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Nikulin 曲面上曲线的双覆盖

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谨以此文献给彼得·纽斯特德，以庆祝他的 80^{th} 岁生日。

摘要。我们综述了关于 Prym 簇的基本结果、由 Welters 开创的 Prym–Brill–Noether 理论，以及亏格为 $g \geq 2$ 的曲线的一般 étale 双覆盖的 Brill–Noether 理论。随后，我们将研究具体化到 Nikulin 曲面上的曲线，并证明，标准型 Nikulin 曲面上的曲线的 étale 双覆盖不满足 Welters 定理。另一方面，通过将研究具体化到非标准型 Nikulin 曲面上的曲线，我们证明，在 $b = 2$，4，6 个点处分歧的曲线的一般双覆盖是 Brill–Noether 一般的； $b = 2$ 的情形已由 Bud [Bu] 使用不同的方法得到。

1. 引言

复曲线的双覆盖是代数几何中一个经典且至今仍十分热门的研究课题。人们对这一课题的兴趣部分源于这样一个事实：任意一个亏格为 $g \geq 2$ 的光滑不可约曲线 C 的 étale 双覆盖 $\pi : \widetilde{C} \to C$，都会自然地定义一个维数为 $g-1$ 的主极化阿贝尔簇（P，Ξ），称为 π 的 Prym 簇。Prym 簇最初由 Schottky 和 Jung [SJ] 在研究 Schottky 问题时引入，随后 Mumford（[M1]）以德国数学家 Prym 的名字为其命名。Mumford 也是第一位从代数角度研究 Prym 簇的学者。尽管这一课题已有大量文献，关于一般双覆盖的几何性质，仍有许多问题尚未解决，无论是在 étale 情形还是在分歧情形下都是如此。本文将重点关注 Brill-Noether 类型的问题。

我们回顾一下，自 20 世纪 80 年代以来，人们对亏格为 g 的一般曲线 C 的布里尔–诺特理论已有相当深入的了解。布里尔–诺特定理指出，参数化曲线上度为 d、全局截面空间维数为 $\geq r+1$ 的线丛的布里尔–诺特簇 $W^r_d(C)$，当且仅当所谓的布里尔–诺特数 $\rho(g,\ r,\ d) := g-(r+1)(g-d+r)$ 非负时才非空。此外，根据 Gieseker-Petri 定理，若 $W^r_d(C)$ 非空，则除 $W^{r+1}_d(C)$ 外，它是维数为 $\rho(g,\ r,\ d)$ 的光滑簇。布里尔–诺特定理中的存在性部分实际上适用于任意亏格为 g 的光滑曲线，这一结果归功于 Kleiman-Laksov [KL] 和 Kempf[Ke]。布里尔–诺特定理中的非存在性部分以及 Gieseker-Petri定理的最初证明均基于退化技术，分别由 Griffiths-Harris[Gr]、Eisenbud-Harris [EH1, EH2]，和 Gieseker [Gi] 完成。Lazarsfeld 提出了 Gieseker-Petri定理的另一种证明 [La, Pa]，避免了任何形式的退化，而改为专门研究位于 $K3$ 曲面上的光滑曲线，这是一项重大突破。

If $\pi : \widetilde{C} \to C$ is a double cover of a general genus $g \geq 2$ curve C, the curve $\widetilde{C}$ 显然在模空间中并非一般情形，因此研究其布里尔–诺特理论是有意义的。在 étale 情形下，$\widetilde{C}$的亏格为 $2g-1$，并且它绝非 Petri 一般的，因为 $W^1_{2g-2}(\widetilde{C})$ 结果是奇异的（参见 [We]以及下文第 3.2 节）。此外，如果 g 为偶数，则 C 具有一个次数为 g 的铅笔，这使其不可能是布里尔–诺特一般的（参见 [AF]以及本文后文第 4 节）。因此， C 的布里尔–诺特性质似乎相当复杂。若转而考虑.....，情况则要好得多。

~所谓的 CPrym-布里尔–诺特理论 V^r，是对 Prym-布里尔–诺特簇几何性质的研究。这些簇由 Welters [We] 引入，与~Prym 簇 P的几何性质相关；事实上，它们是通过将布里尔–诺特簇 $W^r_{2g-2}(C)$与 P 在 Picard 群$^{2g-2}(\widetilde{C})$中适当平移后的子簇相交而得到的。利用 Eisenbud 和 Harris 开发的退化技术，Welters 证明了（参见下文定理 2.3），对于一般的 π，簇 V^r 是光滑的，其维数为 $\rho^-(g,\ r)\colon = g-1-(r+1)r/2$。特别地，当 $\rho^-(g,\ r) < 0$时，它们为空；反过来，根据 Bertram [Ber]的一个结果，对于任意 étale 双覆盖 π，当 $\rho^-(g,\ r) \geq 0$时，V^r 非空。这些结论分别是 Prym-布里尔–诺特理论中布里尔–诺特定理和 Gieseker-Petri定理的对应版本。

或许是因为 étale 情形的复杂图景令人望而却步，直到最近，人们才开始关注分歧情形下 C 的 Brill-Noether~ ~理论。Bud [Bu] 证明，若 $\pi\colon\ C \to C$ 是一条亏格为 g 的一般曲线 C 在两个一般点处的双覆盖，那么 C 既是 Brill-Noether 一般的，也是 Petri 一般的（即同时满足 Brill-Noether 定理和 Gieseker-Petri~ 定理）。这一结果相当惊人，因为从 Brill-Noether 的角度看，分歧双覆盖的表现优于 étale 双覆盖。

本文源于这样一个愿望：让 $K3$ 曲面和 Lazarsfeld 的技巧进入研究视野。通过特化到 Nikulin 曲面即可实现这一点；Nikulin 曲面是带有双覆盖 $\pi_S\colon\ S \to S$ 的原始极化$\backsim K3$ 曲面（S, H），该双覆盖沿八条（-2）曲线分歧，这些曲线两两不交，且与线性系 $|H|$ 中的光滑曲线不交。S 的^ ~极小模型仍是一个带有对合 S 的 $K3$ 曲面，该对合有 8 个不动点。S 和 S 的 Picard 秩均为 ≥ 9；当等号成立时，它们的 Picard 群已在 [VGS, GS] 中得到描述。Nikulin 曲面有两种类型，取决于由 H 和（-2）曲线生成的秩 9 子格嵌入 Pic(S) 时是否为原始嵌入：按照 [KLV1], （S, H）中引入的术语，前一种情形称为标准型，后一种则称为非标准型。注意，π 在任意光滑曲线 $C \in |H|$ 上诱导出一个 étale 双覆盖 $\pi|_C\colon\ \widetilde{C} := \pi^{-1}(C) \to C$，而曲线 $\widetilde{C}$ 可~视为其在 S 中的像。过去十年间，Nikulin 曲面被广泛用于研究曲线的 étale 双覆盖（等价地，Prym 曲线）及其模空间$\mathcal{R}_h$。特别是，Farkas 和 Verra [FV2] 证明，对于 $6 \neq h \leq 7$，亏格为 h 的一般曲线的 étale 双覆盖可嵌入标准型 Nikulin 曲面，并利用这一事实描述了低亏格时 $\mathcal{R}_h$ 的~双有理几何。此外，在标准情形下，曲线 $C \subset S$ 的 gonality 与亏格为 h 的曲线 [的一般 étale 双覆盖相同 |AF。由此自然会想到，是否可以通过特化到嵌入标准型 Nikulin 曲面的 étale 双覆盖，来获得 Welters 定理的另一种证明。遗憾的是，我们将给出否定答案，并证明以下定理。

Theorem. *Let $\pi : \widetilde{C} \to C$ be an étale double cover on a Nikulin surface S of standard type, with C in the primitive linear system $|H|$ of genus h. If $h > 7$ or $h = 6$, the curve $\widetilde{C}$ does not satisfy Welters' Theorem.*

Nikulin surfaces of non-standard type have not received as much attention, and their investigation essentially started in [KLV1, KLV2]. As emerged in [KLV1], in the non-standard case the curve $\widetilde{C}$ is quite special as it possesses two theta characteristics with many sections cut out by two line bundles $R_1, R_2 \in \mathrm{Pic}(\widetilde{S})$. The main result of the paper suggests that Nikulin surfaces of non-standard type are instead the right environment to investigate double covers of curves ramified at $2, 4, 6$ points, which can be realized on a Nikulin surface of non-standard type by restricting π_S to the inverse image of smooth curves in the linear systems $|R_1|$ and $|R_2|$. By specialization to ramified double covers of curves on Nikulin surfaces of non-standard type, we obtain the following generalization of Bud's result.

