depending on ν_∞, δ and n. The condition on n is stated only near the end of the construction, in (11.13), depending on ν_∞ and δ.

1. 如有必要可缩小 ε_1，因此不妨假设 that

$$\eta_{\infty,1}(E_1(\varepsilon_1)\backslash E) < \frac{1}{10}\eta_{\infty,1}(E). \tag{7.3}$$

这可以如下看出。由于 $\eta_{\infty,1}(E_1(\alpha))$ 收敛到 $\eta_{\infty,1}(E)$ 当 $\alpha \to 0$ 时，且假设极限为正，我们有

$$\eta_{\infty,1}(E_1(\alpha)\backslash E) < \frac{1}{10}\eta_{\infty,1}(E) \tag{7.4}$$

对任意足够小的 $\alpha > 0$。根据备注 7.3,

$$E_1(\alpha) \text{ is contained in the } \nu_k^{(n)}\text{-core of } E_1(\varepsilon_1) \tag{7.5}$$

对于每个足够大的 k 和每个足够小的 $\alpha > 0$。显然,

$$\eta_{\infty,1}(\partial E_1(\alpha)) = 0 \tag{7.6}$$

除可数多个 α 的取值外。固定 $\alpha > 0$，它仅取决于 ν_∞ 和 ε_1，并满足 (7.4)、(7.5) 和 (7.6)。

Applying Proposition 6.25 with $X = E_1(\varepsilon_1)$, $U = E_1(\alpha)$, $\nu = \nu_k^{(n)}$ and $\mathcal{T} = \mathcal{T}_{k,1}$, for k large, we get a continuous Markov operator

$$\mathcal{T}'_{k,1} : \mathcal{B}(E_1(\alpha)) \to \mathcal{B}(E_1(\alpha))$$

adapted to $(\nu_k^{(n)}, E_1(\alpha))$ and leaving invariant the normalized restriction $\eta'_{k,1}$ of $\eta_{k,1}$ to $E(\alpha)$. Replace ε_1, $\mathcal{T}_{k,1}$, and $\eta_{k,1}$ with α, $\mathcal{T}'_{k,1}$, and $\eta'_{k,1}$, respectively. Conditions (i) and (ii) in Theorem 7.1 are not affected by this: in particular, observe that (7.6) ensures that $(\eta'_{k,1})_k \to \eta'_{\infty,1}$. Then (7.3) just corresponds to (7.4).

我们还引入了一种适当的连续自耦合。

$$\begin{aligned} &\widetilde{\mathcal{T}}_{k,1} : \mathcal{B}(E_1(\varepsilon_1)^2) \to \mathcal{B}(E_1(\varepsilon_1)^2) \\ &\widetilde{\mathcal{T}}_{k,1}\tilde{\varphi}(x,x') = \int_{E_1(\varepsilon_1)^2} \tilde{\varphi}(y,y')\, d\tilde{\sigma}_{k,1,x,x'}(y,y') \end{aligned} \tag{7.7}$$

Lyapunov 指数乘积算子的$\mathcal{T}_{k,1}$ 使得

$$\widetilde{\mathcal{T}}_{k,1}\tilde{\varphi}(x,x') = \int_G \tilde{\varphi}(gx, gx')\, d\nu_k^{(n)}(g) \tag{7.8}$$

当 x 和 x' 都接近赤道 E 时。

2. Consider $\varepsilon_1''' \in (0, \varepsilon_1)$ and a sequence $(\omega_{k,1})_k$ decreasing to zero. Property (7.3) ensures that

$$\eta_{k,1}(E_1(\varepsilon_1, \varepsilon_1''')) < \frac{2}{10}\eta_{k,1}(E_1(\varepsilon_1)) \tag{7.9}$$

对每个充分大的 k。通过以下方式定义 A_k 和 $B_k = B' \cup B_k''$

$$A_k = \{(x,x') \in E_1(\varepsilon_1''')^2 : d(x+x', E) > \omega_{k,1}\} \tag{7.10}$$

$$B' = \{(x,x') \in E_1(\varepsilon_1)^2 : d(x,E) > \varepsilon_1''' \text{ or } d(x',E) > \varepsilon_1'''\} \tag{7.11}$$

$$B_k'' = \{(x,x') \in E_1(\varepsilon_1)^2 : d(x+x', E) \leqslant \omega_{k,1}\}. \tag{7.12}$$

It is clear that A_k and $B_k = B' \cup B_k''$ are disjoint, and their union is the whole $E_1(\varepsilon_1)^2$. For every large k, we have $\omega_{k,1} < \varepsilon_1'''$ and, in view of the definition (7.1), that ensures that B' and B_k'' are also disjoint. Moreover, (7.2) implies that $B_k'' = \varnothing$ if $\dim E = 1$.

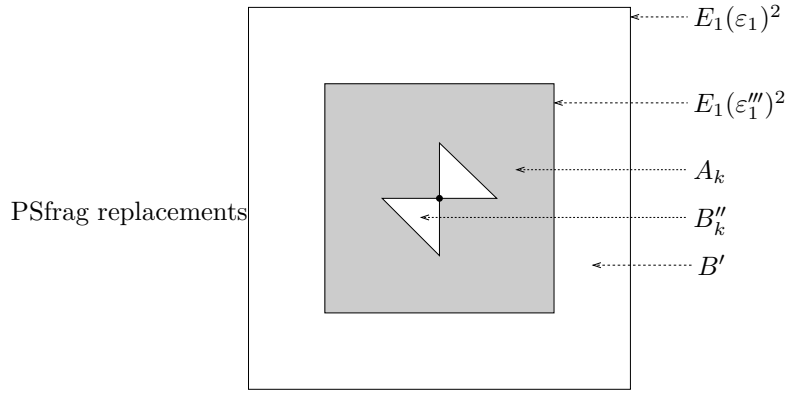

FIGURE 2. 将 $E_1(\varepsilon_1)^2$ 划分为集合 A_k、B'和 B_k''的示意图。当 $k \to \infty$时，后者收敛于中心点 (E、 E) ， 因为$\omega_{k,\ 1} \to 0$。

Suppose that for $n \in \mathbb{N}$ sufficiently large there exist constants $\kappa_A, \kappa_B > 0$ with $\kappa_A > 9\kappa_B$, and for each large $k \in \mathbb{N}$ there exists a lower semi-continuous function $\Psi_{k,1} : E_1(\varepsilon_1)^2 \to \mathbb{R}$ such that

$$\widetilde{\mathcal{T}}_{k,1}\Psi_{k,1}(x,x') \leqslant \Psi_{k,1}(x,x') - \kappa_A \text{ for every } (x,x') \in A_k \tag{7.13}$$

$$\widetilde{\mathcal{T}}_{k,1}\Psi_{k,1}(x,x') \leqslant \Psi_{k,1}(x,x') + \kappa_B \text{ for every } (x,x') \in B_k. \tag{7.14}$$

Then $\Psi_{k,1}$ is a Margulis function for the operator $\widetilde{\mathcal{T}}_{k,1}$ relative to the partition (A_k, B_k) of $E_1(\varepsilon_1)^2$. Combining Proposition 6.23 with Lemma 6.21, we conclude

存在自耦合 $\tilde{\eta}_{k,1,j}$，其属于 $\eta_{k,1}$，使得 $\int_{E_1(\varepsilon_1)^2} \Psi_{k,1}\, d\tilde{\eta}_{k,1,j}$ 是有限的，并且

$$\begin{aligned}\tilde{\eta}_{k,1,j}(B_k) &\geqslant \frac{\kappa_A}{\kappa_A+\kappa_B}\tilde{\eta}_{k,1,j}(E_1(\varepsilon_1)^2)\\ &> \frac{9}{10}\tilde{\eta}_{k,1,j}(E_1(\varepsilon_1)^2) = \frac{9}{10}\eta_{k,1}(E_1(\varepsilon_1)).\end{aligned} \tag{7.15}$$

Moreover, $(\tilde{\eta}_{k,1,j})_j$ may be taken to converge to a $\widetilde{\mathcal{T}}_{k,1}$-invariant self-coupling $\tilde{\eta}_{k,1}$ of $\eta_{k,1}$.

The fact that $\tilde{\eta}_{k,1,j}$ is a self-coupling of $\eta_{k,1}$ together with the inequality (7.9) ensure that

$$\tilde{\eta}_{k,1,j}(B') \leqslant 2\eta_{k,1}(E_1(\varepsilon_1,\varepsilon_1'''')) < \frac{4}{10}\eta_{k,1}(E_1(\varepsilon_1)). \tag{7.16}$$

将 (7.16) 从 (7.15) 中相减，我们得出

$$\tilde{\eta}_{k,1,j}(B_k'') > \frac{5}{10}\eta_{k,1}(E_1(\varepsilon_1)) > 0. \tag{7.17}$$

This yields a contradiction when $\dim E = 1$ because, as observed previously, in that case the set B_k'' is empty.

3. 若 $\dim E \geqslant 2$，考虑该映射。

$$\Sigma : E_1(\varepsilon_1)^2 \to \mathrm{Gr}(2,d), \quad \Sigma(x,x') = x + x'.$$

The fact that Σ is not defined on the diagonal of $E_1(\varepsilon_1)^2$ need not concern us at this stage: we will deal with it in Section 11.3. For each large k, let $\eta_{k,2} = \Sigma_*\tilde{\eta}_{k,1}$ and $\{\hat{\eta}_{k,1,y} : y \in \mathrm{Gr}(2,d)\}$ be a disintegration of $\tilde{\eta}_{k,1}$ with respect to the partition $\{\Sigma^{-1}y : y \in \mathrm{Gr}(2,d)\}$ of $E_1(\varepsilon_1)^2$. Then define

$$\mathcal{T}_{k,2} : \mathcal{B}(\Sigma(E_1(\varepsilon_1)^2)) \to \mathcal{B}(\Sigma(E_1(\varepsilon_1)^2)),\ \mathcal{T}_{k,2}\psi(y) = \int_{\Sigma^{-1}y} \widetilde{\mathcal{T}}_{k,1}(\psi\circ\Sigma)\, d\hat{\eta}_{k,1,y}.$$

The measure $\eta_{k,2}$ is $\mathcal{T}_{k,2}$-invariant. Indeed, since $\tilde{\eta}_{k,1}$ is $\widetilde{\mathcal{T}}_{k,1}$-invariant,

$$\begin{aligned}\int_{\Sigma(E_1(\varepsilon_1)^2)} \mathcal{T}_{k,2}\psi\, d\eta_{k,2} &= \int_{\Sigma(E_1(\varepsilon_1)^2)}\int_{\Sigma^{-1}y} \widetilde{\mathcal{T}}_{k,1}(\psi\circ\Sigma)\, d\hat{\eta}_{k,1,y}\, d\eta_{k,2}(y)\\ &= \int_{E_1(\varepsilon_1)^2} \widetilde{\mathcal{T}}_{k,1}(\psi\circ\Sigma)\, d\tilde{\eta}_{k,1} = \int_{E_1(\varepsilon_1)^2} (\psi\circ\Sigma)\, d\tilde{\eta}_{k,1}\\ &= \int_{\Sigma(E_1(\varepsilon_1)^2)} \psi\, d\eta_{k,2}\end{aligned}$$

for any $\psi \in \mathcal{B}(\Sigma(E_1(\varepsilon_1)^2))$. Taking the limit as $j \to \infty$ in (7.17), we find that

$$\tilde{\eta}_{k,1}(B_k'') \geqslant \frac{5}{10}\eta_{k,1}(E_1(\varepsilon_1)). \tag{7.18}$$

Recall that we take $\omega_{k,1} \to 0$ as $k \to \infty$. Then, in view of the definition of B_k'' in (7.12), the sets $\Sigma(B_k'')$ approach E_2. Taking the limit $k \to \infty$ in (7.18), we find that

$$\eta_{\infty,2}(E_2) \geqslant \frac{5}{10}\eta_{\infty,1}(E_1(\varepsilon_1)) > 0.$$

Now define $n_2 = n$. If y is close to E_2 in $\mathrm{Gr}(2,d)$ then x and x' are close to E_1 in $\mathrm{Gr}(1,d)$ and, in particular, they are in the $\nu_k^{(n_2)}$-core of $E_1(\varepsilon_1)$ for every large k.

然后，利用（7.8），

$$\begin{aligned}\mathcal{T}_{k,2}\psi(y) &= \int_{\Sigma^{-1}y}\int_G \psi\circ\Sigma(gx,gx')\,d\nu_k^{(n_2)}(g)\,d\tilde{\eta}_{k,1,y}(x,x')\\ &= \int_{\Sigma^{-1}y}\int_G \psi(gy)\,d\nu_k^{(n_2)}(g)\,d\tilde{\eta}_{k,1,y}(x,x') = \int_G \psi(gy)\,d\nu_k^{(n_2)}(g),\end{aligned}$$

because each $\tilde{\eta}_{k,1,y}$ is a probability. Pick $\varepsilon_2 > 0$ such that this holds for every $y \in E_2(\varepsilon_2)$. Apply Proposition 6.25 with $X = \Sigma(E_1(\varepsilon_1)^2)$, $U = E_2(\varepsilon_2)$, $\nu = \nu_k^{(n_2)}$, and $\mathcal{T} = \mathcal{T}_{k,2}$, for k large. Replace $\mathcal{T}_{k,2}$ and $\eta_{k,2}$ with this new Markov operator and invariant measure, respectively.

这将完成针对 $r = 1$ 的定理 7.1 的证明。

4. However, in general we cannot construct a Margulis function $\Psi_{k,1}$ as required. Essentially, the problem is that, since we do not control the measures $\tilde{\sigma}_{k,1,x,x'}$ in the border region, the inequality (7.14) cannot be proved to hold unless the Margulis function $\Psi_{k,1}$ is taken to be bounded in the border region. However, outside the border region $\Psi_{k,1}(x,x')$ must be very large if x and x' are close. The only way to reconcile these two requirements is to introduce some drastic discontinuities in $\Psi_{k,1}$ which then cause (7.14) to fail at some points.

This problem is solved as follows. We do indeed create a discontinuity by cutting $\Psi_{k,1}$ off in such a way that it is bounded in the border region. The main idea for dealing with the discontinuity, that we call *recoupling*, involves replacing $\widetilde{\mathcal{T}}_{k,1}$ with another Markov operator $\widehat{\mathcal{T}}_{k,1}$ that still projects to $\mathcal{T}_{k,1}$ on either factor. The recoupling modification is restricted to a region which is disjoint from B_k'' and relatively far from the equator, so that the properties of the measures $\eta_{k,2}$ are not affected. These arguments are detailed in Sections 8 through 11.

In Section 8 we introduce the notion of *vertical projection* $\mathrm{VP}_1(x,x')$ of a pair of points x and x' in P, and we use it to construct a candidate $-\log \mathrm{VP}_1(x,x')$ to a Margulis function for $\widetilde{\mathcal{T}}_{\infty,1}$. The problem with this function is that it refers explicitly to E and, since the equator is not ν_k-invariant for $k \in \mathbb{N}$, the estimates break down for $\widetilde{\mathcal{T}}_{k,1}$ when k is finite.

This is fixed in Section 9.1, where we replace the vertical projection with a kind of cut-off that we call the *stabilized vertical projection* and denote as $\mathrm{SVP}_1(x,x')$. Then, in Section 9.2 we give the precise recipe for the other cut-off, that was mentioned previously. By the end of Section 10 point we will have a much more viable candidate $\Psi_{k,1}$ for Margulis function when k is finite.

The details of the recoupling procedure are described in Section 11.1. Proposition 11.2 essentially states that this function $\Psi_{k,1}$ is a Margulis function for the recoupled Markov operator $\widehat{\mathcal{T}}_{k,1}$. In Sections 11.2 and 11.3 we wrap the arguments up to conclude the step $r = 1$ of the proof.

7.3. **General step.** For $r > 1$ the strategy is similar, except that we need to work with *flag varieties*. Recall that $\mathcal{F}(r,d)$ denotes the space of flags

$$F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r \subset \mathbb{R}^d,$$

where each F_i has dimension i. We use $x = (F_1,\dots,F_r)$ and $x' = (F_1',\dots,F_r')$ to denote generic elements of $\mathcal{F}(r,d)$. Define $E_r^\diamond = \{x \in \mathcal{F}(r,d) : F_r \in E_r\}$ and, for each $\varepsilon > \varepsilon' > 0$,

$$E_r^\diamond(\varepsilon) = \big\{x \in \mathcal{F}(r,d) : F_r \in E_r(\varepsilon)\big\} \text{ and } E_r^\diamond(\varepsilon,\varepsilon') = E_r^\diamond(\varepsilon)\backslash E_r^\diamond(\varepsilon').$$

根据定义，对于任意 $\nu \in \mathcal{P}_c(G)$， $E^\diamond(\varepsilon)$ 的 ν-核是满足 $F_r \in \mathcal{X}_\nu E_r(\varepsilon_r)$ 的旗 x 的子集，而 $E^\diamond(\varepsilon)$ 的 ν-边缘也以类似方式定义。

We consider $E_r^\diamond(\varepsilon_r)^2 = E_r^\diamond(\varepsilon_r) \times E_r^\diamond(\varepsilon_r)$ and we aim to construct a Margulis function $\Psi_{k,r}$ for a suitable Markov operator $\widehat{\mathcal{T}}_{k,r}^\diamond : \mathcal{B}(E_r^\diamond(\varepsilon_r)^2) \to \mathcal{B}(E_r^\diamond(\varepsilon_r)^2)$ relative to a convenient generalization $E_r^\diamond(\varepsilon_r)^2 = A_k \cup B_k' \cup B_k''$ of the partition (7.10)–(7.12). For $i = 1, 2$, define $\theta_i : E_r^\diamond(\varepsilon_r)^2 \to \mathrm{Gr}(r, d)$ to be the projection to either factor followed by the canonical map

$$\mathcal{F}(r, d) \to \mathrm{Gr}(r, d), \quad x \mapsto F_r. \tag{7.19}$$

对 $\mathcal{T}_{k,\,r}^\diamond$ 的一个约束是，在两个 θ_i 下，它都必须投影到 $\mathcal{T}_{k,\,r}$。然后，我们可以找到一个 $\mathcal{T}_{k,\,r}^\diamond$-不变测度 $\eta_{k,\,r}^\diamond$ 使其在两个投影下都映射到 $\eta_{k,\,r}$。

与情形 $r = 1$ 类似，一旦我们构造出 $\mathcal{T}_{k,r}^\diamond$, $\Psi_{k,r}$ 和 $\eta_{k,r}^\diamond$，就可以尝试得到如下形式的估计。

$$\hat{\eta}_{k,r}^\diamond(B_k'') > c\hat{\eta}_{k,r}^\diamond(E_r^\diamond(\varepsilon_r)^2),$$

where $c > 0$ is some absolute constant. Then, if we push $\hat{\eta}_{k,r}^\diamond$ forward by one of the maps

$$\Sigma_r : E_r^\diamond(\varepsilon_r)^2 \to \mathrm{Gr}(r+1, d), \quad \Sigma_r\left(x, x'\right) = F_1' + F_r$$
$$\Sigma_r : E_r^\diamond(\varepsilon_r)^2 \to \mathrm{Gr}(r+1, d), \quad \Sigma_r\left(x, x'\right) = F_1 + F_r'$$

we obtain probability measures $\eta_{k,r+1}$ on $\mathrm{Gr}(r+1, d)$ satisfying the conditions of Theorem 7.1.

However, for $r > 1$ the simple cut-off procedure we use in the initial step of the induction is no longer sufficient to ensure that the Margulis function is bounded on the border region. To fix that, we take advantage of the additional freedom provided by the flag space, which is that we may vary the projections to $\mathrm{Gr}(j, d)$ for $j < r$. More precisely, we modify the dynamics by averaging ("spreading out") the Markov operator over the subspaces of dimension less than r in the flag. Thus we end up with modified Markov operators $\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}$ that still project to the $\mathcal{T}_{k,r}$ under both θ_i. Then the kind of argument we sketched in the previous paragraphs can actually be carried out for $\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}$-invariant measures $\hat{\eta}_{k,r}^{\mathcal{Q}}$ that map to $\eta_{k,r}$ under both projections.

The detailed arguments are structured as follows in Sections 12 through 16. In Section 12 we extend the notions of *vertical angle* and *vertical projection* to $r > 1$, and we use them to exhibit a candidate $-\log \psi_r(x, x')$ to a Margulis function. In Section 13.1 we move to introduce the $r > 1$ version of the *stabilized vertical projection* $\mathrm{SVP}_r(x, x')$, and in Section 13.2 we describe the corresponding version for the cut-off procedure. In Section 14 we extend this analysis to the perturbed random walks, that is, to finite k. The Margulis function $\Psi_{k,r}$ is defined at the end of that section.

The spreading out modification is detailed in Section 15. Essentially, we define $\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}} = \widehat{\mathcal{T}}_{q,r}^\diamond \circ \widetilde{\mathcal{Q}}_r$ where $\widetilde{\mathcal{Q}}_r : Bd(E_r^\diamond(\varepsilon_r)^2) \to \mathcal{B}(E_r^\diamond(\varepsilon_r)^2)$ has the form

$$\widetilde{\mathcal{Q}}_r \psi(x, x') = \int \Psi \, d\tilde{q}_{r,x,x'}$$

where $\tilde{q}_{r,x,x'}$ is a smooth probability measure on the subset of pairs of flags whose r-coordinate is (F_r, F_r'). A relevant feature is that $\widetilde{\mathcal{Q}}_r$ maps to the identity under the canonical map (7.19). The details of the recoupling procedure are described in Section 16.2. In Sections 16.3 and 16.4 we wrap up the proof.

第三部分。归纳法的第一步

8. 准备马古利斯函数

We are going to construct a positive function ψ_1 such that $-\log\psi_1$ has some features of a Margulis function for large iterates of the operator $\mathcal{P}_{\nu_\infty}$:

Proposition 8.1. *There exist $\kappa_1' = \kappa_1'(\nu_\infty) > 0$ and $C_1' = C_1'(\nu_\infty) > 0$ and for any $\delta > 0$ there exists $N_1 = N_1(\nu_\infty, \delta) \in \mathbb{N}$ such that for every $n \geqslant N_1$ there exists $\rho_1' = \rho_1'(\nu_\infty, \delta, n) > 0$ satisfying*

$$\int_G -\log\psi_1(gx, gx')\, d\nu_\infty^{(n)}(g) \leqslant -\log\psi_1(x, x') - (\kappa_1' - C_1'\delta)n \tag{8.1}$$

对于每个 $x \neq x' \in E_1(\rho_1')$。

The conclusion of Proposition 8.1 does not quite fit the definition of a Margulis function because the set $E_1(\rho_1') \times E_1(\rho_1')$ where the estimate holds is not invariant under the G-action restricted to $\operatorname{supp}\nu_\infty$ and thus $\mathcal{P}_{\nu_\infty}$ cannot be considered a Markov operator on this set. Nevertheless, $-\log\psi_1$ is an important ingredient in the definition of the actual Margulis function, which will be completed in Section 11.

The proof of Proposition 8.1 occupies the remainder of this section. Throughout, we think of the equator as being "horizontal" and use the word "vertical" to refer to the orthogonal direction. The numbers $\delta > 0$ and $n \in \mathbb{N}$ will remain as free parameters for most of our arguments. Near the end, in (11.13), we will fix $\delta > 0$ to be small, depending only on ν_∞, and $n \in \mathbb{N}$ large, depending on ν_∞ and $\delta > 0$.

Fix $A = A(\nu_\infty) > 0$ and a compact neighborhood $\mathcal{W}_0 = \mathcal{W}_0(\nu_\infty) \subset G$ of the support of ν_∞ such that

$$-\log d(U, V) - A \leqslant -\log d(fU, fV) \leqslant -\log d(U, V) + A$$

对于任意 $f \in \mathcal{W}_0$、任意一对 $U \neq V \in \mathbf{Gr}(r, d)$ 以及任意 $1 \leqslant r \leqslant d$。此外，令 $B = B(\nu_\infty) > 0$ 定义为

$$B = \sup\left\{\log\|f\| + \log\|f^{-1}\| : f \in \mathcal{W}_0\right\} + 2.$$

由于 $(\nu_k)_k$ 在空间 $\mathcal{P}_c(G)$ 中收敛于 ν_∞，因此不妨假设对于每个 $k \in \mathbb{N}$，都有 $\operatorname{supp}\nu_k \subset \mathcal{W}_0$。特别地，

$$-\log d(fU, fV) \leqslant -\log d(U, V) + An \tag{8.2}$$

$$\log\|f\|\|f^{-1}\| \leqslant Bn \tag{8.3}$$

for every $U \neq V$ in $\operatorname{Gr}(r, d)$, $1 \leqslant r \leqslant d$, $f \in \operatorname{supp}\nu_k^{(n)}$, $k \in \mathbb{N} \cup \{\infty\}$, and $n \in \mathbb{N}$.

8.1. **Vertical angle function.** Given distinct points x and x' in $P = \operatorname{Gr}(1, d)$, let $y = y(x, x') \subset P$ denote the *great circle* through x and x', that is, the subset of P associated with the element $x + x'$ of $\operatorname{Gr}(2, d)$ that contains both x and x'.

垂直角函数 VA_1 是大圆$y = y(x、\ x')$ 与赤道之间夹角的正弦，即（回顾 (7.1) 并查看图 3），

$$\mathrm{VA}_1(x, x') = d(y, E) = \sup_{u \in y} d(u, E) = \sup_{u \in y} \frac{\|u^\perp\|}{\|u\|}. \tag{8.4}$$

对于任意实现上确界的 $u \in y$，

$$\mathrm{VA}_1(gx, gx') \geqslant \frac{\|(gu)^\perp\|}{\|gu\|} = \mathrm{VA}_1(x, x')\frac{\|(gu)^\perp\|}{\|gu\|}\frac{\|u\|}{\|u^\perp\|} \tag{8.5}$$

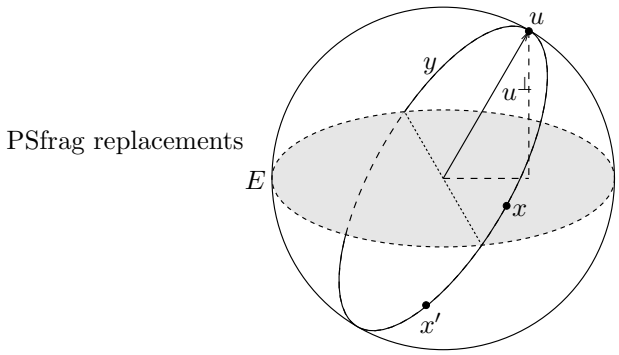

FIGURE 3. 垂直夹角函数 VA_1 的几何意义。通过令 x 和 x' 沿着给定的大圆趋近于 E，可以看出，当 x 和 x' 趋近于赤道时，$\mathsf{VA}_1(x,x')$ 不一定趋于无穷大。

for any $g \in G$. By Remark 3.1, when $g \in \operatorname{supp} \nu_\infty^{(n)}$ this means that

$$\mathrm{VA}_1(gx, gx') \geqslant \frac{\|g^\perp u^\perp\|}{\|gu\|} = \mathrm{VA}_1(x,x') \frac{\|g^\perp u^\perp\|}{\|gu\|} \frac{\|u\|}{\|u^\perp\|}. \tag{8.6}$$

Still for $g \in \operatorname{supp} \nu_\infty^{(n)}$, using Remark 3.1 and (8.3) we get that

$$\frac{\|g^\perp u^\perp\|}{\|gu\|} \frac{\|u\|}{\|u^\perp\|} \geqslant \frac{1}{\|(g^\perp)^{-1}\| \|g\|} \geqslant \frac{1}{\|g^{-1}\| \|g\|} \geqslant e^{-Bn},$$

and so

$$-\log \mathrm{VA}_1(gx, gx') \leqslant -\log \mathrm{VA}_1(x,x') + Bn. \tag{8.7}$$

Lemma 8.2. *There exists $\tilde\kappa_1 = \tilde\kappa_1(\nu_\infty) > 0$ and for each $\delta > 0$ there exist $\tilde\theta_1 = \tilde\theta_1(\nu_\infty, \delta) > 0$ and $\widetilde N_1 = \widetilde N_1(\nu_\infty, \delta) \in \mathbb{N}$ such that for every $n \geqslant \widetilde N_1$ and $x \neq x'$ in P there exists $\widetilde{\mathcal{E}}_1 = \widetilde{\mathcal{E}}_1(\nu_\infty, \delta, n, x, x') \subset \operatorname{supp} \nu_\infty^{(n)}$ with $\nu_\infty^{(n)}(\widetilde{\mathcal{E}}_1^c) < \delta$ and*

$$-\log \mathrm{VA}_1(gx, gx') \leqslant \max\{-\log \mathrm{VA}_1(x,x') - \tilde\kappa_1 n, \tilde\theta_1\} \text{ for every } g \in \widetilde{\mathcal{E}}_1. \tag{8.8}$$

Proof. Let $\kappa_0 = \kappa_0(\nu_\infty) > 0$, $N_0 = N_0(\nu_\infty, \delta) \in \mathbb{N}$, $\mathcal{E}_0 = \mathcal{E}_0(\nu_\infty, \delta, n, u^\perp) \subset \operatorname{supp} \nu_\infty^{(n)}$, and $\tau_0 = \tau_0(\nu_\infty, \delta) > 0$ be as in Proposition 5.5. Given $x \neq x'$ in P, take $u \in y$ realizing the supremum in (8.4). Write $u = u^E + u^\perp$ with $u^E \in E$ and $u^\perp \in E^\perp$. Take

$$\begin{gathered} \tilde\kappa_1 = \kappa_0/2, \quad \tilde\theta_1 = -\log(\tau_0/4), \\ \widetilde N_1 > \max\{N_0, 4/\tilde\kappa_1\}, \quad \text{and } \widetilde{\mathcal{E}}_1 = \mathcal{E}_0(\nu_\infty, \delta, n, u^\perp). \end{gathered} \tag{8.9}$$

设 $n \geqslant \widetilde N_1$ 和 $g \in \widetilde{\mathcal{E}}_1 \subset \operatorname{supp} \nu_\infty^{(n)}$。如果 $\|g^\perp u^\perp\|/\|gu\| \geqslant \tau_0/2$，那么（8.6）中的不等式意味着

$$-\log \mathrm{VA}_1(gx, gx') \leqslant -\log \frac{\tau_0}{2} \leqslant \tilde\theta_1. \tag{8.10}$$

If $\|g^\perp u^\perp\|/\|gu\| < \tau_0/2$ then part (2) of Proposition 5.5 gives that

$$\frac{\|gu^\perp\|}{\|gu\|} < \frac{1}{2}, \text{ which implies } \frac{\|gu^E\|}{\|gu\|} > \frac{1}{2}.$$

Substituting the latter inequality and $\|u\| \geqslant \|u^E\|$ in (8.6), we find that

$$\mathrm{VA}_1(gx, gx') \geqslant \frac{1}{2}\,\mathrm{VA}_1(x, x')\frac{\|g^\perp u^\perp\|}{\|u^\perp\|}\,\frac{\|u^E\|}{\|gu^E\|}. \tag{8.11}$$

因此，回顾定义 (5.3) 和命题 5.5 的第 (1) 部分，

$$\mathrm{VA}_1(gx, gx') \geqslant \frac{1}{2}\,\mathrm{VA}_1(x, x')\frac{\|Dg^\perp_{u^E} u^\perp\|}{\|u^\perp\|} \geqslant \frac{1}{2}\,\mathrm{VA}_1(x, x')e^{\kappa_0 n}.$$

~根据（8.9）中对 $\tilde\kappa_1$ 和 N_1 的选取，这意味着

$$\begin{aligned} -\log \mathrm{VA}_1(gx, gx') &\leqslant -\log \mathrm{VA}_1(x, x') + \log 2 - 2\tilde\kappa_1 n \\ &\leqslant -\log \mathrm{VA}_1(x, x') - \tilde\kappa_1 n. \end{aligned} \tag{8.12}$$

引理的结论包含在 (8.10) 和 (8.12) 中。 □

8.2. **Vertical projection function.** The function $-\log \mathrm{VA}_1(x, x')$ cannot be used as a Margulis function to detect the equator because (unless $\dim E = 1$) it is possible that $-\log \mathrm{VA}_1(x, x')$ remains bounded even as $d(x, E)$ and $d(x', E)$ go to zero: the great circle through points close to the equator need not be close to the equator, as illustrated in Figure 3.

为对此进行（部分）修正，我们引入如下定义的垂直投影函数 VP_1,:

$$\mathrm{VP}_1(x, x') = \mathrm{VA}_1(x, x')d(x, x')^{\gamma_1} \tag{8.13}$$

where $\gamma_1 = \gamma_1(\nu_\infty)$ is a small positive constant to be chosen shortly (Proposition 8.3). Note that if $\gamma_1 = 1$ then $\mathrm{VP}_1(x, x')$ would indeed be a sort of projection on the orthogonal complement to the equator. It is clear from the definitions (8.4) and (8.13) that VA_1 and VP_1 are symmetric functions:

$$\mathrm{VA}_1(x, x') = \mathrm{VA}_1(x', x) \text{ and } \mathrm{VP}_1(x, x') = \mathrm{VP}_1(x', x) \text{ for any } x \neq x'. \tag{8.14}$$

Proposition 8.3. *There exist $\gamma_1 = \gamma_1(\nu_\infty) > 0$ and $\kappa_1' = \kappa_1'(\nu_\infty) > 0$ and for each $\delta > 0$ there exists $N_1 = N_1(\nu_\infty, \delta) \in \mathbb{N}$ such that for every $n \geqslant N_1$ there exists $\rho_1' = \rho_1'(\nu_\infty, \delta, n) > 0$ such that for any $x \neq x'$ in $E_1(\rho_1')$ there exists $\mathcal{E}_1' = \mathcal{E}_1'(\nu_\infty, \delta, n, x, x') \subset \operatorname{supp} \nu_\infty^{(n)}$ with $\nu_\infty^{(n)}((\mathcal{E}_1')^c) < \delta$ and*

$$-\log \mathrm{VP}_1(gx, gx') \leqslant -\log \mathrm{VP}_1(x, x') - \kappa_1' n \text{ for every } g \in \mathcal{E}_1'. \tag{8.15}$$

Proof. The overall strategy goes as follows. If the vertical angle $\mathrm{VA}_1(x, x')$ is small then the conclusion of the present proposition is a consequence of Lemma 8.2, provided that we choose the constant γ_1 small enough to make the variation of $d(x, x')^{\gamma_1}$ negligible. If $\mathrm{VA}_1(x, x')$ is large, let v and v' be unit vectors in the direction of x and x' and such that the angle between them is non-obtuse (the latter may always be obtained by replacing v with $-v$ if necessary). In Lemma 8.4 we check that if x is close to E then the difference $w = v' - v$ is roughly vertical, meaning that the angle between w and the equator is bounded from below. See Figure 4. In Lemma 8.5 we deduce that in this situation $-\log d(x, x')$ decreases under most large iterates: the reason is that the dynamics increases vertical components and, for n sufficiently large, the vertical component of the iterate of w dominates. This implies the conclusion of the proposition because in this regime the variation of $\mathrm{VA}_1(x, x')$ is bounded. Let us fill-in the details.

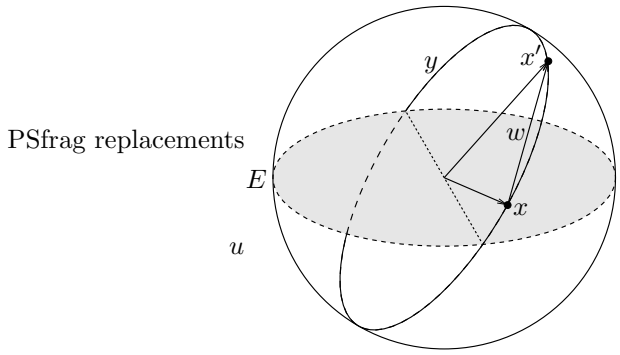

FIGURE 4. 引理 8.4 的解释：如果 $d(x, E)$ 相比 $\mathrm{VA}_1(x, x') = d(y, E)$ 足够小，那么 $w = v' - v$ 大致是竖直的。

引理 8.4。给定 $x \neq x'$ 属于 P，设 $w = v' - v$ 为分别沿着 x 和 x' 方向的单位向量之差。那么

$$d(x, E) < \frac{1}{4}\,\mathrm{VA}_1(x, x') \text{ implies } \frac{\|w^{\perp}\|}{\|w\|} > \frac{1}{4}\,\mathrm{VA}_1(x, x'). \tag{8.16}$$

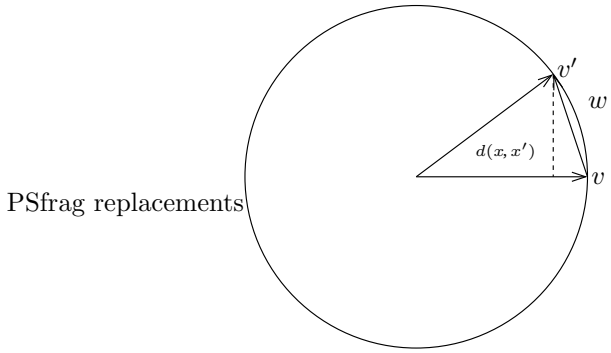

FIGURE 5. Illustrating the estimates in (8.17): the distance from x to x' is given by the length of the vertical dashed segment, which is bounded above by $\|w\|$; the angle between v and w is at least $\pi/4$, as the angle between v and v' is non-obtuse.

Proof. As the angle between v and v' was taken to be non-obtuse (check Figure 5)

$$d(x, x') = |\sin \angle(v, v')| \leqslant \|w\| \text{ and } \angle(w, v) \geqslant \frac{\pi}{4}. \tag{8.17}$$

取 $u \in y$，使其达到（8.4）中的上确界。由于 $y = \mathrm{span}\{v、\ v'\} = \mathrm{span}\{v \text{和 } w\}$，我们可以将 $u = av + bw$ 写为 a，其中 $b \in \mathbb{R}$。（8.17）中的夹角界意味着 $\|av\|$

且 $\|bw\|$ 和 $2\|u\|$均小于 $2\|u\|$。因此,

$$\begin{aligned}\mathrm{VA}_1(x,x') = \frac{\|u^\perp\|}{\|u\|} &\leqslant \frac{\|av^\perp\|}{\|u\|} + \frac{\|bw^\perp\|}{\|u\|} \\ &< 2\frac{\|v^\perp\|}{\|v\|} + 2\frac{\|w^\perp\|}{\|w\|} \leqslant 2d(x,E) + 2\frac{\|w^\perp\|}{\|w\|}.\end{aligned}$$

Thus, $d(x,E)$ and $\|w^\perp\|/\|w\|$ cannot be both less than $\mathrm{VA}_1(x,x')/4$. □

Take $\tilde{\kappa}_1 = \tilde{\kappa}_1(\nu_\infty) > 0$ and $\tilde{\theta}_1 = \tilde{\theta}_1(\nu_\infty, \delta) > 0$ to be as in Lemma 8.2.

Lemma 8.5. *There exists $\hat{\kappa}_1 = \hat{\kappa}_1(\nu_\infty) > 0$ and for each $\delta > 0$ there exists $\widehat{N}_1 = \widehat{N}_1(\nu_\infty, \delta) \in \mathbb{N}$ such that for each $n \geqslant \widehat{N}_1$ there exists $\hat{\rho}_1 = \hat{\rho}_1(\nu_\infty, \delta, n) > 0$ such that for any $x \neq x'$ in $E_1(\hat{\rho}_1)$ with $-\log \mathrm{VA}_1(x,x') \leqslant \tilde{\theta}_1 + \tilde{\kappa}_1 n$ there exists $\widehat{\mathcal{E}}_1 = \widehat{\mathcal{E}}_1(\nu_\infty, \delta, n, x, x') \subset \operatorname{supp} \nu_\infty^{(n)}$ with $\nu_\infty^{(n)}(\widehat{\mathcal{E}}_1^c) < \delta$ and*

$$-\log d(gx, gx') \leqslant -\log d(x,x') - \log \mathrm{VA}_1(x,x') - \hat{\kappa}_1 n \text{ for every } g \in \widehat{\mathcal{E}}_1. \tag{8.18}$$

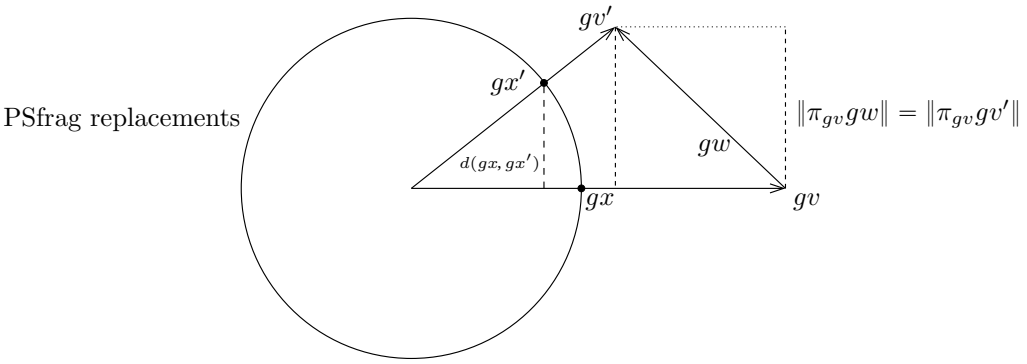

FIGURE 6. Verifying the inequality (8.21), when $\|gv\| \geqslant \|gv'\|$: the distance between gx and gx') is given by the length of the vertical dashed segment on the left, which is greater than the length of the vertical dashed segment on the right divided by $\|gv\|$.

