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尖点奇点的定点商概形与科恩–伦斯特拉级数

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Abstract. The Quot scheme of points $\mathrm{Quot}_{d,n}(X)$ on a variety X over a field k parametrizes quotient sheaves of $\mathcal{O}_X^{\oplus d}$ of zero-dimensional support and length n. It is a rank-d generalization of the Hilbert scheme of n points. When X is a reduced curve with only the cusp singularity $\{x^2 = y^3\}$ and $d \geq 0$ is fixed, the generating series for the motives of $\mathrm{Quot}_{d,n}(X)$ in the Grothendieck ring of varieties is studied via Gröbner bases, and shown to be rational. Moreover, the generating series is computed explicitly when $d \leq 3$. The computational results exhibit surprising patterns (despite the fact that the category of finite length coherent modules over a cusp is wild), which not only enable us to conjecture the exact form of the generating series for all d, but also suggest a general functional equation whose $d = 1$ case is the classical functional equation of the motivic zeta function known for any Gorenstein curve.

作为这一研究的另一个方面，Quot 概形与 Cohen–Lenstra 级数有关。Cohen–Lenstra 级数编码了有限域上一个簇的“交换矩阵点”（或等价地，有限长度的凝聚模）的计数，Huang [38] 针对奇异曲线提出了一个关于其“有理性”的猜想。我们证明了一个一般公式，该公式将 Cohen–Lenstra 级数表示为（尖点）Quot 概形的动机。结合我们的主要有理性定理，这为 Huang 关于尖点的猜想提供了积极的证据。

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1. Introduction

模空间中产生的生成级数普遍存在有理性结果。我们考虑若干个模空间，它们参数化了在 k-簇 X 上具有零维支撑的层，其中k 是任意域:

(a) 格罗滕迪克商概形 $\mathsf{Quot}_{d,n}(X)$，其参数化的是 $\mathcal{O}_X^{\oplus d}$ 的长度为 n 的零维商，其生成级数为

$$Q^{\mathrm{mot}}_{d,X}(t) := \sum_{n\geq 0} [\mathrm{Quot}_{d,n}(X)]\, t^n \in K_0(\mathrm{Var}_k)[[t]] \tag{1.1}$$

其系数取值于 k-簇的格罗滕迪克环。

(b) 参数化 n 上长度为 n 的零维相干层的代数叠 $\mathsf{Coh}_n(X)$，其生成级数为

$$\widehat{Z}^{\mathrm{mot}}_X(t) := \sum_{n\geq 0} [\mathrm{Coh}_n(X)] t^n \in \widehat{K}_0(\mathrm{Var}_k)[[t]] \tag{1.2}$$

系数取自 $\widehat{K}_0(\mathsf{Var}_k) := K_0(\mathsf{Var}_k)[\mathbb{L}^{-1}, (1-\mathbb{L}^{-1})^{-b} : b \geq 0] \subseteq K_0(\mathsf{Var}_k)[[\mathbb{L}^{-1}]]$。代数叠 $[\mathsf{Coh}_n(X)]$（或其局部版本，见下一段）是某个交换簇关于 GL_n 的商叠，参见 §2，因此其动机可按照 Behrend–Dhillon [9] 或 Ekedahl [20] 的意义定义，并且属于 $K_0(\mathsf{Var}_k)$。

When p is a closed point of X, we analogously define the local (or **punctual**) versions $\mathrm{Quot}^{\mathrm{mot}}_{d,n}(X,p)$, $Q^{\mathrm{mot}}_{d,X,p}(t)$, $\mathrm{Coh}^{\mathrm{mot}}_n(X,p)$, $\widehat{Z}^{\mathrm{mot}}_{X,p}(t)$ by requiring the zero-dimensional sheaves to be supported at p. These local series only depend on the completed local ring $\widehat{\mathcal{O}}_{X,p}$. The global series $Q^{\mathrm{mot}}_{d,X}(t)$ and $\widehat{Z}^{\mathrm{mot}}_X(t)$ are determined by the local ones at all closed points $p \in X$. This local-to-global principle can be written

in terms of the notion of power structures due to [33]. If $k=\mathbb{F}_q$, the point-count version of the local-to-global principle can be more elementarily understood as Euler products ([38, Proposition 4.2] and Proposition 2.1). The essential reason behind the local-to-global principle is simple: a zero-dimensional sheaf on X is determined by its stalk at each point in its support.

当 $d=1$ 时，Quot 概形 $\mathrm{Quot}_{d,\ n}(X)$ 是 n 个 X 上的希尔伯特概形 $\mathrm{Hilb}_n(X)$，而级数 $Q^{\mathrm{mot}}_{1\ X,}\ (t)$ 在文献中得到了广泛研究，通常称为（动机）希尔伯特 ζ 函数。当 X 为光滑曲线时，希尔伯特 ζ 函数实际上就是动机 ζ 函数[1]

$$Z_X^{\mathrm{mot}}(t):=\sum_{n\geq 0}[\mathrm{Sym}^n(X)]t^n\in K_0(\mathrm{Var}_k)[[t]], \tag{1.3}$$

which is rational and (when X is projective) satisfies a functional equation analogous to the one in Weil's conjectures [41, 45]. When X is a smooth surface, it is well known that $\mathrm{Hilb}_n(X)$ is a desingularization of $\mathrm{Sym}^n(X)$, and the Hilbert zeta function is given by Göttsche's formula [31] in terms of the motivic zeta function, which depends on a local result by Ellingsrud and Strømme [21] using the Białynicki-Birula decomposition. See [34] for related research on singular surfaces.

对于 $d>1$ 和 X 是光滑曲线，$\mathrm{Quot}_{d,\ n}(X)$ 的动机由 Bifet [11] 于 1989 年用 X 的动机 ζ 函数表示如下：

$$Q^{\mathrm{mot}}_{d,X}(t)=Z_X^{\mathrm{mot}}(t)Z_X^{\mathrm{mot}}(\mathbb{L}t)\dots Z_X^{\mathrm{mot}}(\mathbb{L}^{d-1}t),\text{ if } X \text{ is a smooth curve.} \tag{1.4}$$

根据局部到整体原理，等价的局部公式为

$$Q^{\mathrm{mot}}_{d,\mathbb{A}^1,0}(t)=\frac{1}{(1-t)(1-\mathbb{L}t)\dots(1-\mathbb{L}^{d-1}t)}, \tag{1.5}$$

where $(\mathbb{A}^1,0)$ is the origin on an affine line. (We only list the exact form of these two formula among other known formulas, since they will be helpful later in the introduction section.) Both formulas are in fact essentially equivalent to formulas due to Solomon [56] in 1977 that count full-rank lattices of $\mathbb{Z}^d$ and ${\mathbb{Z}_p}^d$, as is observed and explained in [39]. Bifet's formula is later generalized to arbitrary vector bundles [6] (in place of the trivial vector bundle $\mathcal{O}_X^{\oplus d}$), and the nested Quot schemes [49]. Other works along this line consider high-rank-quotient versions of the Quot scheme [13, 15, 57] (where (equivariant) Chow rings are often computed as well) or the case where X is a smooth surface [40, 53].

The motive of the stack $\mathrm{Coh}_n(X)$, from an elementary point of view, is essentially counting commuting matrices (2.6), at least when X is affine and $k=\mathbb{F}_q$. When X is a smooth curve, we have an infinite product formula for $\widehat{Z}_X^{\mathrm{mot}}(t)$ in terms of $Z_X^{\mathrm{mot}}(t)$, essentially discovered by Cohen and Lenstra [16]. When X is a smooth surface, an infinite product formula for $\widehat{Z}_X^{\mathrm{mot}}(t)$ can be found by "globalizing" a local result of Feit and Fine [23] on counting pairs of commuting matrices. The globalization is treated in [14] using power structures and more elementarily in [38] using Euler products.

The most active part of the story, however, is the Hilbert zeta function $Q^{\mathrm{mot}}_{1,X}(t)$ for a reduced singular curve X, which the present paper attempts to generalize. We partially follow the exposition in the lecture note [28] and references therein. Let X be a reduced singular curve over k, and for simplicity, we assume **for the rest of the paper** that X has a normalization $\pi:\widetilde{X}\to X$ such that $\widetilde{X}$ is a finite disjoint union of geometrically connected smooth curves defined over k, and each singular point p of X is a k-point with $\pi^{-1}(p)$ consisting of only k-points. Whenever we talk about the analytic isomorphism class of a curve singularity p, we assume it has a model in such a curve X. Then the Hilbert zeta function of X (or (X,p)) satisfies the following properties:

(a) (Rationality; [10]) The series $Q^{\mathrm{mot}}_{1,X}(t)$ is rational in t. Moreover, if $\widetilde{X}$ denotes the normalization of X, then $Q^{\mathrm{mot}}_{1,X}(t)/Q^{\mathrm{mot}}_{1,\widetilde{X}}(t)$ is a polynomial in t that only depends on the singularities of X. Equivalently, if p is a k-point of X, then the local factor $Q^{\mathrm{mot}}_{1,X,p}(t)$ is rational with denominator $(1-t)^s$, where s is the number of branches through p.

[1]当 $k=\mathbb{F}_q$ 时，取 $\mathbb{F}_q$ 点计数会将 $Z_X^{\mathrm{mot}}(t)$ 特化为通常的局部ζ函数 [35, 附录 C]，，其中韦伊猜想适用。

Definition 1.1. Define

$$NQ^{\mathrm{mot}}_{1,X,p}(t) := (1-t)^s Q^{\mathrm{mot}}_{1,X,p}(t), \tag{1.6}$$

其中， s 是 p 的分支数。根据上述有理性，$\mathrm{NQ}^{\mathrm{mot}}_{1,X,p}(t)$ 是关于 t 的一个多项式。如果 X 的唯一奇点是 p，那么我们有

$$NQ^{\mathrm{mot}}_{1,X,p}(t) = \frac{Q^{\mathrm{mot}}_{1,X}(t)}{Q^{\mathrm{mot}}_{1,\widetilde{X}}(t)}, \tag{1.7}$$

~其中 X 是 X的正规化。

(b)（函数方程； [32, 47, 54]）如果 X 是 Gorenstein、几何连通且射影的，且其算术亏格为 g_a，那么

$$Q^{\mathrm{mot}}_{1,X}(t) = (\mathbb{L}t^2)^{g_a-1} Q^{\mathrm{mot}}_{1,X}(\mathbb{L}^{-1}t^{-1}) \tag{1.8}$$

as rational functions in t. The proof uses the fact that the dualizing sheaf on a Gorenstein curve is a line bundle, and expresses $Q^{\mathrm{mot}}_{1,X}(t)$ in terms of a certain stratification of the compactified Jacobian $\overline{J}(X)$ in the sense of Altman–Kleiman [3, 2]. The equivalent local statement is that the polynomial $NQ_{1,X,p}(t)$ in t has degree 2δ and satisfies

$$NQ^{\mathrm{mot}}_{1,X,p}(t) = (\mathbb{L}t^2)^{\delta} NQ^{\mathrm{mot}}_{1,X,p}(\mathbb{L}^{-1}t^{-1}), \tag{1.9}$$

~ ~其中 δ = 维数$_k \mathcal{O}_{X,p}/\mathcal{O}_{X,p}$ 是奇点 p 的 δ-不变量，其中 $\mathcal{O}_{X,p}$ 是$\mathcal{O}_{X,p}$的整闭包。

(c) (Exact formulas and Oblomkov–Rasmussen–Shende conjecture [51]) If p is a planar singularity on a reduced curve X over $k = \mathbb{C}$, the Hilbert scheme $\mathrm{Hilb}_n(X,p)$ is expected to have a affine cell decomposition, and [51] conjectured an exact formula for $Q^{\mathrm{mot}}_{1,X,p}(t)$ in terms of the HOMFLY homology of the associated algebraic link. The ORS conjecture is verified for the singularity $\mathbb{C}[[t^m, t^n]], \gcd(m,n) = 1$ with one Puiseux pair [29] and the singularity $\mathbb{C}[[t^{nd}, t^{md} + at^{md+1} + \dots]], \gcd(m,n) = 1, a \neq 0$ with two Puiseux pairs [30]. See also a related computation [42] for the affine Springer fiber associated to the multibranch singularity $\{x^n = y^{dn}\}$. In these examples, the relevant spaces do have affine cell decompositions.

As for planar singularities in general, the Euler-characteristic specialization of the ORS conjecture is proposed by [52] and is proved by Maulik [46]. A recent result [43, Corollary 1.11] by Kivinen and Tsai implies that $\mathrm{Hilb}_n(X,p)$ for any planar singularity is polynomial-count in the sense of Katz [36], i.e., the point counts of $\mathrm{Hilb}_n(X,p)$ are governed by its weight polynomial; moreover, the weight polynomial has nonnegative integer coefficients. The method is based on harmonic analysis and representation theory, and *not* stratification; in particular, it does not imply that the motive of $\mathrm{Hilb}_n(X,p)$ is a polynomial in $\mathbb{L}$.

注 1.2. 文献中引用的许多结果或猜想涉及的量与 $Q^{\mathrm{mot}}_{1,X,p}(t)$ 几乎等价（但并不完全等价）。这些版本在以下方面有所不同：

与 $\mathrm{Hilb}_n(X,p)$ 密切相关的两个空间是紧化雅可比簇 [47, 55] 和 A 型仿射 Springer 纤维 [26, 27]。

与动机密切相关的三个概念是 $\mathbb{F}_q$ 上的点计数、Poincar´e 多项式和权多项式；对于具有仿射胞腔分解的簇，这些概念彼此等价。一般而言，适用于每种概念的方法工具集各不相同。

欧拉示性数的概念比上述所有概念都更加粗略；对于具有胞腔分解的簇，欧拉示性数记录的是胞腔的数量，而不记录它们的维数。

Having discussed $Q^{\mathrm{mot}}_{1,X}(t)$ for reduced singular curves, we now move on to discuss $\widehat{Z}^{\mathrm{mot}}_X(t)$. In [38], the first author discovered an analogous "rationality" phenomenon for $\widehat{Z}^{\mathrm{mot}}_{X,p}(t)$, where (X,p) is a node singularity $\{xy = 0\}$. The "rationality" is put in quote because the "denominator" is in general an infinite product and the "numerator" is in general an infinite series in $\widehat{K}_0(\mathrm{Var}_k)[[t]]$. Lacking a more

suitable notion of when an infinite series qualifies as a "numerator", we work over finite fields and let $\#_q : \widehat{K}_0(\mathrm{Var}_k) \to \mathbb{Q}$ be the natural ring homomorphism induced by taking the $\mathbb{F}_q$-point count. We restate the conjecture formulated by the first author.

猜想 1.3 ~([38]). 设 X 是 $\mathbb{F}_q$ 上的一条约化曲线， X 是它的正规化。那么

$$\#_q\left(\frac{\widehat{Z}_X^{\mathrm{mot}}(t)}{\widehat{Z}_{\widetilde{X}}^{\mathrm{mot}}(t)}\right) \in \mathbb{Q}[[t]] \tag{1.10}$$

是一个关于 t 的幂级数，且具有无穷收敛半径。

The first author proved that Conjecture 1.3 holds when X has only nodal singularities $\{xy = 0\}$, by expressing the "numerator" as an explicit infinite series in $\mathbb{L}^{-1}$ and t. As a consequence of this explicit expression, he showed that for any real number $q > 1$, its specialization at $\mathbb{L} \mapsto q$ has infinite radius of convergence. His method is based on the observation that counting pairs of matrices with $AB = BA = 0$ over a finite field turns out to be achievable by elementary means.

现在我们讨论 $Q^{\mathrm{mot}}_{d\,X,}(t)$，这是本文的重点。我们提出，$Q^{\mathrm{mot}}_{d\,X,}(t)$ 的有理性陈述和函数方程应采用以下形式，从而推广关于 $d = 1$ 的已知结果。为便于查阅，下面分别陈述全局版本和局部版本；二者的等价性可直接利用 (1.4) 和 (1.5) 验证。

猜想 1.4. 设 X 是域 k 上的一条约化奇异曲线，且 $d \geq 0$。那么，对于 $Q^{\mathrm{mot}}_{d\,X,}(t) \in K_0(\mathrm{Var}_k)[[t]]$，我们有以下性质。

(a) (Rationality) *The series $Q^{\mathrm{mot}}_{d,X}(t)$ is rational in t. Moreover, if $\widetilde{X}$ is the normalization of X, then $Q^{\mathrm{mot}}_{d,X}(t)/Q^{\mathrm{mot}}_{d,\widetilde{X}}(t)$ is a polynomial in t that only depends on d and the singularities of X. Equivalently, if p is a k-point of X, then the local factor $Q^{\mathrm{mot}}_{d,X,p}(t)$ is rational with denominator $\big((1-t)(1-\mathbb{L}t)\dots(1-\mathbb{L}^{d-1}t)\big)^s$, where s is the number of branches through p.*

定义 1.5. 定义一个幂级数

$$NQ^{\mathrm{mot}}_{d,X,p}(t) := \big((1-t)(1-\mathbb{L}t)\dots(1-\mathbb{L}^{d-1}t)\big)^s Q^{\mathrm{mot}}_{d,X,p}(t) \in K_0(\mathrm{Var}_k)[[t]], \tag{1.11}$$

其中， s 是 p的分支数。有理性猜想等价于断言 $\mathrm{NQ}^{\mathrm{mot}}_{d\,X,,p}(t)$ 是关于 t 的多项式。如果 X 的唯一奇点是 p，那么我们有

$$NQ^{\mathrm{mot}}_{1,X,p}(t) = \frac{Q^{\mathrm{mot}}_{1,X}(t)}{Q^{\mathrm{mot}}_{1,\widetilde{X}}(t)}, \tag{1.12}$$

其中 $\widetilde{X}$ 是 X的正规化。这个等式不依赖于有理性猜想。

(b)（函数方程）若 X 是具有算术亏格 g_a的 Gorenstein 射影簇，则

$$Q^{\mathrm{mot}}_{d,X}(t) = (\mathbb{L}^{d^2}t^{2d})^{g_a-1}Q^{\mathrm{mot}}_{d,X}(\mathbb{L}^{-d}t^{-1}) \tag{1.13}$$

作为 t中的有理函数。等价地，$\mathrm{NQ}_{d,X,p}(t)$是 t 中的一个次数为 2δ 的多项式，并且满足

$$NQ^{\mathrm{mot}}_{d,X,p}(t) = (\mathbb{L}^{d^2}t^{2d})^{\delta}NQ^{\mathrm{mot}}_{d,X,p}(\mathbb{L}^{-d}t^{-1}), \tag{1.14}$$

其中 δ 是奇点 p的 δ不变量。

As far as we know, no $d \geq 2$ examples for singular curves have been explored or conjectured. Difficulties of some ideas commonly used for the Hilbert zeta function are discussed in Remark 1.12.

1.1. **Main results.** As the main theorem, we prove the following, which is what leads to the form of the functional equation proposed above. Recall that the cusp singularity has branching number $s = 1$ and δ-invariant $\delta = 1$.

定理1.6. 如果 (X, p) 是尖点奇点 $\{x^2 = y^3\}$，那么 $\mathsf{NQ}_{d,X,p}(t)$ 对所有 $d \geq 0$ 满足猜想 1.4(a)，并且对 $0 \leq d \leq 3$ 满足猜想 1.4(b)。此外，我们有

$$\begin{aligned} NQ^{\mathrm{mot}}_{0,X,p}(t) &= 1;\\ NQ^{\mathrm{mot}}_{1,X,p}(t) &= 1 + \mathbb{L}t^2;\\ NQ^{\mathrm{mot}}_{2,X,p}(t) &= 1 + (\mathbb{L}^2 + \mathbb{L}^3)t^2 + \mathbb{L}^4t^4;\\ NQ^{\mathrm{mot}}_{3,X,p}(t) &= 1 + (\mathbb{L}^3 + \mathbb{L}^4 + \mathbb{L}^5)t^2 + (\mathbb{L}^6 + \mathbb{L}^7 + \mathbb{L}^8)t^4 + \mathbb{L}^9t^6. \end{aligned} \tag{1.15}$$

The for（该部分原文不完整，暂译为"该……"） $\mathsf{NQ}_{\mathrm{mot}2,X,p}(t)$ 的公式、$\mathsf{NQ}^{\mathrm{mot}}_{3\,X,,p}(t)$ 以及 $\mathsf{NQ}_{\mathrm{mot}d,X,p}(t)$ 的 t-多项式性均为新的成果。.

The first step of our proof of Theorem 1.6 involves expressing the motive of $\mathrm{Quot}_{d,n}(X, p)$ in terms of the motives of some related moduli spaces. For $0 \leq r \leq \min(d, n)$, consider the scheme $\mathrm{Quot}^r_{d,n}(X, p)$ parametrizing zero-dimensional quotients of $\mathcal{O}_X^{\oplus d}$ of length n supported at p that require exactly r sections to generate. For $0 \leq r \leq n$, consider the stack $\mathrm{Coh}^r_n(X, p)$ parametrizing zero-dimensional coherent sheaves of length n supported at p that require exactly r sections to generate. Their rigorous definitions via functors of points are given in Appendix B. Consider the generating series

$$H^{\mathrm{mot}}_{d,X,p}(t) := \sum_{n\geq 0} [\mathrm{Quot}^d_{d,n+d}(X, p)]\, t^n \in K_0(\mathrm{Var}_k)[[t]] \tag{1.16}$$

and

$$\widehat{Z}^{\mathrm{mot}}_{d,X,p}(t) = \sum_{n\geq 0} [\mathrm{Coh}^d_n(X, p)]\, t^n \in \widehat{K}_0(\mathrm{Var}_k)[[t]]. \tag{1.17}$$

我们使用 q-Pochhammer 符号。

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}) \tag{1.18}$$

$$(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2) \dots \tag{1.19}$$

并考虑 $d \geq r$ 的 q-二项式系数：

$$\begin{bmatrix} d \\ r \end{bmatrix}_q = \frac{(q; q)_d}{(q; q)_r (q; q)_{d-r}}. \tag{1.20}$$

定理 1.7. 设 X 为任意 k-簇，p 为 X 的一个 k-点。对于任意 $d \geq r \geq 0$，我们在 $K_0(\mathsf{Var}_k)$ 中有如下恒等式 ^：

$$[\mathrm{Coh}^r_n(X, p)] = \frac{[\mathrm{Quot}^r_{d,n}(X, p)]}{\mathbb{L}^{d(n-r)}[\mathrm{Gr}(r, d)][\mathrm{GL}_r]}, \tag{1.21}$$

其中，$\mathsf{Gr}(r, d)$ 是格拉斯曼簇。

就生成级数而言，我们有

$$Q^{\mathrm{mot}}_{d,X,p}(t) = \sum_{r=0}^{d} \begin{bmatrix} d \\ r \end{bmatrix}_{\mathbb{L}} t^r\, H^{\mathrm{mot}}_{r,X,p}(\mathbb{L}^{d-r}t) \tag{1.22}$$

and

$$\widehat{Z}^{\mathrm{mot}}_{d,X,p}(t) = \frac{t^d}{\mathbb{L}^{d^2}(\mathbb{L}^{-1}; \mathbb{L}^{-1})_d}\, H^{\mathrm{mot}}_{d,X,p}(\mathbb{L}^{-d}t). \tag{1.23}$$

定理 1.7 的点计数版本实际上是一个初等观察结果，我们将在推论 2.4 和 2.7 中陈述这一结果，并给出一个自成体系的证明。然而，动机版本则需要格外谨慎，我们将在附录 B 中处理这一问题。

Having Theorem 1.7, proving Theorem 1.6 reduces to proving the following result. It is routine to verify that Theorem 1.8 implies Theorem 1.6 given (1.22).

Theorem 1.8. *If (X,p) is the cusp singularity $\{x^2=y^3\}$, then $(1-t)(1-\mathbb{L}t)\dots(1-\mathbb{L}^{d-1}t)H^{\mathrm{mot}}_{d,X,p}(t)$ is a polynomial in t, which we denote by $NH^{\mathrm{mot}}_{d,X,p}(t)$, and we have*

$$\begin{aligned} &NH^{\mathrm{mot}}_{0,X,p}(t)=1;\\ &NH^{\mathrm{mot}}_{1,X,p}(t)=1+\mathbb{L}t;\\ &NH^{\mathrm{mot}}_{2,X,p}(t)=1+(\mathbb{L}^2+\mathbb{L}^3)t+\mathbb{L}^4t^2;\\ &NH^{\mathrm{mot}}_{3,X,p}(t)=1+(\mathbb{L}^3+\mathbb{L}^4+\mathbb{L}^5)t+(\mathbb{L}^6+\mathbb{L}^7+\mathbb{L}^8)t^2+\mathbb{L}^9t^3. \end{aligned} \tag{1.24}$$

1.2. 进一步的猜想。以下假设 (X,p) 是尖点奇点。我们从上述数据中观察到,

$$NQ^{\mathrm{mot}}_{d,X,p}(t)=NH^{\mathrm{mot}}_{d,X,p}(t^2) \tag{1.25}$$

for $0\le d\le 3$. The resemblance between $NH^{\mathrm{mot}}_{d,X,p}(t)$ and $NQ^{\mathrm{mot}}_{d,X,p}(t)$ for $d\le 3$ is not expected, given that $Q^{\mathrm{mot}}_{d,X,p}(t)$ is computed as a weighted summation of $H^{\mathrm{mot}}_{r,X,p}(t)$ over $0\le r\le d$. The relation (1.25) does not hold for the node singularity $\{xy=0\}$, since for this singularity we have $NQ^{\mathrm{mot}}_1(t)=1-t+\mathbb{L}t^2$ and $NH^{\mathrm{mot}}_1(t)=1-t+\mathbb{L}t$. It is unclear for what other curve singularities we have an analogue of (1.25).

In fact, the pattern (1.25) is so strong that if true for all d, it would *overdetermine* the exact formulas of $H^{\mathrm{mot}}_{d,X,p}(t)$ for all d (the word "overdetermine" is in the sense of solving overdetermined linear system); the combinatorics is handled in Proposition 7.4. We conjecture the following formulas of striking simplicity; Proposition 7.4 shows that Conjecture 1.4(b), Theorem 1.7 and the conjectural (1.25) would imply Conjecture 1.9.

猜想 1.9. 如果 (X,p) 是尖点奇点 $\{x^2=y^3\}$，那么我们有

(a) $NQ^{\mathrm{mot}}_{d,X,p}(t):=\sum_{j=0}^{d}\begin{bmatrix}d\\ j\end{bmatrix}_{\mathbb{L}}(\mathbb{L}^dt^2)^j$.

(b) $\widehat{Z}^{\mathrm{mot}}_{X,p}(t)=\dfrac{1}{(\mathbb{L}^{-1}t;\mathbb{L}^{-1})_\infty}\displaystyle\sum_{n=0}^{\infty}\frac{\mathbb{L}^{-n^2}t^{2n}}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_n}$.

In light of Proposition 7.4, the existence of a guess in Conjecture 1.9(a) can be understood as an assertion that Conjecture 1.4(b), Theorem 1.7 and the conjectural (1.25) are consistent in the cusp case. Due to the overdetermined nature of this combination of theorems and conjectures, the mere consistency would be a strong evidence that (1.25) is expected to hold for all d.

Conjecture 1.9(b) gives a guess of $\widehat{Z}^{\mathrm{mot}}_{X,p}(t)$ for the cusp singularity for the first time. Since its "numerator" $\displaystyle\sum_{n=0}^{\infty}\frac{\mathbb{L}^{-n^2}t^{2n}}{(\mathbb{L}^{-1};\mathbb{L}^{-1})_n}$ substituted with $\mathbb{L}\mapsto q$ is an entire power series in t for all $q>1$, Conjecture 1.9(b) would verify the cusp case of Conjecture 1.3. See also (7.31) for a concrete consequence of Conjecture 1.9(b) in matrix counting.

现在我们稍作岔开。注意，若猜想 1.9(a) 成立，则对所有 d，$NQ^{\mathrm{mot}}_{d\,X,,p}(t)$ 都是 $\mathbb{L}$ 中的一个多项式，因此对所有 d、n，商概形$_{d,n}(X,p)$ 都是 $\mathbb{L}$ 中的一个多项式。我们提出一个较弱的猜想:

猜想 1.10（猜想 1.9 的弱形式）。设 (X,p) 是尖点奇点 $\{x^2=y^3\}$。那么商概形$_{d,n}(X,p)$ 的动机是 $\mathbb{L}$ 中的一个多项式。

It turns out that this statement is implied by Conjecture 6.1 about commuting matrices in certain upper triangular matrix algebras determined by posets. This conjecture motivates the following theorem as an important special case we are able to prove. This is an unexpected side product of our main goal and is of separate interest.