Theorem. *Let $\pi : \widetilde{C} \to C$ be a general double cover of a genus $g \geq 2$ curve ramified at 2, 4, or 6 points. Then the curve $\widetilde{C}$ is Brill-Noether general.*

The paper is organized as follows. Section 2 surveys the basic theory of Prym varieties and Prym-Brill-Noether theory, with particular attention to Welters' infinitesimal study of V^r in terms of the Prym-Petri map, which is the anti-invariant part $\mu_{0,L}^-$ of the Petri map $\mu_{0,L}$ of any line bundle $L \in V^r$. The sections ends with a discussion and interpretation of its invariant counterpart $\mu_{0,L}^+$, as well as with a conjectural picture concerning its injectivity. Section 3 recalls some prerequisites on theta characteristics in order to show, following an argument by Beauville [Bea], that for any irreducible étale double cover $\pi : \widetilde{C} \to C$ the curve $\widetilde{C}$ possesses some invariant vanishing thetanulls: in particular, $\widetilde{C}$ is not Petri general. In Section 4 we concentrate on the Brill-Noether theory of $\widetilde{C}$ when π is general. More precisely, we recall Schwarz's non-existence result [Sc] concerning linear series whose Brill-Noether number is negative enough, and Aprodu-Farkas' theorem [AF] on the gonality of $\widetilde{C}$. The latter prevents $\widetilde{C}$ from being Brill-Noether general if the genus of C is even, while the Brill-Noether generality/speciality in the odd genus case is still unknown in full generality, up to our knowledge. We also recall Bud's result concerning the ramified case. Section 5 is focused on Nikulin surfaces of standard and non-standard type and on the proof of the above theorems.

2. Prym-Brill-Noether theory

2.1. **Preliminaries on Prym varieties.** Prym varieties are principally polarized abelian varieties arising from étale double covers of curves, and they are useful to link the geometry of curves to that of abelian varieties.

Let C be a complex smooth irreducible curve, and let $\pi : \widetilde{C} \to C$ be an irreducible étale double cover of C. Denoting by g and $\widetilde{g}$ the genus of C and $\widetilde{C}$, respectively, Hurwitz's formula yields $\widetilde{g} = 2g - 1$. The cover π induces the so-called *Norm map* between the Jacobians $J(C)$ and $J(\widetilde{C})$ of C and $\widetilde{C}$, respectively:

$$\begin{aligned} \mathrm{Nm}\colon J(\widetilde{C}) &\longrightarrow J(C) \\ \mathcal{O}_{\widetilde{C}}(D) &\longmapsto \mathcal{O}_C(\pi(D)). \end{aligned}$$

This fits in the following commutative diagram:

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{a_{y_0}} & J(\widetilde{C}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{Nm}} \\ C & \xrightarrow[a_{x_0}]{} & J(C), \end{array}$$

where a_{x_0} and a_{y_0} are the Abel-Jacobi maps associated with some fixed points $x_0 \in C$ and $y_0 \in \widetilde{C}$ such that $\pi(y_0) = x_0$.

In particular, the principally polarized Abelian varieties $(J(C)\ \Theta)$ and $(J(\widetilde{C}), \widetilde{\Theta})$ are related by two maps

$$\pi^* : J(C) \to J(\widetilde{C}), \quad \mathrm{Nm} : J(\widetilde{C}) \to J(C),$$

使得 π^* 与 Nm 互为对偶，且 $\mathrm{Nm} \circ \pi^*$： $J(C) \to J(C)$ 是乘以二（[M1]）。事实上，若 $N = \mathcal{O}_C(D) \in J(C)$，则 $\pi^* N = \mathcal{O}_{\widetilde{C}}(\pi^{-1}(D)) \in J(\widetilde{C})$，因此 $\mathrm{Nm}(\pi^* N) = \mathcal{O}_C(\pi(\pi^{-1}(D))) \in J(C)$；由于 π 是双覆盖，我们得到 $\pi(\pi^{-1}(D)) = 2D$。

Denote by $\iota : \widetilde{C} \to \widetilde{C}$ the involution that interchanges the sheets of π. For every divisor $\widetilde{D}$ on $\widetilde{C}$, the following equality is straightforward:

$$\pi^{-1}(\pi(\widetilde{D})) = \widetilde{D} + \iota(\widetilde{D}).$$

It follows that

$$\pi^*(\mathrm{Nm}(M)) = M \otimes \iota^*(M), \quad \forall \quad M \in J(\widetilde{C}).$$

Since Nm is surjective, we also get that

$$\iota^*(\pi^*(N)) = \pi^* N, \quad \forall \quad N \in J(C),$$

that is, the involution ι^* acts as the identity on $\pi^*(J(C)) \subset J(\widetilde{C})$. On the other hand, ι^* acts as -1 on $\mathrm{Ker}(\mathrm{Nm}) \subset J(\widetilde{C})$. In [M2] Mumford decomposed the kernel of Nm into two irreducible components:

$$P_0 := \{M \otimes \iota^* M^\vee \mid M \in \mathrm{Pic}^0(\widetilde{C})\},$$

$$P_1 := \{M \otimes \iota^* M^\vee \mid M \in \mathrm{Pic}^1(\widetilde{C})\}.$$

More precisely, he proved the following:

Lemma 2.1 ([M2] Lem. 1)**.** *If L is a line bundle on $\widetilde{C}$ such that $\mathrm{Nm}\, L \simeq \mathcal{O}_C$, then $L \simeq M \otimes \iota^* M^\vee$ for some line bundle M on $\widetilde{C}$. Moreover, M can be chosen of degree 0 or 1.*

A classical theorem by Wirtinger ([Wi]) yields that the dimension of the space of global sections is constant mod 2 on P_0 and P_1 and it has opposite parity on the two components. The component $P := P_0$ of $\mathrm{Ker}(\mathrm{Nm})$ containing the origin is an abelian subvariety of $J(\widetilde{C})$ of dimension

$$\dim P = \dim J(\widetilde{C}) - \dim J(C) = g - 1;$$

furthermore, it turns out that the canonical polarization of $J(\widetilde{C})$ restricts to twice a principal polarization Ξ on P. The principally polarized abelian variety (P, Ξ) is called the *Prym variety* associated with π (cf. [M1, §3] for more details).

By the same argument [M1], the inverse image $\mathrm{Nm}^{-1}(\omega_C) \subset \mathrm{Pic}^{2g-2}(\widetilde{C})$ of the canonical line bundle $\omega_C \in \mathrm{Pic}^{2g-2}(C)$ breaks up in two components

$$P^+ := \{L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) \simeq \omega_C,\ h^0(C, L) \equiv 0 \pmod 2\},$$

$$P^- := \{L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) \simeq \omega_C,\ h^0(C, L) \equiv 1 \pmod 2\},$$

which are both translates of P.

We recall that the cover $\pi : \widetilde{C} \to C$ defines a class of order two $\eta \in J(C)[2]$ such that $\pi_* \mathcal{O}_{\widetilde{C}} = \mathcal{O}_C \oplus \eta$; viceversa, any non-trivial $\eta \in J(C)[2]$ determines an étale double cover of C by setting $\widetilde{C} := \mathrm{Spec}(\mathcal{O}_C \oplus \eta)$. Therefore, the datum of π is equivalent to that of the pair (C, η), which is called a *Prym curve* of genus g. The moduli space

$$\mathcal{R}_g := \{(C, \eta) \,|\, C \text{ smooth curve of genus } g,\ \eta \in J(C)[2]\,,\, \eta \not\simeq \mathcal{O}_C\}\,,$$

在过去几十年中受到了广泛关注；关于其几何学的精彩综述，请参阅 [F3] 。

2.2. **Prym-Brill-Noether 簇与 Welters 定理**。在 [We] 中，Welters 开创性地研究了簇 P^- 和 P^+ 的 Brill-Noether 理论，并引入了 $\mathrm{Nm}^{-1}(\omega_C)$ 的以下闭子集，这些子集对任意整数 $r \geq -1$ 定义：

$$V^r(C,\eta) := \{L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) \simeq \omega_C,\, h^0(C,L) \geq r+1,\, h^0(C,L) \equiv r+1 \pmod 2\}.$$

Scheme-theoretically, these can be realized as the intersections

$$V^r(C,\eta) = W^r_{2g-2}(\widetilde{C}) \cap P^+ \quad \text{if } r \text{ is odd}, \tag{2.1}$$
$$V^r(C,\eta) = W^r_{2g-2}(\widetilde{C}) \cap P^- \quad \text{if } r \text{ is even},$$

where

$$W^r_{2g-2}(\widetilde{C}) = \left\{L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid h^0(C,L) \geq r+1\right\}$$

is the classical Brill-Noether variety; this justifies the name *Prym-Brill-Noether varieties* used for the loci $V^r(C,\eta)$. The following result proved in [M2] and [Ha] concerns their expected dimension.