Proof. Let $\kappa_0 = \kappa_0(\nu_\infty) > 0$, $N_0 = N_0(\nu_\infty, \delta) \in \mathbb{N}$, and $\mathcal{E}_0 = \mathcal{E}_0(\nu_\infty, \delta, n, v^\perp) \subset \operatorname{supp} \nu_\infty^{(n)}$ be as in Proposition 5.5. Given $x \neq x'$ in P, let $w = v' - v$ be the difference between unit vectors v and v' in the directions of x and x', respectively. Take

$$\begin{aligned}&\hat{\kappa}_1 = \kappa_0/2, \quad \widehat{N}_1 > \max\{N_0, 5/\hat{\kappa}_1\}, \\ &\hat{\rho}_1 < e^{-\tilde{\theta}_1 - \tilde{\kappa}_1 n}/10, \text{ and } \widehat{\mathcal{E}}_1 = \mathcal{E}_0(\nu_\infty, \delta, n, w^\perp).\end{aligned} \tag{8.19}$$

Let $n \geqslant \widehat{N}_1$ and $g \in \widehat{\mathcal{E}}_1 \subset \operatorname{supp} \nu_\infty^{(n)}$. As observed in (8.17),

$$d(x,x') \leqslant \|w\| \tag{8.20}$$

不妨设 $\|gv\| \geqslant \|gv'\|$; 情形 $\|gv\| \leqslant \|gv'\|$ 类似, 只需交换 x 和 x' 的角色。于是（参见图 6）

$$d(gx, gx') = \left\| \Pi_{gv} \frac{gv'}{\|gv'\|} \right\| = \left\| \Pi_{gv} \frac{gw}{\|gv'\|} \right\| \geqslant \frac{1}{\|gv\|} \|\Pi_{gv} gw\|, \tag{8.21}$$

因此（请记住，$\|v\| = \|v'\| = 1$），

$$-\log d(gx, gx') \leqslant -\log d(x, x') - \log \frac{\|\Pi_{gv} gw\|}{\|gv\|} \frac{\|v\|}{\|w\|}. \tag{8.22}$$

By the condition on $\hat\rho_1$ in (8.19), if $d(x, E) \leqslant \hat\rho_1$ then

$$4d(x, E) < e^{-\tilde\theta_1 - \tilde\kappa_1 n} < \mathrm{VA}_1(x, x'),$$

然后引理 8.4 给出：

$$\frac{\|w^\perp\|}{\|w\|} > \frac{1}{4}\,\mathrm{VA}_1(x, x') > \frac{1}{4} e^{-\tilde\theta_1 - \tilde\kappa_1 n}. \tag{8.23}$$

在进一步缩小 $\hat{}\rho_1$ 后，我们还可以假设：

$$\frac{\|hz\|}{\|z\|} \geqslant \frac{1}{2}\frac{\|hz^E\|}{\|z^E\|} \text{ and } \|\Pi_{hz} hw\| \geqslant \frac{1}{2}\|\Pi_{hz^E} hw\| \tag{8.24}$$

for any non-zero $z = z^E + z^\perp$ in $E \oplus E^\perp$ with $\|z^\perp\|/\|z\| \leqslant \hat\rho_1$ and any $h \in \operatorname{supp} \nu_\infty^{(n)}$. Indeed, the first part of (8.24) is a simple consequence of continuity; in the second one note also that w is bounded away from the horizontal, by (8.23). This last part of (8.24) implies that

$$\|\Pi_{hz} hw\| \geqslant \frac{1}{2}\|\Pi_{hz^E} hw\| \geqslant \frac{1}{2}\|\Pi_E hw\| = \frac{1}{2}\|(hw)^\perp\| = \frac{1}{2}\|h^\perp w^\perp\|. \tag{8.25}$$

注意到 $\|v^\perp\|/\|v\| = d(x, E) \leqslant \hat{}\rho_1$，在前面两个关系中取 $z = v$ 和 $h = g$。因此，将 (8.24) 和 (8.25) 代入 (8.22)，

$$-\log d(gx, gx') \leqslant -\log d(x, x') + \log 4 - \log \frac{\|g^\perp w^\perp\|}{\|w\|} \frac{\|v^E\|}{\|gv^E\|}.$$

然后，再结合 (8.23) 和 (5.3)，

$$\begin{aligned} -\log d(gx, gx') &\leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') + \log 16 - \log \frac{\|g^\perp w^\perp\|}{\|w^\perp\|} \frac{\|v^E\|}{\|gv^E\|} \\ &= -\log d(x, x') - \log \mathrm{VA}_1(x, x') + \log 16 - \log \frac{\|Dg^\perp_{v^E} w^\perp\|}{\|w^\perp\|}. \end{aligned}$$

By part (1) of Proposition 5.5 and the choice of $\hat\kappa_1$ and $\widehat{N}_1$ in (8.19), it follows that

$$\begin{aligned} -\log d(gx, gx') &\leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') + 5 - 2\hat\kappa_1 n \\ &\leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') - \hat\kappa_1 n, \end{aligned}$$

as claimed. □

We are ready to prove Proposition 8.3. Take $A = A(\nu_\infty)$ as in (8.2) and then define

$$\begin{aligned} &\gamma_1 = \min\{1, \tilde\kappa_1/(2A)\}, \\ &\kappa_1' = \min\{\tilde\kappa_1/2, \gamma_1\hat\kappa_1/2\}, \\ &\rho_1' = \min\{\hat\rho_1(\nu_\infty, \delta), \hat\rho_1(\nu_\infty, \delta/2)\} \\ &N_1 > \max\{\widetilde{N}_1(\nu_\infty, \delta/2), \widehat{N}_1(\nu_\infty, \delta/2), 2\tilde\theta_1/(\gamma\hat\kappa_1)\} \text{ and} \\ &\mathcal{E}_1' = \widetilde{\mathcal{E}}_1(\nu_\infty, \delta/2, n, x, x') \cap \widehat{\mathcal{E}}_1(\nu_\infty, \delta/2, n, x, x'). \end{aligned} \tag{8.26}$$

注意到 $\nu_\infty^{(n)}((\mathcal{E}_1')^c) < \delta$。根据定义,

$$-\log \mathrm{VP}_1(gx, gx') = -\log \mathrm{VA}_1(gx, gx') - \gamma_1 \log d(gx, gx'). \tag{8.27}$$

˜考虑 $x \neq x'$ 在 $E_1(\rho_1')$、 $n \geqslant N_1$ 和 $g \in \mathcal{E}_1'$中的情形。如果 $-\log \mathrm{VA}_1(x, x') \geqslant \theta_1 + \tilde{\kappa}_1 n$那么根据引理 8.2,

$$-\log \mathrm{VA}_1(gx, gx') \leqslant -\log \mathrm{VA}_1(x, x') - \tilde{\kappa}_1 n \tag{8.28}$$

Substituting (8.2) and (8.28) in (8.27) we find that

$$\begin{aligned} -\log \mathrm{VP}_1(gx, gx') &\leqslant -\log \mathrm{VA}_1(x, x') - \tilde{\kappa}_1 n - \gamma_1 \log d(x, x') + \gamma_1 An \\ &\leqslant -\log \mathrm{VP}_1(x, x') - \frac{\tilde{\kappa}_1}{2} n \leqslant -\log \mathrm{VP}_1(x, x') - \kappa_1' n. \end{aligned} \tag{8.29}$$

Now assume that $-\log \mathrm{VA}_1(x, x') \leqslant \tilde{\theta}_1 + \tilde{\kappa}_1 n$. In this case, Lemma 8.2 yields

$$-\log \mathrm{VA}_1(gx, gx') \leqslant \tilde{\theta}_1 \tag{8.30}$$

而引理 8.5 表明,

$$-\log d(gx, gx') \leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') - \hat{\kappa}_1 n. \tag{8.31}$$

将（8.30）和（8.31）代入（8.27），得到

$$\begin{aligned} -\log \mathrm{VP}_1(gx, gx') &\leqslant \tilde{\theta}_1 - \gamma_1 \log d(x, x') - \gamma_1 \log \mathrm{VA}_1(x, x') - \gamma_1 \hat{\kappa}_1 n \\ &\leqslant -\log \mathrm{VP}_1(x, x') + \tilde{\theta}_1 + (1 - \gamma_1) \log \mathrm{VA}_1(x, x') - \gamma_1 \hat{\kappa}_1 n. \end{aligned}$$

˜由于 $\gamma_1 \leqslant 1$、$\mathrm{VA}_1(x、\ x') \leqslant 1$ 和 $n \geqslant N_1 \geqslant 2\theta_1/(\gamma_1 \hat{\kappa}_1)$，因此得到

$$\begin{aligned} -\log \mathrm{VP}_1(gx, gx') &\leqslant -\log \mathrm{VP}_1(x, x') + \tilde{\theta}_1 - \gamma_1 \hat{\kappa}_1 n \\ &\leqslant -\log \mathrm{VP}_1(x, x') - \frac{\gamma_1 \hat{\kappa}_1}{2} n \leqslant -\log \mathrm{VP}_1(x, x') - \kappa_1' n. \end{aligned} \tag{8.32}$$

The relations (8.29) and (8.32) contain the conclusion of Proposition 8.3. □

8.3. **The function $-\log \psi_1$.** Now we are going to prove that $\psi_1 = \mathrm{VP}_1$ satisfies Proposition 8.1. Let $A = A(\nu_\infty)$ and $B = B(\nu_\infty) > 0$ be as in (8.2) and (8.3). Take $\kappa_1' > 0$, $N_1 \in \mathbb{N}$ and $\rho_1' > 0$ as in Proposition 8.3. By definition,

$$-\log \psi_1(x, x') = -\log \mathrm{VP}_1(x, x') = -\log \mathrm{VA}_1(x, x') - \gamma_1 \log d(x, x') \tag{8.33}$$

对每个 $x \neq x'$，在 P 中定义。

$$C_1' = B + \gamma_1 A. \tag{8.34}$$

Substituting (8.7) and (8.2) in (8.33), we find that

$$\begin{aligned} -\log \mathrm{VP}_1(gx, gx') &\leqslant -\log \mathrm{VA}_1(x, x') + Bn - \gamma_1 \log(dx, dx') + \gamma_1 An \\ &\leqslant -\log \mathrm{VP}_1(x, x') + C_1' n \end{aligned} \tag{8.35}$$

对于每个 $x \neq x' \in P$ 且 $g \in \operatorname{supp} \nu_\infty^{(n)}$ 的情形。对 $\mathcal{E}_1'$ 积分 (8.15)，并对补集积分 (8.35)，再利用 $\nu_\infty^{(n)}((\mathcal{E}_1')^c) < \delta$ 这一事实，我们得到:

$$\int_G -\log \mathrm{VP}_1(gx, gx')\, d\nu_\infty^{(n)}(g) \leqslant -\log \mathrm{VP}_1(x, x') - \kappa_1' n + C_1' \delta n.$$

for every $n \geqslant N_1$ and $x \neq x'$ in $E_1(\rho_1')$. This completes the proof of Proposition 8.1.

Another relevant feature is that $-\log \psi_1(x, x')$ goes to infinity when x and x' approach the equator E:

Lemma 8.6. *Given any $R > 0$, there exists $\tilde{\rho}_1 = \tilde{\rho}_1(\nu_\infty, R) > 0$ such that $-\log \psi_1(x, x') > R$ for any $x \neq x'$ in $E_1(\tilde{\rho}_1)$.*

Proof. Consider any $\tilde{\rho}_1 > 0$ and let v and v' be unit vectors in the direction of x and x', respectively. Observe that

$$\frac{\|v^\perp\|}{\|v\|} = d(x, E) \leqslant \tilde{\rho}_1 \text{ and } \frac{\|(v')^\perp\|}{\|v'\|} = d(x', E) \leqslant \tilde{\rho}_1.$$

如果 $d(x,\ x') \leqslant \tilde{\rho}_1^{1/2}$ 那么

$$\mathrm{VP}_1(x, x') = \mathrm{VA}_1(x, x') d(x, x')^{\gamma_1} \leqslant \tilde{\rho}_1^{\gamma_1/2}. \tag{8.36}$$

Now suppose that $d(x, x') = |\sin \angle(v, v')|$ is greater than $\tilde{\rho}_1^{1/2}$. Then every u in the great circle y generated by x and x' may be written as $u = \alpha v + \alpha' v'$ with $\alpha, \alpha' \in \mathbb{R}$ such that $\|u\| \geqslant \tilde{\rho}_1^{1/2} \|\alpha v\|$ and $\|u\| \geqslant \tilde{\rho}_1^{1/2} \|\alpha' v'\|$. Thus,

$$\frac{\|u^\perp\|}{\|u\|} \leqslant \frac{\|\alpha v^\perp\|}{\|u\|} + \frac{\|\alpha'(v')^\perp\|}{\|u\|} \leqslant \frac{1}{\tilde{\rho}_1^{1/2}} \left(\frac{\|v^\perp\|}{\|v\|} + \frac{\|(v')^\perp\|}{\|v'\|} \right) \leqslant 2\tilde{\rho}_1^{1/2}.$$

Since u is arbitrary, this proves that $\mathrm{VA}_1(x, x') \leqslant 2\tilde{\rho}_1^{1/2}$, and so

$$\mathrm{VP}_1(x, x') = \mathrm{VA}_1(x, x') d(x, x')^{\gamma_1} \leqslant 2\tilde{\rho}_1^{1/2} \leqslant 2\tilde{\rho}_1^{\gamma_1/2} \tag{8.37}$$

(because $\gamma_1 \leqslant 1$). The inequalities (8.36) and (8.37) imply that

$$-\log \psi_1(x, x') = -\log \mathrm{VP}_1(x, x') \geqslant -\frac{\gamma_1}{2} \log \tilde{\rho}_1 - \log 2$$

对于任意 $x \neq x' \in E_1(\tilde{\rho}_1)$，由此即可得到所需结论。 □

9. Stabilization and cut-off

此前，我们构造了一个函数 $-\log\ \psi_1 = -\log \mathrm{VP}_1$，它具有马古利斯函数的某些特征。在本节中，我们将对该函数进行修改，以纠正两个重要的缺陷。

One problem with $-\log \psi_1$, that originates from VA_1, is that the definition refers explicitly to E. The reason why this is a problem is that for $k \in \mathbb{N}$ the equator is not necessarily a ν_k-invariant set. Thus, even for k large, for $g \in \operatorname{supp} \nu_k$ there is no uniform upper bound on the absolute value of

$$-\log \psi_1(gx, gx') + \log \psi_1(x, x')$$

because $-\log \mathrm{VA}_1(gx, gx')$ may be wildly different from $-\log \mathrm{VA}_1(x, x')$. To rectify this, in Section 9.1 we define "stabilized" versions of the functions VA_1 and VP_1. This will come at a price: the analogue of Proposition 8.1 will hold only outside a "stabilization region" near E. Still, as we will explain in Section 10, stabilization does allow us to by-pass this first difficulty in a satisfactory way.

Another problem is that, because of the way the Markov operators $\mathcal{T}_{k,1}$ will be constructed, we have little control over the border region. The simplest way to by-pass this is to take the Margulis function to be bounded when either x or x' is in the border region. The function $-\log \psi_1$ does not satisfy this: for instance, it can get arbitrarily large when x and x' are close to each other and, of course, that may occur even if they are both in the border region. Thus, it is necessary to cut that function off in (a neighborhood of) the border region.

However, doing a cut-off creates a discontinuity that translates into a drastic failure of (7.14) near the discontinuity. The main idea to handle this, which we

call *recoupling*, will be explained in Section 11.1. The recoupling technique is far from universal, it can only handle certain types of discontinuities. Thus, it makes sense to make the discontinuity as mild as possible. In Section 9.2 we explain just how to do this. In particular, we only cut $-\log\psi_1$ off if x and x' are both in (a neighborhood of) the border region. In the present ($r = 1$) situation this is fine because if one of the points, x or x', is close to the border region and the other one is far outside then $-\log\psi_1(x, x')$ is bounded anyway. The steps $r > 1$ involve additional issues, which we will discuss in Sections 13 and 15.

We use the following elementary inequalities, whose proof we leave to the reader: given any $C > 0$, $c > 0$ and $\psi > 0$,

$$\log(\Omega + c\psi^{-1}) \leqslant \log(\Omega + \psi^{-1}) + \log c \quad \text{if} \quad c > 1 \tag{9.1}$$

$$\log(\Omega + c\psi^{-1}) \leqslant \log(\Omega + \psi^{-1}) \quad \text{if} \quad c < 1 \tag{9.2}$$

$$\log(\Omega + c\psi^{-1}) \leqslant \log(\Omega + \psi^{-1}) + \log\sqrt{c} \text{ 若 } c < 1 \text{ 且 } \psi^{-1} \geqslant \Omega/\sqrt{c}. \tag{9.3}$$

Let $\kappa_1' = \kappa_1'(\nu_\infty) > 0$, $C_1' = C_1'(\nu_\infty) > 0$, $N_1 = N_1(\nu_\infty, \delta) \in \mathbb{N}$ be as in Propositions 8.1 and 8.3. Keep in mind that $\varepsilon_1 < \rho_1' \leqslant \hat{\rho}_1$ and $C_1' = B + \gamma_1 A$, according to (8.26), (8.34) and (9.12).

9.1. **Stabilization.** Let $B = B(\nu_\infty) > 0$ and $\gamma_1 = \gamma_1(\nu_\infty) > 0$ be as in (8.3) and (8.26), respectively. For each $\omega > 0$, define the *stabilized vertical angle*

$$\mathrm{SVA}_1(x, x'; \omega) = \max\big\{\, \mathrm{VA}_1(x, x'), \omega e^{-Bn}\big\} \tag{9.4}$$

以及稳定化垂直投影乘以

$$\psi_1(x, x'; \omega) = \mathrm{SVP}_1(x, x'; \omega) = \mathrm{SVA}_1(x, x'; \omega) d(x, x')^{\gamma_1} \tag{9.5}$$

for every $x \neq x'$ in $E_1(\varepsilon_1)$. The estimate (8.35) remains valid for the stabilized vertical projection:

Lemma 9.1. *For every $x \neq x'$ in P, $g \in \operatorname{supp}\nu_\infty^{(n)}$, and $\omega > 0$,*

$$-\log\psi_1(gx, gx'; \omega) \leqslant -\log\psi_1(x, x'; \omega) + C_1' n. \tag{9.6}$$

证明。我们首先提出如下主张。

$$-\log\mathrm{SVA}_1(gx, gx'; \omega) \leqslant -\log\mathrm{SVA}_1(x, x'; \omega) + Bn. \tag{9.7}$$

The proof can be split into two cases. First, suppose that $\mathrm{VA}_1(x, x') < \omega$. Then, by the definition (9.4),

$$-\log\mathrm{SVA}_1(x, x'; \omega) \geqslant -\log\omega \text{ and}$$
$$-\log\mathrm{SVA}_1(gx, gx'; \omega) \leqslant -\log\omega + Bn \leqslant -\log\mathrm{SVA}_1(x, x'; \omega) + Bn,$$

as claimed. Now suppose that $\mathrm{VA}_1(x, x') \geqslant \omega$. Then, again by (9.4),

$$-\log\mathrm{SVA}_1(x, x'; \omega) = -\log\mathrm{VA}_1(x, x') \text{ and}$$
$$-\log\mathrm{SVA}_1(gx, gx'; \omega) \leqslant -\log\mathrm{VA}_1(gx, gx').$$

那么，(9.7) 便是 (8.7) 的直接推论。至此，(9.7) 的证明完成。现在，由 (9.7) 和 (8.2) 可容易地得到 (9.6)：回顾 (8.34) 中 C_1' 的定义，

$$\begin{aligned} -\log\mathrm{SVP}_1(gx, gx'; \omega) &= -\log\mathrm{SVA}_1(gx, gx'; \omega) - \gamma_1 \log d(gx, gx') \\ &\leqslant -\log\mathrm{SVA}_1(x, x'; \omega) + Bn - \gamma_1 \log d(x, x') - \gamma_1 An \\ &= -\log\mathrm{SVP}_1(x, x') + (B + \gamma_1 A)n. \end{aligned} \tag{9.8}$$

This completes the argument. □

The *stabilization region* is the set of pairs (x, x') such that $\mathrm{VA}_1(x, x') < \omega$. We have seen in (8.7) that

$$\mathrm{VA}_1(gx, gx') \geqslant \mathrm{VA}_1(x, x') e^{-Bn}$$

for any $x \neq x'$ in P, $g \in \operatorname{supp} \nu_\infty^{(n)}$ and $n \in \mathbb{N}$. Consequently, if (x, x') is not in the stabilization region then

$$\begin{aligned} \mathrm{SVA}_1(x, x'; \omega) &= \mathrm{VA}_1(x, x') \text{ and } \mathrm{SVA}_1(gx, gx'; \omega) = \mathrm{VA}_1(gx, gx') \\ \mathrm{SVP}_1(x, x'; \omega) &= \mathrm{VP}_1(x, x') \text{ and } \mathrm{SVP}_1(gx, gx'; \omega) = \mathrm{VP}_1(gx, gx') \end{aligned} \tag{9.9}$$

对于任意 $g \in \operatorname{supp} \nu_\infty^{(n)}$ 和 $n \in \mathbb{N}$。

Propositions 8.1 and 8.3 immediately yield the following analogues for stabilized vertical angles and vertical projections:

命题 9.2. 对于每个 $\delta > 0$、$n \geqslant N_1$、$x \neq x' \in E_1(\rho_1')$，以及满足 $\mathrm{VA}_1(x, x') \geqslant \omega$ 的 $\omega > 0$，

$$-\log \psi_1(gx, gx'; \omega) \leqslant -\log \psi_1(x, x'; \omega) - \kappa_1' n \text{ for every } g \in \mathcal{E}_1', \tag{9.10}$$

and

$$\int_G -\log \psi_1(gx, gx'; \omega)\, d\nu_\infty^{(n)}(g) \leqslant -\log \psi_1(x, x'; \omega) - (\kappa_1' - C_1'\delta) n. \tag{9.11}$$

证明。由于假设 (x, x') 不在稳定化区域内，(9.10) 是命题 8.3 的另一种表述，而 (9.11) 是命题 8.1 的另一种表述。□

9.2. **Cutoff.** Recall that $\varepsilon_1 = \varepsilon_1(\nu_\infty, \delta, n) > 0$ was chosen in Section 7.2, in the context of (7.3). As observed then, it may be taken to be arbitrarily small. In particular, it is no restriction to suppose that

$$\varepsilon_1 < \min\{\rho_0, \rho_1'\} \tag{9.12}$$

where $\rho_0 = \rho_0(\nu_\infty, n) > 0$ is as in Corollaries 5.6 and 5.7 and $\rho_1' = \rho_1'(\nu_\infty, \delta, n) > 0$ is as in Propositions 8.1 and 8.3.

Arguing twice as in Remark 7.3, we find constants $\varepsilon_1' = \varepsilon_1'(\nu_\infty, \delta, n) > 0$ and $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(\nu_\infty, \delta, n) > 0$ with $0 < \varepsilon_1' < \tilde{\varepsilon}_1 < \varepsilon_1$, and a compact neighborhood $\mathcal{W}_1 = \mathcal{W}_1(\nu_\infty, \delta, n)$ of $\operatorname{supp} \nu_\infty^{(n)}$, such that

$$gx \in E_1(\varepsilon_1/2) \text{ for every } x \in E_1(2\tilde{\varepsilon}_1) \text{ and } g \in \mathcal{W}_1 \text{ and} \tag{9.13}$$

$$g^{-1}x \in E_1(\tilde{\varepsilon}_1/2) \text{ for every } x \in E_1(2\varepsilon_1') \text{ and } g \in \mathcal{W}_1. \tag{9.14}$$

Let $\kappa_0 = \kappa_0(\nu_\infty) > 0$ be as in Proposition 5.5, and define $\varepsilon_1'' = \varepsilon_1''(\nu_\infty, \delta, n)$ by

$$\varepsilon_1'' = 3\varepsilon_1' e^{-\kappa_0 n/2}. \tag{9.15}$$

Taking $\rho = \varepsilon_1''$ in Corollary 5.7, and recalling that ε_1 was chosen smaller than ρ_0, we get that there are $\tilde{k}_1 = \tilde{k}_1(\nu_\infty, \delta, n) \in \mathbb{N}$ and $\mathcal{D}_k(x) = \mathcal{D}_k(\nu_\infty, \delta, n, x) \subset \operatorname{supp} \nu_k^{(n)}$ such that $\nu_k^{(n)}(\mathcal{D}_k(x)^c) < \delta$ and

$$d(gx, E) > e^{\kappa_0 n/2} d(x, E) > e^{\kappa_0 n/2} \varepsilon_1'' > 2\varepsilon_1'$$

for any $g \in \mathcal{D}_k(x)$, $x \in E(\varepsilon_1, \varepsilon_1'')$ and $k \geqslant \tilde{k}_1$. In other words, for $k \geqslant \tilde{k}_1$,

$$x \in E(\varepsilon_1, \varepsilon_1'') \Rightarrow gx \notin E(2\varepsilon_1') \text{ for every } g \in \mathcal{D}_k(x). \tag{9.16}$$

如有必要增大 $\tilde{k}_1$，我们可以假设对每个 $k \geqslant \tilde{k}_1$ 都有 $\operatorname{supp} \nu_k^{(n)} \subset \mathcal{W}_1$。那么 (9.13) 和 (9.14) 推出

$$E_1(2\tilde{\varepsilon}_1) \subset \mathcal{X}_{\nu_k^{(n)}} E_1(\varepsilon_1) \text{ and } E_1(2\varepsilon_1') \subset \mathcal{X}^{\#}_{\nu_k^{(n)}} E_1(\varepsilon_1) \tag{9.17}$$

$$x \notin E_1(\tilde{\varepsilon}_1/2) \Rightarrow gx \notin E_1(2\varepsilon_1') \text{ for every } g \in \operatorname{supp} \nu_k^{(n)}. \tag{9.18}$$

最后，固定 $\Omega_1 = \Omega_1(\nu_\infty, \delta, n) > 1$ 足够大，使得

$$\Omega_1\, \varepsilon_1'(\varepsilon_1' - 2\varepsilon_1'')^{\gamma_1} \geqslant 1 \tag{9.19}$$

and define

(9.20)

$$\Psi_1(x, x'; \omega) = \begin{cases} \log\left(\Omega_1 + \psi_1(x, x'; \omega)^{-1}\right) & \text{if } x \in E_1(2\varepsilon_1'') \text{ or } x' \in E_1(2\varepsilon_1'') \\ \log \Omega_1 & \text{otherwise.} \end{cases}$$

We will refer to the set $E_1(2\varepsilon_1'')^c \times E_1(2\varepsilon_1'')^c$ as the *cut-off region*. See Figure 7.

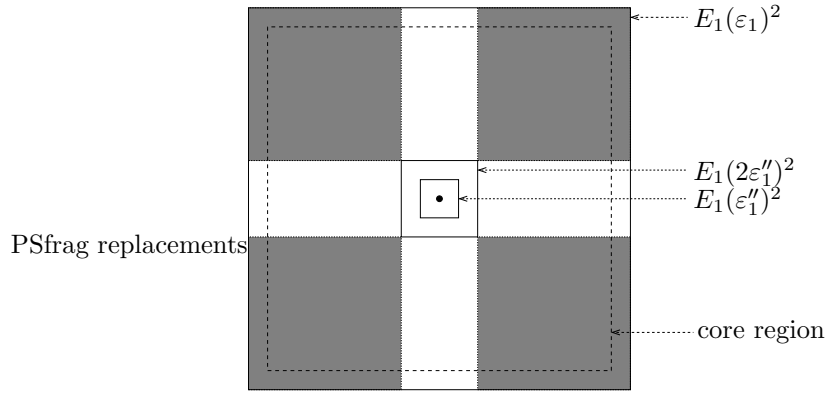

FIGURE 7. Illustrating the cut-off in the definition of the Margulis function. The black dot at the center marks the point (E, E). The dashed lined represents the boundary between the $\nu_k^{(n)}$-core and the $\nu_k^{(n)}$-border of $E_1(\varepsilon_1)$. The shaded area is the cut-off region, where $\Psi_1 \equiv \log \Omega_1$.

命题 9.3. 存在 $\kappa_1''' = \kappa_1'''(\nu_\infty) > 0$，并且对于每个 $\delta > 0$ 和 $n \geqslant N_1$，都存在 $\varepsilon_1''' = \varepsilon_1'''(\nu_\infty, \delta, n) > 0$，使得

(i) *For any $x \neq x'$ in $E_1(\varepsilon_1)$ with $\Psi_1(x, x'; \omega) > \log \Omega_1$,*

$$\int_G \Psi_1(gx, gx'; \omega)\, d\nu_\infty^{(n)}(g) \leqslant \Psi_1(x, x'; \omega) + C_1' n.$$

(ii) 对于任意 $x \neq x'$ in $E_1(\varepsilon_1)$ with $\Psi_1(x, x'; \omega) > \log \Omega_1$ 和 $\mathrm{VA}_1(x, x') \geqslant \omega$,

$$\int_G \Psi_1(gx, gx'; \omega)\, d\nu_\infty^{(n)}(g) \leqslant \Psi_1(x, x'; \omega) + C_1' \delta n.$$

(iii) *For any $x \neq x'$ in $E_1(\varepsilon_1''')$ with $\mathrm{VA}_1(x, x') \geqslant \omega$,*

$$\int_G \Psi_1(gx, gx'; \omega)\, d\nu_\infty^{(n)}(g) \leqslant \Psi_1(x, x'; \omega) - (\kappa_1''' - C_1'\delta) n.$$

Proof. Define $\kappa_1''' = \kappa_1'/2$. Let $n \geqslant N_1$. Part (i) of the proposition is a consequence of the following lemma:

Lemma 9.4. *If $x \neq x'$ in P are such that $\Psi_1(x, x'; \omega) > \log \Omega_1$ then*

$$\Psi_1(gx, gx'; \omega) \leqslant \Psi_1(x, x'; \omega) + C_1' n \text{ for any } g \in \operatorname{supp} \nu_\infty^{(n)}. \tag{9.21}$$

证明。由 (9.6) 和 (9.1),

$$\begin{aligned}\Psi_1(gx, gx'; \omega) &\leqslant \log\left(\Omega_1 + \psi_1(gx, gx'; \omega)^{-1}\right) \leqslant \log\left(\Omega_1 + e^{C_1' n}\psi_1(x, x'; \omega)^{-1}\right) \\ &\leqslant \log\left(\Omega_1 + \psi_1(x, x'; \omega)^{-1}\right) + C_1' n = \Psi_1(x, x'; \omega) + C_1' n.\end{aligned}$$

This gives the claim. □

To prove part (ii) we use Proposition 9.2: given any $\omega > 0$ and $x \neq x'$ in $E_1(\varepsilon_1)$,

$$\log \psi_1(gx, gx';\ \omega) \leqslant -\log \psi_1(x, x';\ \omega) - \kappa_1' n \leqslant -\log \psi_1(x, x';\ \omega) \tag{9.22}$$

对每个 $g \in \mathcal{E}_1'$ 均成立。然后,

$$\begin{aligned}\Psi_1(gx, gx'; \omega) &\leqslant \log\left(\Omega_1 + \psi_1(gx, gx'; \omega)^{-1}\right) \\ &\leqslant \log\left(\Omega_1 + \psi_1(x, x'; \omega)^{-1}\right) = \Psi_1(gx, gx'; \omega),\end{aligned} \tag{9.23}$$

for every $g \in \mathcal{E}_1'$. Integrating (9.23) over $\mathcal{E}_1'$ and (9.21) over the complement, we obtain the estimate in part (ii).

现在取关系 (9.3) 中的 $c = e^{-2\kappa_1''' n}$。根据引理 8.6，存在一个仅依赖于 ν_∞、δ 和 n （通过 c 和 Ω_1）的 $\varepsilon_1''' > 0$，使得

$$-\log \psi_1(x, x'; \omega) = -\log \psi_1(x, x') \geqslant \log\left(\Omega_1/\sqrt{c}\right) \tag{9.24}$$

for any $x \neq x'$ in $E_1(\varepsilon_1''')$. Then, using (9.22), (9.24), and (9.3),

$$\begin{aligned}\Psi_1(gx, gx'; \omega) &\leqslant \log\left(\Omega_1 + \psi_1(gx, gx'; \omega)^{-1}\right) \\ &\leqslant \log\left(\Omega_1 + e^{-2\kappa_1''' n}\psi_1(x, x'; \omega)^{-1}\right) \\ &\leqslant \log\left(\Omega_1 + \psi_1(x, x'; \omega)^{-1}\right) - \kappa_1''' n = \Psi_1(x, x'; \omega) - \kappa_1''' n\end{aligned} \tag{9.25}$$

对每个 $g \in \mathcal{E}_1'$都成立。将 (9.25) 对 $\mathcal{E}_1'$ 积分，并将 (9.21) 对补集积分，我们便得到命题 (iii) 部分中的估计。□

引理 9.5. 对于任意满足 $x \notin E_1(\varepsilon_1')$ or$x' \notin E_1(\varepsilon_1')$的 $E_1(\varepsilon_1)$ 中的 $\omega > 0$ 和 $x \neq x'$,

$$\log \Omega_1 \leqslant \Psi_1(x, x'; \omega) \leqslant \log \Omega_1 + \log 2.$$

Proof. The inequality on the left is an immediate consequence of the definition (9.20), and the same is true for the one on the right if both points x and x' are outside $E_1(2\varepsilon_1'')$. Let us suppose that $x \notin E_1(\varepsilon_1')$ but $x' \in E_1(2\varepsilon_1'')$; the case when $x \in E_1(2\varepsilon_1'')$ but $x' \notin E_1(\varepsilon_1')$ is analogous. Let $y = y(x, x')$ be the great circle associated to x and x'. Then

$$d(x, x') \geqslant \varepsilon_1' - 2\varepsilon_1'' \text{ and } \operatorname{VA}_1(x, x') = d(y, E) \geqslant d(x, E) > \varepsilon_1'$$

and so

$$\psi_1(x, x'; \omega) \geqslant \psi_1(x, x') \geqslant \varepsilon_1'(\varepsilon_1' - 2\varepsilon_1'')^{\gamma_1}.$$

根据（9.19）中 Ω_1 的定义，可得

$$\Psi_1(x, x'; \omega) = \log(\Omega_1 + \psi_1(x, x'; \omega)^{-1}) \leqslant \log 2\Omega_1,$$

as claimed. □

10. 开启扰动

现在我们证明，只要 $k \in \mathbb{N}$ 足够大，命题 9.3 的结论对于 $\nu_k^{(n)}$ 而非 $\nu_\infty^{(n)}$ 也成立。更准确地说，我们证明：

命题 10.1。给定 $\delta > 0$、$n \geqslant N_1$ 和 $\omega > 0$，存在 $k_1 = k_1(\nu_\infty, \delta, n, \omega) \in \mathbb{N}$，使得以下结论对每个 $k \geqslant k_1$ 都成立：

(i) *For any $x \neq x'$ in $E_1(\varepsilon_1)$ with $\Psi_1(x, x'; \omega) > \log \Omega_1$,*

$$\int_G \Psi_1(gx, gx'; \omega)\, d\nu_k^{(n)}(g) \leqslant \Psi_1(x, x'; \omega) + C_1' n.$$

(ii) 对于任意 $x \neq x'$ 属于 $E_1(\varepsilon_1)$ 且满足 $\Psi_1(x, x'; \omega) > \log \Omega_1$ 和 $\mathrm{VA}_1(x, x') \geqslant \omega$,

$$\int_G \Psi_1(gx, gx'; \omega)\, d\nu_k^{(n)}(g) \leqslant \Psi_1(x, x'; \omega) + C_1' \delta n.$$

(iii) *For any $x \neq x'$ in $E_1(\varepsilon_1''')$ with $\mathrm{VA}_1(x, x') \geqslant \omega$,*

$$\int_G \Psi_1(gx, gx'; \omega)\, d\nu_k^{(n)}(g) \leqslant \Psi_1(x, x'; n) - (\kappa_1' - C_1' \delta) n.$$

Keep in mind that we have chosen $\varepsilon_1 < \rho_1' \leqslant \hat{\rho}_1$ and $C_1' = B + \gamma_1 A$. Recall also that $\tilde{k}_1 = \tilde{k}_1(\nu_\infty, \delta, n) \in \mathbb{N}$ was chosen so that the relations (9.16) through (9.18) hold for every $k \geqslant \tilde{k}_1$. Moreover, $N_1 \in \mathbb{N}$ is given by Propositions 8.3 and 9.2.

Proof. We are going to extend to large $k \in \mathbb{N}$ several estimates in the proof of Proposition 9.3. This will require a number of conditions on k, depending on ν_∞, δ, n and ω, that we state along the way. We begin with the following extension of Lemma 9.4:

Lemma 10.2. *Given $n \geqslant N_1$ and $\omega > 0$, there is $\hat{k}_1 = \hat{k}_1(\nu_\infty, n, \omega) \in \mathbb{N}$ such that if $x \neq x'$ in P are such that $\Psi_1(x, x'; \omega) > \log \Omega_1$ then*

$$\Psi_1(gx, gx'; \omega) \leqslant \Psi_1(x, x'; \omega) + C_1' n \tag{10.1}$$

对于任意 $g \in \operatorname{supp} \nu_k^{(n)}$ 和任意 $k \geqslant \hat{k}_1$。

Proof. Let $n \geqslant N_1$ and $\omega > 0$. We claim that there is $\hat{k}_1 = \hat{k}_1(\nu_\infty, n, \omega)$ such that

$$-\log \mathrm{SVA}_1(gx, gx'; \omega) \leqslant -\log \mathrm{SVA}_1(x, x'; \omega) + Bn \tag{10.2}$$

对于任意 $x \neq x'$，$g \in \operatorname{supp} \nu_k^{(n)}$，以及 $k \geqslant \hat{k}_1$。这可以如下看出。如果 $\mathrm{VA}_1(x, x') < \omega$，那么根据定义 (9.4),

$$-\log \mathrm{SVA}_1(x, x'; \omega) \geqslant -\log \omega \text{ and}$$
$$-\log \mathrm{SVA}_1(gx, gx'; \omega) \leqslant -\log \omega + Bn \leqslant -\log \mathrm{SVA}_1(x, x'; \omega) + Bn,$$

正如所声称的那样。现在，假设 $\mathrm{VA}_1(x, x') \geqslant \omega$。关系式 (8.7) 在此处不适用。相反，由 (8.5) 和 (9.4) 可得，

$$-\log \mathrm{SVA}_1(gx, gx'; \omega) \leqslant -\log \mathrm{SVA}_1(x, x'; \omega) - \log \frac{\|(gu)^\perp\|}{\|gu\|} \frac{\|u\|}{\|u^\perp\|} \tag{10.3}$$

对于每个 $g \in \operatorname{supp} \nu_k^{(n)}$，其中 $u = u(x, x')$ 实现了式 (8.4) 中的上确界。假设 $\mathrm{VA}_1(x, x') \geqslant \omega$ 意味着 $\|u^\perp\| \geqslant \omega \|u\|$。由于 $\operatorname{supp} \nu_k$ 收敛于

在豪斯多夫拓扑下的 $\operatorname{supp}\nu_\infty$，我们可以找到 $r_k = r_k(\nu_\infty,\ n) \to 0$，使得每个 $g \in \operatorname{supp}\nu_k^{(n)}$ 都位于某个 $f \in \operatorname{supp}\ \nu_\infty^{(n)}$ 的 r_k-邻域内。然后

$$\|(gu)^\perp - (fu)^\perp\| \leqslant \|gu - fu\| \leqslant r_k\|u\| \leqslant \frac{r_k}{\omega}\|u^\perp\|.$$

Fix $\hat{k}_1 \in \mathbb{N}$ large enough that, for every $g \in \operatorname{supp}\nu_k^{(n)}$ and $k \geqslant \hat{k}_1$,

$$\begin{aligned} -\log\frac{\|(gu)^\perp\|}{\|gu\|}\frac{\|u\|}{\|u^\perp\|} &\leqslant -\log\frac{\|(fu)^\perp\|}{\|fu\|}\frac{\|u\|}{\|u^\perp\|} + \log 2 \\ &= -\log\frac{\|f^\perp u^\perp\|}{\|fu\|}\frac{\|u\|}{\|u^\perp\|} + \log 2 \\ &\leqslant \log\|(f^\perp)^{-1}\| + \log\|f\| + \log 2 \leqslant Bn. \end{aligned}$$

将其代入（10.3），即可完成（10.2）的证明。

Next, substituting (10.2) and (8.2) in the definition (9.5) we find that

$$\begin{aligned} -\log\psi_1(gx, gx'; \omega) &\leqslant -\log \mathrm{SVA}_1(x, x'; \omega) + Bn - \log d(x, x') + \gamma_1 An \\ &= -\log\psi_1(x, x'; \omega) + C_1' n \end{aligned} \tag{10.4}$$

for any $x \neq x'$, $g \in \operatorname{supp}\nu_k^{(n)}$, and $k \geqslant \hat{k}_1$. Using (9.1), it follows that

$$\begin{aligned} \Psi_1(gx, gx'; \omega) &\leqslant \log\left(\Omega_1 + \psi_1(gx, gx'; \omega)^{-1}\right) \\ &\leqslant \log\left(\Omega_1 + e^{C_1' n}\psi_1(x, x'; \omega)^{-1}\right) \\ &\leqslant \log\left(\Omega_1 + \psi_1(x, x'; \omega)^{-1}\right) + C_1' n = \Psi_1(x, x'; \omega) + C_1' n, \end{aligned}$$

as stated. □

接下来，我们证明引理 8.2 的以下推广：

Lemma 10.3. *Given $\delta > 0$, $n \geqslant N_1$ and $\omega > 0$, there is $k_1' = k_1'(\nu_\infty, \delta, n, \omega) \in \mathbb{N}$ and for any $x \neq x'$ in P with $\mathrm{VA}_1(x, x') \geqslant \omega$ there is $\mathcal{E}_{k,1}' = \mathcal{E}_{k,1}'(\nu_\infty, \delta, n, x, x', \omega) \subset \operatorname{supp}\nu_k^{(n)}$ with $\nu_k^{(n)}((\mathcal{E}_{k,1}')^c) < \delta$ and*

$$-\log \mathrm{SVA}_1(gx, gx'; \omega) \leqslant \max\{-\log \mathrm{SVA}_1(x, x'; \omega) - \tilde{\kappa}_1 n, \tilde{\theta}_1\} \tag{10.5}$$

对于所有 $g \in \mathcal{E}_{k,\ 1}'$ 以及 $k \geqslant k_{1}'$。

Proof. Fix $\delta > 0$ and $n \geqslant N_1$ and $\omega > 0$. Let P_ω denote the (compact) subset of all $v \in P$ such that $\|v^\perp\|/\|v\| \geqslant \omega/2$. For $v \in P_\omega$ and g in some compact neighborhood V_ω of $\operatorname{supp}\nu_\infty^{(n)}$, consider

$$(v, g) \mapsto -\log\frac{\|(gv)^\perp\|}{\|gv\|}. \tag{10.6}$$

只要 V_ω 足够小（具体取决于 ν_∞、n 和 ω），映射 (10.6) 就是良定义且（均匀）连续的。因此，存在 $\alpha = \alpha(\nu_\infty,\ n$ 和 $\omega) > 0$，使得

$$-\log\frac{\|(gu)^\perp\|}{\|gu\|} \leqslant -\log\frac{\|(fv)^\perp\|}{\|fv\|} + \log 2 \tag{10.7}$$

whenever $d(u, v) < \alpha$ and $d(g, f) < \alpha$. Reducing α if necessary, depending only on ω, we may also assume that

$$d(u, v) < \alpha \quad \Rightarrow \quad -\log\frac{\|u^\perp\|}{\|u\|} \geqslant -\log\frac{\|v^\perp\|}{\|v\|} - \log 2. \tag{10.8}$$

Fix $v_1, \dots, v_l \in P_\omega$ such that $P_\omega \subset B(v_1, \alpha) \cup \cdots \cup B(v_l, \alpha)$. For each $v \in P_\omega$ choose $j \in \{1, \dots, l\}$ such that $v \in B(v_j, \alpha)$ and then define $\mathcal{E}_{k,0} = \mathcal{E}_{k,0}(\nu_\infty, \delta, n, v, \omega) \subset \operatorname{supp} \nu_k^{(n)}$ by

$$\mathcal{E}_{k,0} = \left[\alpha\text{-neighborhood of } \mathcal{E}_0(\nu_\infty, \delta, n, v_j^\perp)\right] \cap \operatorname{supp} \nu_k^{(n)}, \tag{10.9}$$

where $\mathcal{E}_0(\nu_\infty, \delta, n, v_j^\perp) \subset \operatorname{supp} \nu_\infty^{(n)}$ is as defined in Proposition 5.5. Since $\nu_k^{(n)}$ converges to $\nu_\infty^{(n)}$ in the weak* topology, the limit inferior of the $\nu_k^{(n)}$-measure of (10.9) as $k \to \infty$ is greater than or equal to

$$\nu_\infty^{(n)} \left(\mathcal{E}_0(\nu_\infty, \delta, n, v_j^\perp)\right) > 1 - \delta$$

for every $j = 1, \dots, l$. In particular, there is $k_1' = k_1'(\nu_\infty, \delta, n, \omega) \in \mathbb{N}$ such that

$$\nu_k^{(n)}(\mathcal{E}_{k,0}) > 1 - \delta \text{ for every } k \geqslant k_1' \text{ and } v \in P_\omega. \tag{10.10}$$

给定 $x \neq x'$ with $\mathrm{VA}_1(x, x') \geqslant \omega$，取 $u = u^E + u^\perp$ 为一个非零向量，使其在 (8.4) 中取得上确界。则，

$$\omega \leqslant \mathrm{VA}_1(x, x') = \frac{\|u^\perp\|}{\|u\|} \quad (\text{in particular, } u \in P_\omega) \text{ and} \tag{10.11}$$

$$\mathrm{VA}_1(gx, gx') \geqslant \frac{\|(gu)^\perp\|}{\|gu\|} \geqslant \mathrm{VA}_1(x, x') \frac{\|(gu)^\perp\|}{\|gu\|} \frac{\|u\|}{\|u^\perp\|} \tag{10.12}$$

对于任意 $g \in G$。然后定义

$$\mathcal{E}_{k,1}' = \mathcal{E}_{k,0}(\nu_\infty, \delta, n, u, \omega). \tag{10.13}$$

It follows from (10.10) that $\nu_k^{(n)}((\mathcal{E}_{k,1}')^c) < \delta$ for every $k \geqslant k_1'$.