Theorem 1.11 (Corollary 6.9). *Consider the (reduced) k-variety cut out by*

$$V_d := \left\{ (X,Y) \in \mathrm{Mat}_d(k)^2 \;\middle|\; \begin{array}{l} X,Y \text{ are strictly upper triangular} \\ X^2 = Y^3, XY = YX \end{array} \right\}. \tag{1.26}$$

那么， V_d 的动机是关于 $\mathbb{L}$ 的一个多项式。此外，推论 6.9 给出了 $[V_d]$ 的一个归纳公式。

1.3. 方法与策略。我们首先概述一些用于计算希尔伯特概形和商概形的动机的现有方法。以下几类方法均基于分类或分层（即局部闭分解），这也是动机定义所要求的，尽管它们各自的具体形式有所不同。

(a) Local method: Work on the punctual Quot scheme $\mathrm{Quot}_{d,n}(X,p)$ and solve the commutative algebra problem of classifying ideals of the complete local ring $\widehat{\mathcal{O}}_{X,p}$, or submodules of $\widehat{\mathcal{O}}_{X,p}^{\oplus d}$;
(b) Global method: Choose a projective model X and use global properties for torsion free sheaves on X, such as Riemann–Roch and Serre duality;
(c) Torus action: Choose a model X that admits a "good enough" torus action, and make use of Białynicki-Birula decomposition.

Method (a) was used in example-based computations of $Q^{\mathrm{mot}}_{1,X,p}(t)$ for curve singularities (such as [52]) and the proof of the general rationality result [10], to be elaborated later. Method (b) was used in the proof of rationality and the function equation for $Q^{\mathrm{mot}}_{1,X}(t)$ for Gorenstein curves [32, 47, 54]. Method (c) was used in the computation of $Q^{\mathrm{mot}}_{1,\mathbb{P}^2}(t)$ in [21] and generalizations of $Q^{\mathrm{mot}}_{d,\mathbb{P}^1}(t)$ in [49].

The main theorem we are proving is the rationality statement of Theorem 1.8. Our proof belongs to Method (a).[2] More precisely, we use the theory of Gröbner bases over power series rings (also known as "standard bases" in the terminology of Hironaka [37]) to classify finite-k-codimensional submodules of the rank-d free module over $\widehat{\mathcal{O}}_{X,p} = k[[T^2,T^3]]$. We classify by splitting into cases (i.e., "strata"), and identify a parametrization for each case. In general, the method of Gröbner bases would lead to tremendous computational complications, especially those caused by the Buchberger criterion. The complexity is not only in combinatorics (counting the number of parameters in a stratum), but also in completely determining the relation between the parameters in a given stratum. The computation sensitively depends on some artificial choices, such as the monomial order. We use an "hlex" monomial order such that the power of T is more important than the ordering of the basis vectors of $k[[T^2,T^3]]^d$, see §4. Furthermore, we classify submodules of $(T^2,T^3)k[[T^2,T^3]]$ (which directly leads to $H^{\mathrm{mot}}_{d,X,p}(t)$) instead of submodules of $k[[T^2,T^3]]$ (which directly leads to $Q^{\mathrm{mot}}_{d,X,p}(t)$). Fortunately, these choices lead to manageable computations, and we are able to determine completely the relation between the parameters in a given stratum (Lemmas 4.4 and 4.5).

We compare our method to the proof of the BRV theorem [10] that $Q^{\mathrm{mot}}_{1,X,p}(t)$ is rational for any reduced curve singularity (X,p). For simplicity, say (X,p) is a unibranch singularity, and write $\mathcal{O} = \mathcal{O}_{X,p}$ and $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{X,p} \cong k[[T]]$. Consider a "branch-length stratification" by defining the stratum $\mathrm{Hilb}_n(X,p;e)$ to consist of ideals I of $\mathcal{O}$ such that $\dim_k \mathcal{O}/I = n$ and $\dim_k \widehat{\mathcal{O}}/I\widehat{\mathcal{O}} = e$ (namely, $I\widehat{\mathcal{O}} = (T^e)k[[T]]$). The rationality of $Q^{\mathrm{mot}}_{1,X,p}(t)$ is then reduced to the following stabilization statement:

$$[\mathrm{Hilb}_{n+1}(X,p;e+1)] = [\mathrm{Hilb}_n(X,p;e)] \text{ for } e \gg 0. \tag{1.27}$$

对于多分支奇点，BRV 定理的证明采用关于分支数的归纳法，其归纳步骤基于一个类似的稳定化命题。

Our stratification is based on Gröbner theory discussed above. We denote our strata by $\mathrm{Hilb}(\alpha)$, where α is any d-tuple of nonnegative integers, each decorated by one of two available colors. The datum α encodes a submodule of $(T^2,T^3)k[[T^2,T^3]]$ generated by monomials, and $\mathrm{Hilb}(\alpha)$ consists of submodules of $(T^2,T^3)k[[T^2,T^3]]$ whose leading term submodule corresponds to α. We are able to prove that the motive of $\mathrm{Hilb}(\alpha)$ is eventually stable up to a correction factor (Lemma 5.11 and

[2]关于方法 (b)(c) 的困难，参见注 1.12。

Lemma 5.13) with respect to certain "spiral shifting" operators $\gamma_{(1)}, \dots, \gamma_{(d)}$ that act on the index set of α (illustrated in Figure 1). In particular, the stability is not with respect to raising one of the components of the d-tuple by one, but a mixture of permuting and raising the components. The spiral shifting operators originally arise in a similar stratification of the punctual Quot scheme of $(\mathbb{A}^1, 0)$ due to the authors [39].

To finish the comparison with BRV's proof, note that when $d = 1$ (BRV's situation), the only operator $\gamma_{(1)}$ is $n \mapsto n + 1$, which is, in combinatorial essence, the same as in BRV's proof. In effect, we answered what the high-d generalization of BRV's stabilization statement should look like: in the case of the cusp singularity, the combinatorics of the corresponding stabilization statement involves the operators $\gamma_{(j)}$.

We conclude with a discussion about how our method is expected to generalize to arbitrary reduced curve singularities (X, p). We likely need a stabilization statement for the unibranch case, and then generalize to the multibranch case using induction. It is expected that the stratification is obtained in a way inspired by Gröbner basis, but it is unclear what version of stratification is computationally manageable in general. However, the combinatorial nature of the stabilization seems to be not sensitive to the engineering details of the stratification: Our proof suggests that for a general unibranch singularity (X, p), there should be a similar stratification for $\mathrm{Quot}_{d,n}(X, p)$ with strata indexed by some decorated version of d-tuples of integers (so the operators $\gamma_{(j)}$ are defined), such that an eventual stabilization statement for stratum motives holds with respect to $\gamma_{(j)}$.

注 1.12. 目前尚不清楚如何在我们的情形中使用方法 (c)，因为 $\mathrm{Quot}_{d,n}(X,p)$ 通常并不光滑。尽管对于某些奇异簇，例如结点曲线或尖点有理曲线，也存在 Białynicki-Birula 型分解，但尚不清楚我们的 $\mathrm{Quot}_{d,n}(X,p)$ 是否属于此类。

The conjectured functional equation for $Q^{\mathrm{mot}}_{d,X,p}(t)$ for the cusp singularity suggests a potential proof using Method (b), but it is currently not clear how: a direct attempt to generalize the proof of the $d = 1$ case in [32, 47, 54] would involve identifying the moduli space of rank-d torsion free sheaves on a Gorenstein curve X (as a high-rank generalization of the compactified Jacobian, namely, the compactified $\mathcal{B}un_{\mathrm{GL}_d}$) and putting a suitable stratification on it which behaves well with Serre duality and is simple enough so that one can effectively extract information to determine $Q^{\mathrm{mot}}_{d,X}(t)$. One essential observation of [32] is that the embeddings of a rank 1 torsion free sheaf $\mathcal{L}$ into $\mathcal{O}_X$ are classified by $H^0(X, \mathcal{L}^\vee) - \{0\}$, which leads to a nice stratification of the compactified Jacobian by cohomological invariants. But this fails for higher rank torsion free sheaves, namely, the embeddings of a rank d torsion free sheaf $\mathcal{E}$ into $\mathcal{O}_X^d$ is usually only a Zariski open subset $U_{\mathcal{E}} \subseteq H^0(X, \mathcal{E}^\vee) - \{0\}$. At this moment, it is hard to give a satisfactory (geometrical or cohomological) characterization of $\mathcal{E}$ with fixed isomorphism class of $U_{\mathcal{E}}$. One may expect a stratification over the moduli space of rank-d torsion free sheaves depending on the isomorphism classes of $U_{\mathcal{E}}$. But it seems to behave badly with Serre duality.

We also give a remark on the complexity of torsion free sheaves (with no restraint on the rank) over a cusp. This is relevant to us because Fourier–Mukai transform establishes an equivalence between degree 0 semistable torsion free sheaves over a cuspidal cubic with the category of torsion sheaves, see [58, 12]. The innate non-semisimplicity of the category of torsion free sheaves is merely one aspect of its complexity. Even for indecomposable semistable torsion free sheaves, wild phenomena already happen. For example, the category of higher rank torsion free sheaves over a cuspidal curve is wild, in the sense that there are families of indecomposable torsion free sheaves depending on any prescribed number of parameters, see [18]. The category of torsion sheaves supported at the singular point is also wild, see [17]. These two facts are compatible from the theory of Fourier–Mukai transform. These facts suggest that studying $Q^{\mathrm{mot}}_{d,X}(t)$ from a representation theoretical point of view, or from Fourier–Mukai transform, may be hard.

Remark 1.13. In principle, our method involving Gröbner bases can be applied to any singularity $k[[T^m, T^n]]$ with $\gcd(m, n) = 1$ with one Puiseux pair, but it is unclear whether the Buchberger criterion is manageable in any case other than $(m, n) = (2, 3)$. Our Lemma 4.4 and Lemma 4.5 seem

coincidental and we do not know what their generalizations should look like. It might still be possible, though, to attack the rationality conjecture with an eventual stabilization statement analogous to Lemma 5.13 without understanding the Gröbner strata as explicitly as in Lemmas 4.4 and 4.5.

1.4. **Organization and structure of proof.** We take an unusual approach in presenting the proof, namely, we focus on the set of k-points and disregard the geometry when working with various moduli spaces. In particular, we will give a self-contained proof of the point-count version of our theorems that is accessible to readers with background in commutative algebra and basic algebraic geometry. We do so to avoid distractions and highlight the already complicated details in the Gröbner stratification and the combinatorics.

由于证明是基于逐个情形的参数化，因此除少数需要额外谨慎处理的地方外，证明的大部分内容都可以直接转化为局部闭分层结构，这些特殊情况将在附录 B 中处理。

We introduce the preliminaries of the relevant moduli spaces in §2, followed by an introduction of the needed theorems about Gröbner basis theory in §3 (with proofs in Appendix A). Equipped with these, we give a complete description of our Gröbner stratification of $\mathrm{Quot}^d_{d,n+d}(X,p)$ for the cusp in §4. Next, in §5, we sort out the combinatorics and finish the proof of the first part of Theorem 1.8.

In §6, we take a detour to prove Theorem 1.11; the proof in this section is independent of the rest of the paper. In §7, we perform explicit computations of $Q^{\mathrm{mot}}_{d,X,p}(t)$ for the cusp, where $d \le 3$. This finishes the proof of Theorem 1.8. Further conjectures, including Conjecture 1.9, are discussed in §7.3.

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2. Some moduli spaces

Let k be a field and X be a variety over k (a separated k-scheme of finite type that is not necessarily smooth, proper or reduced). Let p be a closed point of X, and consider the completed local ring $\widehat{\mathcal{O}}_{X,p}$. We recall the set of k-points of several moduli spaces over X (or (X,p)), and their point counting over $k=\mathbb{F}_q$. The geometric definitions of these moduli schemes or stacks are given in Appendix B.

按照 Nakajima [50] 对箭图簇的通常处理方式，我们将模空间分为两类：无框架模空间和带框架模空间。

2.1. **Unframed moduli spaces.** We follow [14] and [38]. Recall that $\mathrm{Coh}_n(X)$ is the moduli stack of coherent sheaves M on X supported at n points counting multiplicity (namely, M has finite support and the space of global section has dimension n). Point counting of $\mathrm{Coh}_n(X)$ over $k=\mathbb{F}_q$ can be understood via **stacky sets**. A stacky set[3] is a finite set where each point x is equipped with a finite group $\mathrm{Aut}\, x$, called the automorphism group of x. The cardinality of a stacky set $\mathcal{X}$ is defined as

$$|\mathcal{X}|^{\mathrm{stck}} := \sum_{x\in\mathcal{X}} \frac{1}{|\mathrm{Aut}\, x|}. \tag{2.1}$$

Say $k=\mathbb{F}_q$. We think of the set of k-points of $\mathrm{Coh}_n(X)$ as a stacky set consisting of coherent sheaves on X supported at n points counting multiplicity up to isomorphism, each equipped with the automorphism group of the sheaf. When $X=\mathrm{Spec}\, R$ is affine, $\mathrm{Coh}_n(X)$ is canonically isomorphic to the stacky set of R-modules M such that $\dim_k M = n$ up to isomorphism, equipped with the group of R-linear automorphisms of M. The point count of $\mathrm{Coh}_n(X)$ over $\mathbb{F}_q$ is defined as the stacky cardinality

$$|\mathrm{Coh}_n(X)(\mathbb{F}_q)| := \sum_{M\in\mathrm{Coh}_n(X)} \frac{1}{|\mathrm{Aut}\, M|}. \tag{2.2}$$

[3]等价地，一个叠集集合是具有有限自同构群的有限群胚的同构类集合。我们选择将群胚的使用推迟到附录 B，以使这里的点计数过程保持透明。

代数叠 $\mathsf{Coh}_n(X)$ 可以实现为一个商叠 $[C_n(X)/\mathrm{GL}_n]$，其中 $C_n(X)$ 是一个定义良好的概形；其几何性质将在附录 B 中讨论。如果 X 是仿射的，那么 $C_n(X)$ 的 k-点集具有如下显式描述：设

$$X = \operatorname{Spec} R = \operatorname{Spec} \frac{k[T_1, \dots, T_m]}{(f_1, \dots, f_r)}, \tag{2.3}$$

那么，集合 of $C_n(X)$ 的 k-点是满足 X 方程的可交换矩阵集合：

$$C_n(X)(k) := \{\underline{A} = (A_1, \dots, A_m) \in \mathrm{Mat}_n(k)^m : [A_i, A_j] = 0, f_i(\underline{A}) = 0\}, \tag{2.4}$$

并且 GL_n 通过同时共轭作用。基于此，我们将 X 上的第 n 个交换簇称为 $C_n(X)$。一个无坐标的描述是

$$C_n(X)(k) = \mathrm{Hom}_{\mathrm{AssoAlg}/k}(R, \mathrm{Mat}_n(k)), \tag{2.5}$$

因此，也可以将 $C_n(X)(k)$ 理解为“$X(\mathsf{Mat}_n(k))$”，或者理解为 X 上的 $n \times n$ 矩阵点。

代数叠商 $\mathsf{Coh}_n(X) = [C_n(X)/\mathrm{GL}_n]$，从点计数的角度来看，表示为

$$|\mathrm{Coh}_n(X)(\mathbb{F}_q)| = \frac{|C_n(X)(\mathbb{F}_q)|}{|\mathrm{GL}_n(\mathbb{F}_q)|}. \tag{2.6}$$

事实上，这是轨道–稳定子定理的一个初等推论。

Now we recall the punctual version of the above. Let $p \in X$ be a closed point, and $R = \widehat{\mathcal{O}}_{X,p}$ be the completed local ring of X at p. Let $\mathrm{Coh}_n(X, p)$ be moduli stack of coherent sheaves on X supported at p with multiplicity n. Let $\mathrm{Coh}_n(R)$ be the moduli stack of R-modules M with $\dim_k M = n$. Then we have a canonical isomorphism $\mathrm{Coh}_n(X, p) \cong \mathrm{Coh}_n(R)$.

If

$$R = \frac{k[[T_1, \dots, T_m]]}{(f_1, \dots, f_r)}, \tag{2.7}$$

对于所有 i，令其满足 $f_i(0, \dots, 0) = 0$，定义

$$C_n(X, p) = C_n(R) := \{\underline{A} = (A_1, \dots, A_m) \in \mathrm{Nilp}_n(k)^m : [A_i, A_j] = 0, f_i(\underline{A}) = 0\}, \tag{2.8}$$

其中，$\mathsf{Nilp}_n(k)$ 表示 k 上的 $n \times n$ 幂零矩阵簇。于是，$\mathsf{Coh}_n(X, p) = [C_n(X, p)/\mathrm{GL}_n]$ 可表示为一个商代数叠。

零维层的全局版本和支撑于一点的版本满足一个局部到整体公式。如果 $k = \mathbb{F}_q$，第一作者定义了科恩–伦斯特拉级数

$$\widehat{Z}_X(t) := \sum_{n \geq 0} |\mathrm{Coh}_n(X)(\mathbb{F}_q)| t^n; \tag{2.9}$$

$$\widehat{Z}_{X,p}(t) := \sum_{n \geq 0} |\mathrm{Coh}_n(X, p)(\mathbb{F}_q)| t^n, \tag{2.10}$$

并证明了一个欧拉乘积公式（[38, 命题 4.2]）

$$\widehat{Z}_X(t) = \prod_{p \in X_{\mathrm{cl}}} \widehat{Z}_{X,p}(t), \tag{2.11}$$

其中，p 遍历 X的所有闭点。

2.2. **Framed moduli spaces.** For $n, d \geq 0$, recall that $\mathrm{Quot}_{d,n}(X) = \mathrm{Quot}_{\mathcal{O}_X^d, n}$ is the Quot scheme parametrizing quotient sheaves of $\mathcal{O}_X^d$ supported at n points counting multiplicity. If $d = 0$, then $\mathrm{Quot}_{d,n}(X)$ is a point if $n = 0$ and empty if $n > 0$. If $d = 1$, then $\mathrm{Quot}_{1,n}(X)$ is the Hilbert scheme $\mathrm{Hilb}_n(X)$. The set of k-points is described as

$$\mathrm{Quot}_{d,n}(X)(k) := \left\{ (M, f) \middle| M \in \mathrm{Coh}_n(X), f : \mathcal{O}_X^d \twoheadrightarrow M \right\} \Big/ \sim, \tag{2.12}$$

当且仅当存在一个同构 $\varphi\colon M_1 \to M_2$ 使得 $\varphi \circ f_1 = f_2$时，$(M_1, f_1) \sim (M_2, f_2)$。满射 f 称为 M的一个 d-框架。通过取商映射的核，我们得到以下等价描述。

$$\mathrm{Quot}_{d,n}(X)(k) = \Big\{\mathcal{I} \subseteq \mathcal{O}_X^d : \mathcal{O}_X^d/\mathcal{I} \in \mathrm{Coh}_n(X)\Big\}. \tag{2.13}$$

这里， $\mathcal{I}$ 是$\mathcal{O}_X^d$的相干子层。若 $d = 1$，则 $\mathcal{I}$ 是X 的闭子概形的理想层，其结构层为 $\mathcal{O}_X/\mathcal{I}$。

$\mathsf{Quot}_{d,n}(X)$ 在 $\mathbb{F}_q$ 上的点计数由 $\mathsf{Quot}_{d,n}(X)(\mathbb{F}_q)$ 的通常基数给出。设 $p \in X$ 为一个闭点。定点 Quot 概形的 k-点

$$\mathrm{Quot}_{d,n}(X,p)(k) := \left\{(M,f)\middle| M \in \mathrm{Coh}_n(X,p), f : \mathcal{O}_X^d \twoheadrightarrow M\right\}\Big/ \sim, \tag{2.14}$$

$\widehat{\ }$只能用完备局部环 $R = \mathcal{O}_{X,p}$ 来描述。令

$$\mathrm{Quot}_{d,n}(R)(k) := \Big\{I \subseteq R^d : \dim_k R^d/I = n\Big\}, \tag{2.15}$$

然后商概形$_{d,n}(X,p)(k) \cong$ 商概形$_{d,n}(R)(k)$ 是典范同构的。

与无框架情形一样，全局商概形和定点 Quot 概形的点计数通过一个局部到整体公式相关联。对于任意整数 $d \geq 0$, 考虑秩为 d 的 Quot ζ 函数。

$$\begin{aligned} Q_{d,X}(t) &:= \sum_{n\geq 0} |\mathrm{Quot}_{d,n}(X)(\mathbb{F}_q)| t^n; \\ Q_{d,X,p}(t) &:= \sum_{n\geq 0} |\mathrm{Quot}_{d,n}(X,p)(\mathbb{F}_q)| t^n. \end{aligned} \tag{2.16}$$

于是我们得到如下欧拉乘积公式。

命题 2.1. 设 X 是 $k = \mathbb{F}_q$ 上的一个簇，且 $d \geq 0$ 是一个整数。那么我们有

$$Q_{d,X}(t) = \prod_{p \in X_{\mathrm{cl}}} Q_{d,X,p}(t). \tag{2.17}$$

证明。与 [38, 命题 4.2], 的证明类似， X 上的每个有限长度相干层 M 都有唯一的分解 $M = \bigoplus_{p\in X_{\mathrm{cl}}} M_p$ ，分解为支撑于 p 处的有限长度相干层 M_p ，并且除有限多个 M_p 外，所有 M_p 都为零。注意， M_p 可以被识别为一个有限长度的 $\mathcal{O}_{X,p}$-模，并令 n_p 表示 M_p 的 k-维数。

假设给定 $(M, f) \in$ 商概形$_{d,n}(X)(k)$，其中 $M \in \mathsf{Coh}_n(X)$ 且 $f\colon \mathcal{O}_X^d \twoheadrightarrow M$。在 p 处局部化后，我们得到一个元素 $(M_p, f_p) \in$ 商概形$_{d,n_p}(X,p)(k)$，其中 M_p, n_p 如上所述，且 $f_p\colon \mathcal{O}_{X,p}^d \twoheadrightarrow M_p$ 是 $\mathcal{O}_{X,p}$-模的满射。

反过来，给定 $(M_p, f_p) \in$ 商概形$_{d,n_p}(X,p)(k)$，对所有满足除有限多个 M_p 外其余均为零的 $p \in X_{\mathrm{cl}}$，构造 $M = \bigoplus_{p\in X_{\mathrm{cl}}} M_p$。考虑复合映射 $\mathcal{O}_X^d \to \mathcal{O}_{X,p}^d \xrightarrow{f_p} M_p$ 以及诱导映射 $f\colon \mathcal{O}_X^d \to \prod_{p\in X_{\mathrm{cl}}} M_p = M$（注意，有限直和就是有限直积）。然后我们恢复 $(M, f) \in$ 商概形$_{d,n}(X)(k)$，其中 $n = \sum n_p$。

不难验证，上述两个构造互为逆映射。因此，我们得到一个双射。

$$\mathrm{Quot}_{d,n}(X)(k) \cong \bigsqcup_{\sum n_p = n} \prod_{p\in X_{\mathrm{cl}}} \mathrm{Quot}_{d,n_p}(X,p)(k), \tag{2.18}$$

这就得到了生成级数中所需的乘积公式。 □

2.3. 忘却框架。定理 1.8 中的第一个关键步骤是利用遗忘框架所定义的映射 $\mathrm{Quot}_{d,n}(X) \to \mathrm{Coh}_n(X)$，将 $\mathrm{Coh}_n(X)$ 上的计数问题转化为 $\mathrm{Quot}_{d,n}(X)$ 上的计数问题。我们需要控制该映射的纤维大小，为此，需要考虑其零维版本。

在本节的其余部分，考虑一个 k-簇 X 和一个闭点 $p \in X$。令 $R = \widehat{\mathcal{O}_{X,\,p}}$ 并将 $\mathfrak{m}$ 记为 R 的极大理想。为简单起见，假设 p 是 X 的一个 k-点，因此 R 的剩余域为 k。

从现在起，R-模 M 的维数始终指 $\dim_k M$。因此，$\mathrm{Coh}_n(R)$ 参数化了 R 上的 n 维模。我们将有限维 R-模 M 的秩（记为 $\mathrm{rk}\ M$）定义为生成 M 作为 R-模所需的最少生成元个数。根据中山引理，我们有

$$\mathrm{rk}\, M = \dim_k M \otimes_R k = \dim_k M/\mathfrak{m}M. \tag{2.19}$$

从现在起，我们假设 $k = \mathbb{F}_q$，并且提到模空间时，总是指 k-点集。考虑

$$\mathrm{Coh}_n^r(R) := \{M \in \mathrm{Coh}_n(R) : \mathrm{rk}\, M = r\}, \tag{2.20}$$

and

$$\mathrm{Quot}_{d,n}^r(R) := \{(M, f) \in \mathrm{Quot}_{d,n}(R) : \mathrm{rk}\, M = r\}. \tag{2.21}$$

我们注意到，$\mathrm{Coh}_n^r(R)$ 为空，除非 $r \le n$，且 $\mathrm{Quot}_{d,\,n}^r(R)$ 为空，除非 $r \le \min\{d, n\}$。我们有 $\mathrm{Coh}_n(R) = \bigsqcup_{r=0}^{n} \mathrm{Coh}_n^r(R)$，以及 $\mathrm{Quot}_{d,\,n}(R) = \bigsqcup_{r=0}^{\min\{d,n\}} \mathrm{Quot}_{d,\,n}^r(R)$。

秩将在确定遗忘映射 $\mathrm{Quot}_{d,\,n}(R) \to \mathrm{Coh}_n(R)$ 的纤维大小方面发挥关键作用。

Remark 2.2. We briefly mention the geometric (scheme or stack theoretical) aspects of these moduli spaces. There is an upper semi-continuous function in Zariski topology on $\mathrm{Quot}_{d,n}(R)$ defined as $\mathrm{rk}(x) = \dim_k M/\mathfrak{m}M$, where M is the module corresponding to the point x. This implies that, at least up to reduced structures, $\mathrm{Quot}_{d,n}^{\ge r}(R)$ is a closed subscheme of $\mathrm{Quot}_{d,n}(R)$ and $\mathrm{Quot}_{d,n}^{\ge r}(R)$ is a locally 闭子概形。因此，我们有一个 Zariski 分层 $\mathrm{Quot}_{d,n}(R) = \bigsqcup^{\mathrm{m}}$ $\mathrm{in}\{d\ n\}$ $r=0$ 商概形$_{d,\,n}^{r}(R)$。

类似地，可以在 $\mathrm{Coh}_n(R)$ 上定义一个上半连续函数 rk。至少在约化结构的意义下，$\mathrm{Coh}_n^{\ge r}(R)$ 是 $\mathrm{Coh}_n(R)$ 的一个闭子代数叠，而 $\mathrm{Coh}_n^r(R)$ 是一个局部闭子代数叠。因此，我们得到一个 Zariski 分层 $\mathrm{Coh}_n(R) = \bigsqcup_{r=0}^{n} \mathrm{Coh}_n^r(R)$。

我们只会在附录 B 中用到这些观察结果。

命题 2.3. 假设上述设定及 $d \ge r$。则遗忘映射 $\Phi : \mathrm{Quot}_{d,n}^r(R) \to \mathrm{Coh}_n^r(R)$ 是满射，且原像基数为

$$|\Phi^{-1}(M)| = \frac{1}{|\mathrm{Aut}\, M|} |\mathrm{Surj}(\mathbb{F}_q{}^d, \mathbb{F}_q{}^r)|\, q^{d(n-r)} \tag{2.22}$$

对于任意 $M \in \mathrm{Coh}_n^r(R)$。

证明。固定一个 R-模 M，其同构类属于 $\mathrm{Coh}_n^r(R)$。那么 $\Phi^{-1}(M)$ 由满射 $f : R^d \twoheadrightarrow M$（模等价关系 $f_1 \sim f_2$）组成，当且仅当存在 $\phi \in \mathrm{Aut}\, M$ 使得$\phi \circ f_2 = f_1$。换言之，我们有

$$\Phi^{-1}(M) = \mathrm{Surj}(R^d, M)/\,\mathrm{Aut}\, M, \tag{2.23}$$

where $\mathrm{Surj}(R^d, M)$ is the set of R-linear surjections from R^d to M, and $\mathrm{Aut}\, M$ acts on $\mathrm{Surj}(R^d, M)$ by composition. We note that $\mathrm{Aut}\, M$ acts on $\mathrm{Surj}(R^d, M)$ freely: if $\phi \circ f = f$ and f is a surjection, we must have $\phi = \mathrm{id}$. Therefore, we have $|\Phi^{-1}(M)| = |\mathrm{Surj}(R^d, M)|/|\mathrm{Aut}\, M|$. It suffices to determine $|\mathrm{Surj}(R^d, M)|$.