Proposition 2.2. *Let $L \in \mathrm{Nm}^{-1}(\omega_C)$ and set $h^0(L) = r+1$. Then, the dimension of $V^r(C,\eta)$ at L satisfies*

$$\dim_L(V^r(C,\eta)) \geq g - 1 - \binom{r+1}{2}.$$

We will refer to the number

$$\rho^-(g,r) := g - 1 - \binom{r+1}{2},$$

as the *Prym-Brill-Noether number*.

A priori one may hope that classical results of Brill-Noether theory carry over to étale double covers of smooth curves and to the varieties $V^r(C,\eta)$, but unfortunately this is not the case. Indeed, many statements of this theory, such as the Brill-Noether Theorem and the Gieseker-Petri Theorem, hold only for a general curve $\widetilde{C}$ of genus $\widetilde{g}$ and they fail in the case where $\widetilde{C}$ is the étale double cover of a genus g curve. Nevertheless, an analogue of the Gieseker-Petri Theorem for Prym varieties was obtained by Welters [We] by a degeneration argument similar to the one used by Eisenbud and Harris [EH2] in their proof of the classical statement.

Since $\omega_{\widetilde{C}} = \pi^*\omega_C$, the push-pull formula yields the decomposition

$$H^0(\widetilde{C}, \omega_{\widetilde{C}}) = H^0(C, \omega_C) \oplus H^0(C, \omega_C(\eta)). \tag{2.2}$$

into invariant and anti-invariant forms under the action of ι. Fix a line bundle $L \in \mathrm{Nm}^{-1}(\omega_C)$ and set $h^0(L) = r+1$. Since the composition $\mathrm{Nm} \circ \pi^*$ is the multiplication by 2, the differential of Nm at L is $2({}^t\pi^*)$, where

$${}^t\pi^* : (H^0(\widetilde{C}, \omega_{\widetilde{C}}))^\vee \to (H^0(C, \omega_C))^\vee$$

is the transpose of the pull-back of differential forms. By taking kernels, we obtain the identification

$$T_L(\mathrm{Nm}^{-1}(\omega_C)) = (H^0(C, \omega_C(\eta)))^\vee \hookrightarrow (H^0(\widetilde{C}, \omega_{\widetilde{C}}))^\vee, \tag{2.3}$$

where the inclusion is the transpose of the projection map

$$\begin{aligned} p\colon H^0(\widetilde{C}, \omega_{\widetilde{C}}) &\longrightarrow H^0(C, \omega_C(\eta)) \\ \lambda &\longmapsto \frac{1}{2}(\lambda - \iota^*\lambda) \end{aligned}.$$

By classical Brill-Noether theory (cf., e.g., [ACGH]), we know that

$$T_L(W^r_{2g-2}(\widetilde{C})) = (\mathrm{Im}\mu_{0,L})^\perp \subset (H^0(\widetilde{C}, \omega_{\widetilde{C}}))^\vee, \tag{2.4}$$

where the so-called *Petri map*

$$\mu_{0,L} : H^0(\widetilde{C}, L) \otimes H^0(\widetilde{C}, \omega_{\widetilde{C}} \otimes L^\vee) \to H^0(\widetilde{C}, \omega_{\widetilde{C}})$$

is multiplication of global sections. In particular, the dimension of the tangent space at L to $W^r_{2g-2}(\widetilde{C})$ is:

$$\dim T_L(W^r_{2g-2}(\widetilde{C})) = \rho(2g-1, r, 2g-2) + \dim(\mathrm{Ker}\mu_{0,L}), \tag{2.5}$$

where the number $\rho(2g-1, r, 2g-2) := 2g - 1 - (r+1)^2$ equals the difference between the dimensions of the codomain and the domain of $\mu_{0,L}$. Recalling (2.1) and (2.3), we obtain:

$$T_L(V^r(C, \eta)) = (\mathrm{Im}\mu_{0,L})^\perp \cap (H^0(C, \omega_C(\eta))^\vee = (\mathrm{Im}(p \circ \mu_{0,L}))^\perp. \tag{2.6}$$

By applying π^* to the isomorphism $\mathrm{Nm}\, L \simeq \omega_C$, one gets $L \otimes \iota^* L \simeq \omega_{\widetilde{C}}$, or equivalently, $\omega_{\widetilde{C}} \otimes L^\vee \simeq \iota^* L$. Consider the following composition of maps:

$$H^0(\widetilde{C}, L) \otimes H^0(\widetilde{C}, L) \xrightarrow{1 \otimes \iota^*} H^0(\widetilde{C}, L) \otimes H^0(\widetilde{C}, \omega_{\widetilde{C}} \otimes L^\vee) \xrightarrow{\mu_{0,L}} H^0(C, \omega_{\widetilde{C}}) \xrightarrow{\quad p \quad} H^0(C, \omega_C(\eta))$$

$$s \otimes t \longmapsto s \otimes \iota^* t \longmapsto s \cdot \iota^* t \longmapsto \tfrac{1}{2}(s \cdot (\iota^* t) - (\iota^* s) \cdot t),$$

which is clearly skew-symmetric. By restriction to $\wedge^2 H^0(\widetilde{C}, L)$, we thus obtain the map

$$\begin{aligned} \mu^-_{0,L} \colon \wedge^2 H^0(\widetilde{C}, L) &\to H^0(C, \omega_C(\eta)) \\ s \wedge t &\mapsto \frac{1}{2}(s \cdot (\iota^* t) - (\iota^* s) \cdot t), \end{aligned}$$

which is called the *Prym-Petri map* of L. We can rewrite (2.6) in terms of $\mu^-_{0,L}$ as

$$T_L(V^r(C, \eta)) = (\mathrm{Im}\mu^-_{0,L})^\perp.$$

Since $\rho^-(g, r)$ equals the difference between the dimensions of the codomain and the domain of $\mu^-_{0,L}$, we conclude that

$$\dim(T_L(V^r(C, \eta))) = \rho^-(g, r) + \dim(\mathrm{Ker}\mu^-_{0,L}), \tag{2.7}$$

and $V^r(C, \eta)$ is smooth of dimension $\rho^-(g, r)$ at L if and only if the Prym-Petri map is injective. Welters proved that this is the case for all $L \in \mathrm{Nm}^{-1}(\omega_C)$ provided that the cover π is general.

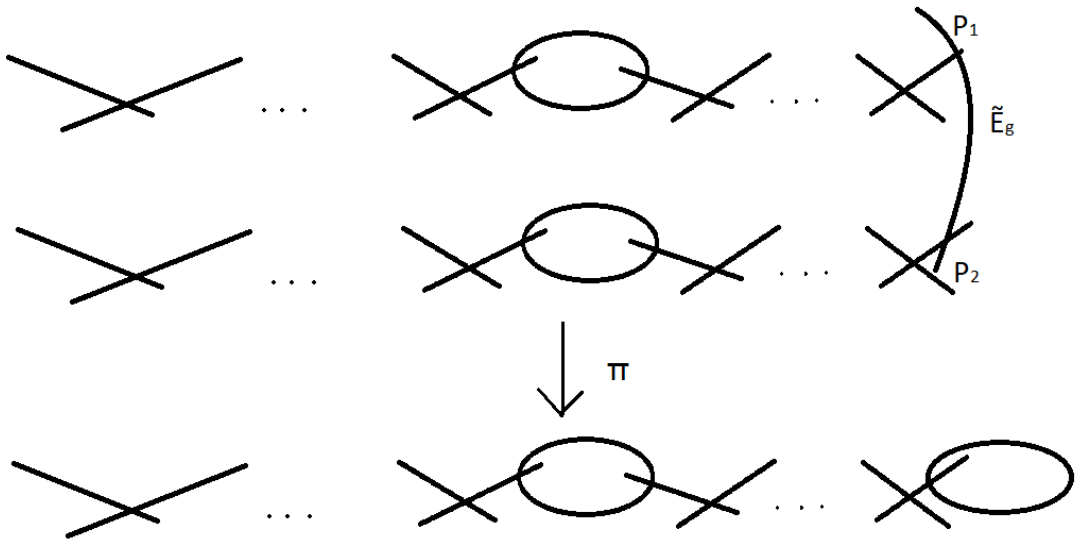
Theorem 2.3 ([We] Thm. 1.11)**.** *Let $(C, \eta) \in \mathcal{R}_g$ be general and let $\pi : \widetilde{C} \to C$ be the étale double cover defined by η. Then the Prym-Petri map $\mu^-_{0,L}$ is injective for all $L \in \mathrm{Nm}^{-1}(\omega_C)$.*