Let $g \in \mathcal{E}_{k,1}'$ and $k \geqslant k_1'$. Then, by definition, there exist $v = v^E + v^\perp$ in P_ω (take $v = v_j$ as in (10.9)) and $f \in \mathcal{E}_0(\nu_\infty, \delta, n, v^\perp) \subset \operatorname{supp} \nu_\infty^{(n)}$ such that $d(u, v) < \alpha$ and $d(g, f) < \alpha$. Thus, substituting (10.7) and (10.8) in (10.12), we find that

$$\mathrm{VA}_1(gx, gx') \geqslant \frac{1}{2} \frac{\|f^\perp v^\perp\|}{\|fv\|} \geqslant \frac{1}{4} \mathrm{VA}_1(x, x') \frac{\|f^\perp v^\perp\|}{\|fv\|} \frac{\|v\|}{\|v^\perp\|}. \tag{10.14}$$

Let $\tau_0 = \tau_0(\nu_\infty, \delta) > 0$ be as in Proposition 5.5. If $\|f^\perp v^\perp\| / \|fv\| \geqslant \tau_0/2$ then the first part of (10.14) gives that (recall (8.9) also)

$$-\log \mathrm{VA}_1(gx, gx') \leqslant -\log \frac{\tau_0}{4} \leqslant \tilde{\theta}_1. \tag{10.15}$$

Now suppose that $\|f^\perp v^\perp\| / \|fv\| < \tau_0/2$. Then part (2) of Proposition 5.5 gives that

$$\frac{\|fv^\perp\|}{\|fv\|} < \frac{1}{2} \text{ and so } \frac{\|fv^E\|}{\|fv\|} > \frac{1}{2}. \tag{10.16}$$

Substituting (10.16) and $\|v\| \geqslant \|v^E\|$ in (10.14), we find that

$$\begin{aligned} \mathrm{VA}_1(gx, gx') &\geqslant \frac{1}{8} \mathrm{VA}_1(x, x') \frac{\|f^\perp v^\perp\|}{\|v^\perp\|} \frac{\|v^E\|}{\|fv^E\|} \\ &= \frac{1}{8} \mathrm{VA}_1(x, x') \frac{\|Df^\perp_{v^E} v^\perp\|}{\|v^\perp\|}. \end{aligned} \tag{10.17}$$

By part (1) of Proposition 5.5 and the choices of $\tilde{\kappa}_1 > 0$ and $\widetilde{N}_1 \in \mathbb{N}$ in (8.9), this implies

$$\begin{aligned} -\log \mathrm{VA}_1(gx, gx') &\leqslant -\log \mathrm{VA}_1(x, x') + \log 8 - 2\tilde{\kappa}_1 n \\ &\leqslant -\log \mathrm{VA}_1(x, x') - \tilde{\kappa}_1 n. \end{aligned} \tag{10.18}$$

引理的结论包含在（10.15）和（10.18）中。 □

接下来，让我们证明引理 8.5 的以下推广：

Lemma 10.4. *Given $\delta > 0$ and $n \geqslant N_1$ there exists $k_1'' = k_1''(\nu_\infty, \delta, n) \in \mathbb{N}$ such that for any $x \neq x'$ in $E_1(\varepsilon_1)$ with $-\log \mathrm{VA}_1(x, x') \leqslant \tilde{\theta}_1 + \tilde{\kappa}_1 n$ there exists $\mathcal{E}''_{k,1} = \mathcal{E}''_{k,1}(\nu_\infty, \delta, n, x, x') \subset \operatorname{supp} \nu_k^{(n)}$ with $\nu_k^{(n)}((\mathcal{E}''_{k,1})^c) < \delta$ and*

$$-\log d(gx, gx') \leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') - \hat{\kappa}_1 n \tag{10.19}$$

对每个 $g \in \mathcal{E}''_{k,1}$ 以及 $k \geqslant k_1''$。

Proof. Let $\hat{P}$ denote the (compact) subset of pairs $(v, w) \in P \times P$ such that

$$\frac{\|v^\perp\|}{\|v\|} \leqslant \hat{\rho}_1 < 2\hat{\rho}_1 \leqslant \frac{\|w^\perp\|}{\|w\|} \tag{10.20}$$

(as before, $v^\perp$ and $w^\perp$ denote the vertical components of v and w). Let $n \geqslant N_1$ and $\hat{V}$ be a compact neighborhood of the support of $\nu_\infty^{(n)}$. Since, $\hat{\rho}_1 = \hat{\rho}_1(\nu_\infty, n)$, $\hat{P}$ and $\hat{V}$ depend only on ν_∞ and n. Condition (10.20) ensures that the angle between v and w is bounded away from zero and, consequently, so is the angle between gv and gw for any $g \in \hat{V}$; both bounds depend only on ν_∞ and n. Thus, the map

$$(v, w, g) \mapsto -\log \frac{\|\Pi_{gv} gw\|}{\|gv\|} \frac{\|v\|}{\|w\|} \tag{10.21}$$

在定义域 $(v、\ w) \in \hat{}P$ 和 $g \in \hat{}V$ 上定义良好且（一致）连续。特别地，存在 $\hat{}\alpha = \hat{}\alpha(\nu_\infty, n) > 0$，使得

$$-\log \frac{\|\Pi_{gv} gw\|}{\|gv\|} \frac{\|v\|}{\|w\|} \leqslant -\log \frac{\|\Pi_{fu} fz\|}{\|fu\|} \frac{\|u\|}{\|z\|} + \log 2 \tag{10.22}$$

当 $d(v, u) < \hat{}\alpha$ 且 $d(z, w) < \hat{}\alpha$ 且 $d(g, f) < \hat{}\alpha$ 时。必要时适当减小 $\hat{}\alpha$，其取值仅取决于 ν_∞ 和 n，我们还可以假设：

$$d(z, w) < \hat{\alpha} \quad \Rightarrow \quad -\log \frac{\|z^\perp\|}{\|z\|} \leqslant -\log \frac{\|w^\perp\|}{\|w\|} + \log 2. \tag{10.23}$$

Fix points $(v_1, w_1), \ldots, (v_l, w_l) \in \hat{P}$ such that the balls of radius $\hat{\rho}$ around these points cover $\hat{P}$. For each $(v, w) \in \hat{P}$ choose $j \in \{1, \ldots, l\}$ such that $v \in B(v_j, \hat{\alpha})$ and $w \in B(w_j, \hat{\alpha})$ and then define $\hat{\mathcal{E}}_{k,0} = \hat{\mathcal{E}}_{k,0}(\nu_\infty, \delta, n, v, w) \subset G$ by

$$\hat{\mathcal{E}}_{k,0} = \left[\hat{\alpha}\text{-neighborhood of } \mathcal{E}_0(\nu_\infty, \delta, n, w_j^\perp)\right] \cap \operatorname{supp} \nu_k^{(n)}, \tag{10.24}$$

where $\mathcal{E}_0(\nu_\infty, \delta, n, w_j^\perp)$ is given by Proposition 5.5. Since $\nu_k^{(n)} \to \nu_\infty^{(n)}$ in the weak* topology, the limit inferior of the $\nu_k^{(n)}$-measure of (10.24) as $k \to \infty$ is greater than or equal to

$$\nu_\infty^{(n)}\left(\mathcal{E}_0(\nu_\infty, \delta, n, w_j^\perp)\right) > 1 - \delta$$

for every $j = 1, \ldots, l$. In particular, there is $k_1'' = k_1''(\nu_\infty, \delta, n) \in \mathbb{N}$ such that

$$\nu_k^{(n)}(\hat{\mathcal{E}}_{k,0}) > 1 - \delta \text{ for every } k \geqslant k_1'' \text{ and } (v, w) \in \hat{P}. \tag{10.25}$$

给定 $x \neq x'$，令 $w = v - v'$ 为分别沿着 x 和 x' 方向的单位向量 v 和 v' 之差。正如在 (8.22) 中一样,

$$-\log d(gx, gx') \leqslant -\log d(x, x') - \log \frac{\|\Pi_{gv} gw\|}{\|gv\|} \frac{\|v\|}{\|w\|} \tag{10.26}$$

对于任意满足 $g \in G$ 的 $\|gv\| \geqslant \|gv'\|$ （情形 $\|gv\| \leqslant \|gv'\|$ 类似，只需交换 x 和 x'的角色）。假设 $x \in E_1(\varepsilon_1)$ 意味着,

$$\frac{\|v^\perp\|}{\|v\|} \leqslant d(x, E) \leqslant \varepsilon_1 < \hat{\rho}_1. \tag{10.27}$$

Now assume that $-\log \mathrm{VA}_1(x, x') < \tilde{\theta}_1 + \tilde{\kappa}_1 n$. Then, using (8.19) and (8.23),

$$\frac{\|w^\perp\|}{\|w\|} > \frac{1}{4} \mathrm{VA}_1(x, x') \geqslant \frac{1}{4} e^{-\tilde{\theta}_1 - \tilde{\kappa}_1 n} > 2\hat{\rho}_1. \tag{10.28}$$

因此， $(v, w) \in \hat{} P$。然后定义

$$\mathcal{E}''_{k,1} = \hat{\mathcal{E}}_{k,0}(\nu_\infty, \delta, n, v, w). \tag{10.29}$$

It follows from (10.25) that $\nu_k^{(n)}((\mathcal{E}''_{k,1})^c) < \delta$ for every $k \geqslant k''_1$.

Take $u = v_j$ and $z = w_j$ as in (10.24). By definition, $(u, z) \in \hat{P}$ and $d(u, v) < \hat{\alpha}$ and $d(z, w) < \hat{\alpha}$. Let $g \in \mathcal{E}''_{k,1}$ and $k \geqslant \hat{k}_1$. The definitions (10.24) and (10.29) imply that there exists $f \in \mathcal{E}_0(\nu_\infty, \delta, n, z^\perp) \subset \operatorname{supp} \nu_\infty^{(n)}$ such that $d(g, f) < \hat{\alpha}$. Replacing z, w and g with u, z and f, respectively, in (8.24) and (8.25), we find that

$$\frac{\|fu\|}{\|u\|} \geqslant \frac{1}{2} \frac{\|fu^E\|}{\|u^E\|} \text{ and } \|\Pi_{fu} fz\| \geqslant \frac{1}{2} \|f^\perp z^\perp\|.$$

结合 (10.26) 和 (10.22)，我们得到

$$\begin{aligned} -\log d(gx, gx') &\leqslant -\log d(x, x') + \log 2 - \log \frac{\|\Pi_{fu} fz\|}{\|fu\|} \frac{\|u\|}{\|z\|} \\ &\leqslant -\log d(x, x') + \log 8 - \log \frac{\|f^\perp z^\perp\|}{\|z\|} \frac{\|u^E\|}{\|fu^E\|}. \end{aligned} \tag{10.30}$$

条件（10.23）和（10.28）表明,

$$-\log \frac{\|z^\perp\|}{\|z\|} \leqslant -\log \frac{\|w^\perp\|}{\|w\|} + \log 2 \leqslant -\log \mathrm{VA}_1(x, x') + \log 8.$$

将其代入（10.30），我们得到

$$\begin{aligned} -\log d(gx, gx') &\leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') + \log 64 - \log \frac{\|f^\perp z^\perp\|}{\|z^\perp\|} \frac{\|u^E\|}{\|fu^E\|} \\ &= -\log d(x, x') - \log \mathrm{VA}_1(x, x') + \log 64 - \log \frac{\|Df^\perp_{u^E} z^\perp\|}{\|z^\perp\|}. \end{aligned}$$

By part (1) of Proposition 5.5 and the choice of $\widehat{N}_1 \in \mathbb{N}$ in (8.19), this implies that

$$\begin{aligned} -\log d(gx, gx') &\leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') + 5 - 2\hat{\kappa}_1 n \\ &\leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') - \hat{\kappa}_1 n, \end{aligned}$$

as claimed. □

现在我们推导出命题 8.3 的以下推广:

Lemma 10.5. *For $\delta > 0$ and $n \geqslant N_1$ and $\omega > 0$ there is $k_1''' = k_1'''(\nu_\infty, \delta, n, \omega) \in \mathbb{N}$ and for $x \neq x'$ in $E_1(\varepsilon_1)$ with $\mathrm{VA}_1(x, x') \geqslant \omega$ there is $\mathcal{E}_{k,1}''' = \mathcal{E}_{k,1}'''(\nu_\infty, \delta, n, x, x', \omega) \subset \operatorname{supp} \nu_k^{(n)}$ such that $\nu_k^{(n)}((\mathcal{E}_{k,1}''')^c) < \delta$ and*

$$-\log \psi_1(gx, gx'; \omega) \leqslant -\log \psi_1(x, x'; \omega) - \kappa_1' n. \tag{10.31}$$

对于任意 $g \in \mathcal{E}_{k,1}'''$ 和 $k \geqslant k_1'''$。

Proof. Fix $\delta > 0$ and $n \geqslant N_1$ and $\omega > 0$. Recall that

$$N_1 > \max\left\{\widetilde{N}_1(\nu_\infty, \delta/2), \widehat{N}_1(\nu_\infty, \delta/2), 2\tilde{\theta}_1/(\gamma\hat{\kappa}_1)\right\},$$

由（8.26）定义。

$$\begin{aligned} k_1''' &= \max\{k_1'(\nu_\infty, \delta/2, n, \omega), k_1''(\nu_\infty, \delta/2, n)\} \\ &\quad \text{and } \mathcal{E}_{k,1}''' = \mathcal{E}_{k,1}'(\nu_\infty, \delta/2, n, x, x', \omega) \cap \mathcal{E}_{k,1}''(\nu_\infty, \delta/2, x, x'n). \end{aligned} \tag{10.32}$$

By construction, $\mathcal{E}_{k,1}'''$ is contained in the support of $\nu_k^{(n)}$ and $\nu_k^{(n)}((\mathcal{E}_{k,1}''')^c) < \delta$.

Consider $x \neq x'$ in $E_1(\varepsilon_1)$ with $\mathrm{VA}_1(x, x') \geqslant \omega$. As observed in (9.9), the latter implies that $\mathrm{SVA}_1(x, x'; \omega) = \mathrm{VA}_1(x, x')$ and $\mathrm{SVA}_1(gx, gx'; \omega) = \mathrm{VA}_1(gx, gx')$. Thus, the claim (10.31) may be rewritten as

$$-\log \mathrm{VP}_1(gx, gx'; \omega) \leqslant -\log \mathrm{VP}_1(x, x'; \omega) - \kappa_1' n. \tag{10.33}$$

˜设 $g \in \mathcal{E}_{k,1}'''$ 且 $k \geqslant k_1'''$。首先假设 $-\log \mathrm{VA}_1(x, x') \geqslant \theta_1 + \tilde{\kappa}_1 n$。那么，由引理 10.3，

$$-\log \mathrm{VA}_1(gx, gx') \leqslant -\log \mathrm{VA}_1(x, x') - \tilde{\kappa}_1 n. \tag{10.34}$$

Substituting (10.34) and (8.2) in the definition (8.13), we find that

$$\begin{aligned} -\log \mathrm{VP}_1(gx, gx') &\leqslant -\log \mathrm{VA}_1(x, x') - \tilde{\kappa}_1 n - \gamma_1 \log d(x, x') + \gamma_1 An \\ &\leqslant -\log \mathrm{VP}_1(x, x') - \frac{\tilde{\kappa}_1}{2} n \end{aligned} \tag{10.35}$$

Now suppose that $-\log \mathrm{VA}_1(x, x') \leqslant \tilde{\theta}_1 + \tilde{\kappa}_1 n$. In this case, Lemma 10.3 yields

$$-\log \mathrm{VA}_1(gx, gx') \leqslant \tilde{\theta}_1, \tag{10.36}$$

而引理 10.4 断言，

$$-\log d(gx, gx') \leqslant -\log d(x, x') - \log \mathrm{VA}_1(x, x') - \hat{\kappa}_1 n. \tag{10.37}$$

Substituting (10.36) and (10.37) in the definition (8.13), we obtain

$$\begin{aligned} -\log \mathrm{VP}_1(gx, gx') &\leqslant \tilde{\theta}_1 - \gamma_1 \log d(x, x') - \gamma_1 \log \mathrm{VA}_1(x, x') - \gamma_1 \hat{\kappa}_1 n \\ &\leqslant -\log \mathrm{VP}_1(x, x') + \tilde{\theta}_1 + (1 - \gamma_1) \log \mathrm{VA}_1(x, x') - \gamma_1 \hat{\kappa}_1 n. \end{aligned}$$

由于 $\mathrm{VA}_1(x、\ x') \leqslant 1、\ \gamma_1 \leqslant 1$ 和 $n \geqslant N_1 \geqslant 2\theta_1/(\gamma_1\hat{\kappa}_1)$，可得】【。

$$\begin{aligned} -\log \mathrm{VP}_1(gx, gx') &\leqslant -\log \mathrm{VP}_1(x, x') + \tilde{\theta}_1 - \gamma_1 \hat{\kappa}_1 n \\ &\leqslant -\log \mathrm{VP}_1(x, x') - \frac{\gamma_1 \hat{\kappa}_1}{2} n \end{aligned} \tag{10.38}$$

鉴于在（8.26）中对常数 κ_1' 的选取，命题（10.33）已包含在（10.35）和（10.38）中。□

让我们回到命题 10.1 的证明。定义

$$k_1 = \max\{\tilde{k}_1, \hat{k}_1, k_1', k_1'', k_1'''\}. \tag{10.39}$$

By construction, k_1 depends only on ν_∞, δ, n and ω. Part (i) of the proposition is an immediate consequence of Lemma 10.2. To prove part (ii), consider the set $\mathcal{E}_{k,1}''' = \mathcal{E}_{k,1}'''(\nu_\infty, \delta, n, x, x', \omega)$ given by Lemma 10.5. Then $\nu_k^{(n)}((\mathcal{E}_{k,1}''')^c) < \delta$ and

$$-\log\psi_1(gx, gx';\omega) \leqslant -\log\psi_1(x, x';\omega) - \kappa_1' n \leqslant -\log\psi_1(x, x';\omega) \tag{10.40}$$

for every $g \in \mathcal{E}_{k,1}'''$ and $k \geqslant k_1'''$. By (9.2), this implies that

$$\Psi_1(gx, gx';\omega) \leqslant \Psi(x, x';\omega) \text{ for every } g \in \mathcal{E}_{k,1}'''. \tag{10.41}$$

Integrating (10.41) over $\mathcal{E}_{k,1}'''$ and (10.1) over the complement, we obtain the estimate in part (ii).

接下来，回忆一下，我们选取了 $\kappa_1''' = \kappa_1'/2$, $c = e^{-2\kappa_1''' n}$, 和 $\varepsilon_1''' > 0$，使得（9.24）成立:

$$-\log\psi_1(x, x';\omega) = -\log\psi_1(x, x') \geqslant \Omega_1/\sqrt{c}$$

for any $x \neq x'$ in $E_1(\varepsilon_1''')$. Then, by (9.3) and the first inequality in (10.40),

$$\Psi_1(gx, gx';\omega) \leqslant \Psi_1(x, x';\omega) + \log\sqrt{c} = \Psi_1(x, x';\omega) - \kappa_1''' n \tag{10.42}$$

for every $g \in \mathcal{E}_{k,1}'''$. Integrating (10.42) over $\mathcal{E}_{k,1}'''$ and (10.1) over the complement, we obtain the estimate in part (iii) of the proposition. This completes the proof of Proposition 10.1. □

It is clear from the statements of Lemma 10.2, Lemma 10.3, and Lemma 10.5 that $\hat{k}_1$, k_1', and k_1''' may be taken to increase to ∞ when ω decreases to zero and ν_∞, δ, n remain fixed. Then the same is true about the map $\omega \mapsto k_1(\nu_\infty, \delta, n, \omega)$ in (10.39). Hence, we may find $\omega_{k,1} = \omega_{k,1}(\nu_\infty, \delta, n) > 0$ such that

$$(\omega_{k,1})_k \text{ decreases to 0 and } k \geqslant k_1(\nu_\infty, \delta, n, \omega_{k,1}) \tag{10.43}$$

for every large $k \in \mathbb{N}$. For instance, $\omega_{k,1} = 2\inf\{\omega > 0 : k_1(\nu_\infty, \delta, n, \omega) \leqslant k\}$. Fix $\check{k}_1 = \check{k}_1(\nu_\infty, \delta, n) \in \mathbb{N}$ such that (10.43) holds for all $k \geqslant \check{k}_1$. Define

$$\psi_{k,1}(x, x') = \psi_1(x, x';\omega_{k,1}) \text{ and } \Psi_{k,1}(x, x') = \Psi_1(x, x';\omega_{k,1}). \tag{10.44}$$

那么，下述结论包含在命题 10.1 中:

Corollary 10.6. *For any $\delta > 0$, $n \geqslant N_1$, and $k \geqslant \check{k}_1$:*

(i) *For any $x \neq x'$ in $E_1(\varepsilon_1)$ with $\Psi_{k,1}(x, x') > \log\Omega_1$,*

$$\int_G \Psi_{k,1}(gx, gx')\, d\nu_k^{(n)}(g) \leqslant \Psi_{k,1}(x, x') + C_1' n.$$

(ii) 对于任意 $x \neq x'$ 属于 $E_1(\varepsilon_1)$ 且 $\Psi_{k,1}(x, x') > \log\Omega_1$ 和 $\mathrm{VA}_1(x, x') \geqslant \omega_{k,1}$,

$$\int_G \Psi_{k,1}(gx, gx')\, d\nu_k^{(n)}(g) \leqslant \Psi_{k,1}(x, x') + C_1'\delta n.$$

(iii) *For any $x \neq x'$ in $E_1(\varepsilon_1''')$ with $\mathrm{VA}_1(x, x') \geqslant \omega_{k,1}$,*

$$\int_G \Psi_{k,1}(gx, gx')\, d\nu_k^{(n)}(g) \leqslant \Psi_{k,1}(x, x') - (\kappa_1' - C_1'\delta)n.$$

11. 重耦合与结论

Here we resolve the difficulty arising out of the discontinuity of the Margulis function $\Psi_{k,1}$. The main issue is that the inequality in Lemma 10.2 may not hold when trajectories leave the cut-off region, that is, when $\Psi_{k,1}(x,x';n) = \log \Omega_1$ but $\Psi(gx, gx'; n) \neq \log \Omega_1$ for some g in the support of $\nu_k^{(n)}$. That may cause the analogue of (6.26) to fail at such points, which is catastrophic for our proof.

From the form of the cut-off, that problem can only happen if both x and x' are outside $E_1(2\varepsilon_1'')$ and at least one of gx or gx' is inside $E_1(2\varepsilon_1'')$. This is a relatively rare occurrence: for instance, Proposition 5.5 shows that, for every x not too far from E, the image gx is further away from E for the majority of g in the support of $\nu_\infty^{(n)}$. However, we still need to handle those rare cases where at least one of the points gx or gx' is in $E_1(2\varepsilon_1'')$.

The idea is to modify the dynamics on the space of pairs, more precisely the self-coupling $\widetilde{\mathcal{T}}_{k,1}$ of the Markov operator $\mathcal{T}_{k,1}$, to allow for the points x and x' to move in a more independent way: instead of the diagonal embedding, we will consider couplings supported on pairs (u,u') such that if one of the components is in $E_1(2\varepsilon_1'')$ then the other lies outside $E_1(\varepsilon_1')$. Then $\Psi_{k,1}(u,u')$ remains bounded above by a constant, which avoids the catastrophe.

11.1. 重新耦合。如第7.2节所示，对于每个 $k \in \mathbb{N}$ let

$$\mathcal{T}_{k,1} : \mathcal{B}(E_1(\varepsilon_1)) \to \mathcal{B}(E_1(\varepsilon_1)), \quad \mathcal{T}_{k,1}\varphi(x) = \int_{E_1(\varepsilon_1)} \varphi(y)\, d\sigma_{k,1,x}(y)$$

be a continuous Markov operator adapted to $(\nu_k^{(n)}, E_1(\varepsilon_1))$, and let $\eta_{k,1}$ be a $\mathcal{T}_{k,1}$-invariant probability measure converging, as $k \to \infty$, to a probability measure $\eta_{\infty,1}$ such that $\eta_{\infty,1}(E) > 0$. Recall that $\varepsilon_1 > 0$ was chosen small enough that

$$\eta_{\infty,1}(E_1(\varepsilon_1)\backslash E_1) < \frac{1}{10}\eta_{\infty,1}(E_1). \tag{11.1}$$

然后，对于每个足够大的 k ,

$$\eta_{k,1}(E_1(\varepsilon_1, \varepsilon_1''')) < \frac{2}{10}\eta_{k,1}(E_1(\varepsilon_1)). \tag{11.2}$$

Consider $X = X' = E_1(\varepsilon_1)$ and $Y = Y' = E_1(\varepsilon_1)$, and $\eta_y = \eta_y' = \sigma_{k,1,y}$ for every $y \in Y$. Moreover, let $K = \mathrm{Diag}_1$ be the diagonal of $E_1(\varepsilon_1)^2$. Clearly, $K(x) = K'(x) = \{x\}$ for every $x \in X$. Since the $\sigma_{k,1,y}$ are non-atomic measures, it follows that $\sigma_{k,1,y}(K(x')) = \sigma_{k,1,y}(K'(x)) = \{0\}$ for every x, x' and y. Thus (6.10) holds in this case, and so we may use Proposition 6.9 to find a continuous family

$$\{\tilde\theta_{k,1,x,x'} : (x,x') \in E_1(\varepsilon_1)^2\}$$

在 $E_1(\varepsilon_1)^2$ 上的一般概率测度，使得每个 $\tilde\theta_{k,1,x,x'}$ 都是 $\sigma_{k,1,x}$ 和 $\sigma_{k,1,x'}$ 的一种耦合，并且在对角线的一致邻域上消失。

Let $\nu_{k,1,x}^{(n)}$ and $\nu_{k,1,x,x'}^{(n)}$ denote the push-forwards of $\nu_k^{(n)}$ under the maps $G \to P$, $g \mapsto gx$ and $G \to P \times P$, $g \mapsto (gx, gx')$, respectively. Let $\tilde\omega : E_1(\varepsilon_1)^2 \to [0,1]$ be a continuous function such that $\tilde\omega(x,x') = 0$ if x and x' are both in $E_1(\tilde\varepsilon_1)$, and $\tilde\omega(x,x') = 1$ if either point is outside $E_1(2\tilde\varepsilon_1)$. Then

$$\tilde\sigma_{k,1,x,x'} = \big(1 - \tilde\omega(x,x')\big)\, \nu_{k,1,x,x'}^{(n)} + \tilde\omega(x,x')\tilde\theta_{k,1,x,x'} \tag{11.3}$$

is a coupling of $\sigma_{k,1,x}$ and $\sigma_{k,1,x'}$ depending continuously on (x, x'), and so

$$\widetilde{\mathcal{T}}_{k,1} : \mathcal{B}(E_1(\varepsilon_1)^2) \to \mathcal{B}(E_1(\varepsilon_1)^2), \quad \widetilde{\mathcal{T}}_{k,1}\varphi(x,x') = \int_{E_1(\varepsilon_1)^2} \varphi(y,y')\, d\tilde{\sigma}_{k,1,x,x'}(y,y'),$$

is a continuous self-coupling of $\mathcal{T}_{k,1}$. Since $\nu_k^{(n)}$ and $\tilde{\theta}_{k,1,x,x'}$ are generic measures, so is $\tilde{\sigma}_{k,1,x,x'}$. We are going to modify these Markov operators on the *recoupling region* $E_1(2\tilde{\varepsilon}_1, \varepsilon_1'')^2$ as follows. See Figure 8.

For $x \in E_1(2\tilde{\varepsilon}_1, \varepsilon_1'')$, it follows from (9.16) that the subset of $g \in \operatorname{supp} \nu_k^{(n)}$ such that $gx \in E_1(2\varepsilon_1')$ is disjoint from the set $\mathcal{D}_k(x)$ given by Corollary 5.7. Hence,

$$\nu_{k,1,x}^{(n)}\left(E_1(2\varepsilon_1')\right) \leqslant \nu_k^{(n)}\left(\mathcal{D}_k(x)^c\right) < \delta \text{ for every } x \in E_1(2\tilde{\varepsilon}_1, \varepsilon_1''). \tag{11.4}$$

Keep in mind that $\sigma_{k,1,x} = \nu_{k,1,x}^{(n)}$ if x is in the $\nu_k^{(n)}$-core of $E_1(\varepsilon_1)$.

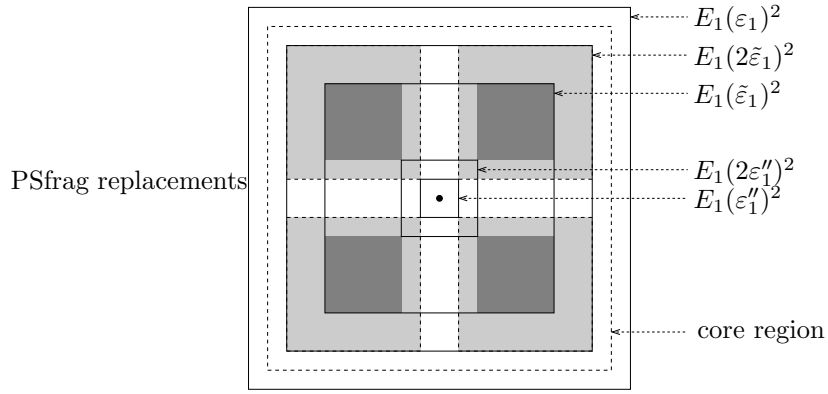

FIGURE 8. Illustrating the recoupling of the Markov operators. The black dot at the center marks the point (E, E). The dashed lined corresponds to the boundary between the $\nu_k^{(n)}$-core and the $\nu_k^{(n)}$-border of $E_1(\varepsilon_1)$. The shaded area is the recoupling region. On the complement, marked in white, $\hat{\sigma}_{k,1,x,x'} = \tilde{\sigma}_{k,1,x,x'}$. On the dark gray area $\tilde{\sigma}_{k,1,x,x'}$ is replaced with $\zeta_{k,1,x,x'}$, and on the light gray area we interpolate between the two.

Consider $X = X' = E_1(\varepsilon_r)$, $Y = Y' = E_1(2\tilde{\varepsilon}_1, \varepsilon_1'')^2$, and $\eta_y = \eta_y' = \sigma_{k,1,y}$ for every $y \in Y$. Moreover, let $K = E_1(\varepsilon_1')^2 \cup \mathrm{Diag}_1$. Then

$$K(x') = E_1(\varepsilon_1') \cup \{x'\} \text{ and } K'(x) = E_1(\varepsilon_1') \cup \{x\}$$

for every $(x, x') \in E_1(\varepsilon_1)^2$. Since the $\sigma_{k,1,y}$ are non-atomic measures, it follows from (11.4) that $\sigma_{k,1,y}(K(x'))$ and $\sigma_{k,1,y}(K'(x))$ are less than $\delta < 1/2$ for every x, x' and y. This ensures that (6.10) holds in this case, and so we may use Proposition 6.9 to find a continuous family

$$\{\zeta_{k,1,x,x'} : (x,x') \in E_1(2\tilde{\varepsilon}_1, \varepsilon_1'')^2\}$$

$E_1(\varepsilon_1)^2$ 上的典型概率测度，使得每个$\zeta_{k,\,1,\,x,\,x'}$ 都是$\sigma_{k,\,1,\,x}$ 和$\sigma_{k,\,1,\,x'}$ 的一个耦合，在对角线的一致邻域上消失，并且满足

$$\zeta_{k,1,x,x'}\left(E_1(\varepsilon_1')^2\right) = 0. \tag{11.5}$$

Fix a continuous function $\tau : E_1(\varepsilon_1)^2 \to [0,1]$ such that $\tau \equiv 1$ on $E_1(\tilde{\varepsilon}_1, 2\varepsilon_1'')^2$ and $\tau \equiv 0$ on the complement of $E_1(2\tilde{\varepsilon}_1, \varepsilon_1'')^2$. Check Figure 8. Then define

$$\hat{\sigma}_{k,1,x,x'} = \big(1 - \tau(x,x')\big)\, \tilde{\sigma}_{k,1,x,x'} + \tau(x,x')\zeta_{k,1,x,x'} \tag{11.6}$$

对于每个 $(x,x') \in E_1(\varepsilon_1)^2$。显然，$\hat{}\sigma_{k,1,x,x'}$ 是 $\sigma_{k,1,x}$ 和 $\sigma_{k,1,x'}$的一个耦合，并且连续依赖于 (x,x')。因此

$$\widehat{\mathcal{T}}_{k,1} : \mathcal{B}(E_1(\varepsilon_1)^2) \to \mathcal{B}(E_1(\varepsilon_1)^2), \quad \widehat{\mathcal{T}}_{k,1}\varphi(x,x') = \int_{E_1(\varepsilon_1)^2} \varphi(y,y')\, d\hat{\sigma}_{k,1,x,x'}(y,y')$$

is also a continuous self-coupling of $\mathcal{T}_{k,1}$. Moreover, $\hat{\sigma}_{k,1,x,x'}$ is a generic measure, since $\tilde{\sigma}_{k,1,x,x'}$ and $\zeta_{k,1,x,x'}$ are generic, and it coincides with $\tilde{\sigma}_{k,1,x,x'}$ outside the *recoupling region* $E_1(2\tilde{\varepsilon}_1, \varepsilon_1'')^2$.

Lemma 11.1. *Let $(x,x') \in E_1(\varepsilon_1)^2$ be such that*

(a) *either at least one of the points x or x' is in the $\nu_k^{(n)}$-border of $E_1(\varepsilon_1)$,*

(b) *or both x and x' are in the $\nu_k^{(n)}$-core of $E_1(\varepsilon_1)$ but outside $E_1(2\varepsilon_1'')$.*

然后 $\hat{}\sigma_{k,\ 1,\ x,\ x'}(E_1(\varepsilon_1')^2) = 0$ 因此 $\widehat{\mathcal{T}}_{k,\ 1}\Psi_{k,\ 1}(x,x') \leqslant \log \Omega_1 + \log 2$。

证明。我们首先证明以下结论：$\hat{}\sigma_{k,1,x,x'}$ 在 $E_1(\varepsilon_1')^2$上消失。若 x 位于 $E_1(\varepsilon_1)$ 的 $\nu_k^{(n)}$ -边缘上，则利用（9.17），

$$\hat{\sigma}_{k,1,x,x'}(E_1(\varepsilon_1')^2) \leqslant \sigma_{k,1,x}(E_1(\varepsilon_1')) \leqslant \sigma_{k,1,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}} E_1(\varepsilon_1)\right) = 0.$$

The same argument applies when x' is in the $\nu_k^{(n)}$-border of $E_1(\varepsilon_1)$. This settles the claim in case (a). Now let x and x' be as in (b). Keep in mind that $\sigma_{k,1,x} = \nu^{(n)}_{k,1,x}$ and $\sigma_{k,1,x'} = \nu^{(n)}_{k,1,x'}$. By (11.5), $\zeta_{k,1,x,x'}$ vanishes on $E_1(\varepsilon_1')^2$, and so (11.6) gives that

$$\hat{\sigma}_{k,1,x,x'}\left(E_1(\varepsilon_1')^2\right) = (1 - \tau(x,x'))\tilde{\sigma}_{k,1,x,x'}\left(E_1(\varepsilon_1')^2\right).$$

If x and x' are both in $E_1(\tilde{\varepsilon}_1)$ then $\tau(x,x') = 1$, and the claim follows. When $x \notin E_1(\tilde{\varepsilon}_1)$ we get from (9.18) that $\nu^{(n)}_{k,1,x}(E_1(\varepsilon_1')) = 0$. Then

$$\tilde{\sigma}_{k,1,x,x'}(E_1(\varepsilon_1')^2) \leqslant \sigma_{k,1,x}(E_1(\varepsilon_1')) = \nu^{(n)}_{k,1,x}(E_1(\varepsilon_1')) = 0.$$

当 $x' \notin E_1(\tilde{\varepsilon}_1)$ 的情形是类似的。我们已经证明，即使在情形 (b) 中，也有 $\hat{}\sigma_{k,\ 1,\ x,\ x'}(E_1(\varepsilon_1')^2) = 0$。

By Lemma 9.5, it follows that $\Psi_{k,1}(u,u') \leqslant \log\Omega_1 + \log 2$ for $\hat{\sigma}_{k,1,x,x'}$-almost every $(u,u') \in E_1(\varepsilon_1)^2$. Integrating with respect to $\hat{\sigma}_{k,1,x,x'}$ we immediately get that $\widehat{\mathcal{T}}_{k,1}\Psi_{k,1}(x,x') \leqslant \log\Omega_1 + \log 2$. □

Proposition 11.2. *There exist $\kappa_1''' = \kappa_1'''(\nu_\infty) > 0$ and $C_1''' = C_1'''(\nu_\infty) > 0$ such that given any $\delta > 0$ and $n \geqslant N_1$ the following holds for every $k \geqslant \check{k}_1$:*

(i) 对任意 $x \neq x' \in E_1(\varepsilon_1)$

$$\widehat{\mathcal{T}}_{k,1}\Psi_{k,1}(x,x') \leqslant \Psi_{k,1}(x,x') + C_1'''n.$$

(ii) *For any $x \neq x'$ in $E_1(\varepsilon_1)$ with $\mathrm{VA}_1(x,x') \geqslant \omega_{k,1}$,*

$$\widehat{\mathcal{T}}_{k,1}\Psi_{k,1}(x,x') \leqslant \Psi_{k,1}(x,x') + C_1''''(1 + \delta n).$$

(iii) *For any $x \neq x'$ in $E_1(\varepsilon_1''')$ with $\mathrm{VA}_1(x,x') \geqslant \omega_{k,1}$,*

$$\widehat{\mathcal{T}}_{k,1}\Psi_{k,1}(x,x') \leqslant \Psi_{k,1}(x,x') - (\kappa_1''' - C_1''''\delta)n.$$

ˇ证明。取 $\kappa_1''' = \kappa_1'$ 和 $C_1''' = \max\{C_1', \log 2\}$，并令 $k \geqslant k_1$。我们将论证分为四种情况（参见图 8）。

First, suppose that both x and x' are in the $\nu_k^{(n)}$-core of $E_1(\varepsilon_1)$, and at least one of them is in $E_1(\varepsilon_1'')$. This is necessarily the case in the setting of (iii). In particular (x, x') is outside the cut-off region,

$$\Psi_{k,1}(x, x') = \log(\Omega_1 + \psi_{k,1}(x, x')^{-1}) > \log \Omega_1,$$

并且也不存在重新耦合：

$$\hat{\sigma}_{k,1,x,x'} = \nu_{k,1,x,x'}^{(n)}, \quad \widehat{\mathcal{T}}_{k,1}\Psi_{k,1}(x, x') = \int_G \Psi_{k,1}(gx, gx')\, d\nu_k^{(n)}(g).$$

Hence the claims in (i), (ii) and (iii) are contained in Corollary 10.6.