确定一个从 R^d 到 M 的 R-线性映射，等价于选取 M 中的一个 d-元组 $(v_1, \ldots, v_d)$。根据 Nakayama 引理，与 $(v_1, \ldots, v_d)$ 相关联的映射是满射，当且仅当

th模 $\mathfrak{m}$ 对 v_1 的约化，……， v_d 生成了 $M/\mathfrak{m}M$ 作为一个 k-向量空间。因此，

$$\mathrm{Surj}(R^d, M) = \mathrm{Surj}(k^d, M/\mathfrak{m}M) \times (\mathfrak{m}M)^d. \tag{2.24}$$

注意到 $\dim_k M/\mathfrak{m}M = r$ 和 $\dim_k M = n$，我们有 $\dim_k \mathfrak{m}M = n - r$，s o

$$|\mathrm{Surj}(R^d, M)| = |\mathrm{Surj}(k^d, k^r)| \cdot |\mathfrak{m}M|^d \tag{2.25}$$

$$= |\mathrm{Surj}({\mathbb{F}_q}^d, {\mathbb{F}_q}^r)|\, q^{d(n-r)}. \tag{2.26}$$

□

推论 2.4. 如果 $d \geq r$，我们有

$$|\mathrm{Coh}^r_n(R)(\mathbb{F}_q)| = \frac{|\mathrm{Quot}^r_{d,n}(R)(\mathbb{F}_q)|}{q^{d(n-r)}|\mathrm{Surj}({\mathbb{F}_q}^d, {\mathbb{F}_q}^r)|}. \tag{2.27}$$

注意，$|\mathrm{Coh}^r_n(R)(\mathbb{F}_q)|$ 是一个代数叠点计数。

由于等式左边不依赖于 d，计数 $|$商概形$^r_{r,n}(R)|$ 确定了所有 $d \geq r$ 对应的 $|$商概形$^r_{d,n}(R)|$。

推论 2.5. 如果 $d \geq r$，我们有

$$|\mathrm{Quot}^r_{d,n}(R)| = q^{(n-r)(d-r)}|\mathrm{Gr}(r,d)(\mathbb{F}_q)| \cdot |\mathrm{Quot}^r_{r,n}(R)|, \tag{2.28}$$

其中，$\mathrm{Gr}(r,d)(\mathbb{F}_q)$ 是格拉斯曼簇，用于参数化 ${\mathbb{F}_q}^d$ 的 r 维子空间。

证明。比较 $d := d$ 和 $d := r$ 的推论 2.4，可得

$$\frac{|\mathrm{Quot}^r_{d,n}(R)|}{|\mathrm{Quot}^r_{r,n}(R)|} = \frac{q^{d(n-r)}|\mathrm{Surj}({\mathbb{F}_q}^d, {\mathbb{F}_q}^r)|}{q^{r(n-r)}|\mathrm{Surj}({\mathbb{F}_q}^r, {\mathbb{F}_q}^r)|} \tag{2.29}$$

$$= q^{(n-r)(d-r)}|\mathrm{Gr}(r,d)(\mathbb{F}_q)|. \tag{2.30}$$

□

上述结果表明，只需研究 $|\mathrm{Quot}^r_{d,n}(R)|$，其中 $r = d$。我们引入如下记号

$$\mathrm{Hilb}_{d,n}(R) := \mathrm{Quot}^d_{d,n+d}(R) \tag{2.31}$$

对于 $d, n \geq 0$。

回顾式 (2.15)，有

$$\mathrm{Quot}^d_{d,n+d}(R) = \{I \subseteq R^d : \dim_k R^d/I = n+d, \mathrm{rk}_k R^d/I = d\}. \tag{2.32}$$

注意到

$$\mathrm{rk}_k \frac{R^d}{I} = \dim_k \frac{R^d}{I} \otimes_R \frac{R}{\mathfrak{m}} = \dim_k \frac{R^d}{\mathfrak{m}R^d + I} \leq \dim_k \frac{R^d}{\mathfrak{m}R^d} = d, \tag{2.33}$$

其中，当且仅当 $I \subseteq \mathfrak{m}R^d$ 时等号成立。因此，我们得到一个简化的描述，供 §4 使用：

$$\mathrm{Hilb}_{d,n}(R) = \{I \subseteq \mathfrak{m}R^d : \dim_k \mathfrak{m}R^d/I = n\}. \tag{2.34}$$

Remark 2.6. We warn the readers that the arguments in Proposition 2.3 and its corollaries do not generalize directly to the motivic setting. Even though the formula (2.22) is true pointwise, we need it to hold stratum-wise in order to treat the motivic case. However, it is not clear how one can find a stratification over which the formula can generalize. We refer the reader to Appendix B for a treatment in the motivic case.

2.4. 一些生成级数。设 $k=\mathbb{F}_q$, X 是一个 k-簇， $p\in X(k)$，且 $R=\mathcal{O}_{X,\,p}$。回顾局部秩-d 商概形 ζ 函数

$$Q_{d,R}(t)=Q_{d,X,p}(t):=\sum_{n\geq 0}|\mathrm{Quot}_{d,n}(R)(\mathbb{F}_q)|t^n. \tag{2.35}$$

我们还定义了秩-d 框架化 Cohen–Lenstra 级数。

$$H_{d,R}(t)=H_{d,X,p}(t):=\sum_{n\geq 0}|\mathrm{Hilb}_{d,n}(R)(\mathbb{F}_q)|t^n \tag{2.36}$$

以及秩为 d （非框架）的科恩–伦斯特拉级数

$$\widehat{Z}_{d,R}(t)=\widehat{Z}_{d,X,p}(t):=\sum_{n\geq 0}|\mathrm{Coh}_n^d(R)(\mathbb{F}_q)|t^n. \tag{2.37}$$

以下推论表明，对于所有 $d\geq 0$， $H_{d,R}(t)$的数据、对于所有 $d\geq 0$ ，以及对于所有 $d\geq 0$ ，$Z_{d,R}(t)$的数据彼此确定。它是推论 2.4 的组合式重新表述。我们回顾一下q-Pochhammer 符号

$$(a;q)_n=(1-a)(1-aq)\dots(1-aq^{n-1}) \tag{2.38}$$

以及关于 $d\geq r$ 的 q-二项式系数

$$\begin{bmatrix}d\\ r\end{bmatrix}_q=\frac{(q;q)_d}{(q;q)_r(q;q)_{d-r}}=|\mathrm{Gr}(r,d)(\mathbb{F}_q)|. \tag{2.39}$$

下面的公式 (2.40) 和 (2.41) 是定理 1.7 的点计数版本，但公式 (2.42) 在本文其余部分中不会被使用。

推论 2.7。

$$\widehat{Z}_{d,R}(t)=\frac{t^d}{q^{d^2}(q^{-1};q^{-1})_d}H_{d,R}(tq^{-d}); \tag{2.40}$$

$$Q_{d,R}(t)=\sum_{r=0}^{d}\begin{bmatrix}d\\ r\end{bmatrix}_q t^r H_{r,R}(tq^{d-r}); \tag{2.41}$$

$$H_{d,R}(t)=t^{-d}\sum_{r=0}^{d}(-1)^{d-r}q^{-\binom{d-r}{2}}\begin{bmatrix}d\\ r\end{bmatrix}_{q^{-1}}Q_{r,R}(tq^{d-r}). \tag{2.42}$$

证明。公式 (2.40) 由推论 2.4 结合 $r=d$ 得出。

$$|\mathrm{Surj}(\mathbb{F}_q{}^d,\mathbb{F}_q{}^d)|=|\mathrm{GL}_d(\mathbb{F}_q)|=q^{d^2}(q^{-1};q^{-1})_d. \tag{2.43}$$

该 f 公式（2.41）由推论 2.5 和 $|\text{商概形}_{d,n}(R)|=\sum_{r=0}^{d}|\text{商概形}_{d,n}^{r}(R\qquad)|$.

我们现在利用一个 q-Pascal 反演公式来证明 (2.42)。注意到，(2.41) 可以重新表述为

$$q_d(t)=\sum_{r=0}^{d}\begin{bmatrix}d\\ r\end{bmatrix}_{q^{-1}}h_r(t), \tag{2.44}$$

其中

$$h_d(t):=t^d q^{-d^2}H_{d,R}(tq^{-d}); \tag{2.45}$$

$$q_d(t):=Q_{d,R}(tq^{-d}). \tag{2.46}$$

这里，我们用到了以下事实： $\begin{bmatrix}d\\ r\end{bmatrix}_q=q^{r(d-r)}\begin{bmatrix}d\\ r\end{bmatrix}_{q^{-1}}$。

众所周知（例如参见 [22]）， q-Pascal 矩阵 $P_{ij}=\begin{bmatrix}i\\ j\end{bmatrix}_{q^{-1}}$ 具有如下求逆公式。

$$[P^{-1}]_{ij}=(-1)^{i-j}q^{-\binom{i-j}{2}}\begin{bmatrix}i\\ j\end{bmatrix}_{q^{-1}}. \tag{2.47}$$

由 (2.44) 可得

$$h_d(t) = \sum_{r=0}^{d} (-1)^{d-r} q^{-\binom{d-r}{2}} \begin{bmatrix} d \\ r \end{bmatrix}_{q^{-1}} q_r(t). \tag{2.48}$$

对 $h_d(tq^d)$ 求值，便得到所需的 (2.42)。 □

根据定义，秩为 d 的 Cohen–Lenstra 级数进一步分解了第一作者此前在 [38] 中研究的 Cohen–Lenstra 级数：

$$\widehat{Z}_R(t) = \sum_{d \geq 0} \widehat{Z}_{d,R}(t). \tag{2.49}$$

因此，对定点 Quot 概形进行点计数，就可以逐项研究 $Z_R(t)$。

3. Gröbner bases for monomial subrings

在自由模 R^d 的子模分类问题中，Gröbner 基理论旨在实现以下目标：

- 给出一种除法算法；
- 为 R^d 的每个子模给出一个规范生成集（约化格罗布纳基）；
- 用于检验一个生成集是否为约化 Gröbner 基的判据（布赫伯格判据）

同构的。

A Gröbner basis theory is classifically known if R is a polynomial ring $k[T_1, \ldots, T_N]$ over a field (see for instance [1, 8, 48]) or a power series ring $k[[T_1, \ldots, T_N]]$ over a field (developed by Hironaka [37] under the terminology of "standard bases"; see also [7]). To approach Theorem 1.8, we shall develop a Gröbner basis theory for the ring $\mathbb{F}_q[[u, v]]/(u^2 - v^3)$. The key observation is that $\mathbb{F}_q[[u, v]]/(u^2 - v^3)$ is isomorphic to $\mathbb{F}_q[[T^2, T^3]]$, a subring of $\mathbb{F}_q[[T]]$.[4] It turns out that the Gröbner basis theory for this subring can be developed in the same way as the classical theory.

For the rest of this section, we will give the relevant statements in the general setting of monomial subrings, after setting up some basic terminology. The statements apply to both the polynomial ring and the power series ring with almost no modifications, but the rest of this paper only needs the theory for the power series ring. For this reason, we only give the proof for the power series ring; moreover, since the proof completely follows the classical treatment, we leave it to Appendix A.

3.1. **Notation setup.** Let k be a field and Ω be either a polynomial ring $k[T_1, \ldots, T_N]$ or a power series ring $k[[T_1, \ldots, T_N]]$. A **monomial subring** of Ω is a k-subalgebra (topologically) generated[5] by finitely many monomials. Let R be a monomial subring. Fix $d \geq 1$, and let

$$F = R^d = Ru_1 \oplus \ldots Ru_d \tag{3.1}$$

是一个具有指定基 $\{u_1, \ldots, u_d\}$ 的自由 R-模。注意， R 是诺特的，因此 F 的每个子模都是有限生成的。 F 的一个单项式是 R 的一个单项式与一个基向量 u_i 的乘积。

F 上的单项式序是 F 的单项式上的全序 $<$，满足对于 R 的任意单项式 $\tau \neq 1$和 F的单项式 $\mu < \nu$，有

- $\mu < \tau\mu$;
- $\tau\mu < \tau\nu$

在本节中，我们使用一个固定的单项式序。如果 $\mu < \nu$，我们称 μ 低于 ν，且 ν 高于 μ。我们回顾标准文献中的结论： $<$ 始终是一个良序。

对于 μ、ν 这两个 F 的单项式，我们定义 $\mu \prec \nu$，其中 $\mu < \nu$ 且 Ω 为幂级数环，或者 $\mu > \nu$ 且 Ω 为多项式环。如果 $\mu \prec \nu$，我们称 μ 在 ν 之前，而 ν 在 μ之后。

[4]Geometrically speaking, by doing this, we are using the fact that the cusp singularity is unibranch.

[5]If Ω is the power series ring, then the monomial subring topologically generated by monomials $\mu_1, \ldots, \mu_r$ is $R = k[[\mu_1, \ldots, \mu_r]]$. Note that R is $(T_1, \ldots, T_N)$-adic closed in Ω.

Any nonzero element f of F can be uniquely written as $f = \sum_i c_i\mu_i$, where $c_i \in k^\times$ and μ_i are distinct monomials of F. We say $c_i\mu_i$ to be a **term** of f. For terms $a\mu$ and $b\nu$ where $a, b \in k^\times$, we say $a\mu \prec b\nu$ if $\mu \prec \nu$. The **leading term** of $f \in F \setminus \{0\}$, denoted by $\mathrm{LT}(f)$, is the **foremost** (i.e., smallest in the $\prec$ ordering) term of f. Note that when Ω is the power series ring, f may have infinitely many terms, but well-ordering ensures that a unique leading term exists. The **leading monomial** of $f \in F \setminus \{0\}$, denoted by $\mathrm{LM}(f)$, is the monomial associated to the leading term of f.

By convention, we say $\mathrm{LT}(0) = 0$, and $0 \succ \mu$ for any monomial μ. Therefore, we can talk about $\mathrm{LT}(f) \succeq \mathrm{LT}(g)$ even when f or g is zero. If $g = 0$, this condition forces $f = 0$. If $g \neq 0$, this condition means f has no term before $\mathrm{LT}(g)$, which holds as well if $f = 0$.

我们采用记号 $(\cdot)$ 来表示由一组元素生成的 F 的子模。给定 F 的一个子模 M， M 的首项子模定义为

$$\mathrm{LT}(M) := (\mathrm{LT}(f) : f \in M), \tag{3.2}$$

由 M 中元素的首项生成的子模。单项式子模是由单项式生成的 F 的子模。任意子模的首项子模都是一个单项式子模。

引理 3.1（引理 A.5）。设 M 是 F 的一个子模。那么存在一个 k-向量空间的同构。

$$\frac{F}{M} \cong \frac{F}{\mathrm{LT}(M)}. \tag{3.3}$$

事实上，所有不属于首项(M)的单项式构成了 F/M的一组 k-基。

每个单项式子模 M 都有唯一的由单项式组成的极小生成集。我们将这个极小生成集记为 $C(M)$，并称其为 M 的角点集。由于 R 是诺特的，因此 M 的角点集总是有限的。集合 $C(M)$ 也是 M 中按整除关系排序的极小单项式集合。

3.2. **Gröbner basis theory.** We will provide a division algorithm, a definition of reduced Gröbner bases and a Buchberger criterion in the above setting of monomial subrings. We first give an overview of what is different in the subring setting. The division algorithm and the definition of reduced Gröbner bases are the same as in the cases of the full polynomial ring or power series ring, except that divisibility must take place in the subring. For example, if $R = k[[T^2, T^3]]$, then T^2 does not divide T^3 in R. The Buchberger criterion now needs to consider a finite set of minimal common multiples rather than a (unique) least common multiple. For example, every two monomials in a polynomial ring or a power series ring have a least common multiple, but in $k[[T^2, T^3]]$, the monomials T^2 and T^3 have two mutually indivisible minimal common multiples: T^5 and T^6.

定义 3.2. 如果存在一个单项式 $\tau \in R$，使得 $\nu = \tau\mu$，则称单项式 $\mu \in F$ 整除单项式 $\nu \in F$。这样的单项式 τ 必须是唯一的，我们记作

$$\frac{\nu}{\mu} := \tau. \tag{3.4}$$

现在我们陈述除法算法，它不仅是发展该理论所必需的，而且在第 4 节应用布赫伯格判据时也需要用到。

命题 3.3（除法算法，命题 A.1）。设 f、g_1、…、g_h 是 F 的元素。那么存在一个（不一定唯一的）表达式

$$f = r + \sum_{i=1}^{h} q_i g_i \tag{3.5}$$

其中 $r \in F$、$q_i \in R$ 满足：

- r 的任何项都不能被任意 i的 $\mathrm{LT}(g_i)$ 整除。
- 首项$(q_i g_i) \succeq$ 首项(f) 对任意 i 均成立，即，没有 $q_i g_i$包含在首项 $\mathrm{LT}(f)$ 之前（即 $\prec$）的项。

这样的表达式称为将 f 除以 g_1、……、g_h 的除法表达式，而 r 是 f 除以 g_1、……、g_h 所得的余式。

此外，一种可能的除法表达式由朴素算法给出，该算法反复消去 f 中由 LT(g_i) 消去的项。

接下来，我们定义（约化）格拉斯曼簇¨obner 基的概念，并陈述唯一性定理。

定义 3.4. 子模 $M \subseteq F$ 的有限生成集 $\{g_1, \ldots, g_h\}$ 称为 M 的格拉斯曼簇¨obner 基，如果

$$\mathrm{LT}(M) = (\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_h)). \tag{3.6}$$

G 中元素的有限集合称为 F 的格拉斯曼簇¨obner 基，如果 G 是由 G 生成的子模的格拉斯曼簇¨obner 基。等价地，$\{g_i\}_i$ 是格拉斯曼簇¨obner 基，当且仅当

(a) $\mathrm{LT}((g_i)_i) = (\mathrm{LT}(g_i))_i$.

如果一个格拉斯曼 Gröbner 基 $G = \{g_1, \ldots, g_h\}$ 是约化的，当且仅当

(b) LT(g_1) . . . , LT(g_h) 是单项式，且彼此不可整除；(c) 对任意 i、j，

g_i 的任何非首项均不能被 LT(g_j) 整除。

注 3.5. 注意，在条件 (c) 中允许 i、j 相等，因此 g_i 不能含有可被首项整除的项（首项本身除外）。

Lemma 3.6 (Lemma A.2)**.** *Let $f, g_1, \ldots, g_h \in F$. If $G = \{g_1, \ldots, g_h\}$ is a Gröbner basis, then the remainder of f in a division expression by G is unique (even though the division expressions may not be unique). Moreover, the remainder of f by G is zero if and only if $f \in (g_1, \ldots, g_h)$.*

命题 3.7 (　　命题 A.4）。M 的 F 的每个子模都有唯一的约化格拉斯曼簇格罗布纳基。　　.

因此，F 的所有子模的集合与约化格拉斯曼簇 ¨obner 基的集合一一对应。注意，有限集 $G = \{g_1, \ldots, g_h\}$ 是约化格拉斯曼簇 ¨obner 基，当且仅当定义 3.4 的条件 (a)(b)(c) 成立。

为便于处理条件 (b)(c)，我们引入以下非标准术语。

Definition 3.8. A finite collection $G = \{g_1, \ldots, g_h\}$ of elements of F is called a **reduced pre-Gröbner basis** (or simply a **prebasis**) if the conditions (b)(c) of Definition of 3.4 hold for G. In this case, we also say G is a prebasis associated to the monomial submodule M, where $M := (\mathrm{LT}(g_1) \ \ldots, \mathrm{LT}(g_h))$.

另一方面，下面的布赫伯格判据确定了对于 F 的任意有限元素集合（不一定是预基），条件 (a) 何时成立。我们需要对单项式序作出一个额外假设，但在 §4 的情形下，该假设是满足的。

假设 3.9。如果 Ω 是多项式环，则无需进一步假设。　　d.

如果 Ω 是幂级数环，我们假设单项式序的序型为 ω，也就是说，F 在 $<$ 下的所有单项式组成的良序集与按自然序排列的自然数集同构。等价地，任意有界的单项式子集都是有限集。

我们还需要处理两个单项式的最小公倍单项式可能不存在的问题。

定义 3.10（最小公倍数集）。对于 F 中的两个单项式 μ_1、μ_2，我们将 LCM(μ_1, μ_2) 定义为所有同时是 μ_1、μ_2 的公倍数的单项式中，按整除关系排序的极小元素构成的集合。LCM(μ_1, μ_2) 的元素称为 μ_1 和 μ_2 的极小公倍单项式。

当且仅当 μ_1 和 μ_2 涉及 F 的同一个基向量 u_i 时，μ_1 和 μ_2 的最小公倍数集才非空。作为一种等价定义，LCM(μ_1, μ_2) 是子模 $R\mu_1 \cap R\mu_2$ 的极小单项式生成元集合，其中 F。因此，由于 R 是诺特的，LCM(μ_1, μ_2) 必定是有限集。

Proposition 3.11 (Buchberger criterion)**.** *Assume that Assumption 3.9 holds. Let $G = \{g_1, \dots, g_h\}$ be a finite collection of nonzero elements of F. Assume without loss of generality that g_i has leading coefficient 1, namely, $\mathrm{LT}(g_i) = \mathrm{LM}(g_i)$ is a monomial, denoted by μ_i. Then G is a Gröbner basis if and only if for any $i \neq j$ and $\nu \in \mathrm{LCM}(\mu_i, \mu_j)$, the element*

$$S_{ij}^{\nu}(g_i, g_j) := \frac{\nu}{\mu_i} g_i - \frac{\nu}{\mu_j} g_j \tag{3.7}$$

在命题 3.3 的意义下，在关于 $g_1, \dots, g_h$ 的某个（或等价地，每个）除法表达式中余式为零。

在实践中，要证明 G 是一个 Gröbner 基，只需为每个 i, j, ν 提供一个余式为零的除法表达式；要证明 G 不是一个 Gröbner 基，只需为某些 i, j, ν 提供一个余式非零的除法表达式。

注 3.12。Becker [7] 证明，如果 R 是完整的幂级数环，那么假设 3.9 并非必要；他的证明利用稳定性论证，将问题化归到序型为 ω 的情形。由于此处并不需要，我们没有尝试推广命题 3.11。

最后，我们需要一个技术性引理，以便处理不一定是约化的格拉斯曼簇 Gröbner 基。

Lemma 3.13 (Lemma A.6)**.** *Let $G = \{g_1, \dots, g_h\}, G' = \{g_1', \dots, g_h'\}$ be Gröbner bases for M with $\mathrm{LT}(g_i) = \mathrm{LT}(g_i') = \mu_i$, and assume G' is reduced. If $g_1 - \mu_1$ has no terms divisible by any μ_i, then we must have $g_1 = g_1'$.*

4. Counting points in Gröbner strata

4.1. 记号约定。固定 $d \geq 1$。令 $k = \mathbb{F}_q$, $R = k[[T^2, T^3]]$, $\mathfrak{m} = (T^2, T^3)R$ 和 $F = R^d = Ru_1 \oplus \cdots \oplus Ru_d$。

We choose to use the lexicographical order induced by $u_1 \prec \cdots \prec u_d \prec T$. Note that this monomial order is of order type ω, so Assumption 3.9 holds. For this reason, we prefer to think of it as the homogeneous lexicographical monomial order on F induced by $u_1 \prec \cdots \prec u_d$ (the position of T does not matter). This choice of monomial order is crucial in our proof.

Our monomial order satisfies that $\mu \in \mathfrak{m}F$ and $\nu \succ \mu$ imply $\nu \in \mathfrak{m}F$. This ensures that doing Gröbner theory never "leaves" $\mathfrak{m}F$: if $f \in F$ satisfies $\mathrm{LT}(f) \in \mathfrak{m}F$, then $f \in \mathfrak{m}F$; if $f, g_1, \dots, g_h \in \mathfrak{m}F$ and r is a remainder of f divided by $g_1, \dots, g_h$, then r is also in $\mathfrak{m}F$. We will often implicitly use this fact later. In the following discussions, the meanings of some notations and terminologies are different from those in §3, in that we work within $\mathfrak{m}F$ instead of F.

Recall from (2.34) that $\mathrm{Hilb}_{d,n}(R) := \mathrm{Quot}_{d,n+d}^{d}(R)$ is the set of n-codimensional[6] R-submodules of $\mathfrak{m}F$. Let $\{M_\alpha : \alpha \in \Xi\}$ denote the set of finite-codimensional monomial submodules of $\mathfrak{m}F$; we will soon identify the index set Ξ concretely. For any $\alpha \in \Xi$, let $\mathrm{Hilb}(\alpha)$ be the set of all R-submodules of $\mathfrak{m}F$ whose leading submodule is M_α. If we denote by $n(\alpha)$ the k-dimension of $\mathfrak{m}F/M_\alpha$, then $\mathrm{Hilb}(\alpha)$ is a subset of $\mathrm{Hilb}_{d,n(\alpha)}(R)$ by Lemma 3.1. Thus, $\mathrm{Hilb}_{d,n}(R)$ is a disjoint union of $\mathrm{Hilb}(\alpha)$ for all $\alpha \in \Xi, n(\alpha) = n$. We call $\mathrm{Hilb}(\alpha)$ the **Gröbner stratum** indexed by the **leading term datum** α.

$\mathfrak{m}F$ 的每个有限余维单项式子模都具有 $M_\alpha = I_1 u_1 \oplus \cdots \oplus I_d u_d$ 的形式，其中 I_i 是 R 的非零真单项式理想。由于 R 的非零真单项式理想有两种类型：

- $J(a) := (T^{a+1})$, $a \geq 1$;
- $K(a) := (T^{a+2}, T^{a+3})$, $a \geq 0$.

Note that $\dim_k \mathfrak{m}/J(a) = \dim_k \mathfrak{m}/K(a) = a$. Let $X = \{J(a)\ (a \geq 1)\ K(a)\ (a \geq 0)\}$ be the set of proper monomial ideals of R. Then we may associate to α the tuple $(I_1, \dots, I_d)$, where $I_i \in X$ and $M_\alpha = I_1 u_1 \oplus \cdots \oplus I_d u_d$. We shall write $\alpha = (I_1, \dots, I_d)$. This way, we identify Ξ as the set X^d.

[6]（余）维数始终指作为 k-向量（子）空间的（余）维数。[7]从现在起，假设 $\mathfrak{m}F$ 的每个子模和 R 的每个理想都具有有限余维。

从现在起，我们通常使用 I 或 I_i 来指代 X 的一般元素，同时注意，$J(a)$ 和 $K(a)$ 指的是 X 中的特定元素。我们通常使用小写希腊字母 α、β，.....来指代 Ξ 的元素；我们应牢记，它们是首项数据，用于标记 $\mathrm{Hilb}_{d,*}(R)$ 的各个分层。

我们将集合 X 上的记号 $n(\cdot)$ 进行推广，定义 $n(I) = \dim_k \mathfrak{m}/I$ 对于 $I \in X$；因此，$n(J(a)) = n(K(a)) = a$。对于单项式子模 $\alpha = (I_1, \dots, I_d)$，我们有 $n(\alpha) = n(I_1) + \cdots + n(I_d)$。

对于单项式理想 $I \in X$，我们将 $\Delta(I)$ 定义为 I 的标准单项式集合，即不属于 I 的非恒定单项式集合；它构成了 $\mathfrak{m}/I$ 的一个 k-基。因此，一般而言，我们有 $|\Delta(I)| = n(I)$，并且

- $\Delta(J(a)) = \{T^2, \dots, T^a\} \cup \{T^{a+2}\}$;
- $\Delta(K(a)) = \{T^2, \dots, T^{a+1}\}$.

我们还将 $C(I)$ 定义为 I 的角点集，即 I 的极小单项式生成集。我们还按照以下特定方式对 $C(I)$ 的角点进行标记：$\mu^0(I)$ .....

- $C(J(a)) = \{\mu^0(J(a))\} = \{T^{a+1}\}$;
- $C(K(a)) := \{\mu^0(K(a)), \mu^1(K(a))\} := \{T^{a+2}, T^{a+3}\}$ (in this order).

For $\alpha = (I_i)_{i=1}^d$, we define the set $\Delta(\alpha)$ of **standard monomials** of α to be the set of monomials in $\mathfrak{m}F$ but not in M_α. Then $\Delta(\alpha) = \bigcup_{i=1}^d \Delta(I_i)u_i$ is the set of monomials in F of the form $\nu_i u_i$ where $\nu_i \in \Delta(I_i)$. Similarly, we define the set of **corners** of α to be the set of minimal monomial generating set of M_α. We have $C(\alpha) = \bigcup_{i=1}^d C(I_i)u_i$.

我们现在描述 $\mathrm{Hilb}(\alpha)$ 中的元素，其中 $\alpha = (I_i)_{i=1}^d$。对于 $c = 0$ 或 1，我们定义

$$\mu_i^c = \mu^c(I_i)u_i, \tag{4.1}$$

回顾 $\mu^0(J(a)) = T^{a+1}$、$\mu^0(K(a)) = T^{a+2}$ 以及 $\mu^1(K(a)) = T^{a+3}$。

当然，μ_i^1 只有在 I_i 具有 $K(a)$的形式时才有意义。对于 F中的任意单项式 μ ，我们定义$\Delta(\alpha)_{\succ\mu}$ 为 $\Delta(\alpha)$中高于 μ 的单项式所组成的集合。于是，与 α 相关联的一个约化的预格拉斯曼簇 Gröbner 基是一个形如 $G = \{g_i^c\}$的集合，其中

$$g_i^c \in \mu_i^c + \mathrm{span}_{\mathbb{F}_q} \Delta(\alpha)_{\succ\mu_i^c}. \tag{4.2}$$

We remark that the set of c where μ_i^c or g_i^c makes sense only depends on α and i; we will implicitly assume that c lies in this set whenever we use the notation μ_i^c or g_i^c. Note also that the monomial μ_i^c is determined by i, c and the leading term datum α, and that the element g_i^c is determined by i, c and the reduced (pre-)Gröbner basis G. For this reason, we use the notation $\mu_i^c(\alpha)$ and $g_i^c(G)$ to extract the corresponding monomial of α and the corresponding basis element of G.