The result is analogous to the Gieseker-Petri Theorem, as it yields the smoothness of the Prym-Brill-Noether varieties $V^r(C, \eta)$ for a general (C, η) and their emptiness in the cases where $\rho^-(g, r) < 0$. The analogue of the existence part of the Brill-Noether Theorem for any Prym curve was instead established by Bertram:

Theorem 2.4 ([Ber] Thm. 1.4)**.** *Let (C, η) be a Prym curve of genus g and let $\pi : \widetilde{C} \to C$ be the étale double cover defined by η. If $\rho^-(g, r) \geq 0$, the Prym-Brill-Noether variety $V^r(C, \eta)$ is nonempty.*

Welters 的证明是通过退化到一条约化的节点曲线 C_0 来完成的，该曲线由一串有理曲线和 g 条椭圆曲线 E_1, ..., E_g 组成，并且满足：如果 x_i 和 y_i 是任意一条 E_i 与两个相邻分支的两个交点，那么线丛 $\mathcal{O}_{E_i}($ $x_i - y_i$) 在 $\mathrm{Pic}^0(E_i)$ 中不是一个挠点。考虑 η_0 上的一个线丛 C_0 ，它在除 E_g 之外的每个分支上都是平凡的，在 E_g 上则.....

限制到 $\mathrm{Pic}^0(E_g)[2]$ 中的一个非零点。由 η_0 定义的 C_0 的 étale 双覆盖 $\widetilde{C}_0$ 因此如下所示：



The points P_1 and P_2 on $\widetilde{E}_g$ are not $\mathbb{Z}$-independent; indeed, $\mathcal{O}_{\widetilde{E}_g}(P_1-P_2) \in \mathrm{Pic}^0(\widetilde{E}_g)$ is a 2-torsion point. This is why Eisenbud and Harris' proof [EH2] of the Gieseker-Petri Theorem fails for $\widetilde{C}_0$. However, using the theory of limit linear series and the Brill-Noether theory of a (non general) 2-pointed elliptic curve, Welters wrote down the explicit form of any element ρ in the kernel of the relevant Petri maps and, thanks to the skew-symmetry of $1 \otimes \iota^*$, concluded that such a ρ never lies in the kernel of the corresponding Prym-Petri map.

2.3. **The map $\mu^+_{0,L}$.** The decomposition (2.2) provides a splitting of the Petri map $\mu_{0,L}$ of any line bundle $L \in \mathrm{Nm}^{-1}(\omega_C)$ into a ι-anti-invariant part, namely, the Prym-Petri map $\mu^-_{0,L}$, and a ι-invariant part

$$\begin{aligned} \mu^+_{0,L} \colon \mathrm{Sym}^2 H^0(\widetilde{C}, L) &\to H^0(C, \omega_C) \\ s \otimes t + t \otimes s &\mapsto s \cdot (\iota^* t) + (\iota^* s) \cdot t. \end{aligned}$$

As already remarked, the Prym-Petri map $\mu^-_{0,L}$ governs the smoothness of the varieties $V^r(C,\eta)$ and has thus been extensively investigated. By contrast, the map $\mu^+_{0,L}$ has been almost ignored so far. Up to our knowledge, it only appeared in the proof of the uniruledness of $\mathcal{R}_8$ due to Farkas and Verra [FV2].

我们通过考虑包含映射 $\mu^+_{0,L}$ $V^r(C,\eta) \subset W^r_{2g-2}(\widetilde{C})$ 以及由此诱导的线性映射短正合序列，给出对该映射的解释。

$$0 \to T_L(V^r(C,\eta)) \to T_L W^r_{2g-2}(\widetilde{C}) \to N_{V^r(C,\eta)/W^r_{2g-2}(\widetilde{C}),L} \to 0,$$

~其中 $N_{V^r(C,\eta)/W^r_{2g-2}(\widetilde{C})\,L}$ 是 $W^r_{2g-2}(C)$ 中 $V^r(C,\eta)$ 的 L 点处的法空间。由 (2.5) 和 (2.7), 可得:

$$\begin{aligned}\dim_{\mathbb{C}}(N_{V^r(C,\eta)/W^r_{2g-2}(\widetilde{C}),L}) &= \\ &= \rho(2g-1,r,2g-2)+\dim(\mathrm{Ker}\mu_{0,L})-\rho^-(g,r)-\dim(\mathrm{Ker}\mu^-_{0,L}) = \\ &= g-\frac{(r+1)(r+2)}{2}+\dim(\mathrm{Ker}\mu^+_{0,L}),\end{aligned}$$

where we have used that $\mathrm{Ker}\mu_{0,L}=\mathrm{Ker}\mu^+_{0,L}\oplus\mathrm{Ker}\mu^-_{0,L}$ by construction. We set

$$\rho^+(g,r):=\rho(2g-1,r,2g-2)-\rho^-(g,r)=g-\frac{(r+1)(r+2)}{2}$$

and notice that this number equals the difference between the dimensions of the codomain and the domain of $\mu^+_{0,L}$. We conclude that the dimension of the normal space $N_{V^r(\widetilde{X})/W^r_{2g-2}(\widetilde{X})\,L}$ equals the dimension of the cokernel of $\mu^+_{0,L}$. More precisely, (2.4) and (2.6) imply that

$$N_{V^r(\widetilde{X})/W^r_{2g-2}(\widetilde{X}),L}=(\mathrm{Im}\mu_{0,L})^\perp\cap(H^0(\omega_C))^\vee=(\mathrm{Im}(q\circ\mu_{0,L}))^\perp,$$

where $q: H^0(\widetilde{C},\omega_{\widetilde{C}})\to H^0(C,\omega_C)$ is the projection mapping a form λ to $\frac{1}{2}(\lambda+\iota^*\lambda)$. The composition

$$q\circ\mu_{0,L}\circ(1\otimes\iota^*): H^0(\widetilde{C},L)\otimes H^0(\widetilde{C},L)\longrightarrow H^0(C,\omega_C)$$

maps a decomposable tensor $s\otimes t$ to the invariant form $s(\iota^*t)+(\iota^*s)t$ and is thus symmetric. Its restriction to $\mathrm{Sym}^2H^0(\widetilde{C},L)$ coincides with the map $\mu^+_{0,L}$ and this provides the identification

$$N_{V^r(\widetilde{X})/W^r_{2g-2}(\widetilde{X}),L}=(\mathrm{Im}\mu^+_{0,L})^\perp.$$

When $(C,\eta)\in\mathcal{R}_g$ is general, it is natural to wonder whether the map $\mu^+_{0,L}$ is injective for all $L\in\mathrm{Nm}^{-1}(\omega_C)$. Unfortunately, this trivially fails for any line bundle $L\in V^r(C,\eta)$ where r is any fixed integer r such that $\rho^-(g,r)\geq 0$ but $\rho(2g-1,r,2g-2)<0$; indeed, the Prym-Petri map $\mu^-_{0,L}$ is injective by Theorem 2.3, but our assumptions prevent $\mu^+_{0,L}$from being injective because $\dim\mathrm{Ker}\mu^+_{0,L}\geq-\rho^+(g,r)>0$. More generally, the map $\mu^+_{0,L}$ fails to be injective for any line bundle $L\in V^r(C,\eta)$ as soon as

$$\rho^-(g,r)>\max\{-1,\rho(2g-1,r,2g-2)\},\tag{2.8}$$

which implies $\rho^+(g,r)<0$. The inequalities (2.8) can be rewritten only in terms of ρ as

$$-r\leq\rho(2g-1,r,2g-2)<r.\tag{2.9}$$

In this range, the equality $\dim\mathrm{Ker}\mu^+_{0,L}=-\rho^+(g,r)$ is the best one may hope for; since $V^r(C,\eta)$ is smooth at L by Theorem 2.3, this hope is equivalent to the expectation that $W^r_{2g-2}(\widetilde{C})$ and $V^r(C,\eta)$ coincide in a neighborhood of L.