Now suppose that both x and x' are in the $\nu_k^{(n)}$-core of $E_1(\varepsilon_1)$ but outside $E_1(\varepsilon_1'')$, and at least one of them is in $E_1(2\varepsilon_1'')$. It is still true that (x, x') is outside the cut-off region, and so $\Psi_{k,1}(x, x') > \log \Omega_1$. Thus the estimates in Corollary 10.6 remain valid for

$$\int \Psi_{k,1}\, d\tilde{\sigma}_{k,1,x,x} = \int_G \Psi_{k,1}(gx, gx')\, d\nu_k^{(n)}(g). \tag{11.7}$$

However, this time (x, x') may be in the recoupling region. That is dealt with as follows. By (11.5), the measure $\zeta_{k,1,x,x'}$ vanishes on $E_1(\varepsilon_1')^2$. Thus, by Lemma 9.5,

$$\int_{E_1(\varepsilon_1)^2} \Psi_{k,1}\, d\zeta_{k,1,x,x} \leqslant \log \Omega_1 + \log 2 \leqslant \Psi_{k,1}(x, x') + \log 2. \tag{11.8}$$

ˆ断言 (i) 和 (ii) 成立，这是因为根据 (11.6)，$\mathcal{T}_{k,1}\Psi_{k,1}(x, x')$ 是 (11.7) 和 (11.8) 中各积分的一个凸组合。

Next suppose that both x and x' are in the $\nu_k^{(n)}$-core of $E_1(\varepsilon_1)$ but outside $E_1(2\varepsilon_1'')$. This corresponds to case (b) of Lemma 11.1: claims (i) and (ii) are contained in the conclusion of that lemma. Finally, suppose that at lest one of the points x and x' is in the $\nu_k^{(n)}$-border of $E_1(\varepsilon_1)$. This is precisely the situation in case (a) of Lemma 11.1, and so claims (i) and (ii) are again contained in the conclusion of that lemma. □

11.2. **Contradicting** $\dim E = 1$**.** We are going to use the following refinement of Lemma 6.21:

Lemma 11.3. *Let $\mathcal{T} : \mathcal{B}(X) \to \mathcal{B}(X)$ be a Markov operator and $\psi : X \to [0, \infty]$ be a measurable function. Suppose that there exist constants κ_A, κ_B', $\kappa_B'' \in \mathbb{R}$ and pairwise disjoint sets A, B', B'' such that $X = A \cup B' \cup B''$ and*

(i) $\mathcal{T}\psi(x) \leqslant \psi(x) - \kappa_A$ 对于 $x \in A$,
(ii) $\mathcal{T}\psi(x) \leqslant \psi(x) + \kappa_B'$ 对于 $x \in B'$, (iii)
$\mathcal{T}\psi(x) \leqslant \psi(x) + \kappa_B''$ 对于 $x \in B''$。

设 ˆη 是 X 上满足 $\int \psi\, d\hat{\eta} < \infty$ 和 $\int_X \mathcal{T}\psi(x)\, d\hat{\eta}(x) \geqslant \int_X \psi(x)\, d\hat{\eta}(x)$ 的测度。那么

$$\hat{\eta}(B'') \geqslant \frac{\kappa_A \hat{\eta}(X) - (\kappa_A + \kappa_B')\hat{\eta}(B')}{\kappa_A + \kappa_B''}. \tag{11.9}$$

证明。条件 (i) - (iii) 表明

$$\begin{aligned}\int_X \psi(x)\, d\hat{\eta}(x) &\leqslant \int_X \mathcal{T}\psi(x)\, d\hat{\eta}(x)\\ &\leqslant \int_X \psi(x)\, d\hat{\eta}(x) - \kappa_A\theta(A) + \kappa'_B\theta(B') + \kappa''_B\theta(B'').\end{aligned}$$

因此， $-\kappa_A\hat{}\eta(A) + \kappa'_B\hat{}\eta(B') + \kappa''_B\hat{}\eta(B'') \geqslant 0$，这意味着 (11.9)。 □

Take $X = E_1(\varepsilon_1)^2$, $\mathcal{T} = \widehat{\mathcal{T}}_{k,1}$, $\psi = \Psi_{k,1}$, $\hat{\eta} = \hat{\eta}_{k,1}$, and

$$A_k = \{(x, x') \in E_1(\varepsilon'''_1)^2 : \mathrm{VA}_1(x, x') > \omega_{k,1}\}, \tag{11.10}$$

$$B' = \{(x, x') \in E_1(\varepsilon_1)^2 : d(x, E) > \varepsilon'''_1 \text{ or } d(x', E) > \varepsilon'''_1\}, \tag{11.11}$$

$$B''_k = \{(x, x') \in E_1(\varepsilon_1)^2 : \mathrm{VA}_1(x, x') \leqslant \omega_{k,1}\}. \tag{11.12}$$

显然， $A = A_k$ 与 $B = B' \cup B''_k$不相交，且它们的并集为整个$E_1(\varepsilon_1)^2$。如果 k 足够大，集合 B' 和 $B'' = B''_k$ 也互不相交，因为

$$\mathrm{VA}_1(x, x') = d(x + x', E) \geqslant \max\{d(x, E), d(x', E)\}$$

is greater than ε'''_1 whereas $(\omega_{k,1})_k$ converges to zero when $k \to \infty$. Also, $A \cup B' \cup B''$ is the whole $E_1(\varepsilon_1)^2$. Moreover, (7.2) implies that $B'' = \varnothing$ when $\dim E = 1$.

Proposition 11.2 shows that, assuming that k is sufficiently large, the hypotheses of Lemma 11.3 are satisfied for these choices, with

$$\kappa_A = (\kappa'''_1 - C'''_1\delta)n, \quad \kappa'_B = C'''_1(1 + \delta n), \text{ and } \kappa''_B = C'''_1 n.$$

Take $\delta > 0$ to be sufficiently small, depending on ν_∞, and $n \in \mathbb{N}$ to be sufficiently 较大，取决于 ν_∞ 和 δ，使得

$$\kappa_A > 9\kappa'_B. \tag{11.13}$$

As in Proposition 6.23, we find a sequence $(\hat{\eta}_{k,1,j})_j$ of probability measures on $E_1(\varepsilon_1)^2$ converging to a $\widehat{\mathcal{T}}_{k,1}$-invariant self-coupling $\hat{\eta}_{k,1}$ of $\eta_{k,1}$ such that every $\hat{\eta}_{k,1,j}$ satisfies $\int_{E_1(\varepsilon_1)^2} \Psi_{k,1}\, d\hat{\eta}_{k,1,j} < \infty$ and

$$\int_{E_1(\varepsilon_1)^2} \widehat{\mathcal{T}}_{k,1}\Psi_{k,1}(x, x')\, d\hat{\eta}_{k,1,j}(x, x') \geqslant \int_{E_1(\varepsilon_1)^2} \Psi_{k,1}(x, x')\, d\hat{\eta}_{k,1,j}(x, x').$$

Applying Lemma 11.3 with $\mathcal{T} = \widehat{\mathcal{T}}_{k,1}$ and $\hat{\eta} = \hat{\eta}_{k,1,j}$ we get that

$$\hat{\eta}_{k,1,j}(B'') \geqslant \frac{\kappa_A\hat{\eta}_{k,1,j}(E_1(\varepsilon_1)^2) - (\kappa_A + \kappa'_B)\hat{\eta}_{k,1,j}(B')}{\kappa_A + \kappa''_B} \tag{11.14}$$

对于每个 j。令 $j \to \infty$ 趋于极限，我们得出：

$$\hat{\eta}_{k,1}(B'') \geqslant \frac{\kappa_A\hat{\eta}_{k,1}(E_1(\varepsilon_1)^2) - (\kappa_A + \kappa'_B)\hat{\eta}_{k,1}(B')}{\kappa_A + \kappa''_B}. \tag{11.15}$$

Observe that $\hat{\eta}_{k,1}(E_1(\varepsilon_1)^2) = \eta_{k,1}(E_1(\varepsilon_1))$ and, using (11.2),

$$\hat{\eta}_{k,1}(B') \leqslant 2\eta_{k,1}(E_1(\varepsilon_1, \varepsilon'''_1)) < \frac{4}{10}\eta_{k,1}(E_1(\varepsilon_1)). \tag{11.16}$$

因此，(11.13) 和 (11.15) 蕴含了以下结论。

$$\hat{\eta}_{k,1}(B'') \geqslant \frac{\kappa_A - \frac{4}{10}(\kappa_A + \kappa'_B)}{\kappa_A + \kappa''_B}\eta_{k,1}(E_1(\varepsilon_1)) \geqslant \frac{5\kappa'_B}{\kappa_A + \kappa''_B}\eta_{k,1}(E_1(\varepsilon_1)) > 0. \tag{11.17}$$

When $\dim E = 1$ this yields a contradiction, because B'' is empty in that case. Thus $\dim E \geqslant 2$.

11.3. 完成第一步。让我们考虑映射。

$$\Sigma : E_1(\varepsilon_1)^2 \setminus \mathrm{Diag}_1 \to \mathrm{Gr}(2,d), \quad \Sigma(x,x') = x + x'.$$

我们希望定义 $\eta_{k,2} = \Sigma_*(\tilde{\eta}_{k,1})$，但存在一个问题：$\Sigma(x,x')$ 在 Diag_1 上未定义，而且我们无法排除 $\tilde{\eta}_{k,1}$ 在对角线上为正的可能性。

为了绕过这一困难，我们引入紧拓扑空间。

$$\mathcal{Y}_1 = \{(x,x',y) \in \mathrm{Gr}(1,d)^2 \times \mathrm{Gr}(2,d) : x \subset y \text{ and } x' \subset y\}$$
$$\mathcal{Y}_1(\varepsilon) = \{(x,x',y) \in \mathcal{Y}_1 : x, x' \in E_1(\varepsilon)\} \text{ for } \varepsilon > 0,$$

together with the canonical projections

$$p_1 : \mathcal{Y}_1 \to \mathrm{Gr}(1,d)^2, \quad (x,x',y) \mapsto (x,x')$$
$$p_2 : \mathcal{Y}_1 \to \mathrm{Gr}(2,d), \quad (x,x',y) \mapsto y.$$

For $(x,x',y) \in \mathcal{Y}_1$, $k \in \mathbb{N}$, and $n \in \mathbb{N}$, denote by $\nu^{(n)}_{k,1,x,x',y}$ the image of $\nu^{(n)}_k$ under the diagonal action

$$G \to \mathcal{Y}_1, \quad (g \mapsto (gx, gx', gy).$$

显然，每个 $\nu^{(n)}_{k\,1\,,,x,x',y}$ 都是相对于 (p_1)：$\mathcal{Y}_1 \to \mathrm{Gr}(1,d)^2$而言的 $\nu^{\,n}_{k\,1\,,,x,x'}$ 的一个提升。$\mathrm{Gr}(1,d)^2$ 中对角线的补集通过以下方式嵌入 $\mathcal{Y}_1$。

$$(x,x') \mapsto (x,x',x+x').$$

In particular, every measure ξ on $\mathrm{Gr}(1,d)^2$ that vanishes on the diagonal has a (unique) lift $\check{\xi}$ to $\mathcal{Y}_1$.

由关系式（**11.3**）和（**11.6**），我们可以看出，

$$\hat{\sigma}_{k,1,x,x'} = \big(1 - \check{\omega}(x,x')\big)\, \nu^{(n)}_{k,1,x,x'} + \check{\omega}(x,x')\hat{\theta}_{k,1,x,x'} \tag{11.18}$$

where $\check{\omega} : E_1(\varepsilon_1)^2 \to [0,1]$ is a continuous function that vanishes identically on $E_1(\varepsilon_1'')^2$, and each $\hat{\theta}_{k,1,x,x'}$ is a coupling of $\sigma_{k,1,x}$ and $\sigma_{k,1,x'}$ vanishing on a uniform neighborhood of the diagonal. In view of the previous remarks, it follows that the $\hat{\sigma}_{k,1,x,x'}$ lift to probability measures

$$\check{\sigma}_{k,1,x,x',y} = \big(1 - \check{\omega}(x,x')\big)\, \nu^{(n)}_{k,1,x,x',y} + \check{\omega}(x,x')\check{\theta}_{k,1,x,x',y} \tag{11.19}$$

on $\mathcal{Y}_1(\varepsilon_1)$, where $\check{\theta}_{k,1,x,x',y}$ is the unique lift of $\hat{\theta}_{k,1,x,x'}$. Since $\nu^{(n)}$ and $\hat{\theta}_{k,1,x,x'}$ are generic measures, so is $\check{\sigma}_{k,1,x,x',y}$ for every $(x,x',y) \in \mathcal{Y}_1(\varepsilon_1)$.

显然，$\check{\nu}^{(n)}_{k\,x\,,,x',y}$ 在 $\mathcal{Y}_1(\varepsilon_1)$ 上连续变化，并且根据唯一性，$\check{}\theta_{k,1,x,x',y}$ 也连续变化。因此，

$$\check{\mathcal{T}}_{k,1} : \mathcal{B}(\mathcal{Y}_1(\varepsilon_1)) \to \mathcal{B}(\mathcal{Y}_1(\varepsilon_1)), \quad \check{\mathcal{T}}_{k,1}\Psi(x,x',y) = \int_{\mathcal{Y}_1(\varepsilon_1)} \Psi \, d\check{\sigma}_{k,1,x,x',y}.$$

defines a continuous Markov operator. From the definition we see that $\check{\mathcal{T}}_{k,1}$ is a lift of $\hat{\mathcal{T}}_{k,1}$, in the sense that

$$\check{\mathcal{T}}_{k,1}(\psi \circ p_1) = \left(\hat{\mathcal{T}}_{k,1}\psi\right) \circ p_1 \text{ for every } \psi \in \mathcal{B}(\mathcal{Y}_1(\varepsilon_1)).$$

Thus, the construction in Proposition 6.23 can be applied simultaneously to the two operators, to yield a sequence of probability measures $\check{\eta}_{k,1,j}$ converging to

a $\check{\mathcal{T}}_{k,1}$-invariant measure $\check{\eta}_{k,1}$ on $\mathcal{Y}_1(\varepsilon_1)$, and whose projections down to $E_1(\varepsilon_1)^2$ are self-couplings $\hat{\eta}_{k,1,j}$ of the $\eta_{k,1}$ vanishing on neighborhoods of the diagonal of $E_1(\varepsilon_1)^2$, and converging to the $\widehat{\mathcal{T}}_{k,1}$-invariant measure $\hat{\eta}_{k,1}$.

Next, define $\eta_{k,2} = p_{2*}\check{\eta}_{k,1}$ and let $\{d\check{\eta}_{k,1,v} : v \in p_2\mathcal{Y}_1(\varepsilon_1)\}$ be a disintegration of $\check{\eta}_{k,1}$ with respect to the partition $\{p_2^{-1}(v) : v \in p_2\mathcal{Y}_1(\varepsilon_1)\}$. Then define

$$\mathcal{T}_{k,2} : \mathcal{B}(p_2\mathcal{Y}_1(\varepsilon_1)) \to \mathcal{B}(p_2\mathcal{Y}_1(\varepsilon_1)),$$
$$\mathcal{T}_{k,2}\Phi(y) = \int_{p_2^{-1}(y)} \check{\mathcal{T}}_{k,1}(\Phi \circ p_2)(x, x', y)\, d\check{\eta}_{k,1,y}(x, x').$$

Equivalently, $\mathcal{T}_{k,2}\Phi(y) = \int_{p_2\mathcal{Y}_1(\varepsilon_1)} \Phi\, d\sigma_{k,2,y}$ with

$$\sigma_{k,2,y} = \int_{p_2^{-1}(y)} p_{2*}\check{\sigma}_{k,1,x,x',y}\, d\check{\eta}_{k,1,y}(x, x'). \tag{11.20}$$

设 $\mathcal{B}_k = p_2 p_1^{-1}(B_k'') = \{x + x' : \ (x, x') \in B_k''\}$, 其中 B_k'' 与前文相同。 s section. Define also $\eta_{k,2,j} = p_{2*}\check{\eta}_{k,1,j}$ for $j \in \mathbb{N}$. Then

$$\eta_{k,2,j}(\mathcal{B}_k) \geqslant \check{\eta}_{k,1,j}\left(p_1^{-1}(B_k'')\right) = \hat{\eta}_{k,1,j}(B_k'').$$

Passing to the limit as $j \to \infty$ and arguing as in (11.14)–(11.17) we find that

$$\eta_{k,2}(\mathcal{B}_k) \geqslant \frac{5\kappa_B'}{\kappa_A + \kappa_B''}\eta_{k,1}(E_1(\varepsilon_1)).$$

现在，定义 (11.12) 意味着当 $k \to \infty$ 时， $\mathcal{B}_k$ 收敛到 E_2 ，因为$\omega_{k,\ 1} \to 0$。因此，$\eta_{k,\ 2}$ 的任何聚点 $\eta_{\infty,\ 2}$ 都必须满足

$$\eta_{\infty,2}(E_2) \geqslant \frac{5\kappa_B'}{\kappa_A + \kappa_B''}\eta_{\infty,1}(E_1) > 0. \tag{11.21}$$

取 $n_2 = n$ 和 $\varepsilon_2 = \varepsilon_1''$。令 $\nu_{k\ 2}^{(n_2)}{}_{,,y}$ 表示 $\nu_k^{(n_2)}$ 在映射 $G \to \mathbf{Gr}(2, d)$, $g \mapsto gy$ 下的推前。

Lemma 11.4.

(i) $\sigma_{k,2,y} = \nu_{k,2,y}^{(n_2)}$ *for every* $y \in E_2(\varepsilon_2)$.

(ii) $\sigma_{k,2,y}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}} E_2(\varepsilon_2)\right) = 0$ *for every* $y \notin E_2(\varepsilon_2)$

(iii) *The measure* $\eta_{k,2}$ *is* $\mathcal{T}_{k,2}$*-invariant.*

证明。显然， $\nu_{k\ 2}^{(n_2)}{}_{,,y}$ 与在投影 p_2下的 $\nu_{k\ 1\ x}^{(n_2)}{}_{,,,x',y}$ 的推前相一致。因此，(11.19) 表明

$$p_{2*}\check{\sigma}_{k,1,x,x',y} = \left(1 - \check{\omega}(x, x')\right)\nu_{k,2,y}^{(n_2)} + \check{\omega}(x, x')p_{2*}\check{\theta}_{k,1,x,x',y},$$

and so,

$$\check{\sigma}_{k,2,y} = \left(1 - \int_{p_2^{-1}(y)} \check{\omega}(x, x')\, d\check{\eta}_{k,1,y}(x, x')\right)\nu_{k,2,y}^{(n_2)} + \int_{p_2^{-1}(y)} \check{\omega}(x, x')p_{2*}\check{\theta}_{k,1,x,x',y}\, d\check{\eta}_{k,1,y}(x, x'),$$

If $y \in E_2(\varepsilon_2)$ then both x and x' are necessarily in $E_1(\varepsilon_2)$, by (7.1), in which case $\check{\omega}(x, x') = 0$. Then $\sigma_{k,2,y} = \nu_{k,2,y}^{(n_2)}$, as claimed in (i).

In view of the expression (11.20), to prove part (ii) it suffices to show that if $y \notin E_2(\varepsilon_2)$ then

$$p_{2*}\check{\sigma}_{k,1,x,x',y}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_2(\varepsilon_2)\right) = 0 \tag{11.22}$$

for any $x, x' \subset y$. If x and x' are both in $E_1(\varepsilon_2)$ then

$$p_{2*}\check{\sigma}_{k,1,x,x',y} = p_{2*}\nu^{(n_2)}_{k,1,x,x',y} = \nu^{(n_2)}_{k,2,y}$$

然后，该结论由注记 6.26 得出。从现在起，我们假设其中一个点（例如 x ）不属于 $E_1(\varepsilon_2)$。根据定义，

$$p_2^{-1}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_2(\varepsilon_2)\right) \subset \mathcal{X}^{\#}_{\nu_k^{(n)}}E_1(\varepsilon_2)^2 \times \mathcal{X}^{\#}_{\nu_k^{(n)}}E_2(\varepsilon_2),$$

and so

$$\begin{aligned} p_{2*}\check{\sigma}_{k,1,x,x',y}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_2(\varepsilon_2)\right) &\leqslant \check{\sigma}_{k,1,x,x',y}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_1(\varepsilon_2)^2 \times \mathcal{X}^{\#}_{\nu_k^{(n)}}E_2(\varepsilon_2)\right) \\ &\leqslant \hat{\sigma}_{k,1,x,x'}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_1(\varepsilon_2)^2\right) \\ &\leqslant \sigma_{k,1,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_1(\varepsilon_2)\right). \end{aligned} \tag{11.23}$$

如果 x 位于 $E_1(\varepsilon_1)$ 的 $\nu_k^{(n)}$-边缘，则

$$\sigma_{k,1,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_1(\varepsilon_2)\right) \leqslant \sigma_{k,1,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_1(\varepsilon_1)\right) = 0,$$

because the operator $\mathcal{T}_{k,1}$ is adapted to $(\nu_k^{(n)}, E_1(\varepsilon_1))$. If x is in the $\nu_k^{(n)}$-core of $E_1(\varepsilon_1)$ then Remark 6.26 gives that

$$\sigma_{k,1,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_1(\varepsilon_2)\right) = \nu^{(n)}_{k,1,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_1(\varepsilon_2)\right) = 0.$$

Thus the right-hand side of (11.23) vanishes in either case. That completes the proof of (11.22) and of part (ii) of the lemma.

Finally, by definition,

$$\begin{aligned} &\int_{p_2\mathcal{Y}_1(\varepsilon_1)} (\mathcal{T}_{k,2}\Phi)\, d\eta_{k,2} \\ &\qquad = \int_{p_2\mathcal{Y}_1(\varepsilon_1)} \int_{p_2^{-1}(y)} \check{\mathcal{T}}_{k,1}\left(\Phi \circ p_2\right)(x, x', y)\, d\check{\eta}_{k,1,y}(x, x')\, d\eta_{k,2}(y) \\ &\qquad = \int_{\mathcal{Y}_1(\varepsilon_1)} \check{\mathcal{T}}_{k,1}(\Phi \circ p_2)(x, x', y)\, d\check{\eta}_{k,1}(x, x', y) \end{aligned}$$

for any $\Phi \in \mathcal{B}(p_2\mathcal{Y}_1(\varepsilon_1))$. Since $\check{\eta}_{k,1}$ is $\check{\mathcal{T}}_{k,1}$-invariant, this gives

$$\begin{aligned} \int_{p_2\mathcal{Y}_1(\varepsilon_1)} (\mathcal{T}_{k,2}\Phi)(y)\, d\eta_{k,2}(y) &= \int_{\mathcal{Y}_1(\varepsilon_1)} (\Phi \circ p_2)(x, x', y)\, d\check{\eta}_{k,1}(x, x', y) \\ &= \int_{p_2\mathcal{Y}_1(\varepsilon_1)} \Phi\, d\hat{\eta}_{k,2}, \end{aligned}$$

这证明了断言 (iii)。 □

Since the $\check{\sigma}_{k,1,x,x',y}$ are generic measures and the projection p_2 is algebraic, it follows readily from (11.20) and Remark 5.1 that every $\sigma_{k,2,y}$ is a generic measure. Then, conclusions (i) and (ii) in Lemma 11.4 allow us to apply Propositions 6.15 and 6.25 to $X = p_2\mathcal{Y}_1(\varepsilon_1)$ and $U = E_2(\varepsilon_2)$. In this way we get a continuous Markov operator adapted to $(\nu_k^{(n_2)}, E_2(\varepsilon_2))$ which leaves invariant the restriction of $\eta_{k,2} \mid E_2(\varepsilon_2)$. Replace $\mathcal{T}_{k,2}$ and $\eta_{k,2}$ with this new Markov operator and invariant measure, respectively. This finishes the first step of the induction.

第四部分：归纳的一般步骤

12. 准备马尔古利斯函数

Let $r > 1$ be fixed. Here we extend the construction in Section 8, to find a positive function ψ_r such that $-\log \psi_r$ has some of the features of a Margulis function. The main result is Proposition 12.8, an extension of Proposition 8.1. Throughout, $\delta > 0$ and $n \in \mathbb{N}$ should be seen as free parameters, whose values are fixed at the end of the construction.

回忆一下，$\mathcal{F}(r,d)$ 表示旗的集合 $F_1 \subset F_2 \subset \cdots \subset F_{r-1} \subset F_r \subset \mathbb{R}^d$，其中每个 F_i 的维数为 i。此外，

$$E_r^\diamond = \{(F_1, \dots, F_r) \in \mathcal{F}(r,d) : F_r \in E_r\} \text{ and,}$$
$$E_r^\diamond(\varepsilon) = \big\{(F_1, \dots, F_r) \in \mathcal{F}(r,d) : F_r \in E_r(\varepsilon)\big\} \text{ for each } \varepsilon > 0.$$

我们用 $x = (F_1、\dots、F_r)$ 和 $x' = (F_1'、\dots、F_r')$ 表示 $\mathcal{F}(r、d)$ 中的典型元素。

Most steps towards Proposition 12.8 are rather straightforward translations of the arguments in Section 8. One significant difference is that Lemma 8.6 no longer holds: for $r > 1$ it is possible to have x and x' with F_r and F_r' arbitrarily close to the equator E without $\mathrm{VP}_r(x,x)'$ getting close to zero. For that reason, we cannot take $\psi_r = \mathrm{VP}_r$. This is dealt with in Section 12.3: instead, we define ψ_r inductively in terms of both VP_r and ψ_{r-1}.

12.1. 垂直角函数。假设 $x、x' \in \mathcal{F}(r、d)$ 满足 $F_1' \not\subset F_r$。根据定义，经过 x 和 x' 的大圆是由 $y = y(x、x')$ 定义的 $\mathrm{Gr}(r、d)$ 的子集，其定义为

$$y = \{\xi \in \mathrm{Gr}(r,d) : F_{r-1} \subset \xi \subset F_1' + F_r\}.$$

This is consistent with the case $r = 1$, as long as we follow the convention that $F_0 = \{0\}$. On the other hand, the great circle y depends on x through F_{r-1} and F_r, whereas it depends on x' through F_1' only. In particular, $y(x,x')$ need not coincide with $y(x',x)$ when $r > 1$. Related to this, the analogue of (8.14) is usually false when $r > 1$: the values of $\mathrm{VA}_r(x,x')$ and $\mathrm{VP}_r(x,x')$ that we define in the following may change when the roles of x and x' are exchanged.

垂直角函数 VA_r 定义为

$$\mathrm{VA}_r(x,x') = \sup_{\xi \in y} d(\xi, E) = \sup_{\xi \in y} \sup_{u \in \xi} d(u,E). \tag{12.1}$$

注意，这与 $r = 1$ 的情形一致。沿用 $r = 1$ 中的直觉，我们将 $\mathrm{VA}_r(x,x')$ 视为大圆 y 与 E_r 之间的夹角。确实，

$$\mathrm{VA}_r(x,x') = \sup_{u \in F_1' + F_r} d(u,E) = d(F_1' + F_r, E), \tag{12.2}$$

and so $\mathrm{VA}_r(x,x') = 0$ if and only if $F_1' + F_r \subset E$, that is, if and only if $y \subset E_r$.

我们将证明命题 8.1 针对 $r > 1$ 的以下推广：

Proposition 12.1. *There exist $\kappa_r' = \kappa_r'(\nu_\infty) > 0$ and $C_r' = C_r'(\nu_\infty) > 0$ and for any $\delta > 0$ there exists $N_r = N_r(\nu_\infty, \delta)$ such that for every $n \geqslant N_r$ there exists $\rho_r' = \rho_r'(\nu_\infty, \delta, n) > 0$ such that*

$$\int_G -\log \mathrm{VP}_r(gx, gx')\, d\nu_\infty^{(n)}(g) \leqslant -\log \mathrm{VP}_r(x, x') - (\kappa_r' - C_r'\delta)n$$

for every $x, x' \in E_r^\diamond(\rho_r')$ with $F_1' \not\subset F_r$.

Begin by noting that the properties (8.5) through (8.7) extend to $r > 1$. More precisely, it follows from (12.2) that

$$\mathrm{VA}_r(x, x') \geqslant d(u, E) = \frac{\|u^\perp\|}{\|u\|} \tag{12.3}$$

对于任意满足 $F_1' \not\subset F_r$ 的 $u \in F_1' + F_r$ 和 x、$x' \in \mathcal{F}(r$、$d)$。此外，如果 $u \in F_1' + F_r$ 在定义 (12.2) 中取到上确界，那么

$$\mathrm{VA}_r(x, x') = d(u, E) = \frac{\|u^\perp\|}{\|u\|}. \tag{12.4}$$

然后，将（12.3）与（12.4）结合，

$$\mathrm{VA}_r(gx, gx') \geqslant \frac{\|(gu)^\perp\|}{\|gu\|} = \mathrm{VA}_r(x, x')\frac{\|(gu)^\perp\|}{\|gu\|}\frac{\|u\|}{\|u^\perp\|} \tag{12.5}$$

for any $g \in G$. By Remark 3.1, when $g \in \operatorname{supp} \nu_\infty^{(n)}$ this means that

$$\mathrm{VA}_r(gx, gx') \geqslant \frac{\|g^\perp u^\perp\|}{\|gu\|} \geqslant \mathrm{VA}_r(x, x')\frac{\|g^\perp u^\perp\|}{\|gu\|}\frac{\|u\|}{\|u^\perp\|}. \tag{12.6}$$

此外，正如对于 (8.7) 的情形一样，

$$-\log \mathrm{VA}_r(gx, gx') \leqslant -\log \mathrm{VA}_r(x, x') + Bn \tag{12.7}$$

for any $x, x' \in \mathcal{F}(r, d)$ with $F_1' \not\subset F_r$ and $g \in \operatorname{supp} \nu_\infty^{(n)}$.

The proof of Proposition 12.1 is analogous to that of Proposition 8.1, replacing $d(x, x')$, $d(gx, gx')$ and $d(x, E)$ with $d(F_r, F_{r-1} + F_1')$, $d(gF_r, g(F_{r-1} + F_1'))$ and $d(F_r, E)$, respectively, and substituting r for 1 in the subscript. The details follow, but the reader may choose to skip them and proceed directly to Section 12.3.

Lemma 12.2. *There exists $\tilde\kappa_r = \tilde\kappa_r(\nu_\infty) > 0$ and for each $\delta > 0$ there exist $\tilde\theta_r = \tilde\theta_r(\nu_\infty, \delta) > 0$ and $\widetilde N_r = \widetilde N_r(\nu_\infty, \delta) \in \mathbb{N}$ such that for every $n \geqslant \widetilde N_r$ and $x, x' \in \mathcal{F}(r, d)$ with $F_1' \not\subset F_r$ there exists $\widetilde{\mathcal{E}}_r = \widetilde{\mathcal{E}}_r(\nu_\infty, \delta, n, x, x') \subset \operatorname{supp} \nu_\infty^{(n)}$ with $\nu_\infty^{(n)}(\widetilde{\mathcal{E}}_r^c) < \delta$ and*

$$-\log \mathrm{VA}_r(gx, gx') \leqslant \max\{-\log \mathrm{VA}_r(x, x') - \tilde\kappa_r n, \tilde\theta_r\} \text{ for every } g \in \widetilde{\mathcal{E}}_r. \tag{12.8}$$

Proof. Analogous to Lemma 8.2. Let $\kappa_0 = \kappa_0(\nu_\infty) > 0$, $N_0 = N_0(\nu_\infty, \delta) \in \mathbb{N}$, and $\mathcal{E}_0 = \mathcal{E}_0(\nu_\infty, \delta, n, u^\perp) \subset \operatorname{supp} \nu_\infty^{(n)}$ and $\tau_0 = \tau_0(\nu_\infty, \delta) > 0$ be as in Proposition 5.5. Given $x, x' \in \mathcal{F}(r, d)$ with $F_1' \not\subset F_r$, take u to be a non-zero vector that realizes the supremum in the definition (12.2). Write $u = u^E + u^\perp$ with $u^E \in E$ and $u^\perp \in E^\perp$. Take

$$\begin{gathered} \tilde\kappa_r = \kappa_0/2, \quad \tilde\theta_r = -\log(\tau_0/4), \\ \widetilde N_r > \max\{N_0, 4/\tilde\kappa_r\}, \quad \text{and } \widetilde{\mathcal{E}}_r = \mathcal{E}_0(\nu_\infty, \delta, n, u^\perp). \end{gathered} \tag{12.9}$$

设 $n \geqslant \tilde{N}_r$ 和 $g \in \tilde{\mathcal{E}}_r$；则 $g \in \operatorname{supp} \nu_\infty^{(n)}$。如果 $\|g^\perp u^\perp\|/\|gu\| \geqslant \tau_0/2$，那么 (12.6) 式中的第一个不等式意味着，

$$-\log \mathrm{VA}_r(gx, gx') \leqslant -\log \frac{\tau_0}{2} \leqslant \tilde{\theta}_r. \tag{12.10}$$

If $\|g^\perp u^\perp\|/\|gu\| < \tau_0/2$ then part (2) of Proposition 5.5 gives that

$$\frac{\|gu^\perp\|}{\|gu\|} < \frac{1}{2}, \text{ and so } \frac{\|gu^E\|}{\|gu\|} > \frac{1}{2}.$$

Substituting the latter inequality and $\|u\| \geqslant \|u^E\|$ in (12.6), we find that

$$\mathrm{VA}_r(gx, gx') \geqslant \frac{1}{2} \mathrm{VA}_r(x, x') \frac{\|g^\perp u^\perp\|}{\|u^\perp\|} \frac{\|u^E\|}{\|gu^E\|}. \tag{12.11}$$

因此，回顾定义 (5.3) 以及命题 5.5 的第 (1) 部分，

$$\mathrm{VA}_r(gx, gx') \geqslant \frac{1}{2} \mathrm{VA}_r(x, x') \frac{\|Dg^\perp_{u^E} u^\perp\|}{\|u^\perp\|} \geqslant \frac{1}{2} \mathrm{VA}_r(x, x') e^{\kappa_0 n}.$$

~根据（12.9）中对 $\tilde{\kappa}_r$ 和 N_r 的选取，这意味着

$$\begin{aligned} -\log \mathrm{VA}_r(gx, gx') &\leqslant -\log \mathrm{VA}_r(x, x') + \log 2 - 2\tilde{\kappa}_r n \\ &\leqslant -\log \mathrm{VA}_r(x, x') - \tilde{\kappa}_r n. \end{aligned} \tag{12.12}$$

引理的结论包含在 (12.10) 和 (12.12) 中。 □

12.2. 垂直投影函数。垂直投影函数 VP_r 定义为。

$$\mathrm{VP}_r(x, x') = \mathrm{VA}_r(x, x') d(F_r, F_{r-1} + F_1')^{\gamma_r} \tag{12.13}$$

where $\gamma_r = \gamma_r(\nu_\infty)$ is a small positive constant chosen through the following result, which extends Proposition 8.3 to $r > 1$:

Proposition 12.3. *There exist $\gamma_r = \gamma_r(\nu_\infty) > 0$ and $\kappa_r' = \kappa_r'(\nu_\infty) > 0$ and for each $\delta > 0$ there exists $N_r = N_r(\nu_\infty, \delta) \in \mathbb{N}$ such that for every $n \geqslant N_r$ there exists $\rho_r' = \rho_r'(\nu_\infty, \delta, n) > 0$ such that for any $x, x' \in E_r^\diamond(\rho_r')$ with $F_1' \not\subset F_r$ there exists $\mathcal{E}_r' = \mathcal{E}_r'(\nu_\infty, \delta, n, x, x') \subset \operatorname{supp} \nu_\infty^{(n)}$ with $\nu_\infty^{(n)}((\mathcal{E}_r')^c) < \delta$ and*

$$-\log \mathrm{VP}_r(gx, gx') \leqslant -\log \mathrm{VP}_r(x, x') - \kappa_r' n \text{ for every } g \in \mathcal{E}_r'. \tag{12.14}$$

为了证明命题12.3，我们需要将引理8.4和引理8.5推广到 $r > 1$，这将在接下来的两个陈述中完成。

Let $v \in F_r$ and $v' \in F_{r-1} + F_1'$ be unit vectors orthogonal to F_{r-1} such that

$$\begin{aligned} F_r &= F_{r-1} + \mathbb{R}v, \\ F_{r-1} + F_1' &= F_{r-1} + \mathbb{R}v', \text{ and} \\ d(F_r, F_{r-1} + F_1') &= |\sin \angle(v, v')|. \end{aligned} \tag{12.15}$$

It is no restriction to take the angle between v and v' to be non-obtuse. Then the vector $w = v' - v$ satisfies (compare Figure 5)

$$d(F_r, F_{r-1} + F_1') \leqslant \|w\| \text{ and } \angle(w, v) \geqslant \frac{\pi}{4}. \tag{12.16}$$

引理 12.4. 给定 x、$x' \in \mathcal{F}(r$、$d)$，其中 $F_1' \not\subset F_r$，令 $w = v' - v$ 如 (12.16) 所示。则

$$d(F_r, E) < \frac{1}{4}\,\mathrm{VA}_r(x, x') \text{ implies } \frac{\|w^\perp\|}{\|w\|} > \frac{1}{4}\,\mathrm{VA}_r(x, x'). \tag{12.17}$$

Proof. Analogous to Lemma 8.4. Let $u \in F_1' + F_r$ be any non-zero vector that realizes the supremum in the definition (12.2). Since $F_1' + F_r = \mathrm{span}\{F_{r-1}, v, w\}$, we may write $u = u_0 + av + bw$ with $u_0 \in F_{r-1}$ and $a, b \in \mathbb{R}$. Recall that v and w are orthogonal to F_{r-1}. Moreover, by (12.16) the angle between them is no less than $\pi/4$. This implies that $\|u_0 + av\|$ and $\|bw\|$ are both less than $2\|u\|$. Thus,

$$\begin{aligned}\mathrm{VA}_r(x, x') = \frac{\|u^\perp\|}{\|u\|} &\leqslant \frac{\|(u_0 + av)^\perp\|}{\|u\|} + \frac{\|bw^\perp\|}{\|u\|} \\ &< 2\frac{\|(u_0 + av)^\perp\|}{\|u_0 + av\|} + 2\frac{\|w^\perp\|}{\|w\|} \leqslant 2d(F_r, E) + 2\frac{\|w^\perp\|}{\|w\|}.\end{aligned}$$

Thus, $d(F_r, E)$ and $\|w^\perp\|/\|w\|$ cannot be both less than $\mathrm{VA}_r(x, x')/4$. □

Take $\tilde{\kappa}_r = \tilde{\kappa}_r(\nu_\infty) > 0$ and $\tilde{\theta}_r = \tilde{\theta}_r(\nu_\infty, \delta) > 0$ to be as in Lemma 12.2.