我们有以下布赫伯格判据，并结合当前的设定对其进行了明确重述。注意，假设 3.9 成立，且 $\mathrm{LCM}(T^{a+2}u_i, T^{a+3}u_i) = \{T^{a+5}u_i, T^{a+6}u_i\}$。

引理 4.1. 设 $\alpha = (I_1, \dots, I_d) \in \Xi$。一个约化的预-Gr ¨obner 基 $G = \{g_i^c\}$ 是约化的 Gr ¨obner 基，当且仅当对于任意满足 I_i 具有 $K(a)$ 形式的 i ，每个元素

$$T^3 g_i^0 - T^2 g_i^1,\ T^4 g_i^0 - T^3 g_i^1 \tag{4.3}$$

除以预基 G 后余式为零。

根据约化格拉斯曼簇¨obner 基的唯一性（命题 3.7），我们现在可以将希尔伯特概形(α)与关联于 α 的约化格拉斯曼簇¨obner 基集合对应起来。

例 4.2。设 $d=3$ 和 $\alpha=(I_1,I_2,I_3)=(K(0)\ K(2)\ J(2))$。那么

$$\begin{aligned} M_\alpha &= \operatorname{span}\{T^2u_1, T^3u_1, T^4u_2, T^5u_2, T^3u_3\},\\ C(\alpha) &= \{T^2u_1, T^3u_1, T^4u_2, T^5u_2, T^3u_3\}\\ &= \{\mu^0(I_1), \mu^1(I_1), \mu^0(I_2), \mu^1(I_2), \mu^0(I_3)\}\\ &= \{\mu_1^0, \mu_1^1, \mu_2^0, \mu_2^1, \mu_3^0\},\\ \Delta(\alpha) &= \{T^2u_2, T^3u_2, T^2u_3, T^4u_3\}, \end{aligned} \tag{4.4}$$

与 α 相关的约化预格拉斯曼簇¨基的形式为 $G=\{g_1^0,g_1^1,g_2^0,g_2^1,g_3^0\}$，其中

$$\begin{aligned} g_1^0 &\in \mu_1^0 + \operatorname{span}\Delta(\alpha)_{\succ\mu_1^0} = T^2u_1 + \operatorname{span}\{T^2u_2, T^3u_2, T^2u_3, T^4u_3\},\\ g_1^1 &\in \mu_1^1 + \operatorname{span}\Delta(\alpha)_{\succ\mu_1^1} = T^3u_1 + \operatorname{span}\{T^3u_2, T^4u_3\},\\ g_2^0 &\in \mu_2^0 + \operatorname{span}\Delta(\alpha)_{\succ\mu_2^0} = T^4u_2 + \operatorname{span}\{T^4u_3\},\\ g_2^1 &\in \mu_2^1 + \operatorname{span}\Delta(\alpha)_{\succ\mu_2^1} = T^5u_2 + \{0\},\\ g_3^0 &\in \mu_3^0 + \operatorname{span}\Delta(\alpha)_{\succ\mu_3^0} = T^3u_3 + \operatorname{span}\{T^4u_3\}. \end{aligned} \tag{4.5}$$

当且仅当这四个元素中的每一个都满足相应条件时，预基 G 才是格拉斯曼簇¨obner 基。

$$T^3g_i^0 - T^2g_i^1,\ T^4g_i^0 - T^3g_i^1\ (i=1,2) \tag{4.6}$$

除以预基 G 时，余式为零。

Remark 4.3. If α is of the form $\alpha = (J(a_1)\ \ldots, J(a_d))$, then every reduced pre-Gröbner basis associated to α is Gröbner because there is only one monomial in $C(\alpha)$ belonging to each basis vector u_i, so the assumption of Lemma 4.1 vacuously holds. However, if α contains terms of the form $K(a)$, the Buchberger criterion will lead to complications and it is the main difficulty we will address in the rest of the section.

4.2. 希尔伯特概形(α)的纤维分解。事实证明，为了计算希尔伯特概形(α)的基数，只需计算满足对所有 $1\le i\le d$， g_i^0(G) 都是单项式的希尔伯特概形(α)中的约化格拉斯曼簇obner 基 G 。更确切地说，

引理 4.4. 设给定 d和 $\alpha\in\Xi$。固定元素 $h_i^0\in\mu_i^0$(α)+张成 $\Delta(\alpha)_{\succ\mu_i^0(\alpha)}$。那么满足 $g_i^0(G)=h_i^0$ 的约化 Gr ¨obner 基的个数与 h_i^0 的选择无关。特别地，若我们定义

$$\operatorname{Hilb}^0(\alpha) := \{G\in\operatorname{Hilb}(\alpha) : g_i^0(G)=\mu_i^0(\alpha),\ 1\le i\le d\}, \tag{4.7}$$

于是我们有

$$|\operatorname{Hilb}(\alpha)| = |\operatorname{Hilb}^0(\alpha)|\cdot q^{\sum_i\left|\Delta(\alpha)_{\succ\mu_i^0(\alpha)}\right|}. \tag{4.8}$$

Fro从现在起，我们将 $\mathbf{Hilb}^0(\alpha)$ 称为由 α 标记的中心格拉斯曼簇 Gröbner 分层。 .

Proof. Let $\operatorname{Hilb}(\{h_i^0\};\alpha)$ be the set of submodules equipped with a reduced Gröbner basis G such that $g_i^0(G)=h_i^0$. It suffices to construct a bijection $\operatorname{Hilb}^0(\alpha)\to\operatorname{Hilb}(\{h_i^0\};\alpha)$. The idea is to use a change of variable $u_i\mapsto\frac{h_i^0}{\mu_i^0/u_i}$, so that μ_i^0 would be mapped to h_i^0. However, $\frac{h_i^0}{\mu_i^0/u_i}$ may involve terms with T^1, in which case it is not an element of $F=k[[T^2,T^3]]^d$.

为使这一想法更加严谨，定义一个 R-模自同态 τ_h ，其在生成元上的取值为

$$\tau_h(T^bu_i) = \frac{T^bh_i^0}{\mu_i^0/u_i},\qquad b\ge 2, \tag{4.9}$$

这里我们注意到，由于 $\mathrm{LT}(h_i^0)=\mu_i^0$ 以及我们在 F 上定义的单项式序，目标最终落在 $\mathfrak{m}F$ 中。

对于一个子模 $M \in$希尔伯特概形$^0(\alpha)$，我们将证明以下映射

$$\iota_h : \mathrm{Hilb}^0(\alpha) \to \mathrm{Hilb}(\{h_i^0\}; \alpha), \;\; M \mapsto \tau_h(M) \tag{4.10}$$

定义良好，且正是我们所需的双射。为了证明 $\tau_h(M)$ 确实属于希尔伯特概形$(\{h_i^0\};\, \alpha)$，令 G andG' 分别为 M 和 $\tau_h(M)$ 的约化格拉斯曼簇¨obner 基，我们需要证明 $g_i^0(G') = h_i^0$。

The morphism τ_h carries the criterion (4.3) satisfied by G to $\tau_h(G)$, because τ_h carries the division expressions involved. (Here, we need to use the elementary observation that $\mathrm{LT}(\tau_h(f)) = \mathrm{LT}(f)$ for any $f \in F$.) Therefore $\tau_h(G)$ is a Gröbner basis for $\tau_h(M)$. Even though $\tau_h(G)$ may not be reduced, thanks to Lemma 3.13, we have $g_i^0(G') = \tau_h(\mu_1^0) = h_i^0$ as desired. Hence ι_h is well defined.

Note that τ_h is an automorphism of $\mathfrak{m}F$, since modulo $\mathfrak{m}^2F$ it is. Indeed, if we choose an ordered basis of $\mathfrak{m}F/\mathfrak{m}^2F$ to be $\{T^2u_1, \ldots, T^2u_d, T^3u_1, \ldots, T^3u_d\}$, then the matrix for the reduction of τ_h mod $\mathfrak{m}^2F$ is lower triangular, with 1's on the diagonal. A similar argument as above shows that there is an inverse map

$$\iota_h^{-1} : \mathrm{Hilb}(\{h_i^0\}; \alpha) \to \mathrm{Hilb}^0(\alpha), \;\; M \mapsto \tau_h^{-1}(M). \tag{4.11}$$

因此 ι_h 是一个双射。 □

4.3. **Explicit parametrization of** $\mathrm{Hilb}^0(\alpha)$**.** In Lemma 4.5 below, we will show that $\mathrm{Hilb}^0(\alpha)$ can be parametrized by an affine variety cut out by explicit matrix equations. The lemma will show in particular that the $J(a)$ terms in α do not complicate $|\mathrm{Hilb}^0(\alpha)|$ too much. For $\alpha = (I_1, \ldots, I_d)$, let J_α (resp., K_α) denote the set of indices in $\{1, \ldots, d\}$ whose corresponding I_i is of the form $J(a)$ (resp., $K(a)$). Note that an element $\{g_i^c\}$ of $\mathrm{Hilb}^0(\alpha)$ is determined by $\{g_i^1\}_{i \in K_\alpha}$, and we will use $\{g_i^1\}_{i \in K_\alpha}$ thereafter to refer to an element of $\mathrm{Hilb}^0(\alpha)$. Of course, the indices in J_α still play a role as the nonleading terms of $g_i^1, i \in K_\alpha$ could involve u_j for $j \in J_\alpha$.

Let $\alpha|_{K_\alpha}$ be the restriction of α to the index set K_α. Working with $\alpha|_{K_\alpha}$ is equivalent to working with the corresponding leading term datum in a lower d (the new d value is the size of K_α), but we choose to keep the index set as K_α instead of $\{1, \ldots, |K_\alpha|\}$ for convenience. Note that K_α inherits the ordering from $[d] = \{1, \ldots, d\}$, and this determines the monomial order when working with the index set K_α. The same discussion applies to $\alpha|_{J_\alpha}$.

引理 4.5. 希尔伯特概形$^0(\alpha)$与满足以下条件的四元组(X, Y, Z, W)之间存在双射。

$$X, Y \in \mathrm{Mat}_{K_\alpha \times K_\alpha}(k), \tag{4.12}$$

$$Z, W \in \mathrm{Mat}_{K_\alpha \times J_\alpha}(k), \tag{4.13}$$

$$X^2 = Y^3, \tag{4.14}$$

$$[X, Y] = 0, \tag{4.15}$$

$$Z = XW, \tag{4.16}$$

$$X_{ij} = 0 \text{ if } \mu_j^0 \prec T^3\mu_i^0, \tag{4.17}$$

$$Y_{ij} = 0 \text{ if } \mu_j^0 \prec T^2\mu_i^0, \tag{4.18}$$

$$Z_{il} = 0 \text{ if } \mu_l^0 \prec T^3\mu_i^0, \tag{4.19}$$

$$W_{il} = 0 \text{ if } \mu_l^0 \prec \mu_i^0, \tag{4.20}$$

其中 (X, Y, Z, W) 对应于约化格拉斯曼簇 ¨obner 基 G，使得

$$g_i^1 = \mu_i^1 + T^{-2} \sum_{j \in K_\alpha} (X_{ij}\mu_j^0 + Y_{ij}g_j^1) + T^{-2} \sum_{l \in J_\alpha} (Z_{il} + W_{il}T^3)\mu_l^0. \tag{4.21}$$

此外，条件 (4.19) 可由其余条件推出。

特别是，

$$\left|\mathrm{Hilb}^0(\alpha)\right| = \left|\mathrm{Hilb}^0(\alpha|_{K_\alpha})\right| \cdot q^{\sum_{i \in K_\alpha} \left| C(\alpha|_{J_\alpha})_{\succ \mu_i^0} \right|}. \tag{4.22}$$

因此，要计算秩为 d的（中心）格拉斯曼簇¨obner分层的大小，只需理解所有秩 $d' \leq d$下具有“纯-K”首项数据的中心格拉斯曼簇¨obner分层。

Proof. We first show that any reduced Gröbner basis can be written into the desired form. Though unnecessary, we will work in the ambient ring $k[T^{-1}][[T]]$, which contains $k[[T^2, T^3]]$; the only reason is to make formulas containing T^{-n} easier to understand. Given a reduced Gröbner basis G, the first formula of (4.3) implies that there are power series $X_{ij}(T)$, $Y_{ij}(T)$,$Z_{il}(T)$,$W_{il}(T) \in k[[T^2]]$ such that

$$g_i^1 = \mu_i^1 + T^{-2} \sum_{j \in K_\alpha} (X_{ij}(T)\mu_j^0 + Y_{ij}(T) g_j^1) + T^{-2} \sum_{l \in J_\alpha} (Z_{il}(T) + W_{il}(T)T^3)\mu_l^0. \tag{4.23}$$

Let $X_{ij} = X_{ij}(0)$, $Y_{ij} = Y_{ij}(0)$, $Z_{il} = Z_{il}(0)$ and $W_{il} = W_{il}(0)$. Since G is reduced, $g_i^1 \in \mu_i^1 + \operatorname{span} \Delta(\alpha)_{\succ \mu_i^1}$. It follows that $X_{ij}(T)$ $Y_{ij}(T)$ $Z_{il}(T)$ $W_{il}(T)$ are all constants. Therefore G can be put in the form of (4.21). Note that the coefficients must satisfy (4.17)∼(4.20) since $\mu_i^1 = T\mu_i^0$ is the term with lowest degree.

We then use the second formula of (4.3) to show that the coefficients of (4.21) satisfy (4.14)∼(4.16). Let $\widetilde{G} = G \cup \{T^3\mu_l^0 | l \in J_\alpha\} = \{\mu_i^0, g_i^1 | i \in K_\alpha\} \cup \{\mu_l^0, T^3\mu_l^0 | l \in J_\alpha\}$. The strategy is looping (4.21) around itself to express $T^3 g_i^1 - T^4\mu_i^0, i \in K_\alpha$ into a k-linear combination of elements in $\bigcup_{m \in \mathbb{Z}} T^{2m}\widetilde{G}$. Note that elements in $\bigcup_{m \in \mathbb{Z}} T^{2m}\widetilde{G}$ are linearly independent, hence such expression is unique. Furthermore, $T^3 g_i^1 - T^4\mu_i^0$ being divisible by G is equivalent to

$$T^3 g_i^1 - T^4\mu_i^0 \in \operatorname{span}_k \left\{ \bigcup_{m \geq 0} T^{2m}\widetilde{G} \right\}. \tag{4.24}$$

注意，(4.21) 导出以下约化公式：

$$T\mu_i^0 = g_i^1 - T^{-2} \sum_{j \in K_\alpha} (X_{ij}\mu_j^0 + Y_{ij} g_j^1) - T^{-2} \sum_{l \in J_\alpha} (Z_{il}\mu_l^0 + W_{il}T^3\mu_l^0). \tag{4.25}$$

$$Tg_i^1 = T^2\mu_i^0 + T^{-2} \sum_{j \in K_\alpha} (X_{ij}(T\mu_j^0) + Y_{ij}(Tg_j^1)) + \sum_{l \in J_\alpha} (Z_{il}T^{-4}(T^3\mu_l^0) + W_{il}T^2\mu_l^0). \tag{4.26}$$

显然，递归地使用这两个公式，可以将 Tg_i^1 写成 $\bigcup_{m \in \mathbb{Z}} T^{2m}\widetilde{G}$ 中元素的 k-线性组合。由于 $T^3 g_i^1 - T^4\mu_i^0 = T^2(Tg_i^1) - T^4\mu_i^0$，也可以将 $T^3 g_i^1 - T^4\mu_i^0$ 写成 $\bigcup_{m \in \mathbb{Z}} T^{2m}\widetilde{G}$ 中元素的 k-线性组合。

To compute the linear combination explicitly, write $X_{ij}, Y_{ij}, Z_{il}, W_{il}$ into matrices X, Y, Z, W satisfying (4.12), (4.13) and (4.17)∼(4.20). Let $\mathbf{J}_0$ *resp.* $\mathbf{K}_0$ *resp.* $\mathbf{K}_1$ be the column vector whose entries are $\mu_l^0, l \in J_\alpha$ *resp.* $\mu_i^0, i \in K_\alpha$ *resp.* $g_i^1, i \in K_\alpha$. For $m \in \mathbb{Z}$, let $\mathbf{J}_m = T^m\mathbf{J}_0$, $\mathbf{K}_{2m} = T^{2m}\mathbf{K}_0$ and $\mathbf{K}_{2m+1} = T^{2m}\mathbf{K}_1$. It follows from definition, formula (4.25) and formula (4.26) that following recursive formulas are true:

$$T\mathbf{J}_m = \mathbf{J}_{m+1},\ T^2\mathbf{K}_m = \mathbf{K}_{m+2}, \tag{4.27}$$

$$T\mathbf{K}_{2m} = \mathbf{K}_{2m+1} - X\mathbf{K}_{2m-2} - Y\mathbf{K}_{2m-1} - Z\mathbf{J}_{2m-2} - W\mathbf{J}_{2m+1}, \tag{4.28}$$

$$T\mathbf{K}_{2m+1} = \mathbf{K}_{2m+2} + XT\mathbf{K}_{2m-2} + YT\mathbf{K}_{2m-1} + Z\mathbf{J}_{2m-1} + W\mathbf{J}_{2m+2}. \tag{4.29}$$

现在，元素为 $T^3g_i^1 - T^4\mu_i^0$ 的列向量恰好为 $T\mathbf{K}_3 - \mathbf{K}_4$。利用 (4.27)∼(4.29) 进行迭代，得到如下结果。

$$\begin{aligned} T\mathbf{K}_3 - \mathbf{K}_4 &= Z\mathbf{J}_1 + W\mathbf{J}_4 + \sum_{m=0}^{\infty} Y^m(Y\mathbf{K}_{2-2m} + X\mathbf{K}_{1-2m}) \\ &- \sum_{m=0}^{\infty} Y^m X(Y\mathbf{K}_{-1-2m} + X\mathbf{K}_{-2-2m}) + \sum_{m=0}^{\infty} Y^{m+1}(Z\mathbf{J}_{-1-2m} + W\mathbf{J}_{2-2m}) \\ &- \sum_{m=0}^{\infty} Y^m X(Z\mathbf{J}_{-2-2m} + W\mathbf{J}_{1-2m}). \end{aligned} \tag{4.30}$$

注意，由于 Y 是幂零的，无穷和实际上是有限和。条件 (4.24) 意味着当 $m \geq 0$ 时，

$$\begin{cases} Y^m Z = Y^m XW, \text{ (since the coefficient of } \mathbf{J}_{1-2m} \text{ is 0)}; \\ Y^{m+3}W = Y^m XZ, \text{ (since the coefficient of } \mathbf{J}_{-2-2m} \text{ is 0)}; \\ Y^m XY = Y^{m+1}X, \text{ (since the coefficient of } \mathbf{K}_{-1-2m} \text{ is 0)}; \\ Y^m X^2 = Y^{m+3}, \text{ (since the coefficient of } \mathbf{K}_{-2-2m} \text{ is 0)}. \end{cases} \tag{4.31}$$

显然，(4.31) 等价于 $[X,Y] = 0, X^2 = Y^3, Z = XW$。注意， Z 完全由 X 和 W 决定，且没有对 W 施加其他条件，因此 X, Y, Z, W 满足 (4.14)∼(4.16)。

反过来，给定满足 (4.12)∼(4.20) 的 (X, Y, Z, W)，只需逆向执行上述计算，即可证明由 (4.21) 得到的 $G = \{g_i^1\}$ 构成一个约化的格拉斯曼簇¨obner 基。□

关于由这些矩阵方程所定义的簇的进一步讨论，请参见 §6。引理 4.4 和引理 4.5 的动机版本在定理 B.8 中进行了讨论。

5. Rationality in the t variable

Denote $H_d(t) = H_{d,R}(t)$, where $R = \mathbb{F}_q[[T^2, T^3]]$. We now prove the point count version of the first part of Theorem 1.8, which says $H_d(t)$ is rational with denominator $(1-t)(1-qt)\dots(1-q^{d-1}t)$. Even though the point count on each Gröbner stratum is not fully understood, Lemma 4.5 implies that the stratum point count satisfies a stability property (Lemma 5.9) when we "raise" the associated leading term datum in a certain way, which is enough to prove the rationality statement. Moreover, we will reach Theorem 5.14, which provides a finite recipe to compute $H_d(t)$ for a fixed d whenever the point count on certain finitely many strata are understood.

我们首先探究所有首项数据集合中的更多结构。

5.1. 首项数据的进一步结构。回顾一下， $X = \{J(a)\ (a \geq 1)\ K(a)\ (a \geq 0)\}$ 是 $R = \mathbb{F}_q[[T^2, T^3]]$的非零真单项式理想集合，而 $\Xi = X^d$是首项数据集合。

给定一个首项数据 $\alpha = (I_1, \dots, I_d)$，我们考虑这些对： $\alpha_1 := (1, I_1) \dots, \alpha_d := (d, I_d)$。令 $i_{(1)}, \dots, i_{(d)}$ 为 $[d]$ 的一个排列，使得 $\mu^0_{i_{(1)}}(\alpha) \prec \cdots \prec \mu^0_{i_{(d)}}(\alpha)$。我们定义

$$\alpha_{(j)} := (i_{(j)}, I_{i_{(j)}}), \tag{5.1}$$

对于 $j \in [d]$，或者更具启发性地，我们写作

$$\alpha_{(j)} = I_{i_{(j)}} u_{i_{(j)}}. \tag{5.2}$$

我们记作 $\alpha = (\alpha_{(1)}, \dots, \alpha_{(d)})$。换言之，我们按照 μ^0 的升序对 α 的分量 $I_i u_i$ 进行排序。（回忆一下， $\mu^0(J(a)u_i) = T^{a+1}u_i$ 和 $\mu^0(K(a)u_i) = T^{a+2}u_i$。）排序顺序可以方便地从

$$K(0) \sim J(1) \prec K(1) \sim J(2) \prec K(2) \sim J(3) \prec \dots \tag{5.3}$$

并使用 u_i 中的索引 i 作为平局时的判定依据。例如，若 $\alpha = (K(1)\ J(1)\ K(0))$，则 $\gamma_{(1)} = J(1)u_2$，$\alpha_{(2)} = K(0)u_3, \alpha_{(3)} = K(1)u_1$。

Note that we reserve the parenthesized subscript for indices associated to the **ranking** of a component in the sorting order above, while the normal subscript often refers to the **seat** of a component. The seat of $I_i u_i$ is i. We will also use the notation $i(\cdot)$ to extract the seat. For example, in the example above, $i(\alpha_{(2)}) = 3$, and this says $\alpha_{(2)} = \alpha_3 = I_3 u_3$. The interplay between the concepts of ranking and seat will be important.

根据排序顺序 5.3，我们将一个分量的层级定义为

$$l(J(a)u_i) = a - 1,\ l(K(a)u_i) = a. \tag{5.4}$$

我们总是有 $\mu^0(\alpha_i) = T^{l(\alpha_i)+2}u_i$。

我们还定义了分量 α_i 的颜色，通过

$$c(J(a)u_i) = J,\ c(K(a)u_i) = K, \tag{5.5}$$

其中，$\{J, K\}$ 是所有两种可能颜色的集合。因此，我们可以通过其分量的层级和颜色来编码一个首项数据。更准确地说，我们有一个双射

$$\Xi \xrightarrow{\cong} \mathbb{N}^d \times \{J, K\}^d \tag{5.6}$$

$$\alpha = (I_1, \ldots, I_d) \longmapsto (l(\alpha), c(\alpha)), \tag{5.7}$$

其中，$l(\alpha) := (l(\alpha_1) \ldots, l(\alpha_d))$ 是层级向量，$c(\alpha) := (c(\alpha_{(1)}) \ldots, c(\alpha_{(d)}))$ 是颜色向量。这里，为了便于后续处理，我们选择将层级排列在位置上，将颜色排列在秩上。

We now define the "spiral raising" operators $\gamma_{(1)}, \ldots, \gamma_{(d)}$ on Ξ. We first define the actions of these operators on $\mathbb{N}^d$, following the work [39] of the authors up to change of notation. Let $x = (l_1, \ldots, l_d) \in \mathbb{N}^d$. Similar to the treatment of $\alpha \in \Xi$, we consider the components $x_i = (i, l_i)$. Sorting the components $x_1, \ldots, x_d$ first by l_i, and then by i, we obtain an ascending list $x_{(1)}, \ldots, x_{(d)}$. Let $i_{(j)} = i(x_{(j)})$ be the seat of the j-th lowest component. For $1 \le j \le d$, we define $\gamma_{(j)}(x)$ as the following, also illustrated in Figure 1:

(a) 我们固定最低的 $j-1$ 个分量，即，

$$\big(\gamma_{(j)}(x)\big)_{(k)} := x_{(k)} \text{ for } 1 \le k \le j-1; \tag{5.8}$$

(b) 考虑剩余位置的集合 $I = \{i_{(j)}, \ldots, i_{(d)}\}$。我们称这些位置为"可用"。每个其他分量都移到右侧下一个可用位置。形式上，写作 $I = \{a_1 < \cdots < a_{d-j+1}\}$。然后

$$\big(\gamma_{(j)}(x)\big)_{a_{k+1}} := (a_{k+1}, l_{a_k}) \text{ for } 1 \le k \le d-j. \tag{5.9}$$

(c) 最右侧的分量（其位置在 I）移至最左侧的位置 I，但层级提高 1。换句话说，

$$\big(\gamma_{(j)}(x)\big)_{a_1} := (a_1, l_{d-j+1} + 1). \tag{5.10}$$

我们通过螺旋提升层级向量，同时保持颜色向量不变，来定义 $\gamma_{(j)}$ 在 Ξ 上的作用。换言之，如果 $\alpha = (l(\alpha)\ c(\alpha))$，那么

$$\gamma_{(j)}(\alpha) = (\gamma_{(j)}(l(\alpha)), c(\alpha)). \tag{5.11}$$

例 5.1。设 $\alpha = (K(1)\ J(1)\ K(0))$，并回顾 $(\alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)}) = (J(1)u_2, K(0)u_3, K(1)u_1)$。那么 $\gamma_{(1)}$ 将每个分量都向右移动。

$$\gamma_{(1)}(\alpha) = (J(1)u_3, K(0+1)u_1, K(1)u_2) = (K(1), K(1), J(1)). \tag{5.12}$$

第二个算子 $\gamma_{(2)}$ 将除最低分量外的所有分量向右移动：

$$\gamma_{(2)}(\alpha) = (J(1)u_2, K(0+1)u_1, K(1)u_3) = (K(1), J(1), K(1)). \tag{5.13}$$

请注意，由于 u_2 已固定且不可用，因此跳过 u_1，转到 u_3。

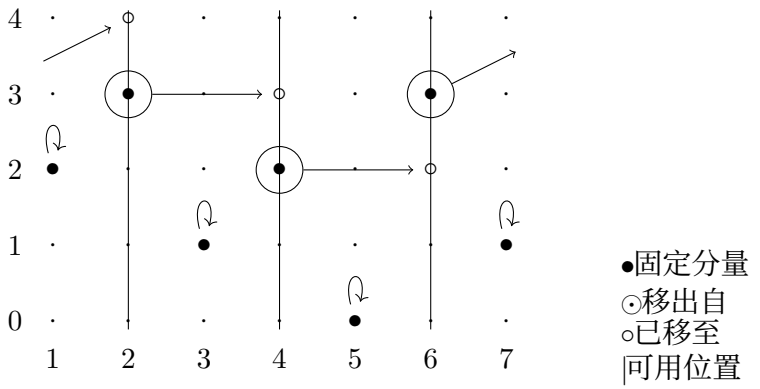

FIGURE 1. $\gamma_{(5)}(2,3,1,2,0,3,1) = (2,4,1,3,0,2,1)$

最后一个算子 $\gamma_{(3)}$ 将最高分量"向右移动"，但由于只有一个可用位置，它只会上升 1：

$$\gamma_{(3)}(\alpha) = (J(1)u_2, K(0)u_3, K(1+1)u_1) = (K(2), J(1), K(0)). \tag{5.14}$$

算子 $\gamma_{(j)}$ 满足以下性质。

命题5.2（[39, 定理1.2]）。对于所有 $1 \le j$、$j' \le d$，我们有

$$\gamma_{(j')} \circ \gamma_{(j)} = \gamma_{(j)} \circ \gamma_{(j')} \tag{5.15}$$

作为从 $\mathbb{N}^d$ 到 $\mathbb{N}^d$的映射。

因此，算子 $\gamma_{(j)}$ 在集合 $\mathbb{N}^d$ 上诱导出自由阿贝尔半群 $\Gamma := \mathbb{N}^d$ 的作用。对于 $a = (a_1, \dots, a_d) \in \Gamma$，该作用定义为

$$a \cdot x := \gamma_{(1)}^{a_1} \circ \cdots \circ \gamma_{(d)}^{a_d}(x) \tag{5.16}$$