Expectation 2.5. *Let $(C,\eta)\in\mathcal{R}_g$ be general and let $\pi:\widetilde{C}\to C$ be the étale double cover defined by η. Then, the following hold:*

(i) *If $-r\leq\rho(2g-1,r,2g-2)<r$, then $W^r_{2g-2}(\widetilde{C})=V^r(C,\eta)$. In particular, for all $L\in V^r(C,\eta)$ one has*

$$\dim(\mathrm{Ker}\mu_{0,L})=\dim(\mathrm{Ker}\mu^+_{0,L})=-\rho^+(g,r).$$

(ii) 如果 $\rho(2g-1,\ r,\ 2g-2)\geq r$，那么对于所有 $L\in V^r(C,\ \eta)$，$\mu_{0,\ L}$ 和 $\mu^+_{0\ ,\ L}$ 都是单射。

~其余情况下的情形 $\rho(2g-1,\ r,\ 2g-2) < -r$ 已经很明确，因为根据 Schwarz 的一项更一般的结果， $W^r_{2g-2}(C)$ 实际上是空的。我们将在第 4 节回顾这一结果。

3. Theta-characteristics and vanishing thetanulls

3.1. **Preliminaries on theta-characteristics.** A *theta-characteristic* on a smooth irreducible curve C is a line bundle $\theta \in \mathrm{Pic}^{g-1}(C)$ such that $\theta^{\otimes 2} = \omega_C$. Theta-characteristics are of two types, called odd and even according to the parity of the dimension of the space of their global sections. The set of theta-characteristics on C, denoted by $\mathrm{Th}(C)$, is a principal homogeneous space for the space $J(C)[2]$ of two-torsion points in the Jacobian of C, that is, $J(C)[2]$ acts freely and transitively on $\mathrm{Th}(C)$. In particular, we have $|\mathrm{Th}(C)| = |J(C)[2]| = 2^{2g}$.

The $\mathbb{F}_2$-vector space $J(C)[2]$ is endowed with a nondegenerate symplectic form

$$\langle \cdot, \cdot \rangle : J(C)[2] \times J(C)[2] \to \mathbb{F}_2,$$

which is called *Weil pairing* and is defined as follows ([M2]). For any pair of points $\eta, \epsilon \in J(C)[2]$, one can write $\eta = \mathcal{O}_C(D)$ and $\epsilon = \mathcal{O}_C(E)$ for two divisors D and E on C with disjoint support, and choose rational functions f and g on C such that $\mathrm{div}(f) = 2D$ and $\mathrm{div}(g) = 2E$. The condition on the supports of D and E ensures the validity of the so-called *Weil Reciprocity Law* (see [Ha]):

$$f((g)) = g((f)),$$

where the evaluation of a rational function h at a divisor Z whose support is disjoint from the set of zeros and poles of h is defined as $h(Z) := \prod_{p \in C} h(p)^{\mathrm{mult}_p Z}$. Hence, we get

$$\frac{f(2E)}{g(2D)} = \left(\frac{f(E)}{g(D)}\right)^2 = 1,$$

and $\frac{f(E)}{g(D)} = \pm 1$. The value of the Weil pairing at the pair (η, ϵ) is given by the following formula:

$$(-1)^{\langle \epsilon, \eta \rangle} = \frac{f(E)}{g(D)},$$

and one can check that this definition is independent of both the divisors D, E and the rational functions f, g.

Given a symplectic vector space $(V, \langle \cdot, \cdot \rangle)$ over $\mathbb{F}_2$, we denote by $Q(V)$ the set of quadratic forms on V with fixed polarity equal to the symplectic form $\langle \cdot, \cdot \rangle$, that is, all functions $q : V \to \mathbb{F}_2$ that satisfy the identity

$$q(x+y) = q(x) + q(y) + \langle x, y \rangle, \quad \forall\ x, y \in V.$$

Having fixed a symplectic basis $(e_1, ..., e_g, f_1, ..., f_g)$ of V, the so-called *Arf invariant* of a quadratic form $q \in Q(V)$ is defined as

$$\mathrm{arf}(q) := \sum_{i=1}^{g} q(e_i) \cdot q(f_i) \in \mathbb{F}_2,$$

this definition being independent of the basis. A quadratic form $q \in Q(V)$ is called *even* if $\mathrm{arf}(q) = 0$, and *odd* if $\mathrm{arf}(q) = 1$. The space of even (respectively, odd) quadratic forms is denoted by $Q(V)^+$ (resp., $Q(V)^-$). An easy computation gives $|Q(V)^+| = 2^{g-1}(2^g + 1)$ and $|Q(V)^-| = 2^{g-1}(2^g - 1)$.

Coming back to theta-characteristics, to any $\theta \in \mathrm{Th}(C)$ one associates the so-called *theta-form* $q_\theta : J(C)[2] \to \mathbb{F}_2$, by setting

$$q_\theta(\eta) := h^0(C, \eta \otimes \theta) + h^0(C, \theta) \bmod 2.$$

对于任意 $\epsilon, \eta \in J(C)[2]$, , ***Riemann-Mumford*** 关系([M2]) 给出:

$$h^0(C, \theta \otimes \epsilon \otimes \eta) + h^0(C, \theta \otimes \epsilon) + h^0(C, \theta \otimes \eta) + h^0(C, \theta) \equiv \langle \eta, \epsilon \rangle \bmod 2,$$

或等价地， $q_\theta \in Q(J(C)[2])$。因此，我们将这两个空间视为相同。

$$\mathrm{Th}(C) = Q(J(C)[2]). \tag{3.1}$$

Since $\mathrm{arf}(q_\theta) = h^0(C,\theta) \bmod 2$ for any $\theta \in \mathrm{Th}(C)$, under this identification even (respectively, odd) theta-characteristics correspond to forms in $Q(J(C)[2])^+$ (resp., $Q(J(C)[2])^-$). Hence, on a genus g curve C there are precisely $2^{g-1}(2^g+1)$ even theta-characteristics and $2^{g-1}(2^g-1)$ odd ones.

Following Atiyah [At], a *spin curve* of genus g is a pair (C,θ), where C is a smooth irreducible curve of genus g and $\theta \in \mathrm{Th}(C)$. Mumford ([M2]) and Atiyah ([At]) proved that the parity of a spin curve, that is, the value of its Arf invariant, is locally constant in families. As a consequence, the moduli space $\mathcal{S}_g$ parametrizing spin curves of genus g splits into two connected components $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$: a pair (C,θ) lies in $\mathcal{S}_g^+$ if θ is an even theta-characteristic, and in $\mathcal{S}_g^-$ otherwise.

For each $g, r \geq 0$ one can define the locus

$$\mathcal{S}_g^r := \{(C,\theta) \in \mathcal{S}_g \mid h^0(C,\theta) \geq r+1 \text{ and } h^0(C,\theta) \equiv r+1 \bmod 2\}.$$

Harris ([Ha]) proved that the dimension of every component of $\mathcal{S}_g^r$ is bounded below by the number $3g-3-\binom{r+1}{2}$, to which we refer as the expected dimension of $\mathcal{S}_g^r$. It may well be the case that $\mathcal{S}_g^r$ is nonempty even when its expected dimension is negative. For instance, $\mathcal{S}_g^{[\frac{g-1}{2}]} \neq \emptyset$ contains hyperelliptic curves and is thus nonempty although in this case $3g-3-\binom{r+1}{2}$ is very negative. Existence of components of $\mathcal{S}_g^r$ having the expected dimension has been established by Farkas [F2] in the range $1 \leq r \leq 11, r \neq 10$ for all $g \geq g(r)$ where $g(r)$ is an explicit integer.

In the case where $r=1$, the locus $\mathcal{S}_g^1$ parametrizes curves having a so-called *vanishing thetanull*, that is, an effective even theta characteristic, and is a divisor in $\mathcal{S}_g^+$. Indeed, every theta characteristic θ on a general curve C of genus g has at most one section. This directly follows from the base-point-free pencil trick, which implies that, as soon as $H^0(C,\theta)$ contains a pencil V, the kernel of the Petri map

$$\mu_{0,V} : V \otimes H^0(C,\theta) \to H^0(C,\omega_C)$$

is nonzero; indeed, one has $\mathrm{Ker}\mu_{0,V} \simeq H^0(C,\mathcal{O}_C(B)) \neq 0$, where B is the base locus of V. The Gieseker-Petri Theorem thus excludes the existence of such a pencil if the curve C is general.