Lemma 12.5. *There exists $\hat{\kappa}_r = \hat{\kappa}_r(\nu_\infty) > 0$ and for each $\delta > 0$ there exists $\widehat{N}_r = \widehat{N}_r(\nu_\infty, \delta) \in \mathbb{N}$ such that for each $n \geqslant \widehat{N}_r$ there exists $\hat{\rho}_r = \hat{\rho}_r(\nu_\infty, \delta, n) > 0$ such that for any $x, x' \in E_r^\diamond(\hat{\rho}_r)$ with $F_1' \not\subset F_r$ and $-\log \mathrm{VA}_r(x, x') \leqslant \tilde{\theta}_r + \tilde{\kappa}_r n$ there exists $\hat{\mathcal{E}}_r = \hat{\mathcal{E}}_r(\nu_\infty, \delta, n, x, x') \subset \mathrm{supp}\,\nu_\infty^{(n)}$ with $\nu_\infty^{(n)}(\hat{\mathcal{E}}_r^c) < \delta$ and*

$$-\text{对数 } d(gF_r, g(F_{r-1} + F_1')) \leqslant -\text{对数 } d(F_r, F_{r-1} + F_1') - \text{对数 } \mathrm{VA}_r(x, x') - \hat{\kappa}_r n \tag{12.18}$$

$\hat{}$ 对于每个 $g \in \mathcal{E}_r$。

Proof. Analogous to Lemma 8.5. Let $\kappa_0 = \kappa_0(\nu_\infty) > 0$, $N_0 = N_0(\nu_\infty, \delta) \in \mathbb{N}$, and $\mathcal{E}_0 = \mathcal{E}_0(\nu_\infty, \delta, n, v^\perp) \subset \mathrm{supp}\,\nu_\infty^{(n)}$ be as in Proposition 5.5. Given x and x' with $F_1' \not\subset F_r$, let $w = v' - v$ be as in (12.16). Take

$$\begin{aligned}&\hat{\kappa}_r = \tilde{\kappa}_r/2, \quad \widehat{N}_r = \max\{N_0, 5/\hat{\kappa}_r\}, \\ &\hat{\rho}_r < e^{-\tilde{\theta}_r - \tilde{\kappa}_r n}/10, \text{ and } \hat{\mathcal{E}}_r = \mathcal{E}_0(\nu_\infty, \delta, n, w^\perp).\end{aligned} \tag{12.19}$$

Let $n \geqslant \widehat{N}_r$ and $g \in \hat{\mathcal{E}}_r \subset \mathrm{supp}\,\nu_\infty^{(n)}$. As observed in (12.16),

$$d(F_r, F_{r-1} + F_1') \leqslant \|w\|. \tag{12.20}$$

不妨设 $\|gv\| \geqslant \|gv'\|$；$\|gv\| \leqslant \|gv'\|$ 的情形类似，只需交换 F_r 和 $F_{r-1} + F_1'$ 的角色。正如在 (8.22) 中一样，

$$-\log d(gF_r, g(F_{r-1} + F_1')) \leqslant -\log d(F_r, F_{r-1} + F_1') - \log \frac{\|\Pi_{gv} gw\|}{\|gv\|}\frac{\|v\|}{\|w\|}. \tag{12.21}$$

for any $g \in G$. By the condition on $\hat{\rho}_r$ in (12.19), if $d(F_r, E) \leqslant \hat{\rho}_r$ then

$$4d(F_r, E) < e^{-\tilde{\theta}_r - \tilde{\kappa}_r n} < \mathrm{VA}_r(x, x'),$$

然后由引理 12.4 可得开奖现场

$$\frac{\|w^\perp\|}{\|w\|} > \frac{1}{4}\,\mathrm{VA}_r(x, x') > \frac{1}{4}e^{-\tilde{\theta}_r - \tilde{\kappa}_r n}. \tag{12.22}$$

通过进一步缩小 $\hat{}\rho_r$，我们还可以假设 that

$$\frac{\|hz\|}{\|z\|} \geqslant \frac{1}{2}\frac{\|hz^E\|}{\|z^E\|} \text{ and } \|\Pi_{hz}hw\| \geqslant \frac{1}{2}\|\Pi_{hz^E}gw\| \tag{12.23}$$

对于任意非零的 $z = z^E + z^\perp$ in $E \oplus E^\perp$ with $\|z^\perp\|/\|z\| \leqslant \hat{}\rho_r$，以及任意 $h \in \operatorname{supp}\nu_\infty^{(n)}$。(12.23) 的第二部分意味着，

$$\|\Pi_{hz}hw\| \geqslant \frac{1}{2}\|\Pi_{hz^E}hw\| \geqslant \frac{1}{2}\|\Pi_E hw\| = \frac{1}{2}\|(hw)^\perp\| = \frac{1}{2}\|h^\perp w^\perp\|. \tag{12.24}$$

注意到 $\|v^\perp\|/\|v\| \leqslant d(F_r, E) \leqslant \hat{}\rho_r$，在前两个关系式中取 $z = v$ 和 $h = g$。因此，将 (12.23) 和 (12.24) 代入 (12.21)，

$$-\log d(gF_r, g(F_{r-1} + F_1')) \leqslant -\log d(F_r, F_{r-1} + F_1') + \log 4 - \log\frac{\|g^\perp w^\perp\|}{\|w\|}\frac{\|v^E\|}{\|gv^E\|}.$$

然后，再利用 (12.22) 和 (5.3)，

$$\begin{aligned}
&-\log d(gF_r, g(F_{r-1} + F_1')) \\
&\quad\leqslant -\log d(F_r, F_{r-1} + F_1') - \log \mathrm{VA}_r(x, x') + \log 16 - \log\frac{\|g^\perp w^\perp\|}{\|w^\perp\|}\frac{\|v^E\|}{\|gv^E\|} \\
&\quad= -\log d(F_r, F_{r-1} + F_1') - \log \mathrm{VA}_r(x, x') + \log 16 - \log\frac{\|Dg^\perp_{v^E} w^\perp\|}{\|w^\perp\|}.
\end{aligned}$$

By part (1) of Proposition 5.5 and the choice of $\hat{\kappa}_r$ and $\widehat{N}_r$ in (12.19), it follows that

$$\begin{aligned}
-\log d(gF_r, g(F_{r-1} + F_1')) &\leqslant -\log d(F_r, F_{r-1} + F_1') - \log \mathrm{VA}_r(x, x') + 5 - 2\hat{\kappa}_r n \\
&\leqslant -\log d(F_r, F_{r-1} + F_1') - \log \mathrm{VA}_r(x, x') - \hat{\kappa}_r n,
\end{aligned}$$

as claimed. □

Proof of Proposition 12.3. Analogous to Proposition 8.3. Take $A = A(\nu_\infty)$ as in (8.2) and then define

$$\begin{aligned}
&\gamma_r = \min\{1, \tilde{\kappa}_r/(2A)\}, \\
&\kappa_r' = \min\{\tilde{\kappa}_r/2, \gamma_r\hat{\kappa}_r/2\}, \\
&\rho_r' = \min\{\hat{\rho}_r(\nu_\infty, \delta), \hat{\rho}_r(\nu_\infty, \delta/2)\}, \\
&N_r = \max\{\widetilde{N}_r(\nu_\infty, \delta/2), \widehat{N}_r(\nu_\infty, \delta/2), 2\tilde{\theta}_r/(\gamma\hat{\kappa}_r)\}, \text{ and} \\
&\mathcal{E}_r' = \widetilde{\mathcal{E}}_r(\nu_\infty, \delta/2, n, x, x') \cap \hat{\mathcal{E}}_r(\nu_\infty, \delta/2, n, x, x').
\end{aligned} \tag{12.25}$$

By construction, $\nu_\infty^{(n)}((\mathcal{E}_r')^c) < \delta$. The definition (12.13) gives that

$$-\log \mathrm{VP}_r(gx, gx') = -\log \mathrm{VA}_r(gx, gx') - \gamma_r \log d(gF_r, g(F_{r-1} + F_1')). \tag{12.26}$$

Consider $x, x' \in E_r^\diamond(\rho_r')$ with $F_1' \not\pitchfork F$, and let $n \geqslant N_r$ and $g \in \mathcal{E}_r'$. First, suppose that $-\log \mathrm{VA}_r(x, x') \geqslant \tilde{\theta}_r + \tilde{\kappa}_r n$. Then, by Lemma 12.2,

$$-\log \mathrm{VA}_r(gx, gx') \leqslant -\log \mathrm{VA}_r(x, x') - \tilde{\kappa}_r n \tag{12.27}$$

将（12.27）和（8.2）代入（12.26），我们得到

$$-\log \mathrm{VP}_r(gx, gx') \leqslant -\log \mathrm{VA}_r(x, x') - \tilde{\kappa}_r n - \gamma_r \log d(F_r, F_{r-1} + F_1') + \gamma_r A n.$$

By (12.26) and the choice of γ_r and κ_r' in (12.25), this yields

$$-\log \mathrm{VP}_r(gx, gx') \leqslant -\log \mathrm{VP}_r(x, x') - \frac{\tilde{\kappa}_r}{2} n \leqslant -\log \mathrm{VP}_r(x, x') - \kappa_r' n. \tag{12.28}$$

Now assume that $-\log \mathrm{VA}_r(x, x') \leqslant \tilde{\theta}_r + \tilde{\kappa}_r n$. In this case, Lemma 12.2 yields

$$-\log \mathrm{VA}_r(gx, gx') \leqslant \tilde{\theta}_r \tag{12.29}$$

并且引理 12.5 表明,

$$-\log d(gF_r, g(F_{r-1} + F_1')) \leqslant -\log d(F_r, F_{r-1} + F_1') - \log \mathrm{VA}_r(x, x') - \hat{\kappa}_r n. \tag{12.30}$$

将（12.29）和（12.30）代入（12.26），我们得到

$$\begin{aligned} -\log \mathrm{VP}_r(gx, gx') &\leqslant \tilde{\theta}_r - \gamma_r \log d(F_r, F_{r-1} + F_1') - \gamma_r \log \mathrm{VA}_r(x, x') - \gamma_r \hat{\kappa}_r n \\ &\leqslant -\log \mathrm{VP}_r(x, x') + \tilde{\theta}_r + (1 - \gamma_r) \log \mathrm{VA}_r(x, x') - \gamma_r \hat{\kappa}_r n. \end{aligned}$$

˜由于 $\mathrm{VA}_r(x、\ x') \leqslant 1$、$\gamma_r \leqslant 1$ 和 $n \geqslant N_r \geqslant 2\theta_r/(\gamma_r \hat{\kappa}_r)$，由此得到

$$\begin{aligned} -\log \mathrm{VP}_r(gx, gx') &\leqslant -\log \mathrm{VP}_r(x, x') + \tilde{\theta}_r - \gamma_r \hat{\kappa}_r n \\ &\leqslant -\log \mathrm{VP}_r(x, x') - \frac{\gamma_r \hat{\kappa}_r}{2} n \leqslant -\log \mathrm{VP}_r(x, x') - \kappa_r' n. \end{aligned} \tag{12.31}$$

The relations (12.28) and (12.31) contain the conclusion of Proposition 12.3. □

12.3. **The function $-\log \psi_r$.** At this point, the proof of Proposition 12.1 is analogous to that of Proposition 8.1. Take $\kappa_r' > 0$, $N_r \in \mathbb{N}$ and $\rho_r' > 0$ as in Proposition 12.3. Let $C_r' = C_r'(\nu_\infty) > 0$ be given by

$$C_r' = B + \gamma_r A. \tag{12.32}$$

Substituting (12.7) and (8.2) in the definition (12.13), we find that

$$\begin{aligned} &-\log \mathrm{VP}_r(gx,gx') \\ &\qquad = -\log \mathrm{VA}_r(gx, gx') - \gamma_r \log d(gF_r, g(F_{r-1} + F_1')) \\ &\qquad \leqslant -\log \mathrm{VA}_r(x, x') + Bn - \gamma_r \log d(F_r, F_{r-1} + F_1') + \gamma_r An \\ &\qquad = -\log \mathrm{VP}_r(x, x') + C_r' n. \end{aligned} \tag{12.33}$$

for any $g \in \operatorname{supp} \nu_\infty^{(n)}$ and $x, x' \in \mathcal{F}(r, d)$ with $F_1' \not\subset F_r$. Integrating (12.14) over $\mathcal{E}_r'$ and (12.33) over the complement, and using the fact that $\nu_\infty^{(n)}((\mathcal{E}_r')^c) < \delta$, we get that

$$\int_G -\log \mathrm{VP}_r(gx, gx')\, d\nu_\infty^{(n)}(g) \leqslant -\log \mathrm{VP}_r(x, x') - \kappa_r' n + C_r' \delta n.$$

for every $n \geqslant N_r$ and $x, x' \in E_r^\diamond(\rho_r')$ with $F_1' \not\subset F_r$. This completes the proof of Proposition 12.1.

However, as mentioned before, for $r > 1$ it is possible to have x and x' with F_r and F_r' arbitrarily close to the equator E without $\mathrm{VP}_r(x, x')$ getting close to zero. Here is a simple example (see also part (2) of Lemma 12.7):

Example 12.6. Denote by (x_1, x_2, x_3, x_4) the elements of $\mathbb{R}^4$ and let $E = \{x_4 = 0\}$, $F_1 = \{x_2 = x_3 = x_4 = 0\}$, $F_2 = \{x_3 = x_4 = 0\}$, $F_1' = \{x_2 = x_3 = 0, x_4 = \varepsilon x_1\}$ and $F_2' = \{x_3 = 0, x_4 = \varepsilon x_1\}$. It is clear that $F_2 \subset E$ and $F_2' \to E$ when $\varepsilon \to 0$. However,

- $\mathrm{VA}_2(x, x') = d(F_2 + F_1', E) = 1$, since $(0, 0, 0, 1) \in (F_2 + F_1') \cap E^\perp$;
- $d(F_2, F_1 + F_1') = 1$, since $(0, 1, 0, 0) \in F_2 \cap (F_1 + F_1')^\perp$.

因此，当 $\varepsilon \to 0$ 时，$\mathrm{VP}_2(x,x')$ 始终远离零。

Thus, $-\log \mathrm{VP}_r(x,x')$ cannot be used as a Margulis function for the equator $E_r^\diamond$ in the flag space. To rectify this problem, we define the function ψ_r inductively in r as follows. Let x_-, $x'_- \in \mathcal{F}_{r-1}(\mathbb{R}^d)$ be the *truncated flags* obtained by dropping the r-dimensional subspaces from x and x', respectively. The assumption $F_1' \not\subset F_r$ implies that $F_1' \not\subset F_{r-1}$, and so we may assume that $\psi_{r-1}(x_-, x'_-)$ has already been defined. Then define

$$\psi_r(x,x') = \psi_{r-1}(x_-,x'_-)^{\beta_{r-1}} \,\mathrm{VP}_r(x,x'), \tag{12.34}$$

where the exponent $\beta_{r-1} = \beta_{r-1}(\nu_\infty)$ is a small constant to be chosen as follows.

由此定义及（12.33）可知，

$$\begin{aligned} &-\log\psi_j(gx,gx') + \log\psi_j(x,x') \\ &\qquad\leqslant \big[-\log\psi_{j-1}(gx,gx') + \log\psi_{j-1}(x,x')\big]\beta_{j-1} + C_j' n \end{aligned}$$

对于每一个 $j = 2,\dots,r$。因此，同时回顾式 (8.35)，

$$\begin{aligned} &-\log\psi_r(gx,gx') \\ &\qquad\leqslant -\log\psi_r(x,x') + \big[C_1'\beta_1\cdots\beta_{r-1} + \cdots + C_{r-1}'\beta_{r-1} + C_r'\big]n \\ &\qquad\leqslant -\log\psi_r(x,x') + C_r'' n, \end{aligned} \tag{12.35}$$

其中，$C_r'' = C_r''(\nu_\infty)$ 定义为

$$C_r'' = C_1' + \cdots + C_{r-1}' + C_r'. \tag{12.36}$$

取（12.34）中的指数 β_{r-1} 足够小，使得

$$\beta_{r-1} C_{r-1}'' \leqslant \frac{1}{2}\kappa_r' \tag{12.37}$$

其中 $\kappa_r' = \kappa_r'(\nu_\infty)$ 如命题 12.1 所述。

The function $-\log\psi_r(x,x')$ thus defined does go to infinity when the flags x and x' approach the equator $E_r^\diamond$:

Lemma 12.7. *Given any $R > 0$, there exists $\tilde\rho_r = \tilde\rho_r(\nu_\infty, R) > 0$ such that for any $x, x' \in E_r^\diamond(\tilde\rho_r)$ with $F_1' \not\subset F_r$,*

(1) $-\log\ \psi_r(x,x') > R$ 以及 **(2)** $-\log \mathrm{VP}_r(x,x') > R$，

除非 $\mathrm{VA}_{r-1}(x_-,x'_-) > \tilde\rho_r$。

Proof. It is clear from the definitions (12.2) and (12.13) that $\mathrm{VP}_j(x,x') \leqslant 1$ for every $1 \leqslant j \leqslant r$ and any $x, x' \in \mathcal{F}(j,d)$ with $F_1' \not\subset F_j$. So, the definition (12.34) implies that

$$-\log\psi_r(x,x') \geqslant -\log\psi_1(F_1,F_1')\beta_1\cdots\beta_{r-1}. \tag{12.38}$$

By definition, the β_j depend only on ν_∞. Then Lemma 8.6 gives that for any $R > 0$ there exists $\rho > 0$ depending only on ν_∞ and R such that the right-hand side of (12.38) is greater than R for any $F_1 \neq F_1'$ in $E_1(\rho)$. Since $d(F_1,E) \leqslant d(F_r,E)$ and $d(F_1',E) \leqslant d(F_r',E)$, because $F_1 \subset F_r$ and $F_1' \subset F_r'$, we get that $-\log\psi_r(x,x') > R$ for any $x, x' \in E_r^\diamond(\rho)$ with $F_1' \not\subset F_r$. This proves part (1).

To prove part (2), consider $x, x' \in \mathcal{F}(r,d)$ with $d(F_r,E) \leqslant \rho$, $d(F_r',E) \leqslant \rho$, and

$$\mathrm{VA}_{r-1}(x_-,x'_-) = d(F_{r-1} + F_1', E) \leqslant \rho. \tag{12.39}$$

若 $d(F_r, F_{r-1}' + F_1') \leqslant \sqrt{\rho}$，则

$$\mathrm{VP}_r(x,x') \leqslant d(F_r, F_{r-1} + F_1')^{\gamma_r} \leqslant \rho^{\gamma_r/2}. \tag{12.40}$$

Now suppose that $d(F_r, F'_{r-1} + F'_1) \geqslant \sqrt{\rho}$. Let v and v' be unit vectors orthogonal to F_{r-1} as in (12.15). Take $u \in F'_1 + F_r$ realizing the supremum in (12.2), and write $u = u_0 + av + bv'$ with $u_0 \in F_{r-1}$ and $a, b \in \mathbb{R}$. Since $|\sin \angle(v', F_r)| = d(F_r, F'_1 + F'_{r-1})$ is taken to be greater than $\sqrt{\rho}$,

$$\|u_0 + av\| \leqslant \frac{2}{\sqrt{\rho}}\|u\| \text{ and } \|bv'\| \leqslant \frac{2}{\sqrt{\rho}}\|u\|.$$

Then,

$$\begin{aligned} \mathrm{VA}_r(x, x') = \frac{\|u^\perp\|}{\|u\|} &\leqslant \frac{\|(u_0 + av)^\perp\|}{\|u\|} + \frac{\|(bv')^\perp\|}{\|u\|} \\ &\leqslant \frac{2}{\sqrt{\rho}} \frac{\|(u_0 + av)^\perp\|}{\|u_0 + av\|} + \frac{2}{\sqrt{\rho}} \frac{\|(v')^\perp\|}{\|v'\|}. \end{aligned}$$

同时回顾（12.39），我们得到♀♀♀♀♀♀

$$\mathrm{VP}_r(x, x') \leqslant \mathrm{VA}_r(x, x') \leqslant \frac{2}{\sqrt{\rho}} d(F_r, E) + \frac{2}{\sqrt{\rho}} d(F'_1 + F'_{r-1}, E)) \leqslant 4\sqrt{\rho}. \tag{12.41}$$

Since γ_r and ρ may be taken to be smaller than 1, both inequalities (12.40) and (12.41) imply that

$$-\log \mathrm{VP}_r(x, x') \geqslant -\frac{\gamma_r}{2} \log \rho - \log 4$$

对于任意满足 $F'_1 \not\subset F_r$ 的 $E_r^\diamond(\rho)$ 中的 x、x'。只要 ρ 取得足够小，右端就大于 R。□

为完整起见，我们给出命题 12.1 和命题 12.3 的以下版本，其中函数为 ψ_r，尽管在下文中并不需要用到它（我们将改为使用相关的命题 13.2）：

Proposition 12.8. *There exists $\kappa''_r = \kappa''_r(\nu_\infty) > 0$ such that for each $\delta > 0$ and $n \geqslant N_r$ there exists $\rho''_r = \rho''_r(\nu_\infty, \delta, n) > 0$ such that for any $x, x' \in E_r^\diamond(\rho''_r)$ with $F'_1 \not\subset F_r$ there exists $\mathcal{E}''_r = \mathcal{E}''_r(\nu_\infty, \delta, n, x, x') \subset \operatorname{supp} \nu_\infty^{(n)}$ with $\nu_\infty^{(n)}((\mathcal{E}''_r)^c) < \delta$ and*

$$-\log \psi_r(gx, gx') \leqslant -\log \psi_r(x, x') - \kappa''_r n \text{ for every } g \in \mathcal{E}''_r \tag{12.42}$$

and

$$\int_G -\log \psi_r(gx, gx')\, d\nu_\infty^{(n)}(g) \leqslant -\log \psi_r(x, x') - (\kappa''_r - C''_r \delta)n. \tag{12.43}$$

Proof. The case $r = 1$ consists of the inequalities (8.15) and (8.1), respectively. with $\kappa''_1 = \kappa'_1$, $C''_1 = C'_1$, $\rho''_1 = \rho'_1$, and $\mathcal{E}''_1 = \mathcal{E}'_1$. Now suppose that $r > 1$. Recall that $C''_r = C''_r(\nu_\infty)$ was defined in (12.36). Define also

$$\kappa''_r = \kappa'_r(\nu_\infty)/2, \quad \rho''_r = \rho'_r(\nu_\infty, \delta, n) \text{ and } \mathcal{E}''_r = \mathcal{E}'_r(\nu_\infty, \delta, n, x, x'). \tag{12.44}$$

Consider $n \geqslant N_r$, and $x, x' \in E_r^\diamond(\rho''_r)$ with $F'_1 \not\subset F_r$. By (12.35),

$$-\log \psi_{r-1}(gx_-, gx'_-) \leqslant -\log \psi_{r-1}(x_-, x'_-) + C'''_{r-1} n \tag{12.45}$$

对每个 $g \in \operatorname{supp} \nu_\infty^{(n)}$。根据命题 12.3,

$$-\log \mathrm{VP}_r(gx, gx') \leqslant -\log \mathrm{VP}_r(x, x') - \kappa'_r n \tag{12.46}$$

对于每个 $g \in \mathcal{E}''_r$。将 (12.45) 和 (12.46) 代入定义 (12.34)，并回顾在 (12.37) 中对 β_{r-1} 的选择，

$$\begin{aligned} -\log\psi_r(gx,gx') - \log\psi_{r-1}(gx_-,gx'_-)\beta_{r-1} - \log \mathrm{VP}_r(gx,gx') \\ \leqslant -\log\psi_{r-1}(x_-,x'_-)\beta_{r-1} + C''_{r-1}n\beta_{r-1} - \log \mathrm{VP}_r(x,x') - \kappa'_r n \\ \leqslant -\log\psi_r(x,x') - (\kappa'_r/2)n \end{aligned}$$

for every $g \in \mathcal{E}''_r$. This gives claim (12.42). Moreover, integrating this inequality on $\mathcal{E}''_r$ and (12.35) on the complement, we obtain

$$\int_G -\log\psi_r(gx,gx')\, d\nu_\infty^{(n)}(g) \leqslant -\log\psi_r(x,x') - (\kappa'_r/2 - C''_r\delta)n,$$

从而得出结论 (12.43)。 □

13. Stabilization and cut-off

Next we present the analogues for $r > 1$ of the two constructions in Section 9. Both are fairly straightforward extensions of the case $r = 1$, but it turns out that for $r > 1$ they do not suffice to deal with the questions discussed at the beginning of Section 9. This difficulty will be handled later, in Section 15. Another issue is that the function ψ_r we construct in the following is not symmetric when $r > 1$. Thus we will use instead the function $\hat\psi_r$ defined by

$$\hat\psi_r(x,x') = \max\{\psi_r(x,x'), \psi_r(x',x)\}.$$

Let $\kappa''_r = \kappa''_r(\nu_\infty) > 0$, $C''_r = C''_r(\nu_\infty) > 0$ and $N_r = N_r(\nu_\infty,\delta) \in \mathbb{N}$ be as in Proposition 12.8. Keep in mind that $\varepsilon_r < \rho''_r = \rho'_r \leqslant \hat\rho_r$, $C'_r = B + \gamma_r A$, and $C''_r = C'_1 + \cdots + C'_r$, according to (12.25), (12.32), (12.36), (12.44), and (13.17).

13.1. **Stabilization.** Let $\gamma_r = \gamma_r(\nu_\infty) > 0$ and $B = B(\nu_\infty) > 0$ be as in (12.25) and (8.3), respectively. For each $\omega_r > 0$ and $n \in \mathbb{N}$, define the *stabilized vertical angle*

$$\mathrm{SVA}_r(x,x';\omega_r) = \max\left\{\mathrm{VA}_r(x,x'), \omega_r e^{-Bn}\right\} \tag{13.1}$$

and the *stabilized vertical projection* by

$$\mathrm{SVP}_r(x,x';\omega_r) = \mathrm{SVA}_r(x,x';\omega_r)\, d(F_r, F_{r-1} + F'_1)^{\gamma_r} \tag{13.2}$$

for every x, x' in $E^\diamond_r(\varepsilon_r)$ with $F'_1 \not\subset F_r$.

给定 $\vec\omega = (\omega_1,\dots,\omega_r) \in \mathbb{R}^d_+$，令 $\vec\omega_- = (\omega_1,\dots,\omega_{r-1})$，并定义稳定化函数 ψ_r 为

$$\psi_r(x,x';\vec\omega) = \psi_{r-1}(x_-,x'_-;\vec\omega_-)^{\beta_{r-1}}\, \mathrm{SVP}_r(x,x';\omega_r). \tag{13.3}$$

The following extension of Lemma 9.1 asserts that the estimate in (12.35) remains valid for these stabilized functions:

引理 13.1。对于每个 $g \in \mathrm{supp}\ \nu_\infty^{(n)}$、任意 x、$x' \in E^\diamond_r(\rho''_r)$（满足 $F'_1 \not\subset F_r$），以及任意$\vec\omega = (\omega_1$、……、$\omega_r) \in \mathbb{R}^d_+$，。

$$-\log\psi_r(gx,gx';\vec\omega) \leqslant -\log\psi_r(x,x';\vec\omega) + C''_r n. \tag{13.4}$$

Proof. The case $r = 1$ was done in Lemma 9.1, so let us consider $r > 1$. By induction,

$$-\log\psi_r(gx_-, gx'_-; \vec{\omega}_-) \leqslant -\log\psi_r(x_-, x'_-; \vec{\omega}_-) + C''_{r-1}n. \tag{13.5}$$

We claim that

$$-\log \mathrm{SVA}_r(gx, gx'; \omega_r) \leqslant -\log \mathrm{SVA}_r(x, x'; \omega_r) + Bn. \tag{13.6}$$

and

$$-\log \mathrm{SVP}_r(gx, gx'; \omega_r) \leqslant -\log \mathrm{SVP}_r(x, x'; \omega_r) + C'_r n. \tag{13.7}$$

The inequality (13.4) follows directly from combining (13.5) and (13.7), and recalling the definition of C''_r in (12.36):

$$\begin{aligned} -\log\psi_r(gx, gx'; \vec{\omega}) &= -\log\psi_r(gx_-, gx'_-; \vec{\omega}_-)\beta_{r-1} \\ &\qquad -\log \mathrm{SVP}_r(gx, gx'; \omega_r) \\ &\leqslant -\log\psi_r(x_-, x'_-; \vec{\omega}_-)\beta_{r-1} + C''_{r-1}n\beta_{r-1} \\ &\qquad -\log \mathrm{SVP}_r(x, x'; \omega_r) + C'_{r-1}n \\ &\leqslant -\log\psi_r(x, x'; \vec{\omega}) + C''_r n. \end{aligned} \tag{13.8}$$

另请回顾，$\beta_{r-1} \leqslant 1$。

We split the proof of (13.6) into two cases. Suppose first that $\mathrm{VA}_r(x, x') < \omega_r$. Then, by the definition (13.1),

$$\begin{aligned} -\log \mathrm{SVA}_r(x, x'; \omega_r) &\geqslant -\log\omega_r \text{ and} \\ -\log \mathrm{SVA}_r(gx, gx'; \omega_r) &\leqslant -\log\omega_r + Bn \leqslant -\log \mathrm{SVA}_r(x, x'; \omega_r) + Bn \end{aligned}$$

as claimed. Now suppose that $\mathrm{VA}_r(x, x') \geqslant \omega_r$. Then, again by the definition (13.1),

$$\begin{aligned} -\log \mathrm{SVA}_r(x, x'; \omega_r) &= -\log \mathrm{VA}_r(x, x') \text{ and} \\ -\log \mathrm{SVA}_r(gx, gx'; \omega_r) &\leqslant -\log \mathrm{VA}_r(gx, gx'). \end{aligned}$$

结合 (12.7)，这就得到了

$$-\log \mathrm{SVA}_r(gx, gx'; \omega_r) \leqslant -\log \mathrm{SVA}_r(x, x'; \omega_r) + Bn,$$

这就完成了（13.6）的证明。

Finally, substituting (13.6) and (8.2) in the definition (12.13), and recalling the definition of C'_r in (12.32),

$$\begin{aligned} -\log \mathrm{SVP}_r(gx, gx'; \omega_r) &= -\log \mathrm{SVA}_r(gx, gx'; \omega_r) \\ &\qquad -\gamma_r \log d(gF_r, g(F_{r-1} + F'_1)) \\ &\leqslant -\log \mathrm{SVA}_r(x, x'; \omega_r) + Bn \\ &\qquad -\gamma_r \log d(gF_r, g(F_{r-1} + F'_1)) - \gamma_r An \\ &= -\log \mathrm{SVP}_r(x, x'; \omega_r) + (B + \gamma_r A)n \end{aligned} \tag{13.9}$$

这就证明了 (13.7)，从而完成了引理的证明。 □

We say that (x, x') is in the *stabilization region* if $\mathrm{VA}_r(x, x') < \omega_r$. As we have seen in (12.7),

$$\mathrm{VA}_r(gx, gx') \geqslant \mathrm{VA}_r(x, x')e^{-Bn}$$

for any $x, x' \in \mathcal{F}(r,d)$ with $F_1' \not\subset F_r$ and $g \in \operatorname{supp} \nu_\infty^{(n)}$. So, if (x,x') is outside the stabilization region then

$$\begin{aligned} &\mathrm{SVA}_r(x,x';\omega_r) = \mathrm{VA}_r(x,x') \text{ and } \mathrm{SVA}_r(gx,gx';\omega_r) = \mathrm{VA}_r(gx,gx') \\ &\mathrm{SVP}_r(x,x';\omega_r) = \mathrm{VP}_r(x,x') \text{ and } \mathrm{SVP}_r(gx,gx';\omega_r) = \mathrm{VP}_r(gx,gx') \end{aligned} \tag{13.10}$$

对任意 $g \in \operatorname{supp} \nu_\infty^{(n)}$ 和 $n \in \mathbb{N}$。

The following analogue of Proposition 12.8 for stabilized vertical angles and the stabilized vertical projections extends Proposition 9.2 to $r > 1$:

Proposition 13.2. *For every $\delta > 0$, $n \geqslant N_r$, x, x' in $E_r^\diamond(\rho_r'')$ with $F_1' \not\subset F_r$, and $\vec{\omega} = (\omega_1, \dots, \omega_r) \in \mathbb{R}_+^d$ with $\mathrm{VA}_r(x,x') \geqslant \omega_r$,*

$$-\log \psi_r(gx, gx'; \vec{\omega}) \leqslant -\log \psi_r(x,x';\vec{\omega}) - \kappa_r'' n \text{ for every } g \in \mathcal{E}_r'', \tag{13.11}$$

and

$$\int_G -\log \psi_r(gx, gx'; \vec{\omega})\, d\nu_\infty^{(n)}(g) \leqslant -\log \psi_r(x,x';\vec{\omega}) - (\kappa_r'' - C_r''\delta)n \tag{13.12}$$

for any

Proof. The case $r = 1$ is given by Proposition 9.2 with $\kappa_1'' = \kappa_1'$, $C_1'' = C_1'$, $\rho_1'' = \rho_1'$, and $\mathcal{E}_1'' = \mathcal{E}_1'$. Now let us consider $r > 1$. Let $\delta > 0$, $n \geqslant N_r$, $x, x' \in E_r^\diamond(\rho_r'')$, and $\vec{\omega} \in \mathbb{R}_+^d$ be as in the statement. By (13.3) and (13.10),

$$\begin{aligned} \psi_r(x,x';\vec{\omega}) &= \psi_{r-1}(x_-, x_-'; \vec{\omega}_-)^{\beta_{r-1}} \mathrm{VP}_r(x,x') \text{ and} \\ \psi_r(gx,gx';\vec{\omega}) &= \psi_{r-1}(gx_-, gx_-'; \vec{\omega}_-)^{\beta_{r-1}} \mathrm{VP}_r(gx,gx'). \end{aligned} \tag{13.13}$$

By (13.4),

$$-\log \psi_{r-1}(gx_-, gx_-'; \vec{\omega}_-) \leqslant -\log \psi_{r-1}(x_-, x_-'; \vec{\omega}_-) + C_{r-1}''' n \tag{13.14}$$

对于每个 $g \in \operatorname{supp} \nu_\infty^{(n)}$。根据命题 12.3,

$$-\log \mathrm{VP}_r(gx,gx') \leqslant -\log \mathrm{VP}_r(x,x') - \kappa_r' n \tag{13.15}$$

对于每一个 $g \in \mathcal{E}_r''$ 均成立。将 (13.14) 和 (13.15) 代入 (13.13) 的第二部分，并回顾 (12.37) 中 β_{r-1} 的选取，

$$\begin{aligned} -\log \psi_r(gx,gx';\vec{\omega}) &= -\log \psi_{r-1}(gx_-, gx_-'; \vec{\omega}_-)\beta_{r-1} - \log \mathrm{VP}_r(gx,gx') \\ &\leqslant -\log \psi_{r-1}(x_-, x_-'; \vec{\omega}_-)\beta_{r-1} + C_{r-1}''' n \beta_{r-1} \\ &\qquad - \log \mathrm{VP}_r(x,x') - \kappa_r' n \\ &\leqslant -\log \psi_r(x,x';\vec{\omega}) - (\kappa_r'/2)n = -\log \psi_r(x,x';\vec{\omega}) - \kappa_r'' n \end{aligned}$$

对每个 $g \in \mathcal{E}_r''$ 均成立。这就证明了断言 (13.11)。此外，在 $\mathcal{E}_r''$ 上对 (13.11) 积分，并在补集上对 (13.4) 积分，我们得到

$$\int_G -\log \psi_r(gx, gx'; \vec{\omega})\, d\nu_\infty^{(n)}(g) \leqslant -\log \psi_r(x,x';\vec{\omega}) - (\kappa_r'' - C_r''\delta)n,$$

正如（13.12）中所声称的那样。 □

13.2. 截断。还要回顾一下，常数 $\varepsilon_2 > 0$ 是在第11.3节中归纳法初始步骤的末尾选取的。还要注意，我们可以根据需要将它取得任意小。

For any $r > 1$, assume that $\varepsilon_r = \varepsilon_r(\nu_\infty, \delta, n) > 0$ has been chosen, satisfying

$$\eta_{\infty,r}(E_r(\varepsilon_r)\backslash E) < \frac{1}{10}\eta_{\infty,r}(E). \tag{13.16}$$

and

$$\varepsilon_r < \min\{\rho_0, \rho_r''\}, \tag{13.17}$$

where $\rho_0 = \rho_0(\nu_\infty, n) > 0$ is as in Corollaries 5.6 and 5.7 and $\rho_r'' = \rho_r''(\nu_\infty, \delta, n) > 0$ is as in Proposition 13.2.

Using Remark 7.3 twice, we find $\varepsilon_r' = \varepsilon_r'(\nu_\infty, \delta, n) > 0$ and $\tilde{\varepsilon}_r = \tilde{\varepsilon}_r(\nu_\infty, \delta, n) > 0$ with $0 < \varepsilon_r' < \tilde{\varepsilon}_r < \varepsilon_r$, and a compact neighborhood $\mathcal{W}_r = \mathcal{W}_r(\nu_\infty, \delta, n)$ of $\operatorname{supp}\nu_\infty^{(n)}$ such that

$$gF_r \in E_r(\varepsilon_r/2) \text{ for every } F_r \in E_r(2\tilde{\varepsilon}_r) \text{ and } g \in \mathcal{W}_r \text{ and} \tag{13.18}$$

$$g^{-1}F_r \in E_r(\tilde{\varepsilon}_r/2) \text{ for every } F_r \in E_r(2\varepsilon_r') \text{ and } g \in \mathcal{W}_r. \tag{13.19}$$

Let $\kappa_0 = \kappa_0(\nu_\infty) > 0$ be as in Proposition 5.5, and define $\varepsilon_r'' = \varepsilon_r''(\nu_\infty, \delta, n)$ by

$$\varepsilon_r'' = 3\varepsilon_r' e^{-\kappa_0 n/2}. \tag{13.20}$$

Taking $\rho = \varepsilon_r''$ in Corollary 5.7, and keeping in mind that $\varepsilon_r < \rho_0$, we get that there are $\tilde{k}_r = \tilde{k}_r(\nu_\infty, \delta, n) \in \mathbb{N}$ and $\mathcal{D}_k(F_r) = \mathcal{D}_k(\nu_\infty, \delta, n, F_r) \subset \operatorname{supp}\nu_k^{(n)}$ such that $\nu_k^{(n)}(\mathcal{D}_k(F_r)^c) < \delta$ and

$$d(gF_r, E) > e^{\kappa_0 n/2} d(F_r, E) > e^{\kappa_0 n/2}\varepsilon_r'' > 2\varepsilon_r'$$

for any $g \in \mathcal{D}_k(F_r)$, $F_r \in E(\varepsilon_r, \varepsilon_r'')$ and $k \geqslant \tilde{k}_r$. In other words, for $k \geqslant \tilde{k}_r$,

$$F_r \in E(\varepsilon_r, \varepsilon_r'') \Rightarrow gF_r \notin E(2\varepsilon_r') \text{ for every } g \in \mathcal{D}_k(F_r). \tag{13.21}$$

如有必要，增大 $\tilde{k}_r$，我们可以假设对于每个 $k \geqslant \tilde{k}_r$，都有 $\operatorname{supp}\nu_k^{(n)} \subset \mathcal{W}_r$。那么，由 (13.18) 和 (13.19) 可得

$$E_r(2\tilde{\varepsilon}_r) \subset \mathcal{X}_{\nu_k^{(n)}} E_r(\varepsilon_r) \text{ and } E_r(2\varepsilon_r') \subset \mathcal{X}^{\#}_{\nu_k^{(n)}} E_r(\varepsilon_r) \tag{13.22}$$

$$F_r \notin E_r(\tilde{\varepsilon}_r/2) \Rightarrow gF_r \notin E_r(2\varepsilon_r') \text{ for every } g \in \operatorname{supp}\nu_k^{(n)}. \tag{13.23}$$

如果 $F_1' \not\subset F_r$ 且 $F_1 \not\subset F_r'$，我们就称 x、$x' \in \mathcal{F}(r$、$d)$ 处于一般位置。对于任意处于一般位置的 x、$x' \in E_r^\diamond(\varepsilon_r)$ 以及 $\vec{\omega} \in \mathbb{R}_+^d$，定义

$$\Psi_r(x, x'; \vec{\omega}) = \begin{cases} \log\left(\Omega_r + \hat{\psi}_r(x, x'; \vec{\omega})^{-1}\right) & \text{if } x \in E_r^\diamond(2\varepsilon_r'') \text{ or } x' \in E_r^\diamond(2\varepsilon_r'') \\ \log\Omega_r & \text{otherwise,} \end{cases} \tag{13.24}$$

where $\Omega_r = \Omega_r(\nu_\infty, \delta, n) > 1$ is a large constant to be chosen in Proposition 15.1, and

$$\hat{\psi}(x, x'; \vec{\omega}) = \max\{\psi(x, x'; \vec{\omega}), \psi(x', x; \vec{\omega})\}. \tag{13.25}$$

It is clear from the definition that $\Psi_r(\cdot, \cdot; \vec{\omega})$ is a symmetric function:

$$\Psi_r(x, x'; \vec{\omega}) = \Psi_r(x', x; \vec{\omega}) \text{ for all } x, x' \in E_r^\diamond(\varepsilon_r).$$

集合 $E_r^\diamond(2\varepsilon_r'')^c \times E_r^\diamond(2\varepsilon_r'')^c$ 是阶段 $r > 1$ 的截断区域。参见图 7。

Proposition 13.3. *There exists $\kappa_r''' = \kappa_r'''(\nu_\infty) > 0$ such that for each $\delta > 0$ and $n \geqslant N_r$ there exists $\varepsilon_r''' = \varepsilon'''(\nu_\infty, \delta, n) > 0$ such that given any $\vec{\omega} = (\omega_1, \dots, \omega_r) \in \mathbb{R}_+^d$:*

(i) *For any $x, x' \in E_r^\diamond(\varepsilon_r)$ in general position with $\Psi_r(x, x'; \vec{\omega}) > \log \Omega_r$,*

$$\int_G \Psi_r(gx, gx'; \vec{\omega})\, d\nu_\infty^{(n)}(g) \leqslant \Psi_r(x, x'; \vec{\omega}) + C_r'' n.$$

(ii) *For any $x, x' \in E_r^\diamond(\varepsilon_r)$ in general position with $\Psi_r(x, x'; \vec{\omega}) > \log \Omega_r$, $\mathrm{VA}_r(x, x') \geqslant \omega_r$ and $\mathrm{VA}_r(x', x) \geqslant \omega_r$,*

$$\int_G \Psi_r(gx, gx'; \vec{\omega})\, d\nu_\infty^{(n)}(g) \leqslant \Psi_r(x, x'; \vec{\omega}) + C_r'' \delta n.$$

(iii) 对于任意处于一般位置的 $x, x' \in E_r^\diamond(\varepsilon_r''')$，满足 $\mathrm{VA}_r(x, x') \geqslant \omega_r$ 和 $\mathrm{VA}_r(x', x) \geqslant \omega_r$,

$$\int_G \Psi_r(gx, gx'; \vec{\omega})\, d\nu_\infty^{(n)}(g) \leqslant \Psi_r(x, x'; \vec{\omega}) - (\kappa_r''' - C_r''\delta) n.$$

Proof. Define $\kappa_r''' = \kappa_r''/2$. Let $n \geqslant N_r$. Part (i) of the proposition is a consequence of the following lemma:

Lemma 13.4. *If $x, x' \in \mathcal{F}(r, d)$ are in general position and $\Psi_r(x, x'; \vec{\omega}) > \log \Omega_r$ then*

$$\Psi_r(gx, gx'; \vec{\omega}) \leqslant \Psi_r(x, x'; \vec{\omega}) + C_r'' n \text{ for any } g \in \operatorname{supp} \nu_\infty^{(n)}. \tag{13.26}$$

证明。由 (13.4) 立即可得,

$$-\log \hat{\psi}_r(gx, gx'; \vec{\omega}) \leqslant -\log \hat{\psi}_r(x, x'; \vec{\omega}) + C_r'' n \tag{13.27}$$

对任意 $g \in \operatorname{supp} \nu_\infty^{(n)}$。然后，利用 (9.1),

$$\begin{aligned}\Psi_r(gx, gx'; \vec{\omega}) &\leqslant \log\left(\Omega_r + \hat{\psi}_r(gx, gx'; \vec{\omega})^{-1}\right) \leqslant \log\left(\Omega_r + e^{C_r'' n} \hat{\psi}_r(x, x'; \vec{\omega})^{-1}\right) \\ &\leqslant \log\left(\Omega_r + \hat{\psi}_r(x, x'; \vec{\omega})^{-1}\right) + C_r'' n = \Psi_r(x, x'; \vec{\omega}) + C_r'' n.\end{aligned}$$

This proves the claim. □

为了证明 (ii) 部分，我们使用命题 13.2：给定任意的 $\vec{\omega} \in \mathbb{R}_+^d$ 和 x, $x' \in E_r^\diamond(\varepsilon_r)$ 满足 $F_1' \neq F_r$,

$$-\log \psi_r(gx, gx'; \vec{\omega}) \leqslant -\log \psi_r(x, x'; \vec{\omega}) - \kappa_r'' n \leqslant -\log \psi_r(x, x'; \vec{\omega}) \tag{13.28}$$

for every $g \in \mathcal{E}_r''$. This remains true if we exchange the roles of x and x', of course. Thus, $-\log \hat{\psi}_r(gx, gx'; \vec{\omega}) \leqslant -\log \hat{\psi}_r(x, x'; \vec{\omega})$, and so

$$\begin{aligned}\Psi_r(gx, gx'; \vec{\omega}) &\leqslant \log\left(\Omega_r + \hat{\psi}_r(gx, gx'; \vec{\omega})^{-1}\right) \\ &\leqslant \log\left(\Omega_r + \hat{\psi}_r(x, x'; \vec{\omega})^{-1}\right) = \Psi_r(x, x'; \vec{\omega})\end{aligned} \tag{13.29}$$

for every $g \in \mathcal{E}_r''$. Integrating (13.29) over $\mathcal{E}_r''$ and (13.26) over the complement, we obtain the estimate in part (ii).