对于 $x \in \mathbb{N}^d$。下一个性质表明，该作用是自由且传递的。

命题 5.3（[39, 定理1.3]）。设 $\mathbf{0} = (0, \dots, 0) \in \mathbb{N}^d$。那么对于任意 $x \in \mathbb{N}^d$，存在唯一的 $a \in \Gamma$ 使得

$$a \cdot \mathbf{0} = x. \tag{5.17}$$

上述性质表明，可以使用螺旋提升算子遍历 Ξ 中的元素；这种枚举方式将适用于我们后续的分析。我们在 $d = 2$ 中对 Γ 作用进行了完整的说明。

例 5.4。我们有如下无限网格。

$$\begin{array}{ccccccc}
(0,0) & \longrightarrow & (0,1) & \longrightarrow & (0,2) & \longrightarrow & (0,3) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(1,0) & \longrightarrow & (2,0) & \longrightarrow & (3,0) & \longrightarrow & (4,0) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(1,1) & \longrightarrow & (1,2) & \longrightarrow & (1,3) & \longrightarrow & (1,4) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(2,1) & \longrightarrow & (3,1) & \longrightarrow & (4,1) & \longrightarrow & (5,1) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

其中，竖直箭头为 $\gamma_{(1)}$，水平箭头为 $\gamma_{(2)}$。$\mathbb{N}^2$ 中的每个元组都恰好位于该网格中的一个位置。

上面的网格与所有可能的颜色向量（K，K）、（K，J）、（J，K）和（J，J）组合后，得到以下网格，它们共同将 Ξ 中的每个元素恰好列出一次。

$$
\begin{array}{ccccccc}
(K(0),K(0)) & \longrightarrow & (K(0),K(1)) & \longrightarrow & (K(0),K(2)) & \longrightarrow & (K(0),K(3)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(K(1),K(0)) & \longrightarrow & (K(2),K(0)) & \longrightarrow & (K(3),K(0)) & \longrightarrow & (K(4),K(0)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(K(1),K(1)) & \longrightarrow & (K(1),K(2)) & \longrightarrow & (K(1),K(3)) & \longrightarrow & (K(1),K(4)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(K(2),K(1)) & \longrightarrow & (K(3),K(1)) & \longrightarrow & (K(4),K(1)) & \longrightarrow & (K(5),K(1)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

$$
\begin{array}{ccccccc}
(K(0),J(1)) & \longrightarrow & (K(0),J(2)) & \longrightarrow & (K(0),J(3)) & \longrightarrow & (K(0),J(4)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(J(2),K(0)) & \longrightarrow & (J(3),K(0)) & \longrightarrow & (J(4),K(0)) & \longrightarrow & (J(5),K(0)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(K(1),J(2)) & \longrightarrow & (K(1),J(3)) & \longrightarrow & (K(1),J(4)) & \longrightarrow & (K(1),J(5)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(J(3),K(1)) & \longrightarrow & (J(4),K(1)) & \longrightarrow & (J(5),K(1)) & \longrightarrow & (J(6),K(1)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

$$
\begin{array}{ccccccc}
(J(1),K(0)) & \longrightarrow & (J(1),K(1)) & \longrightarrow & (J(1),K(2)) & \longrightarrow & (J(1),K(3)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(K(1),J(1)) & \longrightarrow & (K(2),J(1)) & \longrightarrow & (K(3),J(1)) & \longrightarrow & (K(4),J(1)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(J(2),K(1)) & \longrightarrow & (J(2),K(2)) & \longrightarrow & (J(2),K(3)) & \longrightarrow & (J(2),K(4)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(K(2),J(2)) & \longrightarrow & (K(3),J(2)) & \longrightarrow & (K(4),J(2)) & \longrightarrow & (K(5),J(2)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

$$
\begin{array}{ccccccc}
(J(1),J(1)) & \longrightarrow & (J(1),J(2)) & \longrightarrow & (J(1),J(3)) & \longrightarrow & (J(1),J(4)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(J(2),J(1)) & \longrightarrow & (J(3),J(1)) & \longrightarrow & (J(4),J(1)) & \longrightarrow & (J(5),J(1)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(J(2),J(2)) & \longrightarrow & (J(2),J(3)) & \longrightarrow & (J(2),J(4)) & \longrightarrow & (J(2),J(5)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(J(3),J(2)) & \longrightarrow & (J(4),J(2)) & \longrightarrow & (J(5),J(2)) & \longrightarrow & (J(6),J(2)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

5.2. 螺旋提升与格拉斯曼簇 Gröbner 分层。根据引理 4.4 和引理 4.5，Hilb(α) 的大小是三个因子的乘积:

$$|\mathrm{Hilb}(\alpha)| = A(\alpha)\cdot B(\alpha)\cdot D(\alpha), \tag{5.18}$$

其中

$$A(\alpha) := |\mathrm{Hilb}^0(\alpha|_{K_\alpha})|, \tag{5.19}$$

$$B(\alpha) := q^{b(\alpha)} := q^{\sum_{i\in K_\alpha}\left|C(\alpha|_{J_\alpha})_{\succ\mu_i^0}\right|}, \tag{5.20}$$

$$D(\alpha) := q^{\delta(\alpha)} := q^{\sum_i\left|\Delta(\alpha)_{\succ\mu_i^0(\alpha)}\right|}. \tag{5.21}$$

最简单的因子是 $B(\alpha)$，它仅取决于颜色向量。尤其是，螺旋提升不会改变 $B(\alpha)$。

引理 5.5. 设 α 的颜色向量为 $(c_{(1)},\dots,c_{(d)})$。那么

$$b(\alpha) = \#\{(i,j) : 1\le i<j\le d, c_{(i)} = K, c_{(j)} = J\}. \tag{5.22}$$

特别地，对于任意 $j\in[d]$，$B(\gamma_{(j)}(\alpha)) = B(\alpha)$。

证明。这是显然的，因为 $C(\alpha_{(j)}) = \{\mu^0(\alpha_{(j)})\}$ 当且仅当 $c(\alpha_{(j)}) = J$，并且 α 的分量按照其 μ^0 排序。□

The behavior of $A(\alpha)$ and $D(\alpha)$ under spiral raising depends on the "distances" between components of α. We first define a notion of distance between two components $x_i = (i, l_i)$ and $x_j = (j, l_j)$ of an element $x = (l_1,\dots,l_d)$ of $\mathbb{N}^d$. Assume $x_i \prec x_j$, i.e., $l_i + \frac{i}{d} < l_j + \frac{j}{d}$. Then the distance between x_i and x_j is given by

$$\delta(x_i, x_j) := \lfloor l_j + \frac{j}{d} - l_i - \frac{i}{d}\rfloor. \tag{5.23}$$

接下来我们定义 $\delta_{(ij)}(x) := \delta(x_{(i)},x_{(j)})$。我们称所有 $1\le i<j\le d$ 对应的 $\delta_{(ij)}(x)$ 数据为 x 的距离矩阵。

对于 $\alpha\in\Xi$，我们将 α 的距离矩阵定义为 $\delta_{(ij)}(\alpha) := \delta_{(ij)}(l(\alpha))$。换言之，我们只需忽略颜色向量。

等价地,

$$\delta(\alpha_i,\alpha_j) = \max\{N : T^N\mu^0(\alpha_i) \prec \mu^0(\alpha_j)\}. \tag{5.24}$$

以下引理概述了 $\gamma_{(j)}$ 对 α 的距离向量的影响。由于颜色向量无关紧要，我们对任意 $x\in\mathbb{N}^d$ 陈述该引理。

引理 5.6（ [39, 引理 5.2] 的证明）。设 $x\in\mathbb{N}^d$ 和 $j\in[d]$。定义 $\gamma_{(j)}$ 的拉伸集合，记作 $S_{(j)}(x)$，如下:

$$\left\{(b,h), 1\le b<j\le h\le d \;\middle|\; \begin{array}{l} i(x_{(h)}) < i(x_{(b)}) < i(\gamma_{(j)}(x)_{(h)}) \\ \text{or } i(\gamma_{(j)}(x)_{(h)}) < i(x_{(h)}) < i(x_{(b)}) \\ \text{or } i(x_{(b)}) < i(\gamma_{(j)}(x)_{(h)}) < i(x_{(h)}) \end{array}\right\}. \tag{5.25}$$

于是有

(a) 对任意一对 $1\le b<h\le d$，距离的变化

$$\delta_{(bh)}(\gamma_{(j)}(x)) - \delta_{(bh)}(x) \tag{5.26}$$

当 $(b,h)\in S_{(j)}(x)$ 时为 1，否则为 0。

(b) 从 $S_{(j)}(x)$ 到 $[j-1]$ 的映射 $(b,h)\mapsto b$ 是一个双射。特别地，$S_{(j)}(x)$ 有 $j-1$ 个元素。

In other words, the spiral raising operator $\gamma_{(j)}$ fixes the pairwise distances within the lowest $j-1$ (i.e., unmoved) components and the pairwise distances within the remaining (i.e., moved) components, but stretches the distances between unmoved and moved components. One may understand $\gamma_{(j)}$ as raising the moved components "as a whole" away from the unmoved components. The crux of the lemma is the elementary observation that the distance between an unmoved component $x_{(b)}$ and a moved component $x_{(h)}$ is increased by 1 if $x_{(h)}$ moves past the seat of $x_{(b)}$ (cf. the definition of $S_{(j)}(x)$ above), and unchanged otherwise.

给定一个拉伸 $(b,h)\in S_{(j)}(x)$，我们称 $x_{(b)}$（分别称 $x_{(h)}$）为该拉伸的较低分量（分别为较高分量）。我们也称 $\delta(x_{(b)},x_{(h)})$ 为该拉伸的距离；特别地，距离指拉伸之前的距离。$\alpha\in\Xi$ 亦同。

现在我们回到对 $A(\alpha)$ 和 $D(\alpha)=q^{\delta(\alpha)}$ 的分析。

我们称 $S_{(j)}(\alpha)$ 中一个距离为零且其下方分量的颜色为 J 的拉伸为受阻拉伸。下面的引理指出，由 $\gamma_{(j)}$ 引起的 $\delta(\alpha)$ 的增加量等于未受阻拉伸的数量。回顾一下，拉伸总数为 $j-1$ 个。

引理 5.7。固定 $1\le j\le d$，并令

$$s=\#\big\{(b,h)\in S_{(j)}(\alpha):c(\alpha_{(b)})=J,\ \delta_{bh}(\alpha)=0\big\} \tag{5.27}$$

为受阻延展的数量。

那么我们有

$$\delta(\gamma_{(j)}(\alpha))=\delta(\alpha)+j-1-s. \tag{5.28}$$

Proof. Recall that $\delta(\alpha)=\sum_i\left|\Delta(\alpha)_{\succ\mu_i^0(\alpha)}\right|$. We separate the contributions of $\delta(\alpha)$ in terms of basis vectors. For $k\in[d]$, let $\Delta_k(\alpha)$ be the monomials in $\Delta(\alpha)$ involving u_k. Thus,

$$\delta(\alpha)=\sum_{i,k\in[d]}\left|\Delta_k(\alpha)_{\succ\mu_i^0(\alpha)}\right|. \tag{5.29}$$

The observations below can be verified elementarily by writing down the monomials in $\Delta_k(\alpha)$. We first look at the terms with $i=k$. We notice that $\left|\Delta_i(\alpha)_{\succ\mu_i^0(\alpha)}\right|$ is 0 if α_i is of color K, and 1 if α_i is of color J.

对于该 对于含有 $i\neq k$ 的项，我们考虑两种情况：$\alpha_i\succ\alpha_k$ 和 $\alpha_i\prec\alpha_k$。如果 $\alpha_i\succ\alpha_k$，那么

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$$\left|\Delta_k(\alpha)_{\succ\mu_i^0(\alpha)}\right|=\begin{cases}1, & \text{if } \delta(\alpha_k,\alpha_i)=0 \text{ and } c(\alpha_k)=J;\\ 0, & \text{otherwise.}\end{cases} \tag{5.30}$$

如果 $\alpha_i\prec\alpha_k$，那么

$$\left|\Delta_k(\alpha)_{\succ\mu_i^0(\alpha)}\right|=\begin{cases}\delta(\alpha_i,\alpha_k)+1, & \text{if } c(\alpha_k)=J;\\ \delta(\alpha_i,\alpha_k), & \text{if } c(\alpha_k)=K.\end{cases} \tag{5.31}$$

特别地，对于 $\alpha_i\prec\alpha_k$，这两个项的和

$$\left|\Delta_k(\alpha)_{\succ\mu_i^0(\alpha)}\right|+\left|\Delta_i(\alpha)_{\succ\mu_k^0(\alpha)}\right| \tag{5.32}$$

only depends on $\delta(\alpha_i,\alpha_k)$ and the colors of α_i and α_k. Moreover, if $\delta(\alpha_i,\alpha_k)$ is increased by 1, then the two-term sum is increased by 1 (in general) or unchanged (in the special case where $\delta(\alpha_i,\alpha_k)=0$ before the increase and $c(\alpha_i)=J$). Noting that the criterion for the special case is precisely the criterion for an obstructed stretch, the proof of Lemma 5.7 is complete. □

因子 $A(\alpha)$ 在 $\gamma_{(j)}$ 下保持不变，当且仅当每个颜色-K 分量之间的拉伸距离至少为 3。

引理 5.8。若任意 $(b,h)\in S_{(j)}(\alpha)$ 满足 $c(\alpha_{(b)})=c(\alpha_{(h)})=K$，且满足 $\delta_{bh}(\alpha)\neq1,2$，则

$$A(\gamma_{(j)}(\alpha))=A(\alpha). \tag{5.33}$$

证明。回顾引理 4.5， $A(\alpha) = |$希尔伯特概形$^0(\alpha|_{K_\alpha})|$ 是满足 $X^2 = Y^3, XY = YX$ 的矩阵对 $X, Y \in \mathrm{Mat}_{K_\alpha \times K_\alpha}(\mathbb{F}_q)$ 的数量，并且

$$X_{ik} = 0 \text{ if } \mu_k^0 \prec T^3\mu_i^0, \tag{5.34}$$

$$Y_{ik} = 0 \text{ if } \mu_k^0 \prec T^2\mu_i^0. \tag{5.35}$$

对于 $h \in [d]$，令 $i_{(h)}$ 表示 $\alpha_{(h)}$ 的位置。通过定义 $X_{(bh)} = X_{i_{(b)},i_{(h)}}$ 来改变矩阵的指标编排，对于 $Y_{(bh)}$ 也类似。新的指标集为 $H = \{h : c(\alpha_{(h)}) = K\}$。现在，我们将 X、Y 视为 $\mathrm{Mat}_{H\times H}(\mathbb{F}_q)$ 中的矩阵 $(X_{(bh)})$、$(Y_{(bh)})$。

在这种新的指标编排中， X, Y是严格上三角的，满足 $X^2 = Y^3, XY = YX$，并且满足

$$X_{(bh)} = 0 \text{ if } \delta_{(bh)}(\alpha) = 0, 1, 2, \tag{5.36}$$

$$Y_{(bh)} = 0 \text{ if } \delta_{(bh)}(\alpha) = 0, 1 \tag{5.37}$$

对于 $1 \le b < h \le d$。

新指标编排中，关于这对 (X, Y) 的定义方程只有在某个 $\delta_{(bh)}(\alpha)$ 从 1 增加到 2 或从 2 增加到 3 时才会改变。证明至此完成。□

将引理 5.5、5.7 和 5.8 代入因式分解式 (5.18)，我们证明了关于 |希尔伯特概形(α)| 的以下性质。

引理 5.9。设 $j \in [d]$ 和 $\alpha \in \Xi$。假设 $S_{(j)}(\alpha)$ 不存在受阻延展，且引理 5.8 的假设成立。那么

$$|\mathrm{Hilb}(\gamma_{(j)}(\alpha))| = q^{j-1}|\mathrm{Hilb}(\alpha)|. \tag{5.38}$$

在实践中，我们将采用一个更强的假设。

定义 5.10. 如果对于所有满足 $1 \le b < j \le h \le d$ 的数对 (b, h)， α 在 $\gamma_{(j)}$ 下稳定（或称为 $\gamma_{(j)}$-稳定），则称 α 为稳定的。

(a) 每当 $c(\alpha_{(b)}) = J$ 时，我们有 $\delta_{(bh)}(\alpha) \ge 1$；(b) 每当 $c(\alpha_{(b)}) = c(\alpha_{(h)}) = K$ 时，我们有 $\delta_{(bh)}(\alpha) \ge 3$。

关键在于，如果 α 在 $\gamma_{(j)}$ 作用下稳定，那么对于任意 $k \in [d]$， $\gamma_{(k)}(\alpha)$ 在 $\gamma_{(j)}$ 作用下也稳定。因此，我们可以反复应用引理 5.9，并证明以下结果。

Lemma 5.11. 假设 α 在 $\gamma_{(j_1)} \cdots, \gamma_{(j_l)}$ 下稳定，其中 $1 \le j_1 < \cdots < j_l \le d$。Le t

$$\gamma = \gamma_{(j_1)}^{a_1} \circ \cdots \circ \gamma_{(j_l)}^{a_l}. \tag{5.39}$$

Then

$$|\mathrm{Hilb}(\gamma(\alpha))| = q^{a_1(j_1-1)+\cdots+a_l(j_l-1)}|\mathrm{Hilb}(\alpha)|. \tag{5.40}$$

证明。反复应用引理 5.9，并注意到其假设始终成立。 □

5.3. 生成函数。证明 t-有理性的主要思路是

$$H_d(t) = \sum_{\alpha\in\Xi} |\mathrm{Hilb}(\alpha)| \cdot t^{n(\alpha)} \tag{5.41}$$

是将 Ξ 分解为有限个两两不相交的“稳定轨道”的并（其含义见下文的引理），其中每个稳定轨道都会对关于 t 的一个有理函数作出贡献。

对于 $\alpha \in \Xi$，我们将 α 的内容（或 α 对 $H_d(t)$ 的贡献）记为

$$\mathrm{Cont}(\alpha) := |\mathrm{Hilb}(\alpha)| \cdot t^{n(\alpha)}. \tag{5.42}$$

引理 5.12. 假设 β 在 $\gamma_{(j_1)} \cdots, \gamma_{(j_l)}$ 下稳定，且满足 $1 \le j_1 < \cdots < j_l \le d$。令 Γ' 为由 $\gamma_{(j_1)} \cdots, \gamma_{(j_l)}$ 生成的 Γ 的子半群。考虑轨道 $\Gamma' \cdot \beta$。则有

$$\sum_{\alpha\in\Gamma'\cdot\beta} \mathrm{Cont}(\alpha) = \frac{1}{(1-q^{j_1-1}t)\dots(1-q^{j_l-1}t)} \mathrm{Cont}(\beta). \tag{5.43}$$

证明。证明方法与 [39, Theorem 5.4] 相同。我们只需要命题 5.3、引理 5.9，以及以下观察即可。

$$n(\gamma_{(j)}(\alpha)) = n(\alpha) + 1 \tag{5.44}$$

对于任意 $j \in [d]$，这是因为恰好只有一个分量（最右侧移动的分量）的层级增加了 1。□

我们给出了 α 稳定性的一个充分条件。

引理 5.13. 任意 $\alpha \in \Xi$ 都是 $\gamma_{(1)}$-稳定的。如果 $j > 1$，令 $l(\alpha) = a \cdot \mathbf{0}$，$a = (a_1, \ldots, a_d) \in \Gamma$ 为命题 5.3 中的唯一表达式。那么，当 $a_j \geq 3(d - j + 1)$ 时，α 是 $\gamma_{(j)}$-稳定的。

证明。根据定义 (5.25)，对于 $\gamma_{(1)}$ 不存在拉伸，因此任意 α 都是平凡稳定的。若 $j > 1$，我们注意到，对于任意 $x = (x_1, \ldots, x_d) \in \mathbb{N}^d$，

$$\gamma_{(j)}^{d-j+1}(x)_i = \begin{cases} x_i, & i = i(x_{(b)}), b < j; \\ x_i + 1, & i = i(x_{(h)}), h \geq j. \end{cases} \tag{5.45}$$

这是因为 $\gamma_{(j)}$ 会使分量 $d - j + 1$、$x_{(j)}$、……、$x_{(d)}$发生螺旋式旋转，因此，当执行它$d - j + 1$ 次后，这些分量会回到各自原来的位置，并上升一个层级。

因此，对于任意 $\beta \in \Xi$ 和 $b < j \leq h$，我们有 $\delta_{(bh)}(\gamma^{d-j+1}{}_{(j)}(\beta)) = \delta_{(bh)}(\beta) + 1$。因此，对于满足假设的 α，任意 $b < j \leq h$都有 $\delta_{(bh)}(\alpha) \geq 3$，故 α 是 $\gamma_{(j)}$-稳定的。□

这使得可以将 Ξ 分解为有限个稳定轨道的不交并（不一定是最小分解），并由此直接推导出以下定理。

定理 5.14。对于任意颜色向量 $c \in \{J, K\}^d$，令 $\mathbf{0}_c$ 为 Ξ 中层级向量为 0 且颜色向量为 c 的元素。令 B 表示该矩形

$$\{(0, b_2, \ldots, b_d) \in \Gamma : 0 \leq b_j \leq 3(d - j + 1) \text{ for all } 2 \leq j \leq d\}. \tag{5.46}$$

对于任意 $b \in B$，令 Γ_b 为由 $\gamma_{(1)}$ 以及所有满足 $b_j = 3(d - j)$ 的 $\gamma_{(j)}$ 在 Γ 中生成的子半群。则我们有一个稳定轨道分解

$$\Xi = \bigsqcup_{c \in \{J,K\}^d} \bigsqcup_{b \in B} \Gamma_b(b \cdot \mathbf{0}_c). \tag{5.47}$$

特别是，

$$H_d(t) = \sum_{c \in \{J,K\}^d} \sum_{b \in B} \frac{1}{\prod_{j:\gamma_{(j)} \in \Gamma_b} (1 - q^{j-1}t)} \operatorname{Cont}(b \cdot \mathbf{0}_c) \tag{5.48}$$

$$= \sum_{c \in \{J,K\}^d} \sum_{b \in B} \frac{|\operatorname{Hilb}(b \cdot \mathbf{0}_c)|}{\prod_{j:\gamma_{(j)} \in \Gamma_b} (1 - q^{j-1}t)} t^{b_2 + \cdots + b_d + \#_J(c)}, \tag{5.49}$$

其中，$\#_J(c)$ 表示颜色向量 c 中 J 的数量。

证明。唯一需要进一步说明的是最后一行，其表述为

$$n(b \cdot \mathbf{0}_c) = b_2 + \cdots + b_d + \#_J(c). \tag{5.50}$$

根据（5.44），只需证明 $n(\mathbf{0}_c) = \#_J(c)$。事实上，$\mathbf{0}_c$由 $K(0)$ 和 $J(1)$ 组成，并且由于 $n(K(0)) = 0$、$n(J(1)) = 1$，结论得证。□

最后一项断言完成了定理 1.8 有理性部分的证明；注意，所有分母都整除 $(1 - t)(1 - qt) \ldots (1 - q^{d-1}t)$。此外，只要我们理解任意秩不超过 d、纯 K 颜色的 α 的 $\mathbf{Hilb}^0(\alpha)$，就能用它给出计算 $H_d(t)$ 的有限公式。我们将在 §7 中对 $d \leq 3$ 进行计算。

6. Rationality question in the q variable

回顾 $H_d(t) = H_{d,R}(t)$，其中 $R = \mathbb{F}_q[[T^2, T^3]]$。我们想知道 $H_d(t)$ 是否也关于 q 是有理的。这一问题的动机版本正是猜想 1.10。

为了研究猜想 1.10，我们提出以下关于可交换矩阵簇的猜想，该猜想本身也具有独立的研究价值。

猜想 6.1. 设 $\alpha = (K(a_1) \dots, K(a_d))$ 为纯-K 首项数据，并设 k 为任意域。考虑由矩阵方程定义的仿射簇 $V(\alpha) := \{(X, Y)\} \subseteq \mathsf{Mat}_d(k)^2$

$$X^2 = Y^3, \tag{6.1}$$

$$[X, Y] = 0, \tag{6.2}$$

$$X_{ij} = 0 \text{ if } \mu_j^0 \prec T^3 \mu_i^0, \tag{6.3}$$

$$Y_{ij} = 0 \text{ if } \mu_j^0 \prec T^2 \mu_i^0. \tag{6.4}$$

那么，Grothendieck 环 $K_0(\mathsf{Var}_k)$ 中 k-簇 $V(\alpha)$ 的动机 $[V(\alpha)]$ 是关于 $\mathbb{L}$ 的一个多项式。

The set of k-points of $V(\alpha)$ is precisely $\mathrm{Hilb}^0(\alpha)$. By (the motivic versions of) (5.18) and Theorem 5.14, for any $D \geq 0$, Conjecture 6.1 for all $0 \leq d \leq D$ would imply Conjecture 1.10 for $d = D$. In §7, we will show that both conjectures hold when $d \leq 3$.

我们能够证明，对于“最一般”的 α 取值，该猜想成立。

定理 6.2。如果 α 纯粹具有 K 颜色，且对于所有 $j \in [d]$ 在 $\gamma(j)$ 作用下保持稳定（见定义 5.10），那么 $[V(\alpha)]$ 是关于 $\mathbb{L}$ 的一个多项式，且不依赖于 α 的具体选择。此外，它可以通过推论 6.9 中描述的归纳公式计算得到。

α 的一个典型选择是 $\alpha = (K(0)\ K(3)\ K(6) \dots, K(3(d-1))$. 我们马上就会看到，定理 6.2 讨论的是一个单一矩阵方程的解集，与 α 无关。

我们首先按照引理 5.8 的证明中的方式，重新编排矩阵的指标。因此，

$$V(\alpha) \cong \left\{ (X, Y) \in \mathrm{Mat}_d(k)^2 \;\middle|\; \begin{array}{l} X, Y \text{ are strictly upper triangular} \\ X^2 = Y^3, XY = YX \\ X_{ij} = 0 \text{ if } \delta_{(ij)}(\alpha) < 3 \\ Y_{ij} = 0 \text{ if } \delta_{(ij)}(\alpha) < 2 \end{array} \right\}. \tag{6.5}$$

因此，在定理 6.2 的假设下，簇 $V(\alpha)$ 同构于以下仅依赖于 d 的簇：

$$V_d := \left\{ (X, Y) \in \mathrm{Mat}_d(k)^2 \;\middle|\; \begin{array}{l} X, Y \text{ are strictly upper triangular} \\ X^2 = Y^3, XY = YX \end{array} \right\}. \tag{6.6}$$

对于一般的 α 纯颜色 K，簇 $V(\alpha)$ 是 V_d 与若干坐标超平面的交集，这些坐标超平面规定了 X 或 Y 的某些条目为零。令人有些不安的是，尽管一般而言，$V(\alpha)$ 所需的变量比 V_d 更少，但我们无法通过以下这些对 V_d 有效的方法，推导出 $V(\alpha)$ 的 $\mathbb{L}$-有理性。

接下来，本节余下部分将致力于证明关于 V_d 的定理 6.2。

6.1. 关于 V_d 动机的归纳法设置。我们对 d 使用归纳法。令 $(X_{d+1}, Y_{d+1}) \in \mathsf{Mat}_{d+1}(k)^2$ 由以下内容给出

$$X_{d+1} = \begin{bmatrix} X_d & z \\ 0 & 0 \end{bmatrix}, \quad Y_{d+1} = \begin{bmatrix} Y_d & w \\ 0 & 0 \end{bmatrix}, \tag{6.7}$$

其中 $X_d, Y_d \in \mathsf{Mat}_d(k)$ 且 $z, w \in \mathsf{Mat}_{d\times 1}(k)$。通过考察 (6.6)，当且仅当 (X_{d+1}, Y_{d+1}) 属于 V_{d+1}】【。

$$\begin{aligned} &(X_d, Y_d) \in V_d; \\ &X_d\, z = Y_d^2\, w; \\ &X_d\, w = Y_d\, z. \end{aligned} \tag{6.8}$$

Thus, we have a morphism $\Phi_d : V_{d+1} \to V_d$ by sending (X_{d+1}, Y_{d+1}) to (X_d, Y_d), whose fibers are solution spaces of linear equations. Define the constructible function $a : V_d \to \mathbb{N}$ given by $a(X_d, Y_d) := \dim \Phi_d^{-1}(X_d, Y_d)$. In order to compute the motive of V_{d+1}, we need to compute the motive of the stratum $a^{-1}(i)$ for each $i \in \mathbb{N}$. Furthermore, in order to carry out the induction, we also need to understand how $a(X_{d+1}, Y_{d+1})$ depends on the choice of (X_d, Y_d) and (z, w). This is a challenging task in general, but it turns out that we are able to reinterpret (6.8) as equations on modules over a ring, so that we can exploit tools in homological algebra.