3.2. **Invariant vanishing thetanulls on étale double covers.** Let $\pi : \widetilde{C} \to C$ be the irreducible étale double cover associated with a general Prym curve (C,η) of genus g. As explained at the end of Section 2, the Brill-Noether varieties $W^r_{2g-2}(\widetilde{C})$ governing the singularities of the theta divisor of $J(\widetilde{C})$ are nonempty as soon as $\rho(2g-1,r,2g-2) \geq -r$; in particular, for values of g for which there exists an r yielding $-r \leq \rho(2g-1,r,2g-2) < 0$, the curve $\widetilde{C}$ is Brill-Noether special.

We now fix g, r such that $\rho(2g-1,r,2g-2) \geq 0$; under this condition, one may still hope that $W^r_{2g-2}(\widetilde{C})$ is smooth and of the expected dimension. However, this expectation fails even for $r=1$, as observed by Welters himself [We, Rmk. 1.12]; in particular, $\widetilde{C}$ never satisfies the Gieseker-Petri Theorem. By the base-point-free pencil trick, $W^1_{2g-2}(\widetilde{C}) \setminus W^2_{2g-2}(\widetilde{C})$ is singular at any point defining a vanishing thetanull. On the other hand, a vanishing thetanull on $\widetilde{C}$ is easily obtained as pullback π^*M of a theta-characteristics M on C such that both M and $M \otimes \eta$ are odd. The following nice argument taken from [Bea, Prop. 4] counts the number of such Ms and thus in particular proves their existence.

对于任意 $M \in \mathrm{Th}(C)$，推拉公式给出

$$h^0(\widetilde{C}, \pi^*M) = h^0(C, \pi_*\pi^*M) = h^0(C,M) + h^0(C, M\otimes\eta) = q_M(\eta) \bmod 2.$$

我们要寻找满足 $q_M(\eta) = 0$且 $\mathrm{arf}(q_M) = 1$的那些 $M \in \mathrm{Th}(C)$。取 $\epsilon \in J(C)[2]$，使得 Weil 配对 $\langle \eta, \epsilon \rangle = 1$，并将由 η和ϵ张成的平面记为 $\Sigma \subset J(C)[2]$。由于 $J(C)[2] = \Sigma \oplus \Sigma^{\perp}$，任意 $M \in \mathrm{Th}(C)$ 都由其限制 $q_M|_{\Sigma}$和 $q_M|_{\Sigma^{\perp}}$完全确定。条件 $q_M(\eta) = 0$推出 $\mathrm{arf}(q_M|_{\Sigma}) = q_M(\eta)$ $q_M(\epsilon) = 0$。因此， M由值 $q_M(\epsilon) \in \mathbb{F}_2$及限制 $q_M|_{\Sigma^{\perp}}$确定；后者是定义在$\Sigma^{\perp}$上的二次型，其 Arf 不变量为 1（因为 $\mathrm{arf}(q_M) = \mathrm{arf}(q_M|_{\Sigma}) + \mathrm{arf}(q_M|_{\Sigma^{\perp}})$）。由于 $\dim_{\mathbb{F}_2} \Sigma^{\perp} = 2(g-1)$， $2^{g-2}(2^{g-1}-1)$ 种选择可用于 $q_M|_{\Sigma^{\perp}}$；再加上可以选择 $q_M(\epsilon)$，我们得到$2^{g-1}(2^{g-1}-1)$ 个如上所述定义在 C 上的 theta-特征 M。由于 $\pi^*(M \otimes \eta) \simeq \pi^* M$，这一构造恰好给出 $\widetilde{C}$ 上 $2^{g-2}(2^{g-1}-1)$ 个在覆盖对合下不变的消失 theta 常数。

4. Brill-Noether theory for double covers

Given the étale double cover $\pi : \widetilde{C} \to C$ of a general Prym curve $(C, \eta) \in \mathcal{R}_g$, it is natural to investigate not only the Prym-Brill-Noether varieties $V^r(C, \eta)$ but any Brill-Noether variety $W^r_d(\widetilde{C})$. However, not much is known about the Brill-Noether theory of $\widetilde{C}$. The following non-existence result follows from [Sc], where Schwarz proved a more general statement concerning étale cyclic covers of arbitrary degree.

Theorem 4.1 ([Sc])**.** *Let $\pi : \tilde{C} \to C$ be the étale double cover associated with a general Prym curve $(C, \eta) \in \mathcal{R}_g$. Then the Brill-Noether variety $W^r_d(\widetilde{C})$ is empty if*

$$\rho(2g-1, r, d) < -r.$$

The above result can be easily proved using Welters' degeneration and applying [F1, Prop. 4.1] on the Brill-Noether theory of 2-pointed elliptic curves. For $r = 1$ Theorem 4.1 is known to be optimal and this implies that $\widetilde{C}$ is not Brill-Noether general if g is even. Indeed, the existence of a pencil of degree g on $\widetilde{C}$ in this case follows from the surjectivity (cf. [ACGH]) of the difference map

$$\begin{aligned} \phi_{\frac{g}{2}} : C^{\frac{g}{2}} \times C^{\frac{g}{2}} &\longrightarrow J(C) \\ (D, E) &\longmapsto \mathcal{O}_C(D - E), \end{aligned}$$

yielding $\eta = \mathcal{O}_C(D - E)$ for some effective divisors D, E both of degree $g/2$ on C. The pullback $\pi^*(\mathcal{O}_C(E)) \in \mathrm{Pic}^g(\widetilde{C})$ then satisfies

$$h^0(\widetilde{C}, \pi^* \mathcal{O}_C(E)) = h^0(C, \pi_* \pi^* \mathcal{O}_C(E)) = h^0(C, \mathcal{O}_C(E)) + h^0(C, \mathcal{O}_C(D)) \geq 2,$$

where we have again used the push-pull formula. On the other hand, in the odd genus case the gonality of $\widetilde{C}$ is maximal, as first proved by Aprodu and Farkas.

Theorem 4.2 ([AF] Thm. 0.4)**.** *Let $\pi : \tilde{C} \to C$ be the étale double cover associated with a general Prym curve $(C, \eta) \in \mathcal{R}_g$. Then the following hold:*

(i) *if $g \equiv 1$ mod 2, the curve $\tilde{C}$ has maximal gonality, that is,* $\mathrm{gon}(\tilde{C}) = g + 1$*;*
(ii) *if $g \equiv 0$ mod 2, then* $\mathrm{gon}(\tilde{C}) = g$.

In both cases the Clifford index of $\widetilde{C}$ equals $\mathrm{gon}(\tilde{C}) - 2$.

Two different proofs are provided in [AF], one by degeneration and one by specialization to curves on Nikulin surfaces; the latter are particular $K3$ surfaces on which we will focus in the next section. Up to our knowledge, the following natural question remains open:

Question 1. Let g be an odd positive integer such that the inequalities $-r \leq \rho(2g-1, r, 2g-2) < 0$ admit no integral solution $r \geq 1$. If $(C, \eta) \in \mathcal{R}_g$ is general, is the cover $\tilde{C}$ Brill-Noether general?

We stress that, quite surprisingly, ramified double covers of curves seem to behave better than étale ones from a Brill-Noether viewpoint. Following [Bu], let $\mathcal{R}_{g,2n}$ be the moduli space parametrizing irreducible double covers of smooth genus g curves ramified at $2n$ points. The space $\mathcal{R}_{g,2n}$ can be alternatively defined as follows:

$$\mathcal{R}_{g,2n} := \left\{(C, x_1, \ldots, x_{2n}, \eta) \,|\, [C] \in \mathcal{M}_g,\, x_i \in C\, \forall\, i,\, \eta \in \mathrm{Pic}^{-n}(C) \;\; \eta^2 = \mathcal{O}_C(-x_1 - \ldots - x_{2n})\right\}.$$

While studying its birational geometry, Bud proved the following astonishing result, which is in contrast to the étale case.

Theorem 4.3 ([Bu] Thm. 4.1)**.** *Let $\pi : \widetilde{C} \to C$ be the double cover associated with a general $(C, x + y, \eta) \in \mathcal{R}_{g,2}$. Then $\widetilde{C}$ is both Brill-Noether and Petri general.*

Bud's proof relies on the study of divisors on a compactification $\overline{\mathcal{R}_{g,2}}$ of the moduli space. In the next section we will provide a simpler proof of the Brill-Noether generality of $\widetilde{C}$ by specialization to ramified double covers of curves on Nikulin surfaces of non-standard type, and we will extend the result to covers ramified at 4 and 6 points.