现在取关系（9.3）中的 $c = e^{-2\kappa_r''' n}$。根据引理 12.7，存在一个仅取决于 ν_∞、δ 和 n （通过 c 和 Ω_r）的 $\varepsilon_r''' > 0$，使得

$$-\log \hat{\psi}_r(x, x'; \vec{\omega}) = -\log \hat{\psi}_r(x, x') \geqslant \log\left(\Omega_r / \sqrt{c}\right) \tag{13.30}$$

for any $x, x' \in E_r(\varepsilon_r''')$ in general position. Then, using (13.30), (13.28) and (9.3),

$$\begin{aligned}\Psi_r(gx, gx'; \vec{\omega}) &\leqslant \log\left(\Omega_r + \hat{\psi}_r(gx, gx'; \vec{\omega})^{-1}\right)\\ &\leqslant \log\left(\Omega_r + e^{-2\kappa_r''' n}\hat{\psi}_r(x, x'; \vec{\omega})^{-1}\right)\\ &\leqslant \log\left(\Omega_r + \hat{\psi}_r(x, x'; \vec{\omega})^{-1}\right) - \kappa_r''' n = \Psi_r(x, x'; \vec{\omega}) - \kappa_r''' n\end{aligned} \tag{13.31}$$

对于每个 $g \in \mathcal{E}_r''$。将 (13.31) 在 $\mathcal{E}_r''$ 上积分，并将 (13.26) 在补集上积分，我们得到命题 (iii) 部分中的估计。□

14. 开启扰动

We show that the conclusions of Proposition 13.3 hold for $\nu_k^{(n)}$ instead of $\nu_\infty^{(n)}$, as long as $k \in \mathbb{N}$ is sufficiently large. The arguments are close to those in Section 10, but we have to deal with the fact that ψ_r and Ψ_r are not entirely straightforward generalizations of ψ_1 and Ψ_1.

Proposition 14.1. *Given $\delta > 0$, $n \geqslant N_r$, and $\vec{\omega} = (\omega_1, \dots, \omega_r) \in \mathbb{R}_+^d$ there is $k_r = k_r(\nu_\infty, \delta, n, \omega_r) \in \mathbb{N}$ such that the following holds for every $k \geqslant k_r$:*

(i) *For any $x, x' \in E_r^\diamond(\varepsilon_r)$ in general position with $\Psi_r(x, x'; \vec{\omega}) > \log \Omega_r$,*

$$\int_G \Psi_r(gx, gx'; \vec{\omega})\, d\nu_k^{(n)}(g) \leqslant \Psi_r(x, x'; \vec{\omega}) + C_r'' n.$$

(ii) *For any $x, x' \in E_r^\diamond(\varepsilon_r)$ in general position with $\Psi_r(x, x'; \vec{\omega}) > \log \Omega_r$, $\mathrm{VA}_r(x, x') \geqslant \omega_r$, and $\mathrm{VA}_r(x', x) \geqslant \omega_r$,*

$$\int_G \Psi_r(gx, gx'; \vec{\omega})\, d\nu_k^{(n)}(g) \leqslant \Psi_r(x, x'; \vec{\omega}) + C_r'' \delta n.$$

(iii) 对于任意满足 $\mathrm{VA}_r(x, x') \geqslant \omega_r$ 和 $\mathrm{VA}_r(x', x) \geqslant \omega_r$ 的一般位置 x, $x' \in E_r^\diamond(\varepsilon_r''')$,

$$\int_G \Psi_r(gx, gx'; \vec{\omega})\, d\nu_k^{(n)}(g) \leqslant \Psi_r(x, x'; n) - (\kappa_r'' - C_r''\delta) n.$$

Keep in mind that we have chosen $\varepsilon_r < \rho_r'' = \rho_r' \leqslant \hat{\rho}_r$ and $C_r' = B + \gamma_r A$. Recall also that $\tilde{k}_r = \tilde{k}_r(\nu_\infty, \delta, n) \in \mathbb{N}$ was chosen so that the relations (13.21) through (13.23) hold for every $k \geqslant \tilde{k}_r$.

Proof. We are going to extend to large $k \in \mathbb{N}$ several estimates in the proof of Proposition 13.3. This will require a number of conditions on k, depending on ν_∞, δ, n and ω, that we state along the way. We begin with the following extension of Lemma 13.4:

Lemma 14.2. *Given any $n \geqslant N_r$ and $\vec{\omega} = (\omega_1, \dots, \omega_r) \in \mathbb{R}_+^d$, there exists $\hat{k}_r = \hat{k}_r(\nu_\infty, n, \omega_r)$ such that if $x, x' \in E_r^\diamond(\varepsilon_r)$ are in general position and $\vec{\omega} \in \mathbb{R}_+^d$ is such that $\Psi_r(x, x'; \vec{\omega}) > \log \Omega_r$ then*

$$\Psi_r(gx, gx'; \vec{\omega}) \leqslant \Psi_r(x, x'; \vec{\omega}) + C_r'' n \tag{14.1}$$

对于任意 $g \in \operatorname{supp} \nu_k^{(n)}$ 和 $k \geqslant \hat{k}_r$。

证明。考虑 $\delta > 0$、 $n \geqslant N_r$ 和 $\vec{\omega} = (\omega_1, \ldots\ldots, \omega_r) \in \mathbb{R}^d_+$。我们首先声称，存在 $\hat{} k_r = \hat{} k_r(\nu_\infty, n, \vec{\omega}) \in \mathbb{N}$，使得

$$-\log \mathrm{SVA}_r(gx, gx'; \omega_r) \leqslant -\log \mathrm{SVA}_r(x, x'; \omega_r) + Bn \tag{14.2}$$

for any $x, x' \in E_r^\diamond(\varepsilon_r)$ in general position, $g \in \operatorname{supp} \nu_k^{(n)}$, and $k \geqslant \hat{k}_r$. This can be seen as follows. If $\mathrm{VA}_r(x, x') < \omega_r$ then, by the definition (13.1),

$$\begin{aligned} -\log \mathrm{SVA}_r(x, x'; \omega_r) &\geqslant -\log \omega_r \text{ and} \\ -\log \mathrm{SVA}_r(gx, gx'; \omega_r) &\leqslant -\log \omega_r + Bn \leqslant -\log \mathrm{SVA}_r(x, x'; \omega_r) + Bn, \end{aligned}$$

如前所述。现在假设 $\mathrm{VA}_r(x, x') \geqslant \omega_r$。关系式 (12.7) 在这里不适用。相反，由 (12.5) 和 (13.1) 可得

$$-\log \mathrm{SVA}_r(gx, gx'; \omega_r) \leqslant -\log \mathrm{SVA}_r(x, x'; \omega_r) - \log \frac{\|(gu)^\perp\|}{\|gu\|} \frac{\|u\|}{\|u^\perp\|} \tag{14.3}$$

for every $g \in \operatorname{supp} \nu_k^{(n)}$, with $u = u(x, x')$ realizing the supremum in (12.2). The assumption $\mathrm{VA}_r(x, x') \geqslant \omega_r$ means that $\|u^\perp\| \geqslant \omega_r \|u\|$. Since $\operatorname{supp} \nu_k$ converges to $\operatorname{supp} \nu_\infty$ in the Hausdorff topology, we may find $r_k = r_k(\nu_\infty, n) \to 0$ such that every $g \in \operatorname{supp} \nu_k^{(n)}$ is in the r_k-neighborhood of some $f \in \operatorname{supp} \nu_\infty^{(n)}$. Then

$$\|(gu)^\perp - (fu)^\perp\| \leqslant \|gu - fu\| \leqslant r_k \|u\| \leqslant \frac{r_k}{\omega_r} \|u^\perp\|.$$

Then we may take $\hat{k}_r = \hat{k}_r(\nu_\infty, n, \omega_r) \in \mathbb{N}$ large enough so that this inequality implies

$$\begin{aligned} -\log \frac{\|(gu)^\perp\|}{\|gu\|} \frac{\|u\|}{\|u^\perp\|} &\leqslant -\log \frac{\|(fu)^\perp\|}{\|fu\|} \frac{\|u\|}{\|u^\perp\|} + \log 2 \\ &= -\log \frac{\|f^\perp u^\perp\|}{\|fu\|} \frac{\|u\|}{\|u^\perp\|} + \log 2 \\ &\leqslant \log \|(f^\perp)^{-1}\| + \log \|f\| + \log 2 \leqslant Bn \end{aligned}$$

for every $g \in \operatorname{supp} \nu_k^{(n)}$ and $k \geqslant \hat{k}_r$. This proves (14.2).

Substituting (14.2) and (8.2) in the definition (13.2), and recalling the definition of C'_r in (12.32), we find that

$$\begin{aligned} -\log \mathrm{SVP}_r(gx, gx'; \omega_r) &= -\log \mathrm{SVA}_r(gx, gx'; \omega_r) \\ &\qquad - \gamma_r \log d(gF_r, g(F_{r-1} + F'_1) \\ &\leqslant -\log \mathrm{SVA}_1(x, x'; \omega_r) + Bn \\ &\qquad - \gamma_r \log d(F_r, F_{r-1} + F'_1) + \gamma_r An \\ &\leqslant -\log \mathrm{SVP}_r(x, x'; \omega_r) + C'_r n \end{aligned} \tag{14.4}$$

for any $x, x' \in E_r^\diamond(\varepsilon_r)$ in general position, $g \in \operatorname{supp} \nu_k^{(n)}$, and $k \geqslant \hat{k}_r$.

Next we claim that

$$-\log \psi_r(gx, gx'; \vec{\omega}) \leqslant -\log \psi_r(x, x'; \vec{\omega}) + C''_r n \tag{14.5}$$

for any $x, x' \in E_r^\diamond(\varepsilon_r)$ in general position, $g \in \operatorname{supp} \nu_k^{(n)}$, and $k \geqslant \hat{k}_r$. The case $r = 1$ of (14.5) was done in (10.4), so let us consider $r > 1$. By induction,

$$-\log \psi_{r-1}(gx_-, gx'_-; \vec{\omega}_-) \leqslant -\log \psi_{r-1}(x_-, x'_-; \vec{\omega}_-) + C''_{r-1} n. \tag{14.6}$$

将其与（14.4）代入定义（13.3）中，我们得到

$$\begin{aligned}-\log\psi_r(x,x';\vec{\omega}) &= -\beta_{r-1}\log\psi_{r-1}(gx_-,gx'_-;\vec{\omega}_-)\\ &\qquad -\log \mathrm{SVP}_r(gx,gx';\omega_r)\\ &\leqslant -\beta_{r-1}\log\psi_{r-1}(x_-,x'_-;\vec{\omega}_-)+\beta_{r-1}C''_{r-1}n\\ &\qquad -\log \mathrm{SVP}_r(x,x';\omega_r)+C'_r n\\ &= \psi_r(x,x';\vec{\omega})+(\beta_{r-1}C''_{r-1}+C'_r)n.\end{aligned}$$

由于 $C''_r = C''_{r-1}+C'_r$ 和 $\beta_{r-1}<1$，这就证明了 (14.5)。显然，即使交换 x 和 x' 的角色，该估计仍然成立。因此，我们还证明了：

$$-\log\hat{\psi}_r(gx,gx';\omega) \leqslant -\log\hat{\psi}_r(x,x';\vec{\omega})+C''_r n \tag{14.7}$$

for any $x,x'\in E_r^{\diamond}(\varepsilon_r)$ in general position, $g\in\operatorname{supp}\nu_k^{(n)}$, and $k\geqslant\hat{k}_r$.

Combining (14.7) with (9.1) in the definition (13.24), we get that

$$\begin{aligned}\Psi_r(gx,gx';\vec{\omega}) &\leqslant \log\left(\Omega_r+\hat{\psi}_r(gx,gx';\vec{\omega})^{-1}\right)\leqslant \log\left(\Omega_r+e^{C''_r n}\hat{\psi}_r(x,x';\vec{\omega})^{-1}\right)\\ &\leqslant \log\left(\Omega_r+\hat{\psi}_r(x,x';\vec{\omega})^{-1}\right)+C''_r n=\Psi_r(x,x';\vec{\omega})+C''_r n,\end{aligned}$$

as stated. □

接下来，我们证明引理 12.2 的以下推广：

Lemma 14.3. *Given $\delta>0$, $n\geqslant N_r$ and $\omega_r>0$, there is $k'_r=k'_r(\nu_\infty,\delta,n,\omega_r)\in\mathbb{N}$ and for any $x,x'\in\mathcal{F}(r,d)$ with $F'_1\not\subset F_r$ and $\mathrm{VA}_r(x,x')\geqslant\omega_r$ there is $\mathcal{E}''_{k,r}=\mathcal{E}''_{k,r}(\nu_\infty,\delta,n,x,x',\omega_r)\subset\operatorname{supp}\nu_k^{(n)}$ with $\nu_k^{(n)}((\mathcal{E}''_{k,r})^c)<\delta$ and*

$$-\log \mathrm{SVA}_r(gx,gx';\omega_r)\leqslant\max\{-\log \mathrm{SVA}_r(x,x';\omega_r)-\tilde{\kappa}_r n,\tilde{\theta}_r\} \tag{14.8}$$

对于每个 $g\in\mathcal{E}''_{k,r}$ 和 $k\geqslant k'_r$。

Proof. Fix $\delta>0$ and $n\geqslant N_r$ and $\omega_r>0$. Let P_{ω_r} denote the (compact) subset of all $v\in P$ such that $\|v^\perp\|/\|v\|\geqslant\omega_r/2$. For $v\in P_{\omega_r}$ and g in some compact neighborhood V_{ω_r} of $\operatorname{supp}\nu_\infty^{(n)}$, consider

$$(v,g)\mapsto -\log\frac{\|(gv)^\perp\|}{\|gv\|}. \tag{14.9}$$

只要 V_{ω_r} 足够小（其大小取决于 ν_∞、n 和 ω_r），映射 (14.9) 就是良定义且（一致）连续的。因此，存在 $\alpha=\alpha(\nu_\infty$、n 和 $\omega_r)>0$，使得

$$-\log\frac{\|(gu)^\perp\|}{\|gu\|}\leqslant -\log\frac{\|(fv)^\perp\|}{\|fv\|}+\log 2 \tag{14.10}$$

whenever $d(u,v)<\alpha$ and $d(g,f)<\alpha$. Reducing α if necessary, depending only on ω_r, we may also assume that

$$d(u,v)<\alpha \quad\Rightarrow\quad -\log\frac{\|u^\perp\|}{\|u\|}\geqslant -\log\frac{\|v^\perp\|}{\|v\|}-\log 2. \tag{14.11}$$

Fix $v_1,\dots,v_l\in P_{\omega_r}$ such that $P_{\omega_r}\subset B(v_1,\alpha)\cup\cdots\cup B(v_l,\alpha)$. For each $v\in P_{\omega_r}$ choose $j\in\{1,\dots,l\}$ such that $v\in B(v_j,\alpha)$ and define $\mathcal{E}_{k,0}=\mathcal{E}_{k,0}(\nu_\infty,\delta,n,v,\omega_r)\subset$

$\operatorname{supp}\nu_k^{(n)}$ by

$$\mathcal{E}_{k,0} = \big[\alpha\text{-neighborhood of } \mathcal{E}_0(\nu_\infty, \delta, n, v_j^\perp)\big] \cap \operatorname{supp}\nu_k^{(n)}, \tag{14.12}$$

where $\mathcal{E}_0(\nu_\infty, \delta, n, v_j^\perp) \subset \operatorname{supp}\nu_\infty^{(n)}$ is as defined in Proposition 5.5. Since $\nu_k^{(n)}$ converges to $\nu_\infty^{(n)}$ in the weak* topology, the limit inferior of the $\nu_k^{(n)}$-measure of (14.12) as $k \to \infty$ is greater than or equal to

$$\nu_\infty^{(n)}\big(\mathcal{E}_0(\nu_\infty, \delta, n, v_j^\perp)\big) > 1 - \delta$$

for every $j = 1, \dots, l$. In particular, there is $k_r' = k_r'(\nu_\infty, \delta, n, \omega_r) \in \mathbb{N}$ such that

$$\nu_k^{(n)}(\mathcal{E}_{k,0}) > 1 - \delta \text{ for every } k \geqslant k_r' \text{ and } v \in P_{\omega_r}. \tag{14.13}$$

给定 $x, x' \in \mathcal{F}(r, d)$，其中 $F_1' \notin F_r$ 且 $\mathbf{VA}_r(x, x') \geqslant \omega_r$，取 $u = u^E + u^\perp$ 为一个非零向量，使其在定义 (12.2) 中实现上确界。于是，

$$\omega_r \leqslant \mathrm{VA}_r(x, x') = \frac{\|u^\perp\|}{\|u\|} \quad (\text{in particular, } u \in P_{\omega_r}) \text{ and} \tag{14.14}$$

$$\mathrm{VA}_r(gx, gx') \geqslant \frac{\|(gu)^\perp\|}{\|gu\|} \geqslant \mathrm{VA}_r(x, x') \frac{\|(gu)^\perp\|}{\|gu\|} \frac{\|u\|}{\|u^\perp\|} \tag{14.15}$$

对任意 $g \in G$。然后定义

$$\mathcal{E}_{k,r}'' = \mathcal{E}_{k,0}(\nu_\infty, \delta, n, u, \omega_r). \tag{14.16}$$

It follows from (14.13) that $\nu_k^{(n)}((\mathcal{E}_{k,r}'')^c) < \delta$ for every $k \geqslant k_r'$.

Let $g \in \mathcal{E}_{k,r}''$ and $k \geqslant k_r'$. Then, by definition, there exist $v = v^E + v^\perp$ in P_{ω_r} (take $v = v_j$ as in (14.12)) and $f \in \mathcal{E}_0(\nu_\infty, \delta, n, v^\perp) \subset \operatorname{supp}\nu_\infty^{(n)}$ such that $d(u, v) < \alpha$ and $d(g, f) < \alpha$. Thus, substituting (14.10) and (14.11) in (14.15), we find that

$$\mathrm{VA}_r(gx, gx') \geqslant \frac{1}{2}\frac{\|f^\perp v^\perp\|}{\|fv\|} \geqslant \frac{1}{4}\,\mathrm{VA}_r(x, x')\frac{\|f^\perp v^\perp\|}{\|fv\|}\frac{\|v\|}{\|v^\perp\|}. \tag{14.17}$$

Let $\tau_0 = \tau_0(\nu_\infty, \delta) > 0$ be as in Proposition 5.5. If $\|f^\perp v^\perp\|/\|fv\| \geqslant \tau_0/2$ then the first part of (14.17) gives that (recall (12.9) also)

$$-\log \mathrm{VA}_r(gx, gx') \leqslant -\log\frac{\tau_0}{4} \leqslant \tilde{\theta}_r. \tag{14.18}$$

Now suppose that $\|f^\perp v^\perp\|/\|fv\| < \tau_0/2$. Then part (2) of Proposition 5.5 gives that

$$\frac{\|fv^\perp\|}{\|fv\|} < \frac{1}{2} \text{ and so } \frac{\|fv^E\|}{\|fv\|} > \frac{1}{2}. \tag{14.19}$$

Substituting (14.19) and $\|v\| \geqslant \|v^E\|$ in (14.17), we find that

$$\begin{aligned} \mathrm{VA}_r(gx, gx') &\geqslant \frac{1}{8}\,\mathrm{VA}_r(x, x')\frac{\|f^\perp v^\perp\|}{\|v^\perp\|}\frac{\|v^E\|}{\|fv^E\|} \\ &= \frac{1}{8}\,\mathrm{VA}_r(x, x')\frac{\|Df_{v^E}^\perp v^\perp\|}{\|v^\perp\|}. \end{aligned} \tag{14.20}$$

By part (1) of Proposition 5.5 and the choices of $\tilde{\kappa}_r > 0$ and $\widetilde{N}_r \in \mathbb{N}$ in (12.9), this implies

$$\begin{aligned} -\log \mathrm{VA}_r(gx, gx') &\leqslant -\log \mathrm{VA}_r(x, x') + \log 8 - 2\tilde{\kappa}_r n \\ &\leqslant -\log \mathrm{VA}_r(x, x') - \tilde{\kappa}_r n. \end{aligned} \tag{14.21}$$

引理的结论包含在（14.18）和（14.21）中。 □

接下来，我们来证明引理 12.5 的以下推广：

Lemma 14.4. *Given $\delta > 0$ and $n \geqslant N_r$ there exists $\hat{k}_r = \hat{k}_r(\nu_\infty, \delta, n) \in \mathbb{N}$ such that for any $x, x' \in E_r^\diamond(\varepsilon_r)$ with $F_1' \notin F_r$ and $-\log \mathrm{VA}_r(x, x') \leqslant \tilde{\theta}_r + \tilde{\kappa}_r n$ there exists $\mathcal{E}''_{k,r} = \mathcal{E}''_{k,r}(\nu_\infty, \delta, n, x, x') \subset \operatorname{supp} \nu_k^{(n)}$ with $\nu_k^{(n)}((\mathcal{E}''_{k,r})^c) < \delta$ and*

$$-\log d(gF_r, g(F_{r-1} + F_1')) \leqslant -\log d(F_r, F_{r-1} + F_1') - \log \mathrm{VA}_r(x, x') - \hat{\kappa}_r n \tag{14.22}$$

对于每个 $g \in \mathcal{E}''_{k,r}$ 以及 $k \geqslant \hat{k}_r$。

Proof. Let $\hat{P}$ denote the (compact) subset of pairs $(v, w) \in P \times P$ such that

$$\frac{\|v^\perp\|}{\|v\|} \leqslant \hat{\rho}_1 < 2\hat{\rho}_1 \leqslant \frac{\|w^\perp\|}{\|w\|} \tag{14.23}$$

($v^\perp$ and $w^\perp$ denote the components of v and w orthogonal to the equator). Let $\hat{V}$ be some compact neighborhood of the support of $\nu_\infty^{(n)}$. Since, $\hat{\rho}_r = \hat{\rho}_r(\nu_\infty, n)$, both $\hat{P}$ and $\hat{V}$ depend only on ν_∞ and n. Condition (14.23) ensures that the angle between v and w is bounded away from zero and, consequently, so is the angle between gv and gw for any $g \in \hat{V}$; both bounds depend only on ν_∞ and n. Thus, the map

$$(v, w, g) \mapsto -\log \frac{\|\Pi_{gv} gw\|}{\|gv\|} \frac{\|v\|}{\|w\|} \tag{14.24}$$

在定义域 $(v, w) \in \hat{}P$ 和 $g \in \hat{}V$ 上是良定义的且（一致）连续的。特别地，存在 $\hat{}\alpha = \hat{}\alpha(\nu_\infty, n) > 0$，使得

$$-\log \frac{\|\Pi_{gv} gw\|}{\|gv\|} \frac{\|v\|}{\|w\|} \leqslant -\log \frac{\|\Pi_{fu} fz\|}{\|fu\|} \frac{\|u\|}{\|z\|} + \log 2 \tag{14.25}$$

每当 $d(v, u) < \hat{}\alpha$ 且 $d(z, w) < \hat{}\alpha$ 且 $d(g, f) < \hat{}\alpha$ 时。必要时缩小 $\hat{}\alpha$，其取值仅取决于 ν_∞ 和 n，我们还可以假设 that

$$d(z, w) < \hat{\alpha} \quad \Rightarrow \quad -\log \frac{\|z^\perp\|}{\|z\|} \leqslant -\log \frac{\|w^\perp\|}{\|w\|} + \log 2. \tag{14.26}$$

Fix points $(v_1, w_1), \ldots, (v_l, w_l) \in \hat{P}$ such that the balls of radius $\hat{\rho}$ around these points cover $\hat{P}$. For each $(v, w) \in \hat{P}$ choose $j \in \{1, \ldots, l\}$ such that $v \in B(v_j, \hat{\alpha})$ and $w \in B(w_j, \hat{\alpha})$ and then define $\hat{\mathcal{E}}_{k,0} = \hat{\mathcal{E}}_{k,0}(\nu_\infty, \delta, n, v, w) \subset G$ by

$$\hat{\mathcal{E}}_{k,0} = \left[\hat{\alpha}\text{-neighborhood of } \mathcal{E}_0(\nu_\infty, \delta, n, w_j^\perp)\right] \cap \operatorname{supp} \nu_k^{(n)}, \tag{14.27}$$

where $\mathcal{E}_0(\nu_\infty, \delta, n, w_j^\perp)$ is given by Proposition 5.5. Since $\nu_k^{(n)} \to \nu_\infty^{(n)}$ in the weak* topology, the limit inferior of the $\nu_k^{(n)}$-measure of (14.27) as $k \to \infty$ is greater than or equal to

$$\nu_\infty^{(n)}\left(\mathcal{E}_0(\nu_\infty, \delta, n, w_j^\perp)\right) > 1 - \delta$$

for every $j = 1, \ldots, l$. In particular, there is $\hat{k}_r = \hat{k}_r(\nu_\infty, \delta, n) \in \mathbb{N}$ such that

$$\nu_k^{(n)}(\hat{\mathcal{E}}_{k,0}) > 1 - \delta \text{ for every } k \geqslant \hat{k}_r \text{ and } (v, w) \in \hat{P}. \tag{14.28}$$

Given $x, x' \in E_r^\diamond(\varepsilon_r)$ with $F_1' \not\subset F_r$, let $v \in F_r$, $v' \in F_{r-1} + F_1'$ and $w = v - v'$ be as in (12.15) and (12.16). Just as in (12.21),

$$-\log d(gF_r, g(F_{r-1}+F_1')) \leqslant -\log d(F_r, F_{r-1}+F_1') - \log \frac{\|\Pi_{gv} gw\|}{\|gv\|}\frac{\|v\|}{\|w\|} \tag{14.29}$$

对于任意满足 $g \in G$ 且 $\|gv\| \geqslant \|gv'\|$ 的情形（$\|gv\| \leqslant \|gv'\|$ 的情况类似，只需交换 F_r 和 $F_{r-1}+F_1'$的角色）。假设 $x \in E_r(\varepsilon_r)$ 意味着，

$$\frac{\|v^\perp\|}{\|v\|} \leqslant d(x, E) \leqslant \varepsilon_r < \hat{\rho}_r. \tag{14.30}$$

Now assume that $-\log \mathrm{VA}_r(x, x') \leqslant \tilde{\theta}_r + \tilde{\kappa}_r n$. Then, using (12.19) and (12.22),

$$\frac{\|w^\perp\|}{\|w\|} > \frac{1}{4}\,\mathrm{VA}_r(x, x') \geqslant \frac{1}{4} e^{-\tilde{\theta}_r - \tilde{\kappa}_r n} > 2\hat{\rho}_r. \tag{14.31}$$

因此，$(v, w) \in \hat{} P$。然后定义

$$\mathcal{E}''_{k,r} = \hat{\mathcal{E}}_{k,0}(\nu_\infty, \delta, n, v, w). \tag{14.32}$$

It follows from (14.28) that $\nu_k^{(n)}((\mathcal{E}''_{k,r})^c) < \delta$ for every $k \geqslant \hat{k}_r$.

Take $u = v_j$ and $z = w_j$ as in (14.27). By definition, $(u, z) \in \hat{P}$ and $d(u, v) < \hat{\alpha}$ and $d(z, w) < \hat{\alpha}$. Let $g \in \mathcal{E}''_{k,r}$ and $k \geqslant \hat{k}_r$. The definitions (14.27) and (14.32) imply that there exists $f \in \mathcal{E}_0(\nu_\infty, \delta, n, z^\perp) \subset \operatorname{supp} \nu_\infty^{(n)}$ such that $d(g, f) < \theta$. Combining (14.29) with (14.25),

$$-\log d(gF_r, g(F_{r-1}+F_1')) \leqslant -\log d(F_r, F_{r-1}+F_1') + \log 2 - \log \frac{\|\Pi_{fu} fz\|}{\|fu\|}\frac{\|u\|}{\|z\|}.$$

Replacing z, w and g with u, z and f in (12.23) and (12.24), we get that

$$\frac{\|fu\|}{\|u\|} \geqslant \frac{1}{2}\frac{\|fu^E\|}{\|u^E\|} \text{ and} \|\Pi_{fu} fz\| \geqslant \frac{1}{2}\|f^\perp z^\perp\|.$$

将此代入前一个不等式中，

$$-\log d(gF_r, g(F_{r-1}+F_1')) \leqslant -\log d(F_r, F_{r-1}+F_1') + \log 8 - \log \frac{\|f^\perp z^\perp\|}{\|z\|}\frac{\|u^E\|}{\|fu^E\|}.$$

条件（14.26）和（14.31）表明，

$$-\log \frac{\|z^\perp\|}{\|z\|} \leqslant -\log \frac{\|w^\perp\|}{\|w\|} + \log 2 \leqslant -\log \mathrm{VA}_r(x, x') + \log 8.$$

结合前一个不等式，我们得到

$$\begin{aligned} -\log d(gF_r, g(F_{r-1}+F_1')) &\leqslant -\log d(F_r, F_{r-1}+F_1') - \log \mathrm{VA}_r(x, x') \\ &\quad + \log 64 - \log \frac{\|f^\perp z^\perp\|}{\|z^\perp\|}\frac{\|u^E\|}{\|fu^E\|} \\ &= -\log d(F_r, F_{r-1}+F_1') - \log \mathrm{VA}_r(x, x') \\ &\quad + \log 64 - \log \frac{\|Df^\perp_{u^E} z^\perp\|}{\|z^\perp\|}. \end{aligned}$$

By part (1) of Proposition 5.5, and the choice of $\widehat{N}_r \in \mathbb{N}$ in (12.19), this implies that

$$\begin{aligned} -\log d(gF_r, g(F_{r-1}+F_1')) &\leqslant -\log d(F_r, F_{r-1}+F_1') - \log \mathrm{VA}_r(x,x') + 5 - 2\hat{\kappa}_r n \\ &\leqslant -\log d(F_r, F_{r-1}+F_1') - \log \mathrm{VA}_r(x,x') - \hat{\kappa}_r n, \end{aligned}$$

as claimed. □

我们推导出命题 12.3 的以下推广：

Lemma 14.5. *For $\delta > 0$, $n \geqslant N_r$ and $\vec{\omega} = (\omega_1, \dots, \omega_r) \in \mathbb{R}_+^d$ there is $k_r''' = k_r'''(\nu_\infty, \delta, n, \omega_r) \in \mathbb{N}$ and for $x, x' \in E_r^\diamond(\varepsilon_r)$ with $F_1' \not\subset F_r$ and $\mathrm{VA}_r(x,x') \geqslant \omega_r$ there is $\mathcal{E}_{k,r}''' = \mathcal{E}_{k,r}'''(\nu_\infty, \delta, n, x, x', \vec{\omega}) \subset \operatorname{supp} \nu_k^{(n)}$ such that $\nu_k^{(n)}((\mathcal{E}_{k,r}''')^c) < \delta$ and*

$$-\log \psi_r(gx, gx'; \vec{\omega}) \leqslant -\log \psi_r(x, x'; \vec{\omega}) - \kappa_r'' n. \tag{14.33}$$

对于任意 $g \in \mathcal{E}_{k,r}'''$ 和 $k \geqslant k_r'''$。

证明。固定 $\delta > 0$、$n \geqslant N_r$ 和 $\vec{\omega} \in \mathbb{R}_+^d$。回想一下。

$$N_r > \max\left\{\widetilde{N}_r(\nu_\infty, \delta/2), \widehat{N}_r(\nu_\infty, \delta/2), 2\tilde{\theta}_r/(\gamma\hat{\kappa}_r)\right\},$$

由（12.25）定义。

$$\begin{aligned} k_r''' &= \max\{k_r'(\nu_\infty, \delta/2, n, \omega), \hat{k}_r(\nu_\infty, \delta/2, n)\} \\ &\text{and } \mathcal{E}_{k,r}''' = \mathcal{E}_{k,r}''(\nu_\infty, \delta/2, n, x, x', \omega) \cap \mathcal{E}_{k,r}''(\nu_\infty, \delta/2, x, x'n). \end{aligned} \tag{14.34}$$

By construction, $\mathcal{E}_{k,r}'''$ is contained in the support of $\nu_k^{(n)}$ and $\nu_k^{(n)}((\mathcal{E}_{k,r}''')^c) < \delta$.

We claim that given any $x, x' \in E_r^\diamond(\varepsilon_r)$ with $F_1' \not\subset F_r$ and $\mathrm{VA}_r(x,x') \geqslant \omega_r$,

$$-\log \mathrm{SVP}_r(gx, gx'; \omega_r) \leqslant -\log \mathrm{SVP}_r(x, x'; \omega_r) - \kappa_r' n \tag{14.35}$$

for every $g \in \mathcal{E}_{k,r}'''$ and $k \geqslant k_r'''$. As observed in (13.10), the assumption on (x, x') implies that $\mathrm{SVA}_r(x, x'; \omega_r) = \mathrm{VA}_r(x, x')$ and $\mathrm{SVA}_r(x, x'; \omega_r) = \mathrm{VA}_r(gx, gx')$, and so (14.35) may be rewritten as

$$-\log \mathrm{VP}_r(gx, gx') \leqslant -\log \mathrm{VP}_r(x, x') - \kappa_r' n. \tag{14.36}$$

˜令 $g \in \mathcal{E}_{k,r}'''$ 且 $k \geqslant k_r'''$。首先假设 $-\log \mathrm{VA}_r(x,x') \geqslant \theta_r + \tilde{\kappa}_r n$。那么，根据引理 14.3,

$$-\log \mathrm{VA}_r(gx, gx') \leqslant -\log \mathrm{VA}_r(x, x') - \tilde{\kappa}_r n. \tag{14.37}$$

Substituting (14.37) and (8.2) in the definition (12.13) we find that

$$\begin{aligned} -\log \mathrm{VP}_r(gx, gx') &\leqslant -\log \mathrm{VA}_r(x, x') - \tilde{\kappa}_r n \\ &\qquad - \gamma_r \log d(F_r, F_{r-1}+F_1') + \gamma_r A n \\ &\leqslant -\log \mathrm{VP}_r(x, x') - \frac{\tilde{\kappa}_r}{2} n \end{aligned} \tag{14.38}$$

˜（我们在 (12.25) 中选择了 $\gamma_r \leqslant \tilde{\kappa}_r/(2A)$）。现在假设 $-\log \mathrm{VA}_r(x, x') \leqslant \theta_r + \tilde{\kappa}_r n$。在这种情况下，引理 14.3 得出

$$-\log \mathrm{VA}_r(gx, gx') \leqslant \tilde{\theta}_r \tag{14.39}$$

而引理 14.4 得出的是

$$-\log d(gx, gx') \leqslant -\log d(x, x') - \log \mathrm{VA}_r(x, x') - \hat{\kappa}_r n. \tag{14.40}$$

Substituting (14.39) and (14.40) in the definition (12.13), we obtain

$$\begin{aligned}-\log \mathrm{VP}_r(gx,gx') &\leqslant -\gamma_r \log d(F_r, F_{r-1}+F_1') - \gamma_r \log \mathrm{VA}_r(x,x') - \gamma_r\hat\kappa_r n\\ &\leqslant -\log \mathrm{VP}_r(x,x') + \tilde\theta_r + (1-\gamma_r)\log \mathrm{VA}_r(x,x') - \gamma_r\hat\kappa_r n.\end{aligned}$$

由于 $\mathrm{VA}_r(x、\ x') \leqslant 1$、$\gamma_r \leqslant 1$ 和 $n \geqslant N_r \geqslant 2\theta_r/(\gamma_r\hat\kappa_r)$，因此可得

$$\begin{aligned}-\log \mathrm{VP}_r(gx,gx') &\leqslant -\log \mathrm{VP}_r(x,x') + \tilde\theta_r - \gamma_r\hat\kappa_r n\\ &\leqslant -\log \mathrm{VP}_r(x,x') - \frac{\gamma_r\hat\kappa_r}{2} n.\end{aligned} \tag{14.41}$$

Because of the way we chose κ_r' in (12.25), the relations (14.38) and (14.41) contain the claim (14.35).

Now we prove the claim (14.33). The case $r = 1$ was done in (10.31), so let us suppose $r > 1$. By (14.6),

$$-\log \psi_{r-1}(gx_-, gx_-'; \omega) \leqslant -\log \psi_{r-1}(x_-, x_-'; \vec\omega) + C_{r-1}'' n.$$

将此式和 (14.35) 代入定义 (12.34)，

$$\begin{aligned}-\log \psi_r(x,x';\vec\omega) &= -\beta_{r-1}\log \psi_{r-1}(gx_-,gx_-';\omega) - \log \mathrm{SVP}_r(gx,gx';\omega_r)\\ &\leqslant -\beta_{r-1}\log \psi_{r-1}(x_-,x_-';\vec\omega) + \beta_{r-1}C_{r-1}'' n\\ &\qquad - \log \mathrm{SVP}_r(x,x';\omega_r) - \kappa_r' n\\ &\leqslant -\log \psi_r(x,x';\vec\omega) - \kappa_r'',\end{aligned}$$

where the last step uses our choices of β_{r-1}, C_r'', and κ_r'' in (12.37) and (12.44). □

让我们回到命题 14.1 的证明。定义

$$\begin{aligned}k_r = \max\{&\tilde k_r(\nu_\infty,\delta,n), \hat k_r(\nu_\infty,n,\omega_r), k_r'(\nu_\infty,\delta,n\,\omega_r),\\ &\hat k_r(\nu_\infty,\delta,n), k_r'''(\nu_\infty,\delta/2,n,\omega_r)\}.\end{aligned} \tag{14.42}$$

然后，k_r 仅取决于 ν_∞、δ、n 和 ω_r。命题的(i)部分是引理 14.2 的直接推论。为了证明(ii)部分，考虑

$$\mathcal{E} = \mathcal{E}_{k,r}'''(\nu_\infty,\delta/2,n,x,x',\vec\omega) \cap \mathcal{E}_{k,r}'''(\nu_\infty,\delta/2,n,x',x,\vec\omega)$$

where $\mathcal{E}_{k,r}'''$ is as given by Lemma 14.5. Then $\nu_k^{(n)}(\mathcal{E}^c) < \delta$ and

$$-\log\hat\psi_r(gx,gx';\vec\omega) \leqslant -\log\hat\psi_r(x,x';\vec\omega) - \kappa_r' n \leqslant -\log\hat\psi_r(x,x';\vec\omega) \tag{14.43}$$

对于每一个 $g \in \mathcal{E}$ 和 $k \geqslant k_r'''$，根据 (9.2)，这意味着

$$\Psi_r(gx,gx';\omega) \leqslant \Psi_r(x,x';\omega) \text{ for every } g \in \mathcal{E}. \tag{14.44}$$

Integrating (14.44) over $\mathcal{E}$ and (14.1) over the complement, we obtain part (ii).

接下来，回顾一下，我们取 $\kappa_r''' = \kappa_r''/2 = \kappa_r'/2$、$c = e^{-2\kappa_r''' n}$ 和 $\varepsilon_r''' > 0$，使得 (13.30) 成立：

$$-\log\hat\psi_r(x,x';\vec\omega) = -\log\hat\psi(x,x') \geqslant \Omega_r/\sqrt{c}$$

for any $x, x' \in E_r^\diamond(\varepsilon_r''')$ in general position. Then, by (9.3) and the first inequality in (14.43),

$$\Psi_r(gx,gx';\vec\omega) \leqslant \Psi_r(x,x';\vec\omega) + \log\sqrt{c} = \Psi_r(x,x';\vec\omega) - \kappa_r''' n \tag{14.45}$$

for every $g \in \mathcal{E}$. Integrating (14.45) over $\mathcal{E}$ and (14.1) over the complement, we obtain part (iii) of the proposition. □

It is clear from the statements of Lemmas 14.2, 14.3 and 14.5 that one may take $\hat{k}_r$, k'_r and k'''_r to increase to ∞ when ω_r decreases to zero and ν_∞, δ, n remain fixed. Then the same is true about the map $\omega \mapsto k_r(\nu_\infty, \delta, n, \omega_r)$ defined in (14.42). Hence, we may find $\check{k}_r = \check{k}_r(\nu_\infty, \delta, n) \in \mathbb{N}$ and $\omega_{k,r} = \omega_{k,r}(\nu_\infty, \delta, n) > 0$ such that

- the sequence $(\omega_{k,r})_k$ decreases to 0, and
- $k \geqslant k_r(\nu_\infty, \delta, n, \omega_{k,r})$ for every $k \geqslant \check{k}_r$.