6.2. **Some homological algebra.** Let $R = k[[x, y]]/(x^2 - y^3)$[8], $\mathfrak{m} = (x, y)R$, and consider the R-module M_d whose underlying vector space is k^d and the multiplications by x and y are given by the matrices X_d and Y_d, respectively. The condition $(X_d, Y_d) \in V_d$ ensures that the module structure of M_d is well-defined. The module M_d will be where z and w live. Consider the matrix

$$A = \begin{bmatrix} x & -y^2 \\ -y & x \end{bmatrix} \in \mathrm{Mat}_2(R). \tag{6.9}$$

Then (6.8) can be restated as $(z, w) \in \ker A_{M_d}$, where $A_{M_d} \in \mathrm{End}_R(M_d \oplus M_d)$ is the tensor product of $A : R^2 \to R^2$ with M_d. We write $a(M_d) := a(X_d, Y_d) = \dim_k \ker A_{M_d}$. We also write $b(M_d) = \dim_k \mathrm{im}\, A_{M_d}$. By the rank-nullity theorem for A_{M_d}, we have $a(M_d) + b(M_d) = 2d$ and $a(M_d) = \dim_k \mathrm{coker}\, A_{M_d}$.

回想一下，我们需要理解当 $M_d = (X_d, Y_d) \in V_d$ 变化时， $a(M_d)$ 如何变化。一般原理表明，我们应固定 $\mathrm{coker}\, A$ 的一个最小解消：

$$\cdots \xrightarrow{A} R^2 \xrightarrow{A'} R^2 \xrightarrow{A} R^2 \to \mathrm{coker}\, A \to 0, \tag{6.10}$$

其中

$$A' = \begin{bmatrix} x & y^2 \\ y & x \end{bmatrix}. \tag{6.11}$$

One 可以立即验证 $AA' = A'A = 0$，因此上述确实是一个链复形。 .

Remark 6.3. Incidentally, $\mathrm{coker}\, A$ is isomorphic to $\mathfrak{m}$, so $\mathrm{coker}\, A_{M_d} \cong \mathfrak{m} \otimes_R M_d$, so that $a(M_d) = \dim_k(\mathfrak{m} \otimes_R M_d)$. However, we are not able to make use of this observation. In fact, the rest of the proof below does not even use the fact that (6.10) is exact, but only that it is a chain complex.

考虑 2-周期链复形

$$C^\bullet : \cdots \xrightarrow{A} R^2 \xrightarrow{A'} R^2 \xrightarrow{A} R^2 \xrightarrow{A'} \cdots, \tag{6.12}$$

其中，次数约定为使 $C^\bullet$ 的一个部分读作 $C^{-1} \xrightarrow{A'} C^0 \xrightarrow{A} C^1$。我们探讨对 $C^\bullet$ 普遍成立的一些性质，即对于任意 R-模 M，这些性质均适用于链复形 $C^\bullet_M := C^\bullet \otimes_R M$。定义矩阵

$$T = \begin{bmatrix} 0 & y \\ 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H' = -H \tag{6.13}$$

在 $\mathrm{Mat}_2(R)$ 中，以及该矩阵

$$K = \begin{bmatrix} 0 & 1 & 0 & -y \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.14}$$

在 $\mathrm{Mat}_{2\times 4}(R)$ 中。

以下关于 $\mathrm{Mat}_2(R)$ 或 $\mathrm{Mat}_{2\times 4}(R)$ 的矩阵恒等式可利用关于 R 的关系$x^3 = y^2$ 直接验证。

引理 6.4。

(a) $[A, T] = [A', T] = 0$.
(b) $H'A + A'H = HA' + AH' = T^2$.

[8]由（6.6）的性质得到的环 $k[[x, y]]/(x^2 - y^3)$ 恰好与该设定中的奇异环 $k[[T^2, T^3]]$ 同构。

(c) $AK = H\begin{bmatrix}A & O_{2\times 2}\end{bmatrix} - T\begin{bmatrix}T & -A'\end{bmatrix}$.
(d) $\begin{bmatrix}I_{2\times 2} & O_{2\times 2}\end{bmatrix} - TK = H\begin{bmatrix}T & -A'\end{bmatrix} + A'\begin{bmatrix}O_{2\times 2} & H\end{bmatrix}$.
(e) *Let* $g = \mathrm{diag}(1,-1) \in \mathrm{Mat}_2(R)$, *and note that* $g = g^{-1}$. *Then* $gAg = A'$ *and* $gTg = -T$.

由（a），矩阵 T 诱导出链映射 $T\colon C^\bullet \to C^\bullet$。特别地，对于任意 R-模 M，我们有诱导的链映射 $T_M\colon C_M^\bullet \to C_M^\bullet$，以及 $C_M^\bullet$ 的上同调上的诱导映射：

$$T_M^0 : H^0(C_M^\bullet) \to H^0(C_M^\bullet), \quad T_M^1 : H^1(C_M^\bullet) \to H^1(C_M^\bullet). \tag{6.15}$$

引理 6.4(e) 表明，链 $C_M^\bullet$ 与链映射 T_M 在每个次数上都“看起来相同”：

引理 6.5。矩阵 $g\colon C^i = R^2 \to C^{i+1} = R^2$ 诱导链同构 $g\colon C^\bullet \to C^\bullet[1]$，其中 $(C^\bullet[1])^i = C^{i+1}$。此外，我们有如下交换链复形图：

$$\begin{array}{ccc} C^\bullet & \xrightarrow{g} & C^\bullet[1] \\ \big\downarrow{\scriptstyle T} & & \big\downarrow{\scriptstyle -T[1]} \\ C^\bullet & \xrightarrow{g} & C^\bullet[1] \end{array} \tag{6.16}$$

证明。 g 是链映射且该图交换，这只是引理 6.4(e) 的重述。链映射 g 是链同构，因为 g^2 : $C^\bullet \to C^\bullet[2] = C^\bullet$ 是恒等映射，其中等号体现了链复形 $C^\bullet$的 2 周期性。□

引理 6.4(b)(c)(d) 蕴含链映射 T 的一个普遍成立的性质：

引理 6.6. 对任意 R-模 M，序列

$$\ldots \xrightarrow{T_M^i} H^i(C_M^\bullet) \xrightarrow{T_M^i} H^i(C_M^\bullet) \xrightarrow{T_M^i} \ldots \tag{6.17}$$

对于 $i = 0, 1$ 是正合的。

Proof. Lemma 6.4(b) implies that $T^2 : C^\bullet \to C^\bullet$ is chain homotopic to zero, with the chain homotopy map given by $H : C^0 \to C^1$ from even degrees and $H' : C^{-1} \to C^0$ from odd degrees. Therefore, the induced map on the cohomology $(T_M^i)^2 : H^i(C_M^\bullet) \to H^i(C_M^\bullet)$ is zero.

To show the exactness, it suffices to work with $i = 0$ thanks to Lemma 6.5 tensored with M. It remains to prove $\ker T_M^0 \subseteq \mathrm{im}\, T_M^0$, where we recall that $T_M^0 : \ker A_M/\mathrm{im}\, A'_M \to \ker A_M/\mathrm{im}\, A'_M$. Let $\overline{u} \in \ker(T_M^0) \subseteq \ker A_M/\mathrm{im}\, A'_M$. Take a lift $u \in M^2$ with $Au = 0$, then $\overline{u} \in \ker(T_M^0)$ implies $Tu = A'v$ for some $v \in M^2$. Note that

$$\begin{bmatrix}A & O_{2\times 2}\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}T & -A'\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} = 0 \in M^2. \tag{6.18}$$

考虑

$$s = K\begin{bmatrix}u\\v\end{bmatrix}. \tag{6.19}$$

因此，根据引理 6.4(c),

$$As = AK\begin{bmatrix}u\\v\end{bmatrix} = H\begin{bmatrix}A & O_{2\times 2}\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} - T\begin{bmatrix}T & -A'\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} = 0 \in M^2. \tag{6.20}$$

根据引理 6.4(d),

$$\begin{aligned} u - Ts &= \left(\begin{bmatrix}I_{2\times 2} & O_{2\times 2}\end{bmatrix} - TK\right)\begin{bmatrix}u\\v\end{bmatrix} \\ &= H\begin{bmatrix}T & -A'\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} + A'\begin{bmatrix}O_{2\times 2} & H\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} \\ &= A'\begin{bmatrix}O_{2\times 2} & H\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix} \in \mathrm{im}(A'_M). \end{aligned} \tag{6.21}$$

因此 $s \in \ker(A_M)$ 和 $T_M^0(\overline{s}) = \overline{u}$，所以 $\overline{u} \in \mathrm{im}(T_M^0)$。 □

特别地，对于任意 R-模 M，都存在一个特征子空间 $\overline{W_M} \subseteq H^0(C_M^\bullet)$，满足 $\overline{W}_M = \ker(T_M^0) = \operatorname{im}(T_M^0)$。研究与 $H^0(C_M^\bullet)$ 上的幂零算子 T_M^0 相关联的滤过是很有启发性的：

$$0 \subseteq \overline{W}_M \subseteq H^0(C_M^\bullet), \qquad H^0(C_M^\bullet) \xrightarrow{T_M^0} \overline{W}_M \xrightarrow{T_M^0} 0. \tag{6.22}$$

当 M 在 k 上是有限维的时，秩-零化度定理意味着

$$\dim_k H^0(C_M^\bullet) = 2 \dim_k \overline{W}_M. \tag{6.23}$$

关于商映射 $\ker A_M \twoheadrightarrow \ker A_M / \operatorname{im} A'_M$，取 (6.22) 的原像，我们定义

$$W^0(M) := \operatorname{im} A'_M \subseteq W^1(M) := T_M^{-1}(\operatorname{im} A'_M) \subseteq W^2(M) := \ker A_M. \tag{6.24}$$

那么，(6.23) 意味着 $\dim_k W^1(M) = (\dim_k W^0(M) + \dim_k W^2(M))/2$。这种中间维数性质是我们需要引理 6.6 的唯一原因。

我们总结与有限维 R- M 模的 $C_M^\bullet$ 和 T 相关的维数之间的关系。回忆 $a(M) := \dim_k \ker A_M$ 和 $b(M) := \dim_k \operatorname{im} A_M$，并且根据秩-零化度定理，我们有 $a(M) + b(M) = 2 \dim_k M$。

引理 6.7. 对于任意 R-模 M，满足 $\dim_k M < \infty$，我们有

(a) $\dim_k \ker A'_M = \dim_k \ker A_M = \dim_k W^0(M) = a(M)$;
(b) $\dim_k \operatorname{im} A'_M = \dim_k \operatorname{im} A_M = \dim_k W^2(M) = b(M)$.
(c) $\dim_k H^0(C_M^\bullet) = \dim_k H^1(C_M^\bullet) = a(M) - b(M)$.
(d) $\dim_k W^1(M) = \dfrac{a(M) + b(M)}{2}$.

Proof. Identities (a)(b)(c) are immediate from tensoring Lemma 6.5 with M. The identity (d) is restating the middle-dimensionality observation above. Note that $\dim_k H^0(C_M^\bullet)$ is *a priori* $\dim_k \ker A_M - \dim_k \operatorname{im} A'_M$, so it is necessary to observe (b) to express $\dim_k H^0(C_M^\bullet)$ in terms of the rank and the nullity of A_M. □

6.3. 一种分层结构 在 V_d 上。在引理 6.7 的启示下，我们在 V_d 上得到了一个定义良好的分层，其方式为

$$V_d = \bigsqcup_{\substack{a+b=2d \\ a \geq b \geq 0}} V_{(a,b)}, \quad V_{(a,b)} := \{M_d \in V_d : a(M_d) = a, b(M_d) = b\}. \tag{6.25}$$

回忆一下，V_{d+1} 中的点 (X_{d+1}, Y_{d+1}) 由点 $(X_d, Y_d) \in V_d$ 和向量 $u := \begin{bmatrix} z \\ w \end{bmatrix} \in \ker A_{M_d}$ 确定，规则如 (6.7) 所示。接着，我们根据 M_d 和 u 确定 (X_{d+1}, Y_{d+1}) 所属的分层。

引理 6.8. 在上述记号下，假设 $M_d \in V_{(a,b)}$，则

$$(X_{d+1}, Y_{d+1}) \in \begin{cases} V_{(a+2,b)}, & \text{if } u \in W^0(M_d); \\ V_{(a+1,b+1)}, & \text{if } u \in W^1(M_d) \setminus W^0(M_d); \\ V_{(a,b+2)}, & \text{if } u \in W^2(M_d) \setminus W^1(M_d). \end{cases} \tag{6.26}$$

注意到 $W^0(M_d)$ $W^1(M_d)$ $W^2(M_d)$ 的维数分别为 b、$(a+b)/2$、a，引理 6.8 立即推出了关于 $V_{(a,b)}$ 的以下归纳公式，从而完成了定理 6.2 的证明。

推论 6.9。 $V_{(a,b)}$ 的动机可以通过以下方式归纳计算：

$$\begin{aligned} &[V_{(0,0)}] = 1, [V_{(a,b)}] = 0 \ \textit{unless}\ a \geq b \geq 0, a \equiv b \pmod 2; \\ &[V_{(a,b)}] = \mathbb{L}^b [V_{(a-2,b)}] + (\mathbb{L}^{\frac{a+b-2}{2}} - \mathbb{L}^{b-1})[V_{(a-1,b-1)}] + (\mathbb{L}^a - \mathbb{L}^{\frac{a+b-2}{2}})[V_{(a,b-2)}]. \end{aligned} \tag{6.27}$$

$d \geq 0$ 下 V_d 的动机由下式给出：

$$[V_d] = \sum_{b=0}^{d} [V_{2d-b,b}]. \tag{6.28}$$

引理 6.8 的证明。只需确定 $b(M_{d+1})$，此外，我们将使用定义$b(M_{d+1}) = $维$_k$像 $A'_{M_{d+1}}\circ$ $A'_{M_{d+1}}$在 $M_{d+1} \oplus M_{d+1} = M_d \oplus k \oplus M_d \oplus k$上的矩阵为

$$A'_{M_{d+1}} = \begin{bmatrix} X_{d+1} & Y_{d+1}^2 \\ Y_{d+1} & X_{d+1} \end{bmatrix} = \begin{bmatrix} X_d & z & Y_d^2 & Y_d w \\ 0 & 0 & 0 & 0 \\ Y_d & w & X_d & z \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.29}$$

因此， $A'_{M_{d+1}}$ 的像实际上位于 $M_d \oplus M_d$中，并且等于

$$\begin{aligned} \operatorname{im} A'_{M_{d+1}} &= \operatorname{im} \begin{bmatrix} X_d & Y_d^2 \\ Y_d & X_d \end{bmatrix} + \operatorname{span}_k \left\{ \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} Y_d w \\ z \end{bmatrix} \right\} \\ &= \operatorname{im} A'_{M_d} + \operatorname{span}_k\{u, T_{M_d} u\} \in M_d \subseteq M_d. \end{aligned} \tag{6.30}$$

现在我们在模 A'_{M_d}意义下进行计算。由于 $u \in \ker A_{M_d}$，我们实际上是在 $\ker A_{M_d}/\operatorname{im} A'_{M_d} = H^0(C^\bullet_{M_d})$ 中进行计算。因此， b-参数的增量由下式给出。

$$b(M_{d+1}) - b(M_d) = \dim_k \operatorname{span}_k\{\overline{u}, T^0_{M_d}\overline{u}\} \subseteq H^0(C^\bullet_{M_d}). \tag{6.31}$$

我们回顾一下， $T^0_{M_d}$ 是 $H^0(C^\bullet_{M_d})$ 上的一个平方零线性映射。根据关于幂零线性映射的一个标准结论（引理 6.10）， $\dim_k \operatorname{span}_k\{\overline{u}, T^0_{M_d}\overline{u}\}$ 的维数等于 $\{\overline{u}, T^0_{M_d}\overline{u}\}$ 中非零元素的个数。至此，证明完成。□

引理 6.10. 设 V 是域 k 上的有限维向量空间，且 $T\colon V \to V$ 是幂零线性映射。对于任意向量 $v \in V$，考虑序列 $v, Tv, T^2v, \ldots$，该序列最终为零。则该序列中的非零元素线性无关。

Proof. Assume $T^{n-1}v \neq 0$ and $T^n v = 0$. Let $W = \operatorname{span}_k\{v, Tv, \ldots, T^{n-1}v\}$, so T is a nilpotent endomorphism of W. We note that T^{n-1} is not the zero map on W because $T^{n-1}v \neq 0$. Hence $\dim W > n-1$, so that $v, Tv, \ldots, T^{n-1}v$ must be linearly independent. □

例 6.11. 利用推论 6.9，我们计算表 1 中 V_d 关于 $d \leq 8$ 的动机。对于 $d \leq 3$ 的情形，可以直接根据定义 (6.6) 进行验证；参见 §7.1。

d	$[V_d]$
0	1
1	1
2	$\mathbb{L}^2$
3	$3\mathbb{L}^4 - 2\mathbb{L}^3$
4	$2\mathbb{L}^8 + 3\mathbb{L}^7 - 5\mathbb{L}^6 + \mathbb{L}^5$
5	$10\mathbb{L}^{12} - 5\mathbb{L}^{11} - 9\mathbb{L}^{10} + 5\mathbb{L}^9$
6	$5\mathbb{L}^{18} + 21\mathbb{L}^{17} - 30\mathbb{L}^{16} - 9\mathbb{L}^{15} + 15\mathbb{L}^{14} - \mathbb{L}^{12}$
7	$35\mathbb{L}^{24} + 7\mathbb{L}^{23} - 84\mathbb{L}^{22} + 15\mathbb{L}^{21} + 35\mathbb{L}^{20} - 7\mathbb{L}^{18}$
8	$14\mathbb{L}^{32} + 112\mathbb{L}^{31} - 112\mathbb{L}^{30} - 162\mathbb{L}^{29} + 113\mathbb{L}^{28} + 70\mathbb{L}^{27} - 7\mathbb{L}^{26} - 28\mathbb{L}^{25} + \mathbb{L}^{22}$

TABLE 1. $[V_d]$ 的动机（针对 $d \leq 8$）

We observe from the table that as a polynomial in $\mathbb{L}$, the total coefficient of $[V_d]$ is 1. Indeed, if we substitute $\mathbb{L} \mapsto 1$ in Corollary 6.9, we have $[V_{2d,0}]|_{\mathbb{L}\mapsto 1} = 1$ and $[V_{a,b}]|_{\mathbb{L}\mapsto 1} = 0$ if $(a,b) \neq (2d,0)$, so $[V_d]|_{\mathbb{L}\mapsto 1} = 1$. Geometrically speaking, this means that when $k = \mathbb{C}$, the Euler characteristic of V_d in the analytic topology is 1, which is expected because the obvious $\mathbb{C}^\times$-action $t \cdot (X,Y) = (tX, tY)$ on V_d has only one fixed point $(X,Y) = (0,0)$.

7. 显式计算（\(d \leq 3\)）

We assume again that $R = k[[T^2, T^3]]$ and $H_d(t) := H_{d,R}(t)$. We compute $H_d(t)$ explicitly for $d \leq 3$ and prove the second part of Theorem 1.8 using Theorem 5.14. As before, we present our proof in terms of point counting over $k = \mathbb{F}_q$. The same proof implies the motivic formulas in Theorem 1.8 since the motivic version of Theorem 5.14 holds according to Appendix B.

7.1. 纯-K 分层。当 $d \leq 3$ 和 α 是纯-K 首项数据时，描述集合 $\mathrm{Hilb}^0(\alpha)$（事实上，即 (6.5) 中簇 $V(\alpha)$ 的约化结构）并不困难。我们给出一个典型的例子。

例 7.1. 设 $\alpha = (K(0)\ K(2)\ K(9))$。那么 $V(\alpha)$ 由满足以下条件的 (X,Y) 组成。

$$X = \begin{bmatrix} 0 & X_{12} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & Y_{12} & Y_{13} \\ 0 & 0 & Y_{23} \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{12} = 0 \tag{7.1}$$

而矩阵方程 $X^2 = Y^3, XY = YX$ 等价于

$$X_{12}X_{23} = 0, \quad X_{12}Y_{23} = Y_{12}X_{23}. \tag{7.2}$$

由于 $X_{12} = 0$，唯一的要求是 $Y_{12}X_{23} = 0$。因此 $[V(\alpha)] = (2\mathbb{L} - 1)\mathbb{L}^3$，其中 $\mathbb{L}^3$ 由三个自由选取的变量 X_{13}、Y_{13}、Y_{23} 得到。

According to (6.5), the variety $V(\alpha)$ only depends on the distance matrix $\delta_{(ij)}(\alpha)$ of α. Moreover, for each $i < j$, the exact distance $\delta_{(ij)}(\alpha)$ does not matter; all that matters is whether it is at most 1 (labeled 1^-), or 2, or at least 3 (labeled 3^+). For example, the above example corresponds to the case $(\delta_{(12)}(\alpha)\ \delta_{(23)}(\alpha)\ \delta_{(13)}(\alpha)) = (2, 3^+, 3^+)$. We provide the motive of $V(\alpha)$ for every possible distance matrix in $d \leq 3$. If $d = 0, 1$, then $V(\alpha)$ is just a point, so $[V(\alpha)] = 1$. If $d = 2, 3$, see Tables 2 and 3. Note that the last rows of Tables 2 and 3 verify the $d = 2, 3$ entries of Table 1.

$\delta_{(12)}(\alpha)$	$[V(\alpha)]$
1^-	1
2	$\mathbb{L}$
3^+	$\mathbb{L}^2$

TABLE 2. 关于距离的纯-K 分层，$d = 2$

7.2. $d \leq 3$ 时 $H_d(t)$ 的计算。

Case $d = 0$. Clearly $H_d(t) = Q_d(t) = 1$ if $d = 0$.

For $d = 1, 2, 3$, we shall compute $H_d(t)$ by first decomposing the set Ξ of leading term data into a disjoint union of stable orbits, and then computing the total contribution of each stable orbit using Lemma 5.12. The existence of a decomposition is guaranteed by Theorem 5.14, but a more efficient decomposition is often available. We will need to compute the contents of finitely many leading term data along the way. Using (5.18), this is always doable as long as we know $\mathrm{Hilb}^0(\alpha)$ for any pure-K leading term data up to rank 3. From now on, we will provide the quantities $|\mathrm{Hilb}(\alpha)|, A(\alpha)\ B(\alpha)\ D(\alpha), \mathrm{Cont}(\alpha)$ as needed without proof.

如果将 q 替换为 $\mathbb{L}$，那么本节中的所有公式在动机意义下也成立，因为前面的讨论表明，当 α 是秩至多为 3 的纯-K 时，$[V(\alpha)]$ 是关于 $\mathbb{L}$ 的多项式。

$(\delta_{(12)}(\alpha)\ \delta_{(23)}(\alpha)\ \delta_{(13)}(\alpha))$	$[V(\alpha)]$
$(1^-, 1^-, 1^-)$	1
$(1^-, 1^-, 2)$	$\mathbb{L}$
$(1^-, 1^-, 3+)$	$\mathbb{L}^2$
$(1^-, 2, 2)$	$\mathbb{L}^2$
$(1^-, 2, 3+)$	$\mathbb{L}^3$
$(1^-, 3+, 3+)$	$\mathbb{L}^4$
$(2, 1^-, 2)$	$\mathbb{L}^2$
$(2, 1^-, 3^+)$	$\mathbb{L}^3$
$(2, 2, 3^+)$	$\mathbb{L}^4$
$(2, 3^+, 3^+)$	$2\mathbb{L}^4 - \mathbb{L}^3$
$(3^+, 1^-, 3^+)$	$\mathbb{L}^4$
$(3^+, 2, 3^+)$	$2\mathbb{L}^4 - \mathbb{L}^3$
$(3^+, 3^+, 3^+)$	$3\mathbb{L}^4 - 2\mathbb{L}^3$

TABLE 3. 关于距离的纯-K 分层，$d = 3$

情形 $d = 1$。首项数据的集合 Ξ 是两个稳定轨道的不交并：$(K(0))$ 在 $\gamma_{(1)}$ 作用下稳定，而 $(J(1))$ 在 $\gamma_{(1)}$ 作用下稳定。我们有 $\mathsf{Cont}(K(0)) = 1$，且 $\mathsf{Cont}(J(1)) = qt$。因此

$$H_1(t) = \frac{1}{1-t} + \frac{qt}{1-t} = \frac{1+qt}{1-t}. \tag{7.3}$$

由 (2.41)，我们有

$$Q_1(t) = 1 + tH_1(t) = \frac{1+qt^2}{1-t}. \tag{7.4}$$

这与尖点奇点的局部希尔伯特 ζ 函数的著名公式一致；例如参见 [32, Example19, A_2]。

Case $d = 2$. We exhaust the elements of Ξ using the four grids in Example 5.4. We say an arrow is stable (marked as solid) if the leading term datum at the source is stable under the spiral raising operator the arrow represents, and unstable (marked as dashed) otherwise; see Figure 2. We introduce the notation Ξ_c to refer to the set of all $\alpha \in \Xi$ with color vector $c(\alpha) = c$. From this grid, we can read that $\Xi_{(K,K)}$ is a disjoint union of four stable orbits: $(K(0)\ K(0))$ under $\gamma_{(1)}$, $(K(0)\ K(1))$ under $\gamma_{(1)}$, $(K(0)\ K(2))$ under $\gamma_{(1)}$, and $(K(0)\ K(3))$ under $\gamma_{(1)}$, $\gamma_{(2)}$.

$$\begin{array}{ccccccc}
(K(0),K(0)) & \overset{\gamma_{(2)}}{\dashrightarrow} & (K(0),K(1)) & \dashrightarrow & (K(0),K(2)) & \dashrightarrow & (K(0),K(3)) \longrightarrow \cdots \\
\downarrow{\scriptstyle \gamma_{(1)}} & & \downarrow & & \downarrow & & \downarrow \\
(K(1),K(0)) & \dashrightarrow & (K(2),K(0)) & \dashrightarrow & (K(3),K(0)) & \dashrightarrow & (K(4),K(0)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(K(1),K(1)) & \dashrightarrow & (K(1),K(2)) & \dashrightarrow & (K(1),K(3)) & \dashrightarrow & (K(1),K(4)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
(K(2),K(1)) & \dashrightarrow & (K(3),K(1)) & \dashrightarrow & (K(4),K(1)) & \dashrightarrow & (K(5),K(1)) \longrightarrow \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

FIGURE 2. 颜色向量（K，K）中的稳定性

我们将这些信息记录在一个更简单的图示（图 3）中，其中只保留各轨道的起始点。同时，我们省略箭头 $\gamma_{(1)}$，并约定 Ξ 中的每个元素都是稳定的。

under$^{\gamma_{(1)}}$（参见引理 5.13）。因此，可以将稳定箭头理解为图中的实线箭头加上$^{\gamma_{(1)}}$。我们还标注了每个节点的内容。

$$\underset{1}{(K(0),K(0))} \overset{\gamma_{(2)}}{\dashrightarrow} \underset{qt}{(K(0),K(1))} \dashrightarrow \underset{q^3t^2}{(K(0),K(2))} \dashrightarrow \underset{q^5t^3}{(K(0),K(3))} \longrightarrow \cdots$$

FIGURE 3. (K,K)中的稳定性与内容

由此图可得

$$\begin{aligned}\sum_{\alpha\in\Xi_{(K,K)}} \mathrm{Cont}(\alpha) &= \frac{1}{1-t} + \frac{qt}{1-t} + \frac{q^3t^2}{1-t} + \frac{q^5t^3}{(1-t)(1-qt)} \\ &= \frac{1-q^2t^2+q^3t^2-q^4t^3+q^5t^3}{(1-t)(1-qt)}.\end{aligned} \tag{7.5}$$

其他三个颜色向量对应的类似图示见图4、图5和图6。

$$\underset{q^3t}{(K(0),J(1))} \overset{\gamma_{(2)}}{\longrightarrow} \cdots$$

$$\sum_{\alpha\in\Xi_{(K,J)}} \mathrm{Cont}(\alpha) = \frac{q^3t}{(1-t)(1-qt)}.$$

FIGURE 4. (K,J)中的稳定性与内容

$$\underset{q^2t}{(J(1),K(0))} \overset{\gamma_{(2)}}{\dashrightarrow} \underset{q^2t^2}{(J(1),K(1))} \longrightarrow \cdots$$

$$\sum_{\alpha\in\Xi_{(J,K)}} \mathrm{Cont}(\alpha) = \frac{q^2t}{1-t} + \frac{q^2t^2}{(1-t)(1-qt)}.$$

FIGURE 5. (J,K)中的稳定性与内容

$$\underset{q^4t^2}{(J(1),J(1))} \overset{\gamma_{(2)}}{\dashrightarrow} \underset{q^4t^3}{(J(1),J(2))} \longrightarrow \cdots$$

$$\sum_{\alpha\in\Xi_{(J,J)}} \mathrm{Cont}(\alpha) = \frac{q^4t^2}{1-t} + \frac{q^4t^3}{(1-t)(1-qt)}.$$

FIGURE 6. (J,J)中的稳定性与内容

综上所述，我们得到

$$H_2(t) = \sum_{\alpha\in\Xi} \mathrm{Cont}(\alpha) = \frac{1+q^2t+q^3t+q^4t^2}{(1-t)(1-qt)}, \tag{7.6}$$

其中需要注意的是，所有涉及 t^3 的项均已相互抵消。

由（2.41）式可得

$$Q_2(t) = 1 + (1+q)tH_1(qt) + t^2H_2(t) = \frac{1+q^2t^2+q^3t^2+q^4t^4}{(1-t)(1-qt)}, \tag{7.7}$$

其中我们注意到意外的 反复地， Q_2（t）的分子是将 H_2（t）代入 $t \mapsto t^2$ 后得到的分子。

Case $d = 3$. We now compute $H_3(t)$ by constructing similar diagrams for all eight color vectors, see Figures 7 through 14. We put $\gamma_{(2)}$ on verticle arrows, $\gamma_{(3)}$ on horizontal arrows, and omit $\gamma_{(1)}$ with the understanding that every node is stable under $\gamma_{(1)}$. To save space, we also omit the first component in the grid, which is always $K(0)$ when the first component of the color vector is K, and $J(1)$ otherwise. For example, in color (J, K, J), the node $(J(1)\ J(2)\ K(0))$ is denoted by $(J(2)\ K(0))$. For ease of verification, we label the content of each node in the form

$$\mathrm{Cont}(\alpha) = A(\alpha) \cdot B(\alpha) \cdot D(\alpha)t^{n(\alpha)}. \tag{7.8}$$

We note that $A(\alpha)$ is generically unchanged by $\gamma_{(j)}$, unless the distance between some K components are stretched from 1 to 2 or from 2 to 3. The factor $B(\alpha)$ stays the same on each grid, and it only depends on the color vector (Lemma 5.5). The factor $D(\alpha)$ is generically multiplied by q^{j-1} under $\gamma_{(j)}$, unless some stretches are obstructed (see Lemma 5.7). The exponent $n(\alpha)$ is always increased by 1 under $\gamma_{(j)}$. The top-left node $\mathbf{0}_c$ of each color vector c only consists of $K(0)$ and $J(1)$ components, and it has $A(\mathbf{0}_c) = 1$, $D(\mathbf{0}_c) = q^{d\#_J(c)} = q^{3\#_J(c)}$, and $n(\mathbf{0}_c) = \#_J(c)$, where $\#_J(c)$ is the number of J's in c. One can write down every factor but $A(\alpha)$ by starting from the top-left node and tracing the arrows using the rules above. For $A(\alpha)$, we consult Tables 2 and 3.