5. Double covers of curves on Nikulin surfaces

5.1. **Nikulin surfaces and their Picard group.** Nikulin surfaces represent a rather special class of K3 surfaces, which has been studied in relation to various topics, including the theory of automorphisms [VGS], the study of Prym curves [FK] and the birational geometry of their moduli spaces [FV1, FV2, KLV1, KLV2]. We recall their definition and some basic properties.

Definition 1. A polarized *Nikulin* surface of genus $h \geq 2$ is a triple (S, M, H) consisting of a smooth $K3$ surface S and two line bundles $\mathcal{O}_S(M)\;\; H \in \mathrm{Pic}\, S$ that satisfy the following conditions:

- S contains 8 disjoint smooth rational curves $N_1, \ldots, N_8$ such that
$$N_1 + \cdots + N_8 \sim 2M.$$
- H is nef, $H^2 = 2(h-1)$ and $H \cdot M = 0$.

We say that (S, M, H) is *primitively polarized* if the class of H is primitive in $\mathrm{Pic}\, S$.

The line bundle M defines a non-trivial double cover $\pi : \widehat{S} \to S$ branched along $\sum_{i=1}^{8} N_i$. Denoting by $\sigma : \widehat{S} \to \widetilde{S}$ the blow-down of the eight (-1)-curves $E_i := \pi^{-1}(N_i) \subset \widehat{S}$, the surface $\widetilde{S}$ is a minimal $K3$ surface endowed with a so-called *Nikulin involution* $\iota \in \mathrm{Aut}(\widetilde{S})$ having eight fixed points corresponding to the images $\sigma(E_i)$ of the exceptional divisors. The quotient $\bar{S} := \widetilde{S}/\iota$ has eight ordinary double points and the quotient map $\bar{\pi} : \widetilde{S} \to \bar{S}$ fits in the following commutative diagram:

$$\begin{array}{ccc} \widehat{S} & \xrightarrow{\sigma} & \widetilde{S} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ S & \xrightarrow[\bar{\sigma}]{} & \bar{S} \end{array},$$

where $\bar{\sigma}$ is the contraction of the curves N_i to the eight nodes of $\bar{S}$.

Definition 2. Let (S, M, H) be a polarized Nikulin surface of genus h. The rank 8 sublattice of $\mathrm{Pic}\, S$ generated by $N_1, \ldots, N_8$ and M is called *Nikulin lattice* and its denoted by $\mathbf{N} = \mathbf{N}(S, M)$.

Since the Picard group of a polarized Nikulin surface (S, M, H) contains both the Nikulin lattice and the polarization H, its rank is at least 9. Consider the rank 9 lattice

$$\Lambda_h = \Lambda(S, M, H) := \mathbb{Z}[H] \oplus_{\perp} \mathbf{N} \subset \mathrm{Pic}\, S;$$

the surface S is said to be a *Nikulin surface of standard type* if the embedding $\Lambda_h \subset \operatorname{Pic} S$ is primitive, and a *Nikulin surface of non-standard type* otherwise. Garbagnati and Sarti [GS] proved that in the latter case h is forced to be odd. The following result describes $\operatorname{Pic}(S)$ in the case of minimal Picard number.

Proposition 5.1 ([GS], Prop. 2.1)**.** *Let (S, M, H) be a genus h primitively polarized Nikulin surface of Picard number 9. Then either $\operatorname{Pic} S = \Lambda_h$ (standard case), or h is odd and $\Lambda_h \subset \operatorname{Pic} S$ has index two (non-standard case). In the latter situation, possibly after renumbering the curves N_i, one falls in one of these cases:*

- $h \equiv 3 \bmod 4$ *and there are $R_1, R_2 \in \operatorname{Pic} S$ such that*
$$R_1 \sim \frac{H - N_1 - N_2}{2}, \quad R_2 \sim \frac{H - N_3 - \cdots - N_8}{2};$$
in particular, one has $g(R_1) = (h+1)/4$, $g(R_2) = (h-3)/4$.
- $h \equiv 1 \bmod 4$ *and there are $R_1, R_2 \in \operatorname{Pic} S$ such that*
$$R_1 \sim \frac{H - N_1 - N_2 - N_3 - N_4}{2}, \quad R_2 \sim \frac{H - N_5 - N_6 - N_7 - N_8}{2};$$
in particular, one has $g(R_1) = g(R_2) = (h-1)/4$.

The growing interest in Nikulin surfaces is motivated from the fact that some Prym curves live on them. The condition $H \cdot M = 0$ in Definition 1 ensures that the double cover π restricts to an étale double cover $\pi|_C : \widetilde{C} := \pi^{-1}(C) \to C$ of any smooth curve $C \in |H|$; in other words, the pair $(C, M|_C)$ is a Prym curve of genus h. Since $\widetilde{C}$ is disjoint from the (-1)-curves E_i, the curve $\widetilde{C}$ can be identified with its image in $\widetilde{S}$ and we set $\widetilde{H} := \mathcal{O}_{\widetilde{S}}(\widetilde{C})$.

The Picard group of the $K3$ surface $\widetilde{S}$ was described by Van Geemen and Sarti [VGS] by investigating the action of the Nikulin involution ι on the cohomology group $H^2(\widetilde{S}, \mathbb{Z})$. It turns out that $\operatorname{Pic}(\widetilde{S})$ always contains the orthogonal complement $(H^2(\widetilde{S}, \mathbb{Z})^\iota)^\perp$, which is isomorphic to the rank 8 lattice $E_8(-2)$. Since $\widetilde{H} \in E_8(-2)^\perp \subset \operatorname{Pic}(\widetilde{S})$, the Picard number of $\widetilde{S}$ is ≥ 9 and equality holds if and only if the same holds for S. The following proposition summarizes results by Van Geemen and Sarti [VGS, Prop. 2.2 and Prop. 2.7] and Aprodu and Farkas [AF, Prop. 4.3] in the standard case.

Proposition 5.2. *Let (S, M, H) be a genus h polarized Nikulin surface of standard type and Picard number 9, and let $\widetilde{S}$ be the $K3$ surface with a Nikulin involution obtained from S. Then, the lattice*
$$\widetilde{\Lambda}_h := \mathbb{Z}\widetilde{H} \oplus E_8(-2)$$
has index 2 in $\operatorname{Pic}(\widetilde{S})$. The latter is generated by $\widetilde{\Lambda}_h$ and a class $\frac{\widetilde{H}+v}{2}$, where v is an element of $E_8(-2)$ satisfying $v^2 = -4$ if h is even and $v^2 = -8$ if h is odd.

现在我们转向非标准情形。根据 [GS, Prop. 3.5 (2)], , 对于 $i = 1,2$, 只要 R_i 具有非负自交数，线性系$|R_i|$ 就包含一条光滑不可约曲线。我们强调，当 $h \equiv 3$模 4 时， $R_1 \cdot M = 2$ 和 $R_2 \cdot M = 6$，而当 $h \equiv 1$模 4 时， $R_1 \cdot M = R_2 \cdot M = 4$。因此，对于所有光滑曲线 $D_1 \in |R_1|$ 和 $D_2 \in |R_2|$，覆盖 π 诱导出双覆盖π_{D_1}： $\pi^{-1}(D_1) \to D_1$ 和 π_{D_2}： $\pi^{-1}(D_2) \to D_2$； 当 $h \equiv 3$ 模 4 时，它们分别在 2 个和 6 个点处分歧；另一方面，当 $h \equiv 1$ 模 4 时， π_{D_1}和 π_{D_2}均在 4 个点处分歧。无论哪种情形，曲线 $\pi^{-1}(D_1)$ 和 $\pi^{-1}(D_2)$ 与每条例外曲线 E_i 至多相交于 1 个点，因此$\widetilde{\quad}$ $\widetilde{\quad}$ $\widetilde{\quad}$同构于它们在 S 中的像，我们分别将其记为 D_1 和 D_2。显然可以验证，

$$\sigma^*\widetilde{D}_1 - E_1 - E_2 \sim \pi^*D_1, \ \ \sigma^*\widetilde{D}_2 - E_3 - \cdots - E_8 \sim \pi^*D_2 \text{ if } h \equiv 3 \bmod 4, \tag{5.1}$$
$$\sigma^*\widetilde{D}_1 - E_1 - E_2 - E_3 - E_4 \sim \pi^*D_1, \ \ \sigma^*\widetilde{D}_2 - E_5 - E_6 - E_7 - E_8 \sim \pi^*D_2 \text{ if } h \equiv 1 \bmod 4.$$

~ ~注意，Hurwitz 公式始终给出 $g(D_1) = g(D_2) = (h+1)/2$。在非标准情形下，以下结果可直接由 [VGS 的命题 2.2 和命题 2.7] 得出。

Proposition 5.3. *Let (S, M, H) be a genus h polarized Nikulin surface of non-standard type and Picard number 9, and let $\widetilde{S}$ be the $K3$ surface with a Nikulin involution obtained from S. Then*

$$\mathrm{Pic}(\widetilde{S}) = \mathbb{Z}\widetilde{R} \oplus E_8(-2),$$

where $\widetilde{R}$ is a polarization of genus $(h+1)/2$ such that the curves $\widetilde{D}_1, \widetilde{D}_2$ as in (5.1) lie in $|\widetilde{R}|$.