Denote $\vec{\omega}_{k,r} = (\omega_{k,1}, \dots, \omega_{k,r})$ and then define

$$\hat{\psi}_{k,r}(x, x') = \hat{\psi}_r(x, x'; \vec{\omega}_{k,r}) \text{ and } \Psi_{k,r}(x, x') = \Psi_r(x, x'; \vec{\omega}_{k,r}). \tag{14.46}$$

15. Spreading out

For $r > 1$ the cut-off in Section 13.2 is insufficient to ensure that $\Psi_r(x, x'; \vec{\omega})$ is bounded near the border region of $E_r^\diamond(\varepsilon_r)$. To explain why, let us consider $x = (F_1, \dots, F_r)$ and $x = (F'_1, \dots, F'_r)$ such that F_r is near the border and F'_r is far from the border of $E_r(\varepsilon_r)$. Then $-\log \mathrm{VP}_r(x, x')$ is bounded, but the problem is that the term $-\log \psi_{r-1}(x_-, x'_-)$ may be arbitrarily large, which forces $\Psi_r(x, x'; \vec{\omega})$ to be arbitrarily large as well. For instance, $d(F_1, F'_1)$ may be very small, in which case $-\log \psi_1(F_1, F'_1)$ is very large. Thus, Lemma 9.5 as stated does not extend to $r > 1$.

为解决这一困难，我们引入一个“扩散”马尔可夫算子 $\widetilde{\mathcal{Q}}_r$ which leaves F_r and F'_r fixed, but averages the function out over the pairs of flags whose r-dimensional components are F_r and F'_r. This is done only on certain domains far from the equator: elsewhere we just take $\widetilde{\mathcal{Q}}_r = \mathrm{id}$. The main properties of this operator are stated in Proposition 15.1: roughly speaking, $\widetilde{\mathcal{Q}}_r \Psi_r$ is never much bigger than Ψ_r itself, with equality close to the equator, and it is bounded near the border region. That replaces Lemma 9.5 when $r > 1$.

The details of the spreading out construction follow. In Section 16.1 we will incorporate $\widetilde{\mathcal{Q}}_r$ into the definition of our main Markov operators, $\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}$, defined on the space of pairs of flags, and $\check{\mathcal{T}}^{\mathcal{Q}}_{k,r}$, its lift to the blow-up space $\mathcal{Y}_r(\varepsilon_r)$.

15.1. 旗簇上的齐性测度。正交群 $\mathrm{O}(d)$ 在旗空间 $\mathcal{F}(r, d)$ 上传递作用。我们将 μ 记为 $\mathcal{F}(r, d)$ 上相应的齐性测度。它可以描述为正交群的 Haar 概率测度在.....下的像。

$$\mathrm{O}(d) \to \mathcal{F}(r, d), \quad g \mapsto gz = (gH_1, \dots, gH_r),$$

for any choice of $z = (H_1, \dots, H_r)$, and it is invariant under the $\mathrm{O}(d)$-action.

类似地，对于任意 $F_r \in \mathrm{Gr}(r, d)$，记 μ_{F_r} 为.....上的齐次测度。

$$\mathcal{F}(F_r) = \{(G_1, \dots, G_{r-1}, G_r) \in \mathcal{F}(r, d) : G_r = F_r\} \tag{15.1}$$

corresponding to the natural action of the orthogonal group $\mathrm{O}(F_r)$ on $\mathcal{F}(F_r)$. We also consider

$$\mathcal{F}^*(F_r) = \left\{(G_1, \dots, G_{r-1}, F_r) \in \mathcal{F}(F_r) : d(G_1, E) \geqslant \frac{1}{2} d(F_r, E)\right\}. \tag{15.2}$$

显然，存在一个仅取决于 r 的 $a_r > 0$，使得 $\mu_{F_r}(\mathcal{F}^*(F_r)) \geqslant a_r$。参见图 9。我们将 μ_{F_r} 在 $\mathcal{F}^*(F_r)$ 上的归一化限制记为 $\mu^*_{F_r}$。

More generally, there exist $c_r > 0$ and $b_r > 0$, depending only on r, such that

$$\mu^*_{F_r}(\{(G_1, \dots, G_{r-1}, F_r) \in \mathcal{F}^*(F_r) : d(G_1, F) < \rho d(F_r, F)\}) \leqslant c_r \rho^{b_r} \tag{15.3}$$

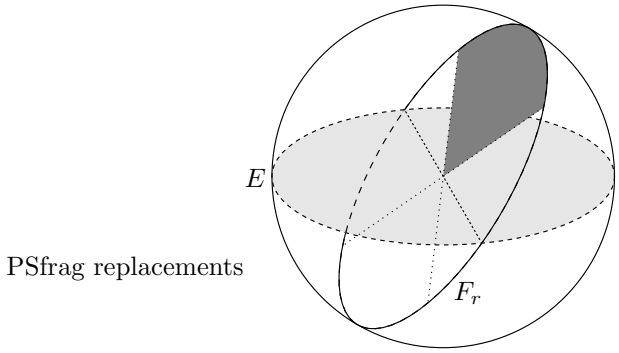

FIGURE 9. The subset $\mathcal{F}^*(F_r)$ of the flags $(G_1, \dots, G_{r-1}, F_r)$ such that $d(G_1, E) \geqslant d(F_r, E)/2$ is represented (for $r = 2$ and $d = 3$) by the dark gray region. It corresponds to a definite fraction of all the flags in $\mathcal{F}(F_r)$, relative to the homogeneous measure μ_{F_r}.

for any $F \neq F_r$ in $\mathrm{Gr}(r, d)$ and $\rho > 0$. Also (increasing c_r and decreasing b_r if necessary),

$$\mu^*_{F_r}(\{(G_1, \dots, G_{r-1}, F_r) \in \mathcal{F}^*(F_r) : d(G_1, F_-) < \rho\}) \leqslant c_r \rho^{b_r} \tag{15.4}$$

对于任意 $F_- \in \mathbf{Gr}(r-1、\ d)$ 和 $\rho > 0$ （因为允许 G_1 在一个维数严格大于 $\dim F_-$ 的定义域中变化）。

15.2. **Spreading out operators.** Let $\delta_{(x,x')}$ denote the Dirac mass at a point $(x, x') \in E_r^\diamond(\varepsilon_r)^2$, and $\tau : E_r(\varepsilon_r)^2 \to [0, 1]$ be a continuous symmetric function such that

$$\begin{aligned} &\tau \equiv 1 \text{ on } E_r(\varepsilon_r, \varepsilon_r') \times E_r(2\varepsilon_r'') \cup E_r(2\varepsilon_r'') \times E_r(\varepsilon_r, \varepsilon_r') \\ &\tau \equiv 0 \text{ outside } E_r(\varepsilon_r, \varepsilon_r'/2) \times E_r(3\varepsilon_r'') \cup E_r(3\varepsilon_r'') \times E_r(\varepsilon_r, \varepsilon_r'/2). \end{aligned} \tag{15.5}$$

See Figure 10. Then

$$\tilde{q}_{r,x,x'} = (1 - \tau(F_r, F_r'))\delta_{(x,x')} + \tau(F_r, F_r')\mu^*_{F_r} \times \mu^*_{F_r'}, \tag{15.6}$$

defines a probability measure on $\mathcal{F}(F_r) \times \mathcal{F}(F_r') \subset E_r^\diamond(\varepsilon_r)^2$ depending continuously on $(x, x') \in E_r^\diamond(\varepsilon_r)^2$. The *spreading out operator* is the corresponding continuous Markov operator

$$\widetilde{\mathcal{Q}}_r : \mathcal{B}(E_r^\diamond(\varepsilon_r)^2) \to \mathcal{B}(E_r^\diamond(\varepsilon_r)^2),\ \widetilde{\mathcal{Q}}_r\tilde{\psi}(x, x') = \int_{\mathcal{F}(F_r)\times\mathcal{F}(F_r')} \tilde{\psi}\, d\tilde{q}_{r,x,x'}. \tag{15.7}$$

It is clear from the definition that $\widetilde{\mathcal{Q}}_r$ is a lift of the identity relative to (f, f), meaning that

$$\widetilde{\mathcal{Q}}_r\big(\tilde{\psi} \circ (f, f)\big) = \tilde{\psi} \circ (f, f) \text{ for every } \psi \in \mathcal{B}(E_r(\varepsilon_r)^2), \tag{15.8}$$

其中 $f\colon\ \mathcal{F}(r,\ d) \to \mathbf{Gr}(r,\ d)$ 表示遗忘映射

$$(F_1, \dots, F_r) \mapsto F_r. \tag{15.9}$$

~由于假设 τ 是对称的，我们还可以得出 $\mathcal{Q}_r$ 保持对称函数空间不变。

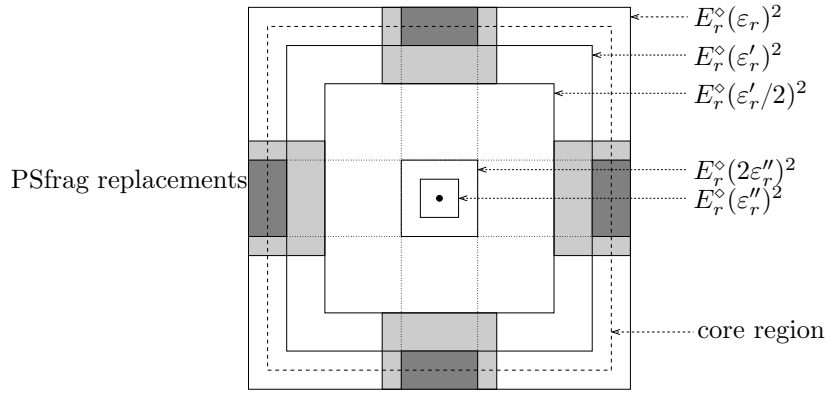

FIGURE 10. Illustrating the spreading out construction. The black dot at the center marks the point (E, E). The dashed lined represents the boundary between the $\nu_k^{(n)}$-core and the $\nu_k^{(n)}$-border of $E_1(\varepsilon_1)$. On the dark gray area we do full averaging. No averaging is needed on the white area. On the light gray area we interpolate between the two. In the cut-off region $E^\diamond(\varepsilon_r, 2\varepsilon_r'')^2$, where $\Psi_{k,r} \equiv \log \Omega_r$, averaging is innocuous.

For each fixed $\vec{\omega}$, denote by $\widetilde{\mathcal{Q}}_r\Psi_r(\cdot,\cdot;\vec{\omega})$ the image of $\Psi_r(\cdot,\cdot;\vec{\omega})$ under the operator $\widetilde{\mathcal{Q}}_r$. Since $\Psi_r(\cdot,\cdot;\vec{\omega})$ is symmetric, by (13.24) and (13.25), the function $\widetilde{\mathcal{Q}}_r\Psi_r(\cdot,\cdot;\vec{\omega})$ is also symmetric.

Proposition 15.1. *There exist $K_r = K_r(\nu_\infty) > 0$ and $\Omega_r = \Omega_r(\nu_\infty, \delta, n) > 1$ such that for every $(x, x') \in E_r^\diamond(\varepsilon_r)^2$ and $\vec{\omega} = (\omega_1, \dots, \omega_r) \in \mathbb{R}_+^d$ with $\omega_r < \varepsilon_r'''$,*

(i) $\widetilde{\mathcal{Q}}_r\Psi_r(x, x';\vec{\omega}) = \Psi_r(x, x';\vec{\omega})$ *if* $(x, x') \in E_r^\diamond(\varepsilon_r'/2)^2$;
(ii) $\widetilde{\mathcal{Q}}_r\Psi_r(x, x';\vec{\omega}) \leqslant \log\Omega_r + K_r \leqslant \Psi_r(x, x';\vec{\omega}) + K_r$ *if* $(x, x') \notin E_r^\diamond(\varepsilon_r')^2$.
(iii) $\widetilde{\mathcal{Q}}_r\Psi_r(x, x';\vec{\omega}) \leqslant \Psi_r(x, x';\vec{\omega}) + K_r$ *if* $(x, x') \notin E_r^\diamond(\varepsilon_r'/2)^2$.

The constants K_r and Ω_r are determined in (15.27) below. The proof of this proposition occupies the remainder of the present section. The first step is the following elementary lemma:

Lemma 15.2. *Let $a, b > 0$ and (Z, θ) be a probability space. Let $f : Z \to (0, \infty)$ be a measurable function such that $\theta(Z \backslash Z_\tau) \leqslant \tau$ for every $\tau \in (0, 1]$, where Z_τ denotes the subset of points $z \in Z$ such that*

$$f(z) \geqslant a\tau^b. \tag{15.10}$$

Then

$$\int_Z \log(\Omega + f(z)^{-1})\, d\theta(z) \leqslant \log(\Omega + a^{-1}) + 10b$$

证明。定义 $q(z) = f(z)a^{-1}$。该假设意味着，对于每个$z \in Z_\tau$, $q(z) \geqslant \tau^b$。然后，由 (9.1) 和 (9.2),

$$\begin{aligned}\log(\Omega + f(z)^{-1}) &\leqslant \log(\Omega + a^{-1}) + \max\{0, \log q(z)^{-1}\} \\ &\leqslant \log(\Omega + a^{-1}) - b\log\tau\end{aligned}$$

对于每个 $z \in Z_\tau$。特别地，

$$\int_{Z_\tau \setminus Z_{e\tau}} \left(\log(\Omega + f(z)^{-1}) - \log(\Omega + a^{-1})\right)\, d\theta(x) \leqslant -e\tau b \log \tau.$$

考虑 $\tau = e^{-j}$ 并对所有 j 求和，我们得到

$$\int_{Z} \left(\log(\Omega + f(z)^{-1}) - \log(\Omega + a^{-1})\right)\, d\theta(x) \leqslant eb \sum_{j=1}^{\infty} j e^{-j},$$

which implies the claim. □

Keep in mind that we write $x = (F_1, \dots, F_{r-1}, F_r)$ and $x' = (F'_1, \dots, F'_{r-1}, F'_r)$. We use $y = (G_1, \dots, G_{r-1}, F_r)$ and $y' = (G'_1, \dots, G'_{r-1}, F'_r)$ to denote the generic elements of $\mathcal{F}(F_r)$ and $\mathcal{F}(F'_r)$, respectively.

Lemma 15.3. *There exist $c'_r = c'_r(\mu_\infty, \delta, n) > 0$, $c''_r = c''_r(\mu_\infty, \delta, n) > 0$, $\alpha'_r = \alpha'_r(\nu_\infty) > 0$, and $\alpha''_r = \alpha''_r(\nu_\infty) > 0$ such that for any $\tau \in (0, 1]$, $x \in \mathcal{F}(r, d)$, $x' \in E^\diamond_r(\varepsilon_r, \varepsilon'_r/2)$, and $\vec{\omega} = (\omega_1, \dots, \omega_r)$ with $\omega_j \in (0, \varepsilon'''_j)$ for $j = 1, \dots, r$:*

(i) 对于集合 $X'_\tau \subset \mathcal{F}^*(F'_r)$ 中满足 $\mu^*_{F'_r}(\mathcal{F}^*(F'_r) \setminus X'_\tau) < \tau$ 的每个 y'，

$$-\log \psi_r(x, y'; \vec{\omega}) \leqslant -\gamma_r \log d(G'_1, F_r) + c'_r - \alpha'_r \log \tau. \tag{15.11}$$

(ii) *For every y' in a set $X''_\tau \subset \mathcal{F}^*(F'_r)$ with $\mu^*_{F'_r}(\mathcal{F}^*(F'_r) \setminus X''_\tau) < \tau$,*

$$-\log \psi_r(x, y'; \vec{\omega}) \leqslant -\gamma_r \log d(F'_r, F_r) + c''_r - \alpha''_r \log \tau. \tag{15.12}$$

Proof. By assumption, $x' \notin E^\diamond_r(\varepsilon'_r/2)$, that is, $F'_r \notin E_r(\varepsilon'_r/2)$. Thus,

$$d(G'_1, E) \geqslant \frac{1}{2} d(F'_r, E) > \frac{1}{4}\varepsilon'_r$$

对于任意 $y' \in \mathcal{F}^*(F'_r)$。因此，回顾（12.2），

$$\mathrm{SVA}_r(x, y'; \omega_r) \geqslant \mathrm{VA}_r(x, y') = d(F_r + G'_1, E) \geqslant d(G'_1, E) > \frac{1}{4}\varepsilon'_r$$

对于任意 $y' \in \mathcal{F}(F'_r)$。因此，利用 (12.13),

$$\begin{aligned} -\log \mathrm{SVP}_r(x, y'; \omega_r) &\leqslant -\log \mathrm{VP}_r(x, y') \\ &\leqslant -\gamma_r \log d(F_{r-1} + G'_1, F_r) - \log(\varepsilon'_r/4) \\ &\leqslant -\gamma_r \log d(G'_1, F_r) - \log(\varepsilon'_r/4) \end{aligned} \tag{15.13}$$

对于任意 $y' \in \mathcal{F}(F'_r)$（请记住，$d(\cdot, \cdot)$ 是限制在 $\mathbf{Gr}(r, d)$ 上的距离）。特别地，(15.13) 包含了命题 (i) 中 $r = 1$ 的情形：

$$-\log \psi_1(x, x'; \omega_1) = -\log \mathrm{SVP}_1(x, x'; \omega_1) \leqslant -\gamma_1 \log d(y', x) + c'_1 \tag{15.14}$$

with $c'_1 = -\log(\varepsilon'_1/4)$ and $\alpha'_1 = 0$ (and $X'_\tau = \mathcal{F}^*(x') = \{x'\}$).

现在令 $r > 1$。根据归纳法，给定 $y'_- = (G'_1, \dots, G'_{r-1})$ 属于 $E^\diamond_{r-1}(\varepsilon_{r-1}, \varepsilon'_{r-1}/2)$，有：

$$-\log \psi_{r-1}(x_-, z'; \vec{\omega}_-) \leqslant -\gamma_{r-1} \log d(H'_1, F_{r-1}) + c'_{r-1} - \alpha'_{r-1} \log(\tau/2) \tag{15.15}$$

对于每个 $z' = (H'_1, \dots, H'_{r-2}, G'_{r-1})$，属于一个集合 $Z' \subset \mathcal{F}^*(G'_{r-1})$，且具有

$$\mu^*_{G'_{r-1}}(\mathcal{F}^*(G'_{r-1}) \setminus Z') < \tau/2.$$

对所有允许的 G'_{r-1} 积分这一估计，我们得到

$$-\log \psi_{r-1}(x_-, y'_-; \vec{\omega}_-) \leqslant -\gamma_{r-1} \log d(G'_1, F_{r-1}) + c'_{r-1} - \alpha'_{r-1} \log(\tau/2) \tag{15.16}$$

对于集合 $Y' \subset \mathcal{F}^*(F_r')$ 中的每个 y'，满足

$$\mu^*_{F_r'}(\mathcal{F}^*(F_r')\backslash Y') < \tau/2.$$

On the other hand, taking $\tau = 2c_r\rho^{b_r}$ and $F_- = F_{r-1}$ in (15.4), we find that

$$-\log d(G_1', F_{r-1}) \leqslant \frac{1}{b_r}\left(-\log\tau + \log c_r\right) \tag{15.17}$$

对于每个属于集合 $Y'' \subset \mathcal{F}^*(F_r')$ 的 y'，具有

$$\mu^*_{F_r'}(\mathcal{F}^*(F_r')\backslash Y'') < \tau/2.$$

定义 $X'_\tau = Y' \cap Y''$。那么 $\mu^*_{F_r'}(\mathcal{F}^*(F_r')\backslash X'_\tau) < \tau$，并且根据 (15.16) 和 (15.17),

$$-\log\psi_{r-1}(x_-, y'_-; \vec{\omega}_-) \leqslant \tilde{c}_r - \tilde{\alpha}_r\log\tau \tag{15.18}$$

对于每个 $y' \in X'_\tau$，其中

$$\tilde{c}_r = \frac{\gamma_{r-1}}{b_r}\log c_r + c'_{r-1} + \alpha'_r\log 2 \text{ and } \tilde{\alpha}_r = \frac{\gamma_{r-1}}{b_r} + \alpha'_{r-1}. \tag{15.19}$$

结合（15.13）和（15.18），我们得到:

$$\begin{aligned} -\log\psi_r(x, y'; \vec{\omega}) &= -\beta_{r-1}\log\psi_{r-1}(x_-, y'_-; \vec{\omega}_-) - \log \mathrm{SVP}_r(x, y'; \omega_r) \\ &\leqslant \beta_{r-1}\left(\tilde{c}_r - \tilde{\alpha}_r\log\tau\right) - \gamma_r\log d(G_1', F_r) - \log(\varepsilon'_r/4) \end{aligned} \tag{15.20}$$

对于每个 $y' \in X''$。这就证明了断言 (ii)，其中

$$c'_r = \beta_{r-1}\tilde{c}_r - \log(\varepsilon'_r/4) \text{ and } \alpha'_r = \beta_{r-1}\tilde{\alpha}_r. \tag{15.21}$$

现在我们推导结论 (ii)。根据 (i) 部分，存在 $X' \subset \mathcal{F}(F_r')$ 使得 $\mu^*_{F_r'}(\mathcal{F}^*(F_r')\backslash X') < \tau/2$并且

$$-\log\psi_r(x, y'; \vec{\omega}) \leqslant -\gamma_r\log(G_1', F_r) + c'_r - \alpha'_r\log(\tau/2). \tag{15.22}$$

for every $y' \in X'$. Taking $\tau = 2c_r\rho^{b_r}$ and $F = F_r'$ in (15.3), we get that

$$\begin{aligned} -\log d(G_1', F_r) &\leqslant -\log d(F_r', F_r) - \log\rho \\ &= -\log d(F_r', F_r) + \frac{1}{b_r}\left(\log 2c_r - \log\tau\right) \end{aligned} \tag{15.23}$$

for every y' in a set $Y \subset \mathcal{F}(F_r')$ with $\mu^*_r(\mathcal{F}^*(F_r')\backslash Y) < \tau/2$. Define $X''_\tau = X' \cap Y$. Then $\mu^*_{F_r'}(\mathcal{F}^*(F_r')\backslash X''_\tau) < \tau$ and the relations (15.22) and (15.23) imply that (15.12) holds for every $y' \in X''_\tau$, with

$$c''_r = c'_r + \frac{\gamma_r}{b_r}\log 2c_r + \alpha'_r\log 2 \text{ and } \alpha''_r = \alpha_r + \frac{\gamma_r}{b_r}. \tag{15.24}$$

引理的证明至此完成。 □

Corollary 15.4. *There exist $K_r = K_r(\nu_\infty) > 0$ and $\Omega_r = \Omega_r(\nu_\infty, \delta, n) > 1$ such that for all $(x, x') \in E^\diamond_r(\varepsilon_r)^2$ such that $x \notin E^\diamond_r(\varepsilon'_r/2)$ or $x' \notin E^\diamond_r(\varepsilon'_r/2)$, and for all $\vec{\omega} = (\omega_1, \ldots, \omega_r) \in \mathbb{R}^d_+$ with $\omega_r < \varepsilon'''_r$,*

$$\int_{\mathcal{F}(F_r')} \Psi_r(x, y'; \vec{\omega})\, d\mu^*_{F_r'}(y') \text{ and } \int_{\mathcal{F}(F_r)} \Psi_r(y, x'; \vec{\omega})\, d\mu^*_{F_r}(y) \tag{15.25}$$

两者均以上界 $\log\Omega_r + K_r$ 为界。

Proof. Since the function $\Psi_r(\cdot,\cdot;\vec{\omega})$ is symmetric, it suffices to consider the first integral in (15.25). Initially, suppose that $x \notin E_r^\diamond(2\varepsilon_r'')$ and $x' \notin E_r^\diamond(2\varepsilon_r'')$. Then $\Psi_r(x,x';\vec{\omega}) = \log\Omega_r$ and, in fact, $\Psi_r(\cdot,\cdot;\vec{\omega}) \equiv \log\Omega_r$ on $\mathcal{F}(F_r)\times\mathcal{F}(F_r')$. Hence the integral is equal to $\log\Omega_r$, and so the claim is trivial in this case.

从现在起，假设点 x 和 x' 中有一个位于 $E_r^\diamond(2\varepsilon_r'')$ 中。根据假设，另一个点必然位于 $E_r^\diamond(\varepsilon'/2)$, 之外，因此可以得出：

$$d(F_r,F_r') \geqslant \varepsilon_r'/2 - 2\varepsilon_r''. \tag{15.26}$$

我们将把引理 15.2 应用于

$$Z = \mathcal{F}(F_r'), \quad \theta = \mu_{F_r'}^*, \quad \Omega = \Omega_r,$$
$$f(y') = \psi_r(x,y';\vec{\omega}), \quad a = d(F_r,F_r')^{\gamma_r} e^{-c_r''}, \text{ and } b = \alpha_r''.$$

引理 15.2 的假设 (15.10) 与引理 15.3 结论的第 (ii) 部分完全对应：对于任意的 $\tau \in (0,1]$,

$$-\log\psi_r(x,y';\vec{\omega}) \leqslant -\gamma_r \log d(F_r',F_r) + c_r'' - \alpha_r''\log\tau$$

for every y' in a set $Z_\tau'' \subset Z$ with $q_{r,x'}(Z\backslash Z_\tau'') < \tau$. The conclusion of Lemma 15.2 asserts that

$$\begin{aligned}\int_{\mathcal{F}(F_r')} \Psi_r(x,y';\vec{\omega})\, d\mu_{F_r'}^*(y') &= \int_{\mathcal{F}(F_r')} \log(\Omega_r + \psi_r(x,y';\vec{\omega})^{-1})\, d\mu_{F_r'}^*(y') \\ &\leqslant \log(\Omega_r + e^{c_r''} d(F_r,F_r')^{-\gamma_r}) + 10\alpha_r''\end{aligned}$$

Define

$$\Omega_r = e^{c_r''}(\varepsilon_r'/2 - 2\varepsilon_r'')^{-\gamma_r} \text{ and } K_r = 10\alpha_r'' + \log 2. \tag{15.27}$$

将 (15.26) 代入前面的不等式，我们得到棋牌游戏官网

$$\begin{aligned}\int_{\mathcal{F}(F_r')} \Psi_r(x,y';\vec{\omega})\, d\mu_{F_r'}^*(y') &\leqslant \log(\Omega_r + e^{c_r''}(\varepsilon_r'/2 - 2\varepsilon_r'')^{-\gamma_r}) + 10\alpha_r'' \\ &\leqslant \log\Omega_r + K_r\end{aligned}$$

as claimed. □

Proof of Proposition 15.1. It follows immediately from the definitions (15.5) and (15.6) that $\widetilde{\mathcal{Q}}_r\Psi_r = \Psi_r$ outside $E_r^\diamond(\varepsilon_r,\varepsilon_r'/2)\times E_r^\diamond(3\varepsilon_r'') \cup E_r^\diamond(3\varepsilon_r'')\times E_r^\diamond(\varepsilon_r,\varepsilon_r'/2)$ This contains part (i) of the proposition. We also have that $\widetilde{\mathcal{Q}}_r\Psi_r = \Psi_r = \log\Omega_r$ on the cut-off region $E_r^\diamond(\varepsilon_r,2\varepsilon_r'')^2$. Thus (check Figure 10), to complete the proof of part (ii) we only have to consider the case when $(x,x') \in E_r^\diamond(\varepsilon_r,\varepsilon_1')\times E_r^\diamond(2\varepsilon_r'') \cup E_r^\diamond(2\varepsilon_r'')\times E_r^\diamond(\varepsilon_r,\varepsilon_r')$. In this case $\tilde{q}_{r,r,x'} = \mu_{F_r}^* \times \mu_{F_r'}^*$, and so Corollary 15.4 gives that

$$\begin{aligned}\widetilde{\mathcal{Q}}_r\Psi_r(x,x';\omega) &= \int_{\mathcal{F}(F_r)}\int_{\mathcal{F}(F_r')} \Psi_r(u,u';\vec{\omega})\, d\mu_{F_r'}^*(u')\, d\mu_{F_r}^*(u) \\ &\leqslant \int_{\mathcal{F}(F_r)} (\log\Omega_r + K_r)\, d\mu_{F_r}^*(u) = \log\Omega_r + K_r.\end{aligned}$$

This proves the first inequality in part (ii). The second one is an immediate consequence of the definition of Ψ_r.

We are left to proving part (iii) of the proposition. Combining the definition (15.6) with Corollary 15.4, we see that

$$\begin{aligned}\int_{\mathcal{F}(F_r)\times\mathcal{F}(F_r')} &\Psi_r(u,u';\vec{\omega})\,d\tilde{q}_{r,x,x'}(u,u')\\ &\leqslant \big(1-\tau(F_r,F_r')\big)\,\Psi_r(x,x';\vec{\omega})+\tau(F_r,F_r')\,(\log\Omega_r+K_r)\\ &\leqslant \Psi_r(x,x';\vec{\omega})+\tau(F_r,F_r')K_r\leqslant\Psi_r(x,x';\vec{\omega})+K_r,\end{aligned}\tag{15.28}$$

as claimed. □

Corollary 15.5. *For any $\delta>0$, $n\geqslant N_r$, the following holds for any $k\geqslant\check{k}_r$:*

(i) *For any $x,x'\in E_r^\diamond(\varepsilon_r)$ in general position with $\Psi_{k,r}(x,x')>\log\Omega_r$,*

$$\int_G\widetilde{\mathcal{Q}}_r\Psi_{k,r}(gx,gx')\,d\nu_k^{(n)}(g)\leqslant\Psi_{k,r}(x,x')+C_r''n+K_r.$$

(ii) 对于任意满足 $\Psi_{k,\,r}(x,\ x')>\log\Omega_r$、$\mathsf{VA}_r(x,\ x')\geqslant\omega_{k,\,r}$ 和 $\mathsf{VA}_r(x',x)\geqslant\omega_{k,\,r}$ 的处于一般位置的 x、$x'\in E_r^\diamond(\varepsilon_r)$,

$$\int_G\widetilde{\mathcal{Q}}_r\Psi_{k,r}(gx,gx')\,d\nu_k^{(n)}(g)\leqslant\Psi_{k,r}(x,x')+C_r''\delta n+K_r.$$

(iii) 对于任意满足一般位置条件的 $x,x'\in E_r^\diamond(\varepsilon_r''')$，且 $\mathsf{VA}_r(x,x')\geqslant\omega_{k,\ r}$ 和 $\mathsf{VA}_r(x',x)\geqslant\omega_{k,\ r}$,

$$\int_G\widetilde{\mathcal{Q}}_r\Psi_{k,r}(gx,gx')\,d\nu_k^{(n)}(g)\leqslant\Psi_{k,r}(x,x')-(\kappa_r''-C_r''\delta)n.$$

证明。命题 15.1 的 (i) 和 (iii) 部分表明,

$$\widetilde{\mathcal{Q}}_r\Psi_r(x,x')\leqslant\Psi_r(x,x')+K_r\text{ for every }(x,x')\in E_r^\diamond(\varepsilon_r)^2,$$

Then the claims in parts (i) and (ii) of the corollary follow immediately from the corresponding statements in Proposition 14.1. In the context of part (iii) of the corollary, we even have that $\widetilde{\mathcal{Q}}_r\Psi_r(x,x')=\Psi_r(x,x')$, and so the claim corresponds exactly to part (iii) of Proposition 14.1. □

16. 重耦合与结论

By induction, there are constants $\varepsilon_r>0$ and $n_r\in\mathbb{N}$, continuous Markov operators

$$\mathcal{T}_{k,r}:\mathcal{B}(E_r(\varepsilon_r))\to\mathcal{B}(E_r(\varepsilon_r)),\quad\mathcal{T}_{k,r}\varphi(F_r)=\int_{E_r(\varepsilon_r)}\psi\,d\sigma_{k,r,F_r}\tag{16.1}$$

adapted to $(\nu_k,E_r(\varepsilon_r))$, and $\mathcal{T}_{k,r}$-invariant probability measures $\eta_{k,r}$ on $E_r(\varepsilon_r)$ such that the sequence $\eta_{\infty,r}=\lim_k\eta_{k,r}$ exists and satisfies $\eta_{\infty,r}(E_r)>0$. Up to "localizing" the Markov operators as described in Section 6.5, if necessary, we may assume that $\varepsilon_r>0$ is small enough that

$$\eta_{\infty,r}(E_r(\varepsilon_r)\backslash E_r)<\frac{1}{10}\eta_{k,r}(E_r).\tag{16.2}$$

然后，对于每个充分大的 k,

$$\eta_{k,r}(E_r(\varepsilon_r,\varepsilon_r'''))<\frac{2}{10}\eta_{k,r}(E_r(\varepsilon_r)).\tag{16.3}$$

We are going to show that this leads to a contradiction when $\dim E=r$, and to recover all this information for $r+1$ when $\dim E>r$.

16.1. **Markov operators on flag varieties.** Let k and r be fixed. We are going to extend $\mathcal{T}_{k,r}$ to a suitable Markov operator $\mathcal{T}^{\diamond}_{k,r}$ in the space of flags, as follows.

The first step is to find a suitable lift of $\sigma_{k,r,x}$ to a probability measure on the group $G = \mathrm{GL}(\mathbb{R}^d)$. For each $F_r \in \mathrm{Gr}(r,d)$ define

$$\mathfrak{E}_{F_r} : G \to \mathrm{Gr}(r,d), \quad g \mapsto gF_r.$$

Lemma 16.1. *There exists a continuous family $\{\mu_{k,r,F_r} : F_r \in E_r(\varepsilon_r)\}$ of probability measures on G such that $(\mathfrak{E}_{F_r})_*\mu_{k,r,F_r} = \sigma_{k,r,F_r}$ for every $F_r \in E_r(\varepsilon_r)$.*

Proof. Write $\mathbb{R}^d = E \oplus E^\perp$. Every $F \in E_r(\varepsilon_r)$ is the graph of a linear map $u_F : E \to E^\perp$. Define

$$h_F : \mathbb{R}^d \to \mathbb{R}^d, \quad v^E + v^\perp \mapsto v^E + \left(v^\perp + u_F(v^E)\right).$$

Then $F \mapsto h_F$ is a continuous injective map from $E_r(\varepsilon_r)$ to $G =$ with $h_F(E) = F$. For each fixed $F_r \in E_r(\varepsilon_r)$, define

$$\mathfrak{L}_{F_r} : E_r(\varepsilon_r) \to G, \quad F \mapsto h_F \circ h_{F_r}^{-1}.$$

那么， $\mathfrak{L}_{F_r}$ 是 $\mathfrak{E}_{F_r}$的连续单射和右逆:

$$\mathfrak{E}_{F_r}\left(\mathfrak{L}_{F_r}(F)\right) = \mathfrak{L}_{F_r}(F)F_r = \left(h_F \circ h_{F_r}^{-1}\right)(F_r) = h_F(E) = F \tag{16.4}$$

对于每个 $F \in E_r(\varepsilon_r)$。定义 $\mu_{k,\,r,\,F_r} = (\mathfrak{L}_{F_r})_*\sigma_{k,\,r,\,F_r}$. 显然，这随 F_r 连续变化。引理中的结论可直接由 (16.4) 得出。□

Since $\mathcal{T}_{k,r}$ is adapted to $(\nu_k^{(n)}, E_r(\varepsilon_r))$, there exists a neighborhood V of the $\nu_k^{(n)}$-core of $E_r(\varepsilon_r)$ such that σ_{k,r,F_r} coincides with $\nu^{(n)}_{k,F_r} = (\mathfrak{E}_{F_r})_*\nu_k^{(n)}$ for every $F_r \in V$. Let $\tau : E_r(\varepsilon_r) \to [0,1]$ be a continuous function vanishing on a neighborhood $U \subset V$ of the $\nu_k^{(n)}$-core of $E_r(\varepsilon_r)$, and constant equal to 1 outside U. Define

$$\nu_{k,r,F_r} = (1 - \tau(F_r))\nu_k^{(n)} + \tau(F_r)\mu_{k,r,F_r}.$$

Observe that $(\mathfrak{E}_{F_r})_*\nu_{k,r,F_r} = \sigma_{k,r,F_r}$ for all $F_r \in E_r(\varepsilon_r)$, and so the operator $\mathcal{T}_{k,r} : \mathcal{B}(E_r(\varepsilon_r)) \to \mathcal{B}(E_r(\varepsilon_r))$ may be rewritten as

$$\mathcal{T}_{k,r}\psi(F_r) = \int_G \psi(gF_r)\, d\nu_{k,r,F_r}(g).$$

We extend this to

$$\mathcal{T}^{\diamond}_{k,r} : \mathcal{B}(E^{\diamond}_r(\varepsilon_r)) \to \mathcal{B}(E^{\diamond}_r(\varepsilon_r)), \quad \mathcal{T}^{\diamond}_{k,r}\psi(x) = \int_G \psi(gx)\, d\nu_{k,r,F_r}(g), \tag{16.5}$$

where $x = (F_1, \dots, F_r)$ and $gx = (gF_1, \dots, gF_r)$. It is clear that $\mathcal{T}^{\diamond}_{k,r}$ projects to $\mathcal{T}_{k,r}$ under the forgetfulness map (15.9). Let $\eta^{\diamond}_{k,r}$ be a $\mathcal{T}^{\diamond}_{k,r}$-invariant probability measure projecting to $\eta_{k,r}$.

16.2. **Recoupling.** We move to construct suitable self-couplings for the Markov operators $\mathcal{T}^{\diamond}_{k,r}$. Begin by writing the definition in (16.5) as

$$\mathcal{T}^{\diamond}_{k,r}\psi(x) = \int_{E^{\diamond}_r(\varepsilon_r)} \psi\, d\sigma^{\diamond}_{k,r,x} \text{ for } \psi \in \mathcal{B}(E^{\diamond}_r(\varepsilon_r)), \tag{16.6}$$

where $\sigma^{\diamond}_{k,r,x}$ is the push-forward of ν_{k,r,F_r} under the map $G \to \mathcal{F}(r,d)$, $g \mapsto gx$. By construction, $\sigma^{\diamond}_{k,r,x}$ coincides with the push-forward $\nu^{(n)}_{k,r,x}$ of $\nu_k^{(n)}$ whenever F_r is in the neighborhood U of $\nu_k^{(n)}$-core of $E_r(\varepsilon_r)$. We denote by $\nu^{(n)}_{k,r,x,x}$ the push-forward of $\nu_k^{(n)}$ under the diagonal embedding $G \to \mathcal{F}(r,d)^2$, $g \mapsto (gx, gx')$.

Consider $X = X' = E_r^\diamond(\varepsilon_r)$, $Y = Y' = E_r^\diamond(\varepsilon_r)$, and $\eta_y = \eta'_y = \sigma^\diamond_{k,r,y}$ for every $y \in Y$. Let K be the (compact) subset Diag_r of pairs $(x, x') \in E_r^\diamond(\varepsilon_r)^2$ which are *not* in general position, that is, such that either $F'_1 \subset F_r$ or $F_1 \subset F'_r$. It is clear that $K(x')$ and $K'(x)$ are algebraic subvarieties of $E_r^\diamond(\varepsilon_r)$, and so

$$\sigma^\diamond_{k,r,y}(K(x')) = \sigma^\diamond_{k,r,y}(K'(x)) = \{0\}$$

for every x, x' and y. This means that (6.10) holds in this setting, and so we may use Proposition 6.9 to find a continuous family

$$\left\{\tilde\theta_{k,r,x,x'} : (x, x') \in E_r^\diamond(\varepsilon_r)^2\right\}$$

$E_r^\diamond(\varepsilon_r)^2$ 上的典型概率测度，使得每个 $\theta_{k,\ r,\ x,\ x'}$ 都是$\sigma^\diamond_{k,\ r,\ x}$ 与 $\sigma^\diamond_{k,\ r,\ x'}$ 的一个耦合，并且在 Diag_r 的某个一致邻域上消失。

Let $\tilde\omega : E_r^\diamond(\varepsilon_r)^2 \to [0, 1]$ be a continuous function such that $\tilde\omega(x, x') = 0$ if F_r and F'_r are both in $E_r^\diamond(\tilde\varepsilon_r)$ and $\tilde\omega(x, x') = 1$ if either of them is outside $E_r^\diamond(2\tilde\varepsilon_r)$). Then

$$\tilde\sigma^\diamond_{k,r,x,x'} = \left(1 - \tilde\omega(x, x')\right) \nu^{(n)}_{k,r,x,x'} + \tilde\omega(x, x')\tilde\theta_{k,r,x,x'} \tag{16.7}$$

is a coupling of $\sigma^\diamond_{k,r,x}$ and $\sigma^\diamond_{k,r,x'}$ depending continuously on (x, x'), and so

$$\widetilde{\mathcal{T}}^\diamond_{k,r} : \mathcal{B}(E_r^\diamond(\varepsilon_r)^2) \to \mathcal{B}(E_r^\diamond(\varepsilon_r)^2),\ \widetilde{\mathcal{T}}^\diamond_{k,r}\tilde\psi(x, x') = \int_{E_r^\diamond(\varepsilon_r)^2} \tilde\psi\, d\tilde\sigma^\diamond_{k,r,x,x'},$$

is a continuous self-coupling of $\mathcal{T}^\diamond_{k,r}$. Just as we did for $r = 1$, we must modify these operators, by recoupling the measures $\sigma^\diamond_{k,r,x}$ and $\sigma^\diamond_{k,r,x'}$ in a suitable way on the region $E_r^\diamond(2\tilde\varepsilon_r, \varepsilon''_r)^2$.