为了获得 $H_3(t)$ 的有理公式，我们将每个网格分解为若干个由包围框标记的不相交稳定轨道的并。注意，我们的分解不同于定理 5.14 中的分解，因为我们利用了引理 5.13 并未保证存在的稳定箭头；这样做仅仅是为了减少计算量。还需注意，图 12 中“最高效”分解的选择并非规范的。每个轨道对 $H_3(t)$ 的贡献由引理 5.12 计算。例如，图 7 右上角的方框贡献为 $q^9t^4(1-t)(1-q^2t)$。

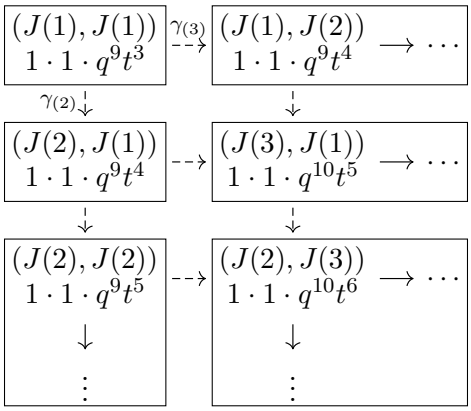

FIGURE 7. (J, J, J) 中的稳定性与内容

将八个图表中 78 个轨道的贡献相加，我们得到

$$H_3(t) = \frac{1+q^3t+q^4t+q^5t+q^6t^2+q^7t^2+q^8t^2+q^9t^3}{(1-t)(1-qt)(1-q^2t)} \tag{7.9}$$

由（2.41）可得

$$Q_3(t) = \frac{1+q^3t^2+q^4t^2+q^5t^2+q^6t^4+q^7t^4+q^8t^4+q^9t^6}{(1-t)(1-qt)(1-q^2t)}, \tag{7.10}$$

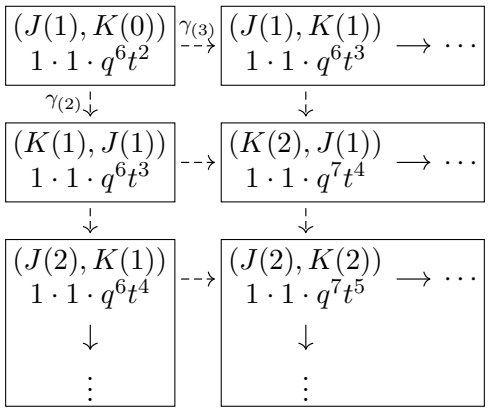

FIGURE 8. (J,J,K) 中的稳定性与内容

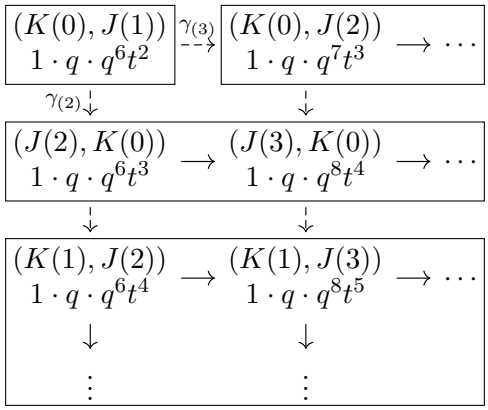

FIGURE 9. (J,K,J) 中的稳定性与内容

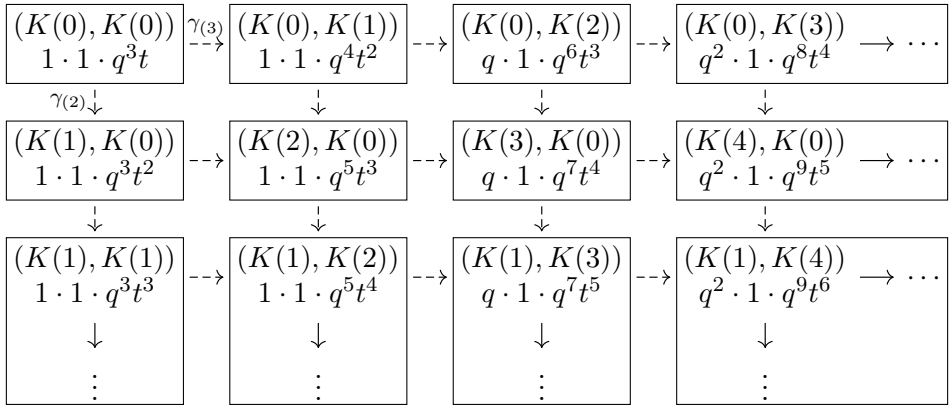

FIGURE 10. (J,K,K)中的稳定性与内容

这里再次指出， $Q_3(t)$ 的分子是将 $H_3(t)$ 在 $t\mapsto t^2$ 处取值所得的分子。

$H_3(t)$ 的公式之简洁令人瞩目。就它们对 $H_3(t)$ 分子贡献的 t-次数而言，各个轨道的贡献最高可达 t^9，而一个颜色向量的总贡献最高可达 t^7，但 $H_3(t)$ 分子的 t-次数却仅为 3。参见

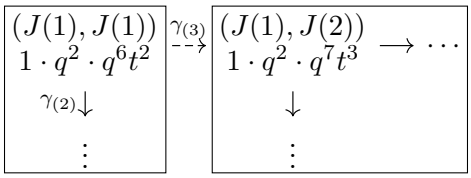

FIGURE 11. (K,J,J)中的稳定性与内容

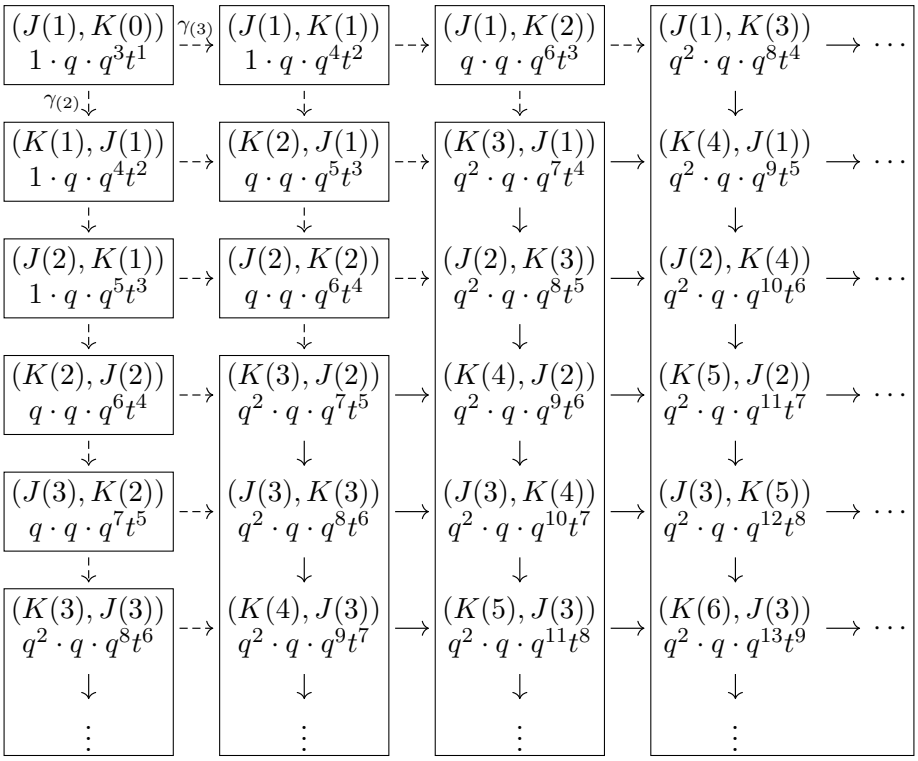

FIGURE 12. (K,J,K)中的稳定性与内容

各个颜色中 $H_3(t)$ 的分解见表 4，其中我们也标出了计算中使用的轨道数。我们将内容乘以 $(t;q)_3=(1-t)(1-qt)(1-q^2t)$，以提取 $H_3(t)$ 的分子中的贡献。

7.3. 一般 d 的猜想公式。沿用定理 1.8 中的记号，令 $\mathrm{NH}_d(t)=(t;q)_dH_d(t)$ 和 $\mathrm{NQ}_d(t)=(t;q)_dQ_d(t)$。根据定理 5.14，现在已知它们是关于 t 的多项式。利用 (2.41)，我们将猜想 1.4(b) 与猜想性的 (1.25) 转化为适用于 $\mathrm{NH}_d(t)$ 的一个版本。

猜想 7.2. 我们有

(a) $NH_d(t^2)=\sum_{r=0}^{d}t^r\begin{bmatrix}d\\r\end{bmatrix}_q(t;q)_{d-r}NH_r(tq^{d-r})$;

(b) (*Functional Equation*) $q^{d^2}t^dNH_d(q^{-2d}t^{-1})=NH_d(t)$.

我们刚刚验证了上述猜想对 $d\le 3$ 成立。事实证明，猜想 7.2 足够强，能够对所有 d 的 $\mathrm{NH}_d(t)$ 施加过多约束。下面这个关于 $d=4$ 的例子将展示如何归纳地求出 $\mathrm{NH}_d(t)$，并验证猜想 7.2 对 d 在给定界限内的一致性。

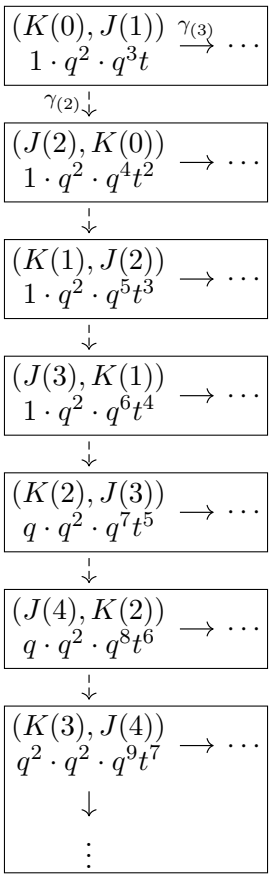

FIGURE 13. (K, K, J) 中的稳定性与内容

c	# 轨道（t; q） $_3\sum_{\alpha\in\Xi_c}$ Cont(α)
(J, J, J)	6	$q^6t^2 + 2q^6t^3 - q^7t^3 - q^8t^3 + q^6t^4 - q^7t^4 - q^8t^4 + q^9t^4 + q^7t^5 - 2q^8t^5 + q^9t^5$
(J, J, K)	6	$q^9t^3+2q^9t^4-q^{10}t^4-q^{11}t^4+q^9t^5-q^{10}t^5-q^{11}t^5+q^{12}t^5+q^{10}t^6-2q^{11}t^6+q^{12}t^6$
(J, K, J)	4	$q^7t^2 + q^7t^3 - q^9t^3 + q^7t^4 - q^8t^4 - q^9t^4 + q^{10}t^4$
(J, K, K)	12	$q^3t + q^3t^2 - q^5t^2 + q^3t^3 - q^4t^3 - q^5t^3 + q^7t^3 - q^9t^4 + q^{10}t^4 - q^7t^5 + 2q^8t^5 - q^9t^5 - q^{10}t^6 + 2q^{11}t^6 - q^{12}t^6$
(K, J, J)	2	$q^8t^2 + q^9t^3 - q^{10}t^3$
(K, J, K)	13	$q^4t + q^5t^2 - q^6t^2 - q^6t^3 + q^8t^3 - q^7t^4 + 2q^8t^4 - 2q^9t^4 + q^{11}t^4 + q^{11}t^5 - q^{12}t^5 - q^{10}t^6 + 2q^{11}t^6 - q^{12}t^6 + q^{12}t^7 - q^{13}t^7$
(K, K, J)	7	$q^5t - q^9t^5 + q^{10}t^5 - q^{12}t^7 + q^{13}t^7$
(K, K, K)	28	$1 - q^3t^2 + q^6t^2 - q^3t^3 + q^4t^3 + q^5t^3 - q^6t^3 - q^7t^3 + q^{10}t^3 - q^6t^4 + q^7t^4 + q^9t^4 - q^{10}t^4 + q^{10}t^6 - 2q^{11}t^6 + q^{12}t^6$
总计	78	$1 + q^3t + q^4t + q^5t + q^6t^2 + q^7t^2 + q^8t^2 + q^9t^3$

TABLE 4. $H_3(t)$ 来自每种颜色

假设猜想 7.2 成立。猜想 7.2(a) 随即推出 NH $_d(t)$ 的常数项为 1。根据猜想 7.2(b)， NH $_d(t)$ 关于 t 的首项为 $q^{d^2}t^d$。

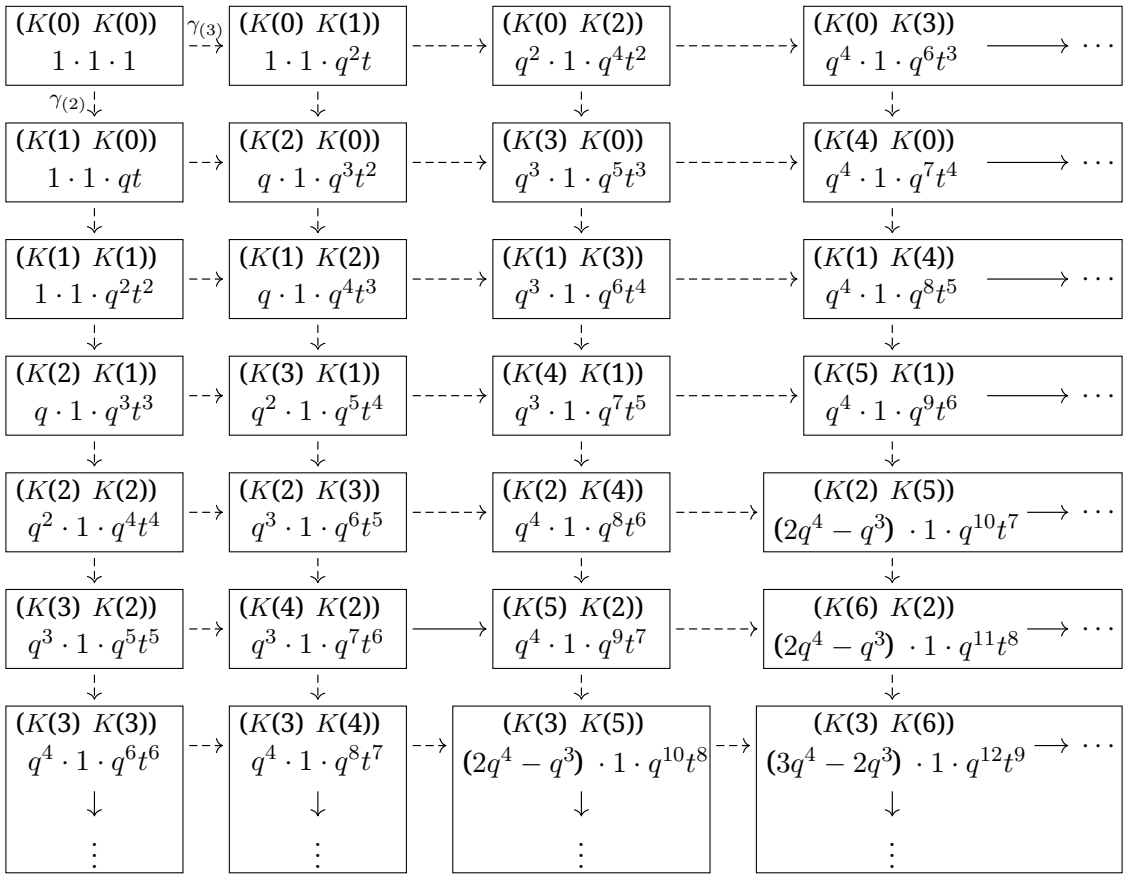

FIGURE 14. (K, K, K)中的稳定性与内容

对于任意含幺交换环 A，设 $A_r[t]$ 为次数至多为r的 A多项式构成的 A-模。考虑由下式定义的 A-线性映射 Θ_d： $A_d[t] \to A_{2d-1}[t]$

$$\Theta_d(f(t)) := f(t^2) - t^d f(t). \tag{7.11}$$

那么，猜想 7.2(a) 可以改写为

$$\Theta_d(NH_d(t)) = \sum_{r=0}^{d-1} t^r \begin{bmatrix} d \\ r \end{bmatrix}_q (t;q)_{d-r} NH_r(tq^{d-r}). \tag{7.12}$$

例 7.3. 当 $d = 4$ 时，根据 $d \le 3$ 的已知表达式，方程 (7.12) 变为

$$\Theta_4(NH_4(t)) = 1 + (q^4+q^5+q^6+q^7)t^2 + (-1+q^8+q^9+2q^{10}+q^{11}+q^{12})t^4 + (-q^4-q^5-q^6-q^7)t^5 + (-q^8-q^9-2q^{10}-q^{11}+q^{13}+q^{14}+q^{15})t^6 + (-q^{12}-q^{13}-q^{14}-q^{15})t^7. \tag{7.13}$$

匹配 $\Theta_4(\mathrm{NH}_4(t))$ 的 t^4 至 t^7 次系数，$\mathrm{NH}_4(t)$ 的唯一可能性 is

$$\mathrm{NH}_4(t) = 1+(q^4+q^5+q^6+q^7)t+(q^8+q^9+2q^{10}+q^{11}+q^{12})t^2+(q^{12}+q^{13}+q^{14}+q^{15})t^3+q^{16}t^4. \tag{7.14}$$

这里，t^4 的系数由函数方程确定，而不是由 (7.13) 确定。可以验证，$\mathrm{NH}_4(t)$ 确实满足 (7.13) 和猜想 7.2(b)。因此，我们得出结论：(7.14) 是唯一一个使猜想 7.2 在 $d \le 4$ 之前保持一致的 $\mathrm{NH}_4(t)$。

我们还注意到，$\mathrm{NH}_4(t)$ 是关于 q、t 的一个多项式，尽管我们并未作出这样的假设。

在猜想 7.2 成立的假设下，我们能够确定 $\mathrm{NH}_d(t)$ 的精确公式。

命题 7.4. 假设 A 是一个含幺交换环，且 q 是 A 的一个元素。那么存在唯一的多项式 $\mathrm{NH}_d(t) \in A[t]$，对于 $d \geq 0$，使得

(a) $NH_0(t) = 1$;

(b) $NH_d(t^2) = \sum_{r=0}^{d} t^r \begin{bmatrix} d \\ r \end{bmatrix}_q (t;q)_{d-r} NH_r(tq^{d-r})$;

(c) (*Functional Equation*) $NH_d(t) = \sum_{i=0}^{d} a_i t^i$, *with* $a_{d-i} = q^{d(d-2i)} a_i$ *for* $0 \leq i \leq \lfloor d/2 \rfloor$.

此外，$\mathrm{NH}_d(t)$ 由一个关于 t, q 的具有非负整数系数的多项式 $\mathrm{NH}_d(t;\, q)$ 给出，其定义如下:

$$NH_d(t;q) := \sum_{j=0}^{d} \begin{bmatrix} d \\ j \end{bmatrix}_q (q^d t)^j \tag{7.15}$$

此外，设 $H_d(t;\, q) := \mathrm{NH}_d(t;\, q)/(t;\, q)_d$，且 $\widehat{Z}(t;\, q) = \sum_{d=0}^{\infty} \underline{q^{-d^2} t^d (q^{-1};\, q^{-1})_d} H_d(q^{-d}t;\, q)$（比较式 (2.40)），则

$$\widehat{Z}(t;q) = \frac{1}{(tq^{-1};q^{-1})_\infty} \sum_{n=0}^{\infty} \frac{q^{-n^2} t^{2n}}{(q^{-1};q^{-1})_n}. \tag{7.16}$$

证明。唯一性的部分已在前文论证。为了证明存在性，只需按照 (7.15) 定义 $\mathrm{NH}_d(t;\, q)$，并证明 $\mathrm{NH}_d(t) := \mathrm{NH}_d(t;\, q)$ 满足方程 (a)(b)(c) 和 (7.16)。

方程 (a)(c) 是显然的。现在我们从右侧开始证明 (b)。我们将使用变量代换 $b = r - j$:

$$\sum_{r=0}^{d} t^r \begin{bmatrix} d \\ r \end{bmatrix}_q (t;q)_{d-r} NH_r(tq^{d-r}) \tag{7.17}$$

$$= \sum_{0 \leq j \leq r \leq d} t^r (q^d t)^j (t;q)_{d-r} \begin{bmatrix} d \\ r \end{bmatrix}_q \begin{bmatrix} r \\ j \end{bmatrix}_q \tag{7.18}$$

$$= \sum_{j=0}^{d} t^{2j} q^{dj} \begin{bmatrix} d \\ j \end{bmatrix}_q \sum_{b=0}^{d-j} t^b (t;q)_{d-j-b} \begin{bmatrix} d-j \\ b \end{bmatrix}_q. \tag{7.19}$$

现在令 $m = d - j$，只需验证 $\sum_{b=0}^{m} t^{m-b} (t;q)_b \begin{bmatrix} m \\ b \end{bmatrix}_q = 1$。令 $a = m - b$，则有

$$\sum_{b=0}^{m} t^{m-b} (t;q)_b \begin{bmatrix} m \\ b \end{bmatrix}_q = (q;q)_m \sum_{a+b=m} \frac{t^a}{(q;q)_a} \frac{(t;q)_b}{(q;q)_b} \tag{7.20}$$

$$= (q;q)_m [z^m] \left(\sum_{a=0}^{\infty} \frac{(tz)^a}{(q;q)_a} \sum_{b=0}^{\infty} \frac{(t;q)_b}{(q;q)_b} z^b \right), \tag{7.21}$$

其中 $[z^m]$ 表示提取 z^m 系数。

根据欧拉恒等式 [4, 推论 2.2] 和柯西 [4, 定理 2.1], ，我们有

$$\sum_{a=0}^{\infty} \frac{(tz)^a}{(q;q)_a} = \frac{1}{(tz;q)_\infty} \tag{7.22}$$

and

$$\sum_{b=0}^{\infty} \frac{(t;q)_b}{(q;q)_b} z^b = \frac{(tz;q)_\infty}{(z;q)_\infty}. \tag{7.23}$$

再次根据欧拉恒等式，我们有

$$[z^m] \left(\sum_{a=0}^{\infty} \frac{(tz)^a}{(q;q)_a} \sum_{b=0}^{\infty} \frac{(t;q)_b}{(q;q)_b} z^b \right) = [z^m] \frac{1}{(z;q)_\infty} = \frac{1}{(q;q)_m}, \tag{7.24}$$

因此结论成立，(b) 的证明也就完成了。

最后，我们证明 (7.16)，其中将使用变量替换 $i = d - j$ 和 $\begin{bmatrix} d \\ j \end{bmatrix}_q = q^{ij} \begin{bmatrix} d \\ j \end{bmatrix}_{q^{-1}}$：

$$\widehat{Z}(t;q) = \sum_{d=0}^{\infty} \frac{q^{-d^2} t^d}{(q^{-1};q^{-1})_d} H_d(q^{-d}t;q) \tag{7.25}$$

$$= \sum_{j=0}^{\infty} \frac{t^{2j} q^{-j^2}}{(q^{-1};q^{-1})_j} \sum_{i=0}^{\infty} \frac{q^{-i(i+j)}}{(q^{-1};q^{-1})_i (tq^{-1};q^{-1})_{i+j}}. \tag{7.26}$$

But

$$\sum_{i=0}^{\infty} \frac{q^{-i(i+j)}}{(q^{-1};q^{-1})_i (tq^{-1};q^{-1})_{i+j}} = \frac{1}{(tq^{-1};q^{-1})_\infty} \tag{7.27}$$

采用涉及分拆的 j-Durfee 矩形的标准论证；例如参见 [25]。 □

推论 7.5. 猜想 7.2 的动机版本蕴含猜想 1.9。

证明。将命题 7.4 应用于 $A = K_0(\mathrm{Var}_k)$ 和 $q = \mathbb{L}$。 □

The above lemma suggests a "conceptual proof" of Conjecture 1.9 without explicit computations and without knowing in advance whether the relevant motives are polynomials of $\mathbb{L}$. Since Conjecture 1.9 implies Conjecture 1.10, this provides a plan to attack Conjecture 1.10 without computing the Gröbner strata and proving Conjecture 6.1. Of course, Conjecture 1.10 does not imply the stratum-wise polynomiality statement of Conjecture 6.1, so the latter conjecture would not be obsolete even if the former is proved to be true.

我们对公式 (7.16) 再作进一步说明。令

$$F(t;q) = \sum_{n=0}^{\infty} \frac{q^{n^2} t^n}{(q;q)_n} \tag{7.28}$$

是罗杰斯–拉曼努金级数（见 [4, 第7章1]），那么 $\widehat{Z}_R(t) = \frac{1}{(tq^{-1};q^{-1})_\infty} F(t^2, q^{-1})$，因此罗杰斯–拉曼努金恒等式将意味着

$$\widehat{Z}_R(1) = \frac{1}{(q^{-1};q^{-1})_\infty (q^{-1};q^{-5})_\infty (q^{-4};q^{-5})_\infty}; \tag{7.29}$$

$$\widehat{Z}_R(-1) = \frac{(q^{-1};q^{-2})_\infty}{(q^{-1};q^{-5})_\infty (q^{-4};q^{-5})_\infty}. \tag{7.30}$$

If true, this would provide another evidence for an observation in [38] that $\widehat{Z}_R(\pm 1)$ tend to have special values that admit infinite product formulas, if R is the local ring at a point on a curve. The conjectured formula (7.16) also gives the first guess for the number of matrix pairs $\{(A,B) \in \mathrm{Mat}_n(\mathbb{F}_q)^2 : AB = BA, A^2 = B^3\}$. Via (2.6) and (2.9), our guess (7.16) is equivalent to

$$\left|\{(A,B) \in \mathrm{Mat}_n(\mathbb{F}_q)^2 : AB = BA, A^2 = B^3\}\right| = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j q^{\frac{1}{2}(3j^2 - j) + n(n-2j)} \frac{(q;q)_n}{(q;q)_j (q;q)_{n-2j}}. \tag{7.31}$$

与其节点型对应情形（目前已有两种通过 [24, 38] 矩阵计数 $AB = BA = 0$给出的证明）不同，矩阵计数问题（7.31）尚不知晓有直接的求解方法。

我们对式 (7.15) 中定义的 $\mathrm{NH}_d(t;\, q)$ 再作一项观察。

命题 7.6. 设 ζ_r 是 $\mathbb{C}$ 中的一个本原 r 次单位根。那么

$$NH_d(t;\zeta_r) = (1+t^r)^{d/r} = NH_1(t^r;1)^{d/r}. \tag{7.32}$$

如果 r 整除 d。等价地，对于任意 $i \in \mathbb{Z}$，令 $e = \gcd(i,d)$，则有

$$NH_d(t;\zeta_d^i) = (1+t^{d/e})^e. \tag{7.33}$$

证明。我们首先断言，$\mathrm{NH}_d(t;\,q)$ 等于一个“移位的 Pochhammer 符号”：

$$NH_d(t;q) = \sum_{j=0}^{d} q^{\binom{j+1}{2}+j(d-j)} c_j(q) t^j, \tag{7.34}$$

其中 $\sum_{j=0}^{d} c_j(q)t^j = (-t;\,q)_d$。为了验证这一结论，我们只需注意到，根据柯西二项式定理，$c_j(q) = q^{\binom{j}{2}} d\ brack j_q(-t;q)_d = \sum_{j=0}^{d} q^{\binom{j}{2}} d\ brack j_q$。

假设 $r>0$ 整除 d。将 $q=\zeta_r$ 代入 (7.15)，我们得到

$$NH_d(t;\zeta_r) = \sum_{j=0}^{d} \zeta_r^{\binom{j+1}{2}+j(d-j)} c_j(\zeta_r) t^j, \tag{7.35}$$

其中 $\sum_{j=0}^{d} c_j(\zeta_r)t^j = (-t;\,\zeta_r)_d$。由于 r 整除 d，我们有 $\zeta_r^d = 1$，所以因子 $\zeta_r^{\binom{j+1}{2}+j(d-j)}$ 就是 $\zeta_r^{\binom{j}{2}}$。利用

$$(t;\zeta_r)_d = ((1-t)(1-\zeta_r t)\dots(1-\zeta_r^{r-1}t))^{d/r} = (1-t^r)^{d/r} \tag{7.36}$$

及替代。代入 $t \mapsto -t$，我们得到 $(-t;\,\zeta_r)_d = (1-(-t)^r)^{d/r}$，因此当 $r \nmid j$ 时，$c_j(\zeta_r) = 0$，并且

$$c_{ir}(\zeta_r) = \binom{d/r}{i}(-(-1)^r)^i = \binom{d/r}{i}(-1)^{i(r-1)} \tag{7.37}$$

对于 $0 \le i \le d/r$。我们有

$$NH_d(t;\zeta_r) = \sum_{i=0}^{d/r} \zeta_r^{-\binom{ir}{2}} \binom{d/r}{i} (-1)^{i(r-1)} t^{ir} \tag{7.38}$$

$$= \sum_{i=0}^{d/r} \zeta_{2r}^{-2\binom{ir}{2}} \binom{d/r}{i} \zeta_{2r}^{r\cdot i(r-1)} t^{ir} \tag{7.39}$$

$$= \sum_{i=0}^{d/r} \zeta_{2r}^{-ir(ir-1)+ir(r-1)} \binom{d/r}{i} t^{ir} \tag{7.40}$$

$$= \sum_{i=0}^{d/r} \zeta_{2r}^{-i(i-1)r^2} \binom{d/r}{i} t^{ir}. \tag{7.41}$$

由于 $i(i-1)$ 是偶数，$\zeta_{2r}^{-i(i-1)r^2} = 1$，因此我们有

$$NH_d(t;\zeta_r) = \sum_{i=0}^{d/r} \binom{d/r}{i} t^{ir} = (1+t^r)^{d/r}. \tag{7.42}$$

□

We note a consequence that has potential generalizations outside the cusp case. Let $NQ_d(t;q) = NH_d(t^2;q)$ and $Q_d(t;q) = NQ_d(t;q)/(t;q)_d$. Then $NQ_d(t;q)$ clearly satisfies an analogous property $NQ_d(t;\zeta_r) = NQ_1(t^r;1)^{d/r}$ for $r|d$. Because the q,t-polynomial $(t;q)_d$ satisfies a similar property $(t;\zeta_r)_d = (1-t^r)^{d/r} = (t^r;1)_1^{d/r}$ when $r|d$, we have

$$\begin{aligned} H_d(t;\zeta_r) &= H_1(t^r;1)^{d/r};\\ Q_d(t;\zeta_r) &= Q_1(t^1;1)^{d/r} \end{aligned} \tag{7.43}$$

如果 r 能整除 d。

对于曲线上的一个光滑有理点，利用 $H_{d,\ \mathbb{F}_q[[T]]}(t) = Q_{d,\ \mathbb{F}_q[[T]]}(t) = 1/(t;q)_d$ 以及上文关于 $(t;q)_d$ 的性质，也有类似的结论。若猜想 1.9 中的公式成立，那么上述命题的 $r=1$ 情形将推出：如果 $k=\mathbb{C}$，且 X 是仅有尖点奇点的约化曲线，则商概形 $\mathrm{Quot}_{d,\ n}(X)$ 的欧拉示性数满足

$$\sum_{n=0}^{\infty}\chi(\mathrm{Quot}_{d,n}(X))t^n=\left(\sum_{n=0}^{\infty}\chi(\mathrm{Hilb}_n(X))t^n\right)^d. \tag{7.44}$$

For a reduced curve X with planar singularities, the right-hand side is known in terms of the HOMFLY polynomial of the associated algebraic link. Therefore, if (7.44) is true for X, it would give a formula for the Euler characteristics of $\mathrm{Quot}_{d,n}(X)$. The truth of (7.44) for the cusp singularity is unknown, and one may attempt it as another weak form of Conjecture 1.9.

我们以关于 $\mathrm{NH}_d(t;q)$ 在有理数域上的因式分解的一条有趣观察作为本节的结尾，这一观察已针对 $d\le 30$ and $t=\pm1,\pm2$ 得到验证。在下述陈述中，我们也称常数多项式 1 为不可约。

猜想 7.7. 定义

$$P_d(t;q):=\begin{cases}NH_d(t;q), & \text{if } d \text{ is even;}\\ NH_d(t;q)/(1+q^dt), & \text{if } d \text{ is odd.}\end{cases} \tag{7.45}$$

Then

(a) $P_d(t;q)$ 是关于 $\mathbb{Z}[t,q]$ 的一个具有非负系数的多项式，且它是不可约的（或为 1）；(b) 对于任意整数 $m\neq -1$，多项式 $P_d(m;q)$ 在 $\mathbb{Z}[q]$ 中是不可约的；(c) 多项式 $P_d(-1;q)$ 具有如下形式

$$P_d(-1;q)=I_d(q)\prod_{\substack{1\le r\le d\\ r\ odd}}\Phi_r(q)^{\lfloor\frac{d+r-1}{2r}\rfloor}, \tag{7.46}$$

其中，$I_d(q)$ 是 $\mathbb{Z}[q]$ 中的一个不可约多项式，而 $\Phi_r(q)$ 是 r 阶分圆多项式。

换言之，当 m 是一个不同于 -1 的整数时，q 中的多项式 $\mathrm{NH}_d(m;q)$ 只有由 $\mathrm{NH}_d(t;q)$ 的一般因式分解所决定的因式分解。如果 $m=-1$，那么 $\mathrm{NH}_d(m;q)$ 含有大量奇数次数的分圆因子，但除此之外没有其他因子。

A. Appendix: Proofs in the Gröbner theory for monomial subrings

我们在 Ω 是域 k 上的幂级数环这一情形下证明 §3 中的结论。注意，较低单项式是首项，即 $<=\prec$。回顾一下，$R\subseteq\Omega$ 是一个单项式子环，并且 $F=Ru_1\oplus\cdots\oplus Ru_d$ 是一个自由的 R-模。我们在 F 上有一个单项式序 $<$。

命题 A.1（命题 3.3）。设 f、g_1、…、g_h 是 F 的元素。那么存在一个（不一定唯一的）表达式

$$f=r+\sum_{i=1}^{h}q_ig_i \tag{A.1}$$

其中 $r\in F$、$q_i\in R$ 满足这样的条件：

- r 中没有任何项能被任何 i 的 LT（g_i）整除。
- 首项$(q_ig_i)\succeq$首项(f) 对于任意 i 均成立，即，没有 q_ig_i 包含位于（即 $\prec$）首项(f) 之前的项。

证明。假设 f、g_1、……、$g_h\in F-\{0\}$，不失一般性。若 f 中没有任何一项能被任意 $\mathrm{LT}(g_i)$ 整除，则取 $r=f$、$q_i=0$ 即可完成证明。否则，我们令 $r=r^{(1)}=f$、$q_i=q_i^{(1)}=0$ 作为初始值，并逐步修改表达式 $f=r+\sum q_ig_i$。其基本思路是选取一个 g_j

并使用 LT(g_j) 同时消去 r 的项，然后我们循环进行，使得每个 j 都会被无穷多次选取。在步骤 l 中，令 $j = j_l$ 满足 $j \equiv l \bmod h$。令 s 表示所有能被 LT(g_j) 整除的项之和。我们按照以下方式修改 q_i 和 r。

$$\begin{aligned} q_j^{(l+1)} &= q_i^{(l)} + \frac{s}{\mathrm{LT}(g_j)}, \\ q_i^{(l+1)} &= q_i^{(l)}, \quad i \neq j, \\ r^{(l+1)} &= r^{(l)} - \frac{s}{\mathrm{LT}(g_j)} g_j. \end{aligned} \tag{A.2}$$

该构造 pr 保留性质 $f = r + \sum_{i=1}^{h} q_i g_i$，且 LT($q_i g_i$) $\succeq$ LT(f)。如果 $s \neq 0$，重写

$$r^{(l+1)} = r^{(l)} - \frac{s}{\mathrm{LT}(g_j)} \mathrm{LT}(g_j) - \frac{s}{\mathrm{LT}(g_j)} (g_j - \mathrm{LT}(g_j)), \tag{A.3}$$

我们可以将从 $r^{(l)}$ 到 $r^{(l+1)}$ 的过程理解为：首先消去 s 中的项，然后再引入一些具有 $-\frac{s}{\mathrm{LT}(g_j)}$ ($g_j - \mathrm{LT}(g_j)$) 的项。一些重新引入的单项式可能与 s 中出现的单项式相同，但我们始终有

$$\mathrm{LT}\left(-\frac{s}{\mathrm{LT}(g_j)} (g_j - \mathrm{LT}(g_j)) \right) \succ \mathrm{LT}(s), \tag{A.4}$$

根据单项式序的定义性质。因此，项 LT(s) 被消去，但不会重新引入。

We claim that $q_i^{(l)}$ and $r^{(l)}$ converge (in the usual power series topology in Ω or Ω^d) as $l \to \infty$. If not, then there exists a monomial $\mu \in F$ that is reintroduced (i.e., appears in $-\frac{s}{\mathrm{LT}(g_j)}(g_j - \mathrm{LT}(g_j))$) infinitely many times. We may assume μ is the lowest with such property, by well ordering. Reintroduction of μ must result from killing a term involving a monomial ν of $r^{(l)}$ such that

$$\mu = \frac{\nu}{\mathrm{LT}(g_j)} \lambda, \tag{A.5}$$

where λ is a nonleading term of $\mathrm{LT}(g_j)$. In particular, ν precedes μ, and ν equals $\frac{\mu}{\lambda} \mathrm{LT}(g_j)$. Let u_{i_0} be the basis vector involved in μ. Then ν divides $\frac{\mu}{u_{i_0}} \mathrm{LT}(g_j)$. We note that there are only finitely many monomials that divide a given monomial, so ν must belong to a finite list of monomials, namely, monomials dividing one of $\frac{\mu}{u_{i_0}} \mathrm{LT}(g_1) \ \ldots, \frac{\mu}{u_{i_0}} \mathrm{LT}(g_h)$. In particular, there is a monomial $\nu \prec \mu$ that occurs in $r^{(l)}$ for infinitely many l's. By construction, every time ν occurs, it will be killed immediately, so in order for ν to appear in $r^{(l)}$ infinitely many times, ν must be reintroduced infinitely many times. Since ν is lower than μ, this contradicts the minimality of μ.

Now, let the limits of $q_i^{(l)}$ and $r^{(l)}$ be $q_i^{(\infty)}$ and $r^{(\infty)}$ respectively. Since R is closed in Ω, we have $q_i^{(\infty)} \in R$ and $r^{(\infty)} \in F$. To show that $f = r^{(\infty)} + \sum_{i=1}^{h} q_i^{(\infty)} g_i$ is a desired division expression, it suffices to show that $r^{(\infty)}$ contains no term divisible by any $\mathrm{LT}(g_i)$. Suppose a monomial μ involved in $r^{(\infty)}$ is divisible by $\mathrm{LT}(g_j)$. In the above discussion, we see that μ is only reintroduced finitely many times. After the last time μ is reintroduced (if any), it will be killed when we deal with g_j next time (which will happen since we cycle through $g_1, \ldots, g_h$), if not earlier, and μ will be never introduced again. Hence μ does not appear in $r^{(\infty)}$, a contradiction. □

引理 A.2（引理 3.6）。设 $f, g_1, \ldots, g_h \in F$。若 $G = \{g_1, \ldots, g_h\}$ 是格拉斯曼簇 ¨obner 基，则 f 除以 G 的除法表达式中的余式是唯一的（尽管除法表达式可能不唯一）。此外，f 除以 G 的余式为零，当且仅当 $f \in (g_1, \ldots, g_h)$。

Proof. Let r and r' be two possible remainders of f divided by G. Then $r - r'$ lies in the submodule $M = (g_1, \ldots, g_h)$. Moreover, since no term of r or r' is divisible by any of $\mathrm{LT}(g_i)$, no term of $r - r'$ is divisible by any of $\mathrm{LT}(g_i)$. If $r - r'$ is nonzero, then it has a leading term $\mathrm{LT}(r - r')$, which lies in $\mathrm{LT}(M)$. Since $g_1, \ldots, g_h$ form a Gröbner basis, $\mathrm{LT}(M) = (\mathrm{LT}(g_1) \ \ldots, \mathrm{LT}(g_h))$. As a property of

单项式子模中，LT(r − r′) 必须能被某个 LT(g_i) 整除，这与假设矛盾。因此，r − r′ = 0，从而证明了余式的唯一性。

在第二个断言中，我们只需证明"若"这一方向。令 f ∈ M = (g_1, ..., g_h)S，并令 r 为 f 除以 G所得的余式。由于 f ∈ M，我们有 r ∈ M，因此可以重复上述相同的论证，将 r − r′ 替换为 r，从而得出 r = 0。□

引理 A.3. 设 M ⊆ F 是一个子模，且 g_1, ..., g_h ∈ M。如果 LT(M) = (LT(g_1) ..., LT(g_h))，那么必有 M = (g_1, ..., g_h)，因此 {g_1, ..., g_h} 是 M 的 Gr¨obner 基。

证明。设 f ∈ M，我们将证明 f ∈ (g_1, ..., g_h)。令

$$f = \sum_{i=1}^{h} q_i g_i + r. \tag{A.6}$$

be a division expression of f divided by $g_1, \ldots, g_h$. It suffices to show that $r = 0$. If not, then r has an initial term $\mathrm{LT}(r)$. Since $r \in M$, we have $\mathrm{LT}(r) \in \mathrm{LT}(M) = (\mathrm{LT}(g_1) \ \ldots, \mathrm{LT}(g_h))$ by the definition of $\mathrm{LT}(M)$. As a result, $\mathrm{LT}(r)$ is divisible by one of $\mathrm{LT}(g_i)$, a contradiction to the requirement of a division expression. □

命题 A.4 (　　命题 3.7）。 M 的 F 的每个子模都有唯一的约化格拉斯曼簇 Gröbner 基。 .

证明。设 C(首项(M)) = {μ_1, ..., μ_h}为首项(M)的角点集合。

我们首先证明存在关于 M 的约化格拉斯曼簇基。根据 LT(M) 的定义，存在 g_1,... , g_h ∈ M，使得 LT(g_i) = μ_i。根据引理 A.3， M 由 G:= {g_1, ..., g_h} 生成，并且 G 是关于 M 的格拉斯曼簇基。

我们现在 修改 G。对于每个 i，将 μ_i 除以 G，并将（唯一的）余式记为 r_i。定义　　e

$$g_i' := \mu_i - r_i; \tag{A.7}$$

从除法表达式可知， g_i′ 属于 M。

We claim that $g_1', \ldots, g_h'$ form a reduced Gröbner basis for M. Fix i. From the division algorithm, we observe that (1) no term of r_i is divisible by any μ_j, and (2) no term of r_i is before μ_i. In particular, μ_i is not involved in any term of r_i, so every term of r_i is strictly behind μ_i. Therefore, we have $\mathrm{LT}(g_i') = \mu_i$, so $g_1', \ldots, g_h'$ form a Gröbner basis for M. It remains to check the requirement (c) of Definition 3.4, namely, non nonleading term of g_i' is divisible by any μ_j. But this is just the observation (1).

Finally, we prove the uniqueness of the reduced Gröbner basis. Say $g_1, \ldots, g_h$ and $g_1', \ldots, g_h'$ are two reduced Gröbner bases for M such that $\mathrm{LT}(g_i) = \mathrm{LT}(g_i') = \mu_i$ for all i. Fix i, and we claim that $g_i' = g_i$. Consider $r = g_i - g_i'$. The leading terms of g_i and g_i' cancel each other, and the nonleading terms of g_i and g_i' are not divisible by any μ_j (by the requirement (c) of a reduced Gröbner basis), so no term of r is divisible by any μ_j. If $r \neq 0$, noting that r is in M, we must have $\mathrm{LT}(r) \in \mathrm{LT}(M)$, so r has a term $\mathrm{LT}(r)$ that is divisible by some μ_j, a contradiction. Hence $r = 0$ and the proof is complete. □

引理 A.5（引理 3.1）。设 M 是 F 的一个子模。那么存在一个 k-向量空间的同构。

$$\frac{F}{M} \cong \frac{F}{\mathrm{LT}(M)}. \tag{A.8}$$

事实上，不属于 LT(M) 的单项式集合是 F/M 的 k-基。

证明。令 G = {g_1, ..., g_h} 为 M 的 Gr¨obner 基，其中 LT(g_i) 是单项式 μ_i。令 Δ 为 F 中不属于 LT(M) 的单项式集合，它等于不被任何 μ_i 整除的单项式集合。考虑映射

$$\begin{aligned} r_G : F &\to \mathrm{span}_k \Delta, \\ f &\mapsto (f \bmod G), \end{aligned} \tag{A.9}$$

其中 $(f \bmod G)$ 表示 f 除以 G 所得的余式，由引理 A.2 可知该余式唯一。我们断言 r_G 是 k-线性的且为满射，其核为 M，从而诱导一个 k-同构

$$\frac{F}{M} \overset{r_G}{\cong} \operatorname{span}_k \Delta = \frac{F}{\operatorname{LT}(M)}. \tag{A.10}$$

由引理 A.2 的第二个断言可知，r_G 的核是 M。为证明 r_G 是满射，令 $f \in$ 张成 $_k \Delta$。那么表达式 $f = f + \sum 0 \cdot g_i$ 本身就是一个除法表达式，因此 $r_G(f) = f$。

还需证明 r_G 是 k-线性的。显然，对于任意标量 $c \in k$，有 $r_G(cf) = cr_G(f)$。我们将证明，当 $x + y + z = 0$ 时，$r_G(x) + r_G(y) + r_G(z) = 0$。

不失一般性，假设 x、y、$z \neq 0$ 和 $\operatorname{LT}(z) \preceq \operatorname{LT}(x)$、$\operatorname{LT}(y)$。令

$$x = r_G(x) + \sum_{i=1}^{h} q_i^x g_i \tag{A.11}$$

and

$$y = r_G(y) + \sum_{i=1}^{h} q_i^y g_i \tag{A.12}$$

是 x 和 y 关于 G 的除法表达式。由于 $\operatorname{LT}(q_i^x g_i) \succeq \operatorname{LT}(x) \succeq \operatorname{LT}(z)$ 且 $\operatorname{LT}(q_i^y g_i) \succeq \operatorname{LT}(y) \succeq \operatorname{LT}(z)$，以下表达式

$$-z = r_G(x) + r_G(y) + \sum_{i=1}^{h} (q_i^x + q_i^y) g_i \tag{A.13}$$

是关于 G 的 z 的除法表达式。因此，$r_G(-z) = r_G(x) + r_G(y)$，所以 r_G 是 k-线性的。□

Lemma A.6 (Lemma 3.13)**.** *Let $G = \{g_1, \dots, g_h\}, G' = \{g_1', \dots, g_h'\}$ be Gröbner bases for M with $\operatorname{LT}(g_i) = \operatorname{LT}(g_i') = \mu_i$, and assume G' is reduced. If $g_1 - \mu_1$ has no terms divisible by any μ_i, then we must have $g_1 = g_1'$.*

证明。我们使用算法 (A.7) 将 G 化简为一个约化的格拉斯曼簇 ¨obner 基。特别地，$g_1' = \mu_1 - r_1$，其中 r_1 是将 μ_1 除以 G 所得的余式。然而，如果我们令 $r = \mu_1 - g_1$，则该表达式

$$\mu_1 = 1 \cdot g_1 + r \tag{A.14}$$

这是一个除法表达式，因为根据我们的假设，$\operatorname{LT}(g_1) = \mu_1$ 和 $r = \mu_1 - g_1$ 没有任何可被某个 μ_i 整除的项。因此，$r_1 = r$ 和 $g_1' = \mu_1 - r = g_1$。□

A.1. **Buchberger criterion.** Let $M = (g_1, \dots, g_h) \subseteq F$ be a submodule, and assume $\operatorname{LT}(g_i) = \mu_i$ is a monomial. Denote $G = \{g_1, \dots, g_h\}$. We shall prove Proposition 3.11 that decides when is G a Gröbner basis. We shall divide it into two independent steps, Lemma A.9 and Lemma A.10.

回顾一下，$\operatorname{LCM}(\mu_i, \mu_j)$ 是 μ_i 和 μ_j 的极小公倍单项式集合，而命题 3.11 的陈述涉及检验所有形如以下形式的关系。

$$S_{ij}^{\nu}(g_i, g_j) := \frac{\nu}{\mu_i} g_i - \frac{\nu}{\mu_j} g_j, \tag{A.15}$$

对于所有 i、$j \in [h]$ 和 $\mu \in$，μ_i 与 μ_j 的最小公倍数。

事实上，我们将证明，对于集合 S_{ij}^{ν}，我们唯一需要的性质是：它们生成了单项式 μ_1、.....、μ_h 之间的“关系模”。现在我们来精确定义这一概念。

定义 A.7. 对于任意有限指标集 I，令 $W_I := \bigoplus_{i \in I} R w_i$。我们始终将 W_I 中的元素视为算子：若 f_i、$i \in I$ 是任意 R-模 N 的元素，且 $\Theta = \sum_{i \in I} a_i w_i$ 是 W_I 中的元素，则记作

$$\Theta(f_i : i \in I) := \sum_{i \in I} a_i f_i. \tag{A.16}$$

特别地，我们有 $\Theta = \Theta(w_i : i \in I)$。

如果 $I=[h]=\{1,\dots,h\}$，那么我们可以将 W_I 简写为 W_h。如果 $I\subseteq J$，那么我们将通过自然包含映射（将 w_i 映射到 w_i），把 W_I 视为 W_J 的一个子模。

For 单项式 $\underline{\mu}=\{\mu_i:\ i\in I\}$ 在 F 中，我们定义 $\underline{\mu}$ 到 b 的关系模 e

$$\mathrm{Rel}(\underline{\mu})=\big\{\Theta\in W_I:\Theta(\underline{\mu})=0\big\}. \tag{A.17}$$

在以下证明中，我们需要在 W_I 上定义一个分次。

定义 A.8. 设 $\underline{\mu}=\{\mu_i:\ i\in I\}$ 是 F 中的单项式。考虑 W_I 上的一种分次，其次数由单项式 F 编号，定义为

$$\deg_{\underline{\mu}}(\tau w_i):=\tau\mu_i \tag{A.18}$$

对于任意单项式 $\tau\in R$ 和指标 $i\in I$。我们称 $\deg_{\underline{\mu}}$ 为由 $\underline{\mu}$ 在 W_I 上诱导的分次。

我们注意到，R-线性映射 $W_I\to F,\ \Theta\mapsto\Theta(\underline{\mu})$ 保持分次结构，其中 F 配备了由 F 中的单项式自然诱导的分次结构。

我们使用记号 $[\cdot]_\nu$ 提取次数为 ν 的部分。这将适用于由 F 中的单项式分次的 W_I 和 F，以及由 R 中的单项式分次的 R。

严格来说，这种分次并非通常意义上的分次：给定 Θ，可能存在无穷多个 ν，使得 $[\Theta]_\nu$ 非零，这与幂级数环的情形类似。然而，我们确实有 $\Theta=0$ 当且仅当对于所有 ν，$[\Theta]_\nu=0$。

引理 A.9。

(a) 对于 F 中的单项式 μ_1、μ_2，关系子模 $\mathrm{Rel}(\mu_1,\mu_2)$ 由以下元素生成。

$$S^\nu:=\frac{\nu}{\mu_1}w_1-\frac{\nu}{\mu_2}w_2\in W_{\{1,2\}} \tag{A.19}$$

对于所有 $\nu\in\mathrm{LCM}(\mu_1,\mu_2)$。

(b) 设 $\mu_1,\dots,\mu_h$ 为 F 中的单项式。对于每一对 $1\le i<j\le h$，令 $\left\{\Theta_{ij}^\lambda:\ \lambda\in\Lambda_{ij}\right\}$ 为 $W_{\{i,j\}}$ 中 $\mathrm{Rel}(\mu_i,\mu_j)$ 的一个生成集。那么集合

$$\left\{\Theta_{ij}^\lambda:i<j,\lambda\in\Lambda_{ij}\right\} \tag{A.20}$$

在 W_h 中生成 $\mathrm{Rel}(\mu_1,\dots,\mu_h)$，其中我们通过自然嵌入 $W_{\{i,j\}}\subseteq W_h$ 将 Θ_{ij}^λ 视为 W_h 的一个元素。

因此，形如以下形式的元素

$$S_{ij}^\nu:=\frac{\nu}{\mu_i}w_i-\frac{\nu}{\mu_j}w_j \tag{A.21}$$

对于所有 i、$j\in[h]$、$\nu\in$，$\mathrm{LCM}(\mu_i,\ \mu_j)$ 生成 W_h 中的关系模 $\mathrm{Rel}(\mu_1,\dots,\mu_h)$。证明。

(a) Let $\mathrm{LCM}(\mu_1,\mu_2)=\{\nu^1,\dots,\nu^l\}$, and consider an element $\Theta=a_1w_1+a_2w_2$ of $\mathrm{Rel}(\mu_1,\mu_2)$, where $a_i\in R$. Thus $a_1\mu_1+a_2\mu_2=0$ by definition. Use the grading on $W_{\{1,2\}}$ induced by μ_1,μ_2. Let ν be a monomial in F, then it suffices to show that $[\Theta]_\nu$ is generated by S^{ν^λ}, $1\le\lambda\le l$. There are two cases:

(i) 单项式 ν 可被 μ_1 和 μ_2整除。对于 $i=1,\ 2,$ 写作 $\tau_i=\nu/\mu_i\in R$。提取 $a_1\mu_1+a_2\mu_2=0$两边的 ν 次部分，我们有

$$[a_1]_{\tau_1}\mu_1+[a_2]_{\tau_2}\mu_2=0. \tag{A.22}$$

Write $[a_i]_{\tau_i}=c_i\tau_i$ for $i=1,2$, where $c_i\in k$, then we have

$$0=c_1\tau_1\mu_1+c_2\tau_2\mu_2=(c_1+c_2)\mu, \tag{A.23}$$

因此 $c_2=-c_1$。所以,

$$[\Theta]_\nu=c_1\tau_1w_1+c_2\tau_2w_2=c_1(\frac{\nu}{\mu_1}w_1-\frac{\nu}{\mu_2}w_2). \tag{A.24}$$

根据 LCM(μ_1,μ_2) 的定义，存在 λ 使得 ν^λ 整除 ν。因此

$$[\Theta]_\nu = c_1\frac{\nu}{\nu^\lambda}\left(\frac{\nu^\lambda}{\mu_1}w_1 - \frac{\nu^\lambda}{\mu_2}w_2\right) = a_1\frac{\nu}{\nu^\lambda}S^{\nu^\lambda}, \tag{A.25}$$

因此，一个 S^{ν^λ} 就足以生成 $[\Theta]_\nu$。

(ii) 单项式 ν 不能被 μ_1 或 μ_2 中至少一个整除。

在这种情况下，$[\Theta]_\nu$ 至多只有一项，因此在 $\Theta(\mu_1,\mu_2)$的ν 次部分中不会发生抵消。因此，为了使 $[\Theta(\mu_1,\mu_2)]_\nu = 0$成立，我们必须有 $[\Theta]_\nu = 0$。

(b) Use the grading on W_h induced by $\mu_1,\dots,