5.2. **Standard Nikulin surfaces.** As highlighted in [KLV1, Prop. 2.3], a general curve $C \in |H|$ on a very general genus h primitively polarized Nikulin surface (S, M, H) of standard type is Brill-Noether general . Furthermore, it was proved by Aprodu and Farkas [AF, Thm. 1.5] that the étale double cover $\widetilde{C} \subset \widetilde{S}$ of C has the gonality of a general curve of genus $2h-1$ that covers a genus h curve, namely, $h+1$ if h is odd and h otherwise. It is thus natural to ask whether $\widetilde{C}$ is general from a Prym-Brill-Noether viewpoint, that is, if it satisfies Welters' Theorem. A positive answer would provide an alternative proof of Welters' result avoiding degeneration, in analogy with Lazarsfeld's proof of the Gieseker-Petri Theorem by specialization to curves on $K3$ surfaces. Unfortunately, the answer turns out to be negative as soon $h > 7$ and $h = 6$. This bound on the genus agrees with the following theorem by Farkas and Verra.

Theorem 5.4 ([FV2], Thm. 0.2)**.** *A general Prym curve $(C, \eta) \in \mathcal{R}_h$ lies on a Nikulin surface if and only if $h \leq 7$ and $h \neq 6$.*

We thus prove the following result.

Theorem 5.5. *Let (S, M, H) be a genus h primitively polarized Nikulin surface of standard type with either $h > 7$, or $h = 6$. For any smooth curve $C \in |H|$, the double cover $\widetilde{C} \in |\widetilde{H}|$ of C defined by $M|_C$ does not satisfy Welter's Theorem.*

Proof. Set $A := \frac{\widetilde{H}+v}{2} \in \mathrm{Pic}(\widetilde{S})$ with $v \in E_8(-2)$ as in Proposition 5.2. Since $\widetilde{C} \cdot v = 0$, the restriction $A|_{\widetilde{C}}$ is a line bundle on $\widetilde{C}$ of degree $2h-2$ such that:

$$r := h^0(\widetilde{C}, A|_{\widetilde{C}}) - 1 = h^0(\widetilde{S}, A) - 1 \geq \chi(A) - 1 = 1 + \frac{1}{2}\left(\frac{\widetilde{H}+v}{2}\right)^2 = \left\lfloor \frac{h}{2} \right\rfloor;$$

here, the first equality follows from the strong version of Bertini's Theorem due to Saint-Donat [SD] yielding $h^1(\widetilde{S}, A(-\widetilde{C})) = 0$, while in the last equality we have used that $v^2 = -4$ when h is even and $v^2 = -8$ otherwise. Since $E_8(-2) = (H^2(\widetilde{S}, \mathbb{Z})^\iota)^\perp \subset \mathrm{Pic}(\widetilde{S})$, one has $\iota^*\widetilde{H} = \widetilde{H}$ and $\iota^* v = -v$. This implies $\iota^* A|_{\widetilde{C}} \simeq \omega_{\widetilde{C}} \otimes A|_{\widetilde{C}}^\vee$, or equivalently, $\mathrm{Nm}(A|_{\widetilde{C}}) = \omega_C$. Hence, $A|_{\widetilde{C}}$ defines an element of the Prym-Brill-Noether variety $V^r(C, M|_C)$. If h is odd, an easy computation gives

$$\rho^-(h, r) = -\frac{(h-1)(h-7)}{8},$$

which is negative when $h > 7$. Analogously, for even h one computes the Prym-Brill-Noether number

$$\rho^-(h, r) = -\frac{(h-2)(h-4)}{8},$$

对于 $h \geq 6$ 为负。因此，当 $h > 7$ 或 $h = 6$ 时， $A|_{\widetilde{C}}$ 的 Prym-Petri映射 不可能是单射的，这就完成了证明。□

5.3. 非标准型 Nikulin 曲面。设 (S, M, H) 为一个非常一般的非标准型亏格为 h 的本原极化 Nikulin 曲面。正如 [KLV1，命题 2.3 和注记 2.4], 中所指出的，在这种情况下，线丛 $R_1, R_2 \in \mathrm{Pic}(S)$ 使得其中的曲线不满足 Brill-Noether 一般性；事实上，它们在任意光滑曲线 $C \in |H|$ 上的限制定义了两个具有负 Brill-Noether 数的 theta-特征。 [GS] 中的命题 3.5 (2) 表明，只要 $c_1(R_i)^2 \geq 0$，对于 $i = 1, 2$，线性系 $|R_i|$ 中就存在一条光滑不可约曲线。我们选取一条属于 $|R_1|, |R_2|$ 中任一线性系的光滑曲线 D，并用 $g \geq 1$ 表示其亏格。若 $h \equiv 1$ 模 4，则 $g = (h-1)/4$ 且 $(D, M|_C) \in \mathcal{R}_{g,4}$。若 $h \equiv 3$ 模 4，则根据 D 属于 $|R_1|$ 还是 $\sim$ $\backsim |R_2|$，有两种情况：或者 $g = (h+1)/4$ 且 $(D, M|_C) \in \mathcal{R}_{g,2}$，或者 $g = (h-3)/4$ 且 $(D, M|_C) \in \mathcal{R}_{g,6}$。无论哪种情况，由 $M|_C$ 定义的 D 的双覆盖 $D \subset S$ 的亏格均为 $(h+1)/2$。

命题 5.6. 设 (S, M, H) 是一个一般的奇亏格非标准 *Nikulin* 曲面，其中亏格为 h，且 D 是 $|R_1|$ 或 $|R_2|$中的一条光滑曲线。那么有以下结论：

(i) D 始终是 *Brill–Noether* 一般的；若其在所属线性系中一般，则它是 *Petri* 一般的；

$\widetilde{\text{(ii)}}$ 由 $M|_C$ 定义的、以 D 为底的分歧双覆盖 D 是 *Brill–Noether* 一般的。

证明。 容易验证， R_1 和 R_2 都不能分解为 S 上两个均满足 $h^0 \geq 2$的线丛的张量积，因此，按照 Lazarsfeld 对 Petri 定理 [La, Pa] 的证明方法，即可推出 (i)。由命题 5.3 得到 $D \in |R|$，其中 R 是 $\widetilde{\quad}$ $\widetilde{\quad}$ $\widetilde{\quad}$Pic(S) 的一个生成元。同样容易排除存在满足 $R \simeq L_1 \otimes L_2$ 且$h^0(\widetilde{S}, L_i) \geq 2$的 $L_1, L_2 \in \mathrm{Pic}(S)$ 的可能性。因此，Lazarsfeld 定理表明，线性系$\sim$ $\backsim |R|$ 中的所有光滑曲线（尤其是 D）都是 Brill–Noether 一般的。□

A general curve $\widetilde{D}$ as in the above statement is highly expected to be Petri general, as well. However, this does not follow directly from Lazarsfeld's Theorem. Indeed, the latter only implies Petri generality for general curves in $|\widetilde{R}|$; however, a double cover $\widetilde{D}$ of a curve D in $|R_1|$ or $|R_2|$ is never general in the linear system $|\widetilde{R}|$, as it lies in either the ι-invariant or the ι-anti-invariant part of it.

By varying h, one obtains all possible values for the genus g of D. Recalling that the moduli space $\mathcal{R}_{g,2n}$ is irreducible for every $g \geq 2$ and $n \geq 0$ (cf. [Bu, Prop. 2.5]), Proposition 5.6 then implies the following generalization of Bud's result.

Theorem 5.7. *Fix integers $g \geq 2$ and $n = 1, 2, 3$. Let $(D, x_1, \ldots, x_{2n}, \eta) \in \mathcal{R}_{g,2n}$ be general and let $\widetilde{D}$ the double cover of D defined by η. Then the curve $\widetilde{D}$ is Brill-Noether general.*

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