For $x \in E_r^\diamond(2\tilde\varepsilon_r, \varepsilon''_r)$, it follows from (9.16) that the subset of $g \in \operatorname{supp} \nu_k^{(n)}$ such that $gx \in E_r^\diamond(2\varepsilon'_r)$ is disjoint from the set $\mathcal{D}_k(F_r)$ given by Corollary 5.7. Hence,

$$\nu^{(n)}_{k,r,x}\left(E_r^\diamond(2\varepsilon'_r)\right) \leqslant \nu_k^{(n)}\left(\mathcal{D}_k(F_r)^c\right) < \delta \text{ for every } x \text{ in } E_r^\diamond(2\tilde\varepsilon_r, \varepsilon''_r). \tag{16.8}$$

Note that $\sigma^\diamond_{k,r,x} = \nu^{(n)}_{k,r,x}$ if F_r is in the $\nu_k^{(n)}$-core of $E_r(\varepsilon_r)$.

Take $X = X' = E_r^\diamond(\varepsilon_r)$, $Y = Y' = E_r^\diamond(2\tilde\varepsilon_r, \varepsilon''_r)^2$, and $\eta_y = \eta'_y = \sigma^\diamond_{k,r,y}$ for every $y \in Y$. Moreover, let $K = E_r^\diamond(\varepsilon'_r)^2 \cup \mathrm{Diag}_r$. On the one hand, (16.8) implies that $\sigma^\diamond_{k,r,y}(E_r^\diamond(\varepsilon'_r))$ and $\sigma^\diamond_{k,r,y}(E_r^\diamond(\varepsilon'_r))$ are less than $\delta < 1/2$ for every $y \in Y$. On the other hand,

$$\mathrm{Diag}_r(x') = \{x \in E_r^\diamond(\varepsilon_r) : F'_1 \subset F_r \text{ or } F_1 \subset F'_r\} \text{ and}$$
$$\mathrm{Diag}'_r(x) = \{x' \in E_r^\diamond(\varepsilon_r) : F'_1 \subset F_r \text{ or } F_1 \subset F'_r\}$$

are algebraic subvarieties of $E_r^\diamond(\varepsilon_r)$, and so they have zero $\sigma^\diamond_{k,r,y}$-measure for every $y \in Y$. These two observations show that

$$K(x') = E_r^\diamond(\varepsilon'_r) \cup \mathrm{Diag}_r(x') \text{ and } K'(x) = E_r^\diamond(\varepsilon'_r) \cup \mathrm{Diag}'_r(x)$$

满足 (6.10)。因此，我们可以利用命题 6.9 找到一个连续族

$$\{\zeta_{k,r,x,x'} : (x, x') \in E_r^\diamond(2\tilde\varepsilon_r, \varepsilon''_r)^2\}$$

一般概率测度的集合，其中每个 $\zeta_{k,r,x,x'}$ 都是$\sigma^\diamond_{k,r,x}$ 与$\sigma^\diamond_{k,r,x'}$ 的一个耦合，在 Diag_r 的一致邻域上消失，并且使得

$$\zeta_{k,r,x,x'}\left(E_r^\diamond(\varepsilon'_r)^2\right) = 0. \tag{16.9}$$

Fix a continuous function $\tau : E_r^\diamond(\varepsilon_r)^2 \to [0,1]$ such that $\tau \equiv 1$ on $E_r^\diamond(\tilde\varepsilon_r, 2\varepsilon_r'')^2$ and $\tau \equiv 0$ on the complement of $E_r^\diamond(2\tilde\varepsilon_r, \varepsilon_r'')^2$, and then define

$$\hat\sigma^\diamond_{k,r,x,x'} = \big(1 - \tau(x,x')\big)\, \tilde\sigma^\diamond_{k,r,x,x'} + \tau(x,x')\zeta_{k,r,x,x'} \tag{16.10}$$

for every $(x,x') \in E_r^\diamond(\varepsilon_r)^2$. Then $\hat\sigma^\diamond_{k,r,x,x'}$ is a coupling of $\sigma^\diamond_{k,r,x}$ and $\sigma^\diamond_{k,r,x'}$ depending continuously on (x,x'), and so

$$\widehat{\mathcal{T}}^\diamond_{k,r} : \mathcal{B}(E_r^\diamond(\varepsilon_r)^2) \to \mathcal{B}(E_r^\diamond(\varepsilon_r)^2), \quad \widehat{\mathcal{T}}^\diamond_{k,r}\tilde\psi(x,x') = \int_{E_r^\diamond(\varepsilon_r)^2} \tilde\psi \, d\hat\sigma^\diamond_{k,1,x,x'}$$

~是$\mathcal{T}^\diamond_{k,\,r}$的另一个连续自耦合，在重新耦合区域 $E_r^\diamond(2\tilde\varepsilon_r, \varepsilon_r'')^2$之外与 $\mathcal{T}^\diamond_{k,\,r}$ 重合。

最后，定义 $\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}$: $\mathcal{B}(E_r^\diamond(\varepsilon_r)^2) \to \mathcal{B}(E_r^\diamond(\varepsilon_r)^2)$，其中 $\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r} = \widehat{\mathcal{T}}^\diamond_{k,r} \circ \widetilde{\mathcal{Q}}_r$，即，

$$\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}\tilde\varphi(x,x') = \int_{E_r^\diamond(\varepsilon_r)^2} (\widetilde{\mathcal{Q}}_r\tilde\varphi)\, d\hat\sigma^\diamond_{k,1,x,x'}. \tag{16.11}$$

Let $\pi_i : E_r(\varepsilon_r)^2 \to E_r(\varepsilon_r)$ and $\pi_i^\diamond : E_r^\diamond(\varepsilon_r)^2 \to E_r^\diamond(\varepsilon_r)$ denote the projections to the ith factor, $i = 1, 2$, and f be the forgetfulness map (15.9). By (16.11) and (15.8),

$$\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}\big(\tilde\psi \circ (f,f)\big) = \widehat{\mathcal{T}}^\diamond_{k,r}\big(\widetilde{\mathcal{Q}}_r\big(\tilde\psi \circ (f,f)\big)\big) = \widehat{\mathcal{T}}^\diamond_{k,r}\big(\tilde\psi \circ (f,f)\big)$$

for any $\tilde\psi \in \mathcal{B}(E_r(\varepsilon_r)^2)$. Take $\tilde\psi = \psi \circ \pi_i$ for any $\psi \in \mathcal{B}(E_r(\varepsilon_r))$. Observing that $\pi_i \circ (f,f) = f \circ \pi_i^\diamond$, and keeping in mind that $\widehat{\mathcal{T}}^\diamond_{k,r}$ is a self-coupling of $\mathcal{T}^\diamond_{k,r}$ and the latter projects to $\mathcal{T}_{k,r}$ under the forgetfulness map, we get that

$$\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}\big(\psi \circ f \circ \pi_i^\diamond\big) = \widehat{\mathcal{T}}^\diamond_{k,r}\big(\psi \circ f \circ \pi_i^\diamond\big) = \mathcal{T}^\diamond_{k,r}\big(\psi \circ f\big) \circ \pi_i^\diamond = \big(\mathcal{T}_{k,r}\psi\big) \circ f \circ \pi_i^\diamond. \tag{16.12}$$

In other words, $\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}$ projects to $\mathcal{T}_{k,r}$ under $f \circ \pi_i^\diamond$ for any $i = 1, 2$.

Remark 16.2. Unlike $\widehat{\mathcal{T}}^\diamond_{k,r}$, this $\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}$ is not a coupling of operators on $\mathcal{B}(E_r^\diamond(\varepsilon_r))$ because $\widetilde{\mathcal{Q}}_r$ itself is not a coupling. That could be remedied by choosing $\widetilde{\mathcal{Q}}_r$ differently. However, our choice is convenient for lifting these operators to the blow-up space $\mathcal{Y}_r(\varepsilon_r)$, as we will see in Section 16.4.

我们将需要以下引理 **11.1** 在 $r > 1$ 上的推广：

Lemma 16.3. *Let $(x,x') \in E_r^\diamond(\varepsilon_r)^2$ be such that*

(a) *either at least one of the points x or x' is in the $\nu_k^{(n)}$-border of $E_r^\diamond(\varepsilon_r)$,*
(b) *or both x and x' are in the $\nu_k^{(n)}$-core of $E_r^\diamond(\varepsilon_r)$ but outside $E_r^\diamond(2\varepsilon_r'')$.*

Then $\hat\sigma^\diamond_{k,r,x,x'}(E_r^\diamond(\varepsilon_r')^2) = 0$ and $\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}\Psi_{k,r}(x,x') \leqslant \log\Omega_r + K_r$.

Proof. Let us begin by proving the claim that $\hat\sigma^\diamond_{k,r,x,x'}$ vanishes on $E_r^\diamond(\varepsilon_r')^2$. If x is in the $\nu_k^{(n)}$-border of $E_r^\diamond(\varepsilon_r)$ then, using (13.22),

$$\hat\sigma^\diamond_{k,r,x,x'}(E_r^\diamond(\varepsilon_r')^2) \leqslant \sigma^\diamond_{k,r,x}(E_r^\diamond(\varepsilon_r')) \leqslant \sigma^\diamond_{k,r,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}} E_r^\diamond(\varepsilon_r)\right) = 0.$$

The same argument applies when x' is in the $\nu_k^{(n)}$-border of $E_r^\diamond(\varepsilon_r)$. This settles the claim in case (a). Now let x and x' be as in (b). Keep in mind that $\sigma^\diamond_{k,r,x} = \nu^{(n)}_{k,r,x}$ and $\sigma^\diamond_{k,r,x'} = \nu^{(n)}_{k,r,x'}$. By (16.9), $\zeta_{k,r,x,x'}$ vanishes on $E_r^\diamond(\varepsilon_r')^2$, and so (16.10) gives that

$$\hat\sigma^\diamond_{k,r,x,x'}\left(E_r^\diamond(\varepsilon_r')^2\right) = (1 - \tau(x,x'))\tilde\sigma^\diamond_{k,r,x,x'}\left(E_r^\diamond(\varepsilon_r')^2\right).$$

If x and x' are both in $E_r^\diamond(\tilde\varepsilon_r)$ then $\tau(x,x')=1$, and the claim follows. When $x\notin E_r^\diamond(\tilde\varepsilon_r)$ we get from (13.23) that $\nu_{k,r,x}^{(n)}(E_r^\diamond(\varepsilon_r'))=0$. Then

$$\tilde\sigma_{k,r,x,x'}^\diamond(E_r^\diamond(\varepsilon_r')^2)\leqslant\sigma_{k,r,x}^\diamond(E_r^\diamond(\varepsilon_r'))=\nu_{k,r,x}^{(n)}(E_r^\diamond(\varepsilon_r'))=0.$$

$x'\notin E_r^\diamond(\tilde\varepsilon_r)$ 的情形与此类似。因此，在情形 (b) 中也有 $\hat\sigma_{k,\,r,\,x,\,x'}^\diamond(E_r^\diamond(\varepsilon_r')^2)=0$。

By part (ii) of Proposition 15.1, it follows that $\widetilde{\mathcal{Q}}_r\Psi_{k,r}(u,u')\leqslant\log\Omega_r+K_r$ for $\hat\sigma_{k,r,x,x'}^\diamond$-almost every $(u,u')\in E_r^\diamond(\varepsilon_r)^2$. Integrating with respect to $\hat\sigma_{k,r,x,x'}^\diamond$ we get that $\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}\Psi_{k,r}(x,x')\leqslant\log\Omega_r+K_r$ as claimed. □

Proposition 16.4. *There exist $\kappa_r'''=\kappa_r'''(\nu_\infty)>0$ and $C_r'''=C_r'''(\nu_\infty)>0$ such that given any $\delta>0$ and $n\geqslant N_r$ the following holds for every $k\geqslant\check k_r$:*

(i) *For any $(x,x')\in E_r^\diamond(\varepsilon_r)^2\setminus\mathrm{Diag}_r$,*

$$\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}\Psi_{k,r}(x,x')\leqslant\Psi_{k,r}(x,x')+C_r'''n.$$

(ii) *For any $(x,x')\in E_r^\diamond(\varepsilon_r)^2\setminus\mathrm{Diag}_r$ with $\mathrm{VA}_r(x,x')\geqslant\omega_{k,r}$,*

$$\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}\Psi_{k,r}(x,x')\leqslant\Psi_{k,r}(x,x')+C_r'''(1+\delta n).$$

(iii) *For any $(x,x')\in E_r^\diamond(\varepsilon_r''')\setminus\mathrm{Diag}_r$ with $\mathrm{VA}_r(x,x')\geqslant\omega_{k,r}$,*

$$\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}\Psi_{k,r}(x,x')\leqslant\Psi_{k,r}(x,x')-(\kappa_r'''-C_r'''\delta)n.$$

证明。取 $\kappa_r'''=\kappa_r''$ 和 $C_r'''=C_r''+K_r$，并令 $k\geqslant\check k_r$。我们将论证分为四种情况（参见图 8）。

First, suppose that both x and x' are in the $\nu_k^{(n)}$-core of $E_r^\diamond(\varepsilon_r)$, and at least one of them is in $E_r^\diamond(\varepsilon_r'')$. This is necessarily the case in the setting of (iii). In particular (x,x') is outside the cut-off region, which means that $\Psi_{k,r}(x,x')>\log\Omega_r$, and there is no recoupling either:

$$\hat\sigma_{k,r,x,x'}^\diamond=\nu_{k,r,x,x'}^{(n)},\quad\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}\Psi_{k,r}(x,x')=\int_G\widetilde{\mathcal{Q}}_r\Psi_{k,r}(gx,gx')\,d\nu_k^{(n)}(g).$$

Hence the claims in (i), (ii) and (iii) are contained in Corollary 15.5.

Now suppose that both x and x' are in the $\nu_k^{(n)}$-core of $E_r^\diamond(\varepsilon_r)$ but outside $E_r^\diamond(\varepsilon_r'')$, and at least one of them is in $E_r^\diamond(2\varepsilon_r'')$. It is still true that (x,x') is outside the cut-off region, and so $\Psi_{k,r}(x,x')>\log\Omega_r$. Thus the estimates in Corollary 15.5 remain valid for

$$\int_{E_r^\diamond(\varepsilon_r)^2}\widetilde{\mathcal{Q}}_r\Psi_{k,r}\,d\tilde\sigma_{k,r,x,x}^\diamond=\int_G\widetilde{\mathcal{Q}}_r\Psi_{k,r}(gx,gx')\,d\nu_k^{(n)}(g).\tag{16.13}$$

根据 (16.9)，测度 $\zeta_{k,r,x,x'}$ 在 $E_r^\diamond(\varepsilon_r')^2$ 上消失。因此，命题 15.1 的 (ii) 部分给出：

$$\int_{E_r^\diamond(\varepsilon_r)^2}\widetilde{\mathcal{Q}}_r\Psi_{k,r}\,d\zeta_{k,r,x,x}\leqslant\log\Omega_r+K_r\leqslant\Psi_{k,r}(x,x')+K_r.\tag{16.14}$$

根据定义（16.10），$\widehat{\mathcal{T}}_{k\,,\,r}^{\mathcal{Q}}\Psi_{k,\,r}(x,x')$ 是（16.13）和（16.14）中的积分的凸组合。因此，在这种情况下，结论（i）和（ii）成立。

Next suppose that both x and x' are in the $\nu_k^{(n)}$-core of $E_r^\diamond(\varepsilon_r)$ but outside $E_r^\diamond(2\varepsilon_r'')$. This corresponds to case (b) of Lemma 16.3: claims (i) and (ii) are contained in the conclusion of that lemma. Finally, suppose that at lest one of

the points x and x' is in the $\nu_k^{(n)}$-border of $E_r^\diamond(\varepsilon_r)$. This is precisely the situation in case (a) of Lemma 16.3, and so claims (i) and (ii) are again contained in the conclusion of that lemma. □

16.3. **Contradicting** $\dim E = r$**.** Now we are going to apply Lemma 11.3 with $X = E_r^\diamond(\varepsilon_r)^2$, $\mathcal{T} = \widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}$, $\psi = \Psi_{k,r}$, $\hat\eta = \hat\eta_{k,r}$,

$$A_k = \big\{(x,x') \in E_r^\diamond(\varepsilon_r)^2 : \mathrm{VA}_r(x,x') > \omega_{k,r},\ \mathrm{VA}_r(x',x) > \omega_{k,r}, \\ d(F_r, E) \leqslant \varepsilon_r''',\ \text{and } d(F_r', E) \leqslant \varepsilon_r'''\big\}, \tag{16.15}$$

$$B_k' = \big\{(x,x') \in E_r^\diamond(\varepsilon_r)^2 : \mathrm{VA}_r(x,x') > \omega_{k,r},\ \mathrm{VA}_r(x',x) > \omega_{k,r}, \\ \text{and } d(F_r, E) > \varepsilon_r''' \text{ or } d(F_r', E) > \varepsilon_r'''\big\}, \tag{16.16}$$

$$B_k^1 = \big\{(x,x') \in E_r^\diamond(\varepsilon_r)^2 : \mathrm{VA}_r(x,x') \leqslant \omega_{k,r}\big\}, \tag{16.17}$$

$$B_k^2 = \big\{(x,x') \in E_r^\diamond(\varepsilon_r)^2 : \mathrm{VA}_r(x',x) \leqslant \omega_{k,r}\big\}. \tag{16.18}$$

The sets $A = A_k$, $B' = B_k'$, and $B'' = B_k^1 \cup B_k^2$ are pairwise disjoint, and their union is the whole $E_r^\diamond(\varepsilon_r)^2$. Moreover, (7.2) implies that $B'' = \varnothing$ when $\dim E = r$.

Proposition 16.4 shows that, as long as k is sufficiently large, the hypotheses of Lemma 11.3 are satisfied for these choices, with

$$\kappa_A = (\kappa_r''' - C_r'''\delta)n, \quad \kappa_B' = C_r'''(1+\delta n), \text{ and } \kappa_B'' = C_r'''n.$$

Take $\delta > 0$ to be sufficiently small, depending on ν_∞, and $n \in \mathbb{N}$ to be sufficiently 较大，取决于 ν_∞ 和 δ，以至于

$$\kappa_A > 9\kappa_B'. \tag{16.19}$$

Using Proposition 6.6, we find a self-coupling $\hat\eta_0^\diamond$ of $\eta_{k,r}^\diamond$ vanishing on a neighborhood of $\mathrm{Diag}_r \subset E_r^\diamond(\varepsilon_r)^2$. Observe that $\hat\eta_0^\diamond$ projects to $\eta_{k,r}$ under $f \circ \pi_i^\diamond$ for any $i = 1, 2$. Then the same is true about every $\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}$-iterate of $\eta_0^\diamond$, by (16.12). Starting from $\hat\eta_0^\diamond$ and arguing as in the proof of Proposition 6.23, we find a sequence $(\hat\eta_{k,r,j}^{\mathcal{Q}})_j$ of probability measures on $E_r^\diamond(\varepsilon_r)^2$ projecting to $\eta_{k,r}$ under $f \circ \pi_i^\diamond$ for any $i = 1, 2$, converging to a $\widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}$-invariant measure $\hat\eta_{k,r}^{\mathcal{Q}}$, and satisfying $\int_{E_r^\diamond(\varepsilon_r)^2} \Psi_{k,r}\, d\hat\eta_{k,r,j}^{\mathcal{Q}} < \infty$ and

$$\int_{E_r^\diamond(\varepsilon_r)^2} \widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}} \Psi_{k,r}(x,x')\, d\hat\eta_{k,r,j}^{\mathcal{Q}}(x,x') \geqslant \int_{E_r^\diamond(\varepsilon_r)^2} \Psi_{k,r}(x,x')\, d\hat\eta_{k,r,j}^{\mathcal{Q}}(x,x')$$

for every j. Applying Lemma 11.3 with $\mathcal{T} = \widehat{\mathcal{T}}_{k,r}^{\mathcal{Q}}$ and $\hat\eta = \hat\eta_{k,r,j}^{\mathcal{Q}}$, we get that

$$\hat\eta_{k,r,j}^{\mathcal{Q}}(B'') \geqslant \frac{\kappa_A \hat\eta_{k,r,j}^{\mathcal{Q}}(E_r^\diamond(\varepsilon_r)^2) - (\kappa_A + \kappa_B')\hat\eta_{k,r,j}^{\mathcal{Q}}(B')}{\kappa_A + \kappa_B''} \tag{16.20}$$

对于每个 j。当 $j \to \infty$ 时取极限，我们得出：

$$\hat\eta_{k,r}^{\mathcal{Q}}(B'') \geqslant \frac{\kappa_A \hat\eta_{k,r}^{\mathcal{Q}}(E_r^\diamond(\varepsilon_r)^2) - (\kappa_A + \kappa_B')\hat\eta_{k,r}^{\mathcal{Q}}(B')}{\kappa_A + \kappa_B''}. \tag{16.21}$$

根据定义，B' 包含于原像 $(f \circ \pi_i^\diamond)^{-1}(E_r(\varepsilon_r, \varepsilon_r'''))$、$i = 1$的并集中。因此，利用（16.3），

$$\hat\eta_{k,r}^{\mathcal{Q}}(B') \leqslant 2\eta_{k,r}(E_r(\varepsilon_r, \varepsilon_r''')) < \frac{4}{10}\eta_{k,r}(E_r(\varepsilon_r)). \tag{16.22}$$

显然，$\hat{\eta}^{\mathcal{Q}}_{k,\,r}(E^{\diamond}_r(\varepsilon_r)^2) = \eta_{k,\,r}(E_r(\varepsilon_r))$。将这些关系代入 (16.21)，并利用 (16.19)，我们得到

$$\hat{\eta}^{\mathcal{Q}}_{k,r}(B'') \geqslant \frac{\kappa_A - \frac{4}{10}(\kappa_A + \kappa'_B)}{\kappa_A + \kappa''_B}\eta_{k,r}(E_r(\varepsilon_r)) \geqslant \frac{5\kappa'_B}{\kappa_A + \kappa''_B}\eta_{k,r}(E_r(\varepsilon_r)) > 0. \tag{16.23}$$

When $\dim E = r$ this is a contradiction, because B'' is empty in that case. Thus $\dim E \geqslant r+1$.

16.4. **Completing step r.** By (16.23), there exists $i = 1, 2$ such that

$$\hat{\eta}^{\mathcal{Q}}_{k,r}(B^i_k) \geqslant \frac{2\kappa'_B}{\kappa_A + \kappa''_B}\eta_{k,r}(E_r(\varepsilon_r)). \tag{16.24}$$

It is no restriction to assume that $i = 1$, as the other case can be deduced just by exchanging the roles of x and x'.

Consider the map

$$\Sigma : E^{\diamond}_r(\varepsilon_r)^2 \setminus \mathrm{Diag}_r \to \mathrm{Gr}(r+1, d), \quad \Sigma(x, x') = F'_1 + F_r$$

and the compact topological spaces

$$\mathcal{Y}_r = \{(x, x', y) \in \mathcal{F}(r, d)^2 \times \mathrm{Gr}(r+1, d) : F'_1 \subset y \text{ and } F_r \subset y\}$$
$$\mathcal{Y}_r(\varepsilon) = \{(x, x', y) \in \mathcal{Y}_r : x, x' \in E^{\diamond}_r(\varepsilon)\} \text{ for } \varepsilon > 0,$$

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$$p_1 : \mathcal{Y}_r \to \mathcal{F}(r, d)^2, \quad (x, x', y) \mapsto (x, x')$$
$$p_2 : \mathcal{Y}_r \to \mathrm{Gr}(r+1, d), \quad (x, x', y) \mapsto y.$$

For $(x, x', y) \in \mathcal{Y}_r$ and $n \in \mathbb{N}$, denote by $\nu^{(n)}_{k,r,x,x',y}$ the image of $\nu^{(n)}_k$ under the diagonal action

$$G \to \mathcal{Y}_r, \quad (g \mapsto (gx, gx', gy).$$

显然，每个 $\nu^{(n)}_{k\ r\ x'',,x',y}$ 都是相对于 p_1 的 $\nu^{(n)}_{k\ r\ x'',,x'}$ 的一个提升： $\mathcal{Y}_r \to \mathcal{F}(r, d)^2$。$\mathrm{Diag}_r$ 在 $\mathcal{F}(r,d)^2$ 中的补集通过.....嵌入于 $\mathcal{Y}_r$ 中。

$$(x, x') \mapsto (x, x', F'_1 + F_r).$$

In particular, every measure ξ on $\mathcal{F}(r, d)^2$ that vanishes on Diag_r has a (unique) lift $\check{\xi}$ to $\mathcal{Y}_r$.

由关系式 (11.3) 和 (16.10)，我们可以看出，

$$\hat{\sigma}^{\diamond}_{k,r,x,x'} = \big(1 - \check{\omega}(x, x')\big)\,\nu^{(n)}_{k,r,x,x'} + \check{\omega}(x, x')\hat{\theta}_{k,r,x,x'} \tag{16.25}$$

where $\check{\omega} : E^{\diamond}_r(\varepsilon_r)^2 \to [0, 1]$ is a continuous function that vanishes identically on $E^{\diamond}_r(\varepsilon''_r)^2$, and each $\hat{\theta}_{k,r,x,x'}$ is a coupling of $\sigma^{\diamond}_{k,1,x}$ and $\sigma^{\diamond}_{k,1,x'}$ vanishing on a uniform neighborhood of Diag_r. In view of the previous remarks, it follows that the $\hat{\sigma}^{\diamond}_{k,r,x,x'}$ lift to probability measures

$$\check{\sigma}_{k,r,x,x',y} = \big(1 - \check{\omega}(x, x')\big)\,\nu^{(n)}_{k,r,x,x',y} + \check{\omega}(x, x')\check{\theta}_{k,r,x,x',y} \tag{16.26}$$

on $\mathcal{Y}_r(\varepsilon_r)$, where $\check{\theta}_{k,r,x,x',y}$ is the unique lift of $\hat{\theta}_{k,r,x,x',y}$. This lift is continuous: it is clear that $\check{\nu}^{(n)}_{k,r,x,x',y}$ varies continuously and, by uniqueness, so does $\check{\theta}_{k,r,x,x',y}$.

We claim that the spreading out measures $\tilde{q}_{r,x,x'}$ in (15.6) also lift continuously to measures $\check{q}_{r,x,x',y}$ on $\mathcal{Y}_r(\varepsilon_r)$. Indeed, it is clear that $\{\delta_{(x,x',y)} : (x, x',) \in \mathcal{Y}_r(\varepsilon_r)\}$

is a continuous lift of $\{\delta_{(x,x')} : (x,x') \in E_r^\diamond(\varepsilon_r)^2\}$, and so it suffices to show that the family

$$\{\mu^*_{F_r} \times \mu^*_{F'_r} : (x,x') \in E_r^\diamond(\varepsilon_r, \varepsilon'_r/2) \times E_r^\diamond(3\varepsilon''_r) \cup E_r^\diamond(3\varepsilon''_r) \times E_r^\diamond(\varepsilon_r, \varepsilon'_r/2)\}$$

唯一地提升到 $\mathcal{Y}_r(\varepsilon_r)$。后者直接源于这样一个事实:

$$\left(\mu^*_{F_r} \times \mu^*_{F'_r}\right)(\mathrm{Diag}_r) = 0 \tag{16.27}$$

for any $(F_r, F'_r) \in E_r(\varepsilon_r, \varepsilon'_r/2) \times E_r(3\varepsilon''_r) \cup E_r(3\varepsilon''_r) \times E_r(\varepsilon_r, \varepsilon'_r/2)$, To prove (16.27), let us write $u = (G_1, \dots, G_r)$ and $u' = (G'_1, \dots, G'_r)$. By definition,

$$\begin{aligned}\left(\mu^*_{F_r} \times \mu^*_{F'_r}\right)(\mathrm{Diag}_r) \leqslant \mu^*_{F'_r}(\{u' \in \mathcal{F}(F'_r) : G'_1 \subset F_r\}) \\ + \mu^*_{F_r}(\{u \in \mathcal{F}(F_r) : G_1 \subset F'_r\})\end{aligned} \tag{16.28}$$

The key point is that in this setting we always have $F_r \neq F'_r$. Thus the set of $u' \in \mathcal{F}(F'_r)$ such that $G'_1 \subset F_r$ is a subvariety of $\mathcal{F}(F'_r)$ of strictly smaller dimension, and so it has zero $\mu^*_{F'_r}$-measure. Thus the first term on the right-hand side of (16.28) vanishes identically, and then so does the second term, by symmetry. This proves the claim.

These observations ensure that the Markov operator $\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r} = \widehat{\mathcal{T}}_{k,r} \circ \widetilde{\mathcal{Q}}_r$ admits a continuous lift

$$\check{\mathcal{T}}^{\mathcal{Q}}_{k,r} : \mathcal{B}(\mathcal{Y}_r(\varepsilon_r)) \to \mathcal{B}(\mathcal{Y}_r(\varepsilon_r)),\ \check{\mathcal{T}}^{\mathcal{Q}}_{k,r}\Phi(x,x',y) = \int_{\mathcal{Y}_r(\varepsilon_r)} \Phi\, d\check{\sigma}^{\mathcal{Q}}_{k,r,x,x',y},$$

given by

$$\check{\sigma}^{\mathcal{Q}}_{k,r,x,x',y} = \int_{\mathcal{Y}_r(\varepsilon_r)} \check{q}_{r,u,u',v}\, d\check{\sigma}_{k,r,x,x',y}(u,u',v).$$

Since the measure $\hat{\eta}^\diamond_0$ was taken to vanish on a neighborhood of $\mathrm{Diag}_r \subset E_r^\diamond(\varepsilon_r)^2$, it also admits a (unique) lift $\check{\eta}^\diamond_0$ to $\mathcal{Y}_r(\varepsilon_r)$. Applying the construction in Proposition 6.23 simultaneously to the operators $\widehat{\mathcal{T}}^{\mathcal{Q}}_{k,r}$ and $\check{\mathcal{T}}^{\mathcal{Q}}_{k,r}$, starting from $\hat{\eta}^\diamond_0$ and $\check{\eta}^\diamond_0$ respectively, we find a sequence $(\check{\eta}^{\mathcal{Q}}_{k,r,j})_j$ of probability measures on $\mathcal{Y}_r(\varepsilon_r)$ converging to a $\check{\mathcal{T}}^{\mathcal{Q}}_{k,r}$-invariant probability measure $\check{\eta}^{\mathcal{Q}}_{k,r}$ and such that (up to restricting to a subsequence) each $\check{\eta}^{\mathcal{Q}}_{k,r,j}$ projects to $\hat{\eta}^{\mathcal{Q}}_{k,r,j}$ under p_1. Observe that $\check{\eta}^{\mathcal{Q}}_{k,r}$ and each $\check{\eta}^{\mathcal{Q}}_{k,r,j}$ project to $\eta_{k,r}$ under $f \circ \pi^\diamond_i \circ p_1$ for $i = 1, 2$.

Next, define $\eta_{k,r+1} = p_{2*}\check{\eta}^{\mathcal{Q}}_{k,r}$ and let $\{d\check{\eta}_{k,r,v} : v \in p_2\mathcal{Y}_r(\varepsilon_r)\}$ be a disintegration of $\check{\eta}^{\mathcal{Q}}_{k,r}$ with respect to the partition $\{p_2^{-1}(v) : v \in p_2\mathcal{Y}_r(\varepsilon_r)\}$. Then define

$$\begin{aligned}&\mathcal{T}_{k,r+1} : \mathcal{B}(p_2\mathcal{Y}_r(\varepsilon_r)) \to \mathcal{B}(p_2\mathcal{Y}_r(\varepsilon_r)),\\ &\mathcal{T}_{k,r+1}\Phi(y) = \int_{p_2^{-1}(y)} \check{\mathcal{T}}^{\mathcal{Q}}_{k,r}(\Phi \circ p_2)(x,x',y)\, d\check{\eta}_{k,r,y}(x,x').\end{aligned}$$

Equivalently, $\mathcal{T}_{k,r+1}\Phi(y) = \int_{p_2\mathcal{Y}_r(\varepsilon_r)} \Phi\, d\sigma_{k,r+1,y}$ with

$$\sigma_{k,r+1,y} = \int_{p_2^{-1}(y)} p_{2*}\check{\sigma}^{\mathcal{Q}}_{k,r,x,x',y}\, d\check{\eta}_{k,r,y}(x,x'). \tag{16.29}$$

Let $\mathcal{B}_k = p_2 p_1^{-1}(B^1_k) = \{F'_1 + F_r : (x,x') \in B^1_k\}$, where B^1_k is as in the previous section. Define also $\eta_{k,r+1,j} = p_{2*}\check{\eta}^{\mathcal{Q}}_{k,r,j}$ for $j \in \mathbb{N}$. Then

$$\eta_{k,r+1,j}(\mathcal{B}_k) \geqslant \check{\eta}^{\mathcal{Q}}_{k,r,j}\left(p_1^{-1}(B^1_k)\right) = \hat{\eta}^{\mathcal{Q}}_{k,r,j}(B^1_k).$$

Passing to the limit as $j \to \infty$, and arguing as in (16.20)–(16.23), we find from (16.24) that

$$\eta_{k,r+1}(\mathcal{B}_k) \geqslant \frac{5\kappa'_B}{\kappa_A + \kappa''_B}\eta_{k,r}(E_r(\varepsilon_r)).$$

现在，由于$\omega_{k,\ r} \to 0$，定义 (16.17) 表明，$\mathcal{B}_k$ 收敛于 E_r ，当$k \to \infty$时。因此，$\eta_{k,\ r+1}$ 的任意聚点$\eta_{\infty,\ r+1}$ 都必须满足

$$\eta_{\infty,r+1}(E_{r+1}) \geqslant \frac{5\kappa'_B}{\kappa_A + \kappa''_B}\eta_{\infty,r}(E_r(\varepsilon_r)) \geqslant \frac{5\kappa'_B}{\kappa_A + \kappa''_B}\eta_{\infty,r}(E_r) > 0. \tag{16.30}$$

取 $n_{r+1} = n$ 和 $\varepsilon_{r+1} = \varepsilon''_r$。令 $\nu_{k\ r+1}^{(n_{r+1})}{}_{,,y}$ 表示 $\nu_k^{(n_{r+1})}$ 在映射 $G \to \mathbf{Gr}(r+1, d)$, $g \mapsto gy$ 下的推前。

引理 16.5。

(i) $\sigma_{k,r+1,y} = \nu_{k,r+1,y}^{(n_{r+1})}$ *for every* $y \in E_{r+1}(\varepsilon_{r+1})$.

(ii) $\sigma_{k,r+1,y}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}} E_{r+1}(\varepsilon_{r+1})\right) = 0$ *for every* $y \notin E_{r+1}(\varepsilon_{r+1})$

(iii) *The measure* $\eta_{k,r+1}$ *is* $\mathcal{T}_{k,r+1}$*-invariant.*

证明。显然，$\nu_{k\ r+1}^{(n_{r+1})}{}_{,,y}$ 与 $\nu_{k\ r\ x\ x'}^{(n_{r+1})}{}_{,,,,y}$ 在投影 p_2下的推前相一致。因此，由（16.26）可得，

$$p_{2*}\check{\sigma}_{k,r,x,x',y} = \left(1 - \check{\omega}(x,x')\right)\nu_{k,r+1,y}^{(n_{r+1})} + \check{\omega}(x,x')p_{2*}\check{\theta}_{k,r,x,x',y},$$

and so,

$$\begin{aligned}\check{\sigma}_{k,r+1,y} = &\left(1 - \int_{p_2^{-1}(y)} \check{\omega}(x,x')\, d\check{\eta}_{k,r,y}(x,x')\right)\nu_{k,r+1,y}^{(n_{r+1})} \\ &+ \int_{p_2^{-1}(y)} \check{\omega}(x,x')p_{2*}\check{\theta}_{k,r,x,x',y}\, d\check{\eta}_{k,r,y}(x,x'),\end{aligned}$$

If $y \in E_{r+1}(\varepsilon_{r+1})$ then both x and x' are necessarily in $E_r^{\diamond}(\varepsilon_{r+1})$, by (7.1), in which case $\check{\omega}(x,x') = 0$. Then $\sigma_{k,r+1,y} = \nu_{k,r+1,y}^{(n_{r+1})}$, as claimed in (i).

In view of (16.29), to prove part (ii) it suffices to show that if $y \notin E_{r+1}(\varepsilon_{r+1})$ then

$$p_{2*}\check{\sigma}_{k,r,x,x',y}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}} E_{r+1}(\varepsilon_{r+1})\right) = 0 \tag{16.31}$$

for any $x, x' \subset y$. If x and x' are both in $E_r^{\diamond}(\varepsilon_{r+1})$ then

$$p_{2*}\check{\sigma}_{k,r,x,x',y} = p_{2*}\nu_{k,r,x,x',y}^{(n_{r+1})} = \nu_{k,r+1,y}^{(n_{r+1})}$$

然后，由注记 6.26 可得该结论。以下，我们假设其中一个点（不妨设为 x ）不属于 $E_r^{\diamond}(\varepsilon_{r+1})$。根据定义，

$$p_2^{-1}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}} E_{r+1}(\varepsilon_{r+1})\right) \subset \mathcal{X}^{\#}_{\nu_k^{(n)}} E_r^{\diamond}(\varepsilon_{r+1})^2 \times \mathcal{X}^{\#}_{\nu_k^{(n)}} E_{r+1}(\varepsilon_{r+1}),$$

and so

$$
\begin{aligned}
& p_{2*}\check{\sigma}_{k,r,x,x',y}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_{r+1}(\varepsilon_{r+1})\right) \\
& \qquad \leqslant \check{\sigma}_{k,r,x,x',y}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_r^{\diamond}(\varepsilon_{r+1})^2 \times \mathcal{X}^{\#}_{\nu_k^{(n)}}E_{r+1}(\varepsilon_{r+1})\right) \\
& \qquad \leqslant \hat{\sigma}^{\diamond}_{k,r,x,x'}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_r^{\diamond}(\varepsilon_{r+1})^2\right) \\
& \qquad \leqslant \sigma^{\diamond}_{k,r,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_r^{\diamond}(\varepsilon_{r+1})\right).
\end{aligned} \tag{16.32}
$$

如果 x 位于 $E_r^{\diamond}(\varepsilon_r)$ 的 $\nu_k^{(n)}$ 边缘，则

$$
\sigma^{\diamond}_{k,r,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_r^{\diamond}(\varepsilon_{r+1})\right) \leqslant \sigma^{\diamond}_{k,r,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_r^{\diamond}(\varepsilon_r)\right) = 0,
$$

because the operator $\mathcal{T}_{k,1}$ is adapted to $(\nu_k^{(n)}, E_r^{\diamond}(\varepsilon_r))$. If x is in the $\nu_k^{(n)}$-core of $E_r^{\diamond}(\varepsilon_r)$ then Remark 6.26 gives that

$$
\sigma^{\diamond}_{k,r,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_r^{\diamond}(\varepsilon_{r+1})\right) = \nu^{(n)}_{k,r,x}\left(\mathcal{X}^{\#}_{\nu_k^{(n)}}E_r^{\diamond}(\varepsilon_{r+1})\right) = 0.
$$

Thus the right-hand side of (16.32) vanishes in either case. That completes the proof of (16.31) and of part (ii) of the lemma.

Finally, by definition,

$$
\begin{aligned}
& \int_{p_2\mathcal{Y}_r(\varepsilon_r)} (\mathcal{T}_{k,r+1}\Phi)\, d\eta_{k,r+1} \\
& \qquad = \int_{p_2\mathcal{Y}_r(\varepsilon_r)} \int_{p_2^{-1}(y)} \check{\mathcal{T}}_{k,r}\left(\Phi \circ p_2\right)(x,x',y)\, d\check{\eta}_{k,r,y}(x,x')\, d\eta_{k,r+1}(y) \\
& \qquad = \int_{\mathcal{Y}_r(\varepsilon_r)} \check{\mathcal{T}}_{k,r}(\Phi \circ p_2)(x,x',y)\, d\check{\eta}_{k,r}(x,x',y)
\end{aligned}
$$

for any $\Phi \in \mathcal{B}(p_2\mathcal{Y}_r(\varepsilon_r))$. Since $\check{\eta}_{k,r}$ is $\check{\mathcal{T}}_{k,r}$-invariant, this gives

$$
\begin{aligned}
\int_{p_2\mathcal{Y}_r(\varepsilon_r)} (\mathcal{T}_{k,r+1}\Phi)\,(y)\, d\eta_{k,r+1}(y) &= \int_{\mathcal{Y}_r(\varepsilon_r)} (\Phi \circ p_2)(x,x',y)\, d\check{\eta}_{k,r}(x,x',y) \\
&= \int_{p_2\mathcal{Y}_r(\varepsilon_r)} \Phi\, d\hat{\eta}_{k,r+1},
\end{aligned}
$$

这就证明了断言 **(iii)**。 □

Since the $\check{\sigma}_{k,r,x,x',y}$ are generic measures and the projection p_2 is algebraic, it follows readily from (16.29) and Remark 5.1 that every $\sigma_{k,r+1,y}$ is a generic measure. Then, conclusions (i) and (ii) in Lemma 16.5 allow us to apply Propositions 6.15 and 6.25 with $E_{r+1}(\varepsilon_{r+1})$ and $p_2\mathcal{Y}_r(\varepsilon_r)$ in the roles of X and U, respectively. Thus we get a continuous Markov operator adapted to $(\nu_k^{(n_{r+1})}, E_{r+1}(\varepsilon_{r+1}))$ and which leaves the restriction of $\eta_{k,r+1} \mid E_{r+1}(\varepsilon_{r+1})$ invariant. Replace $\mathcal{T}_{k,r+1}$ and $\eta_{k,r+1}$ with these new Markov operator and invariant measure, respectively. This finishes step r of the induction.

定理 **7.1** 的证明现已完成。

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IMPA, Est. D. Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro, Brazil and Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland
Email address: avila@impa.br

Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
Email address: eskin@math.uchicago.edu

IMPA, Est. D. Castorina 110, Jardim Botânico, 22460-320 Rio de Janeiro, Brazil
Email address: viana@impa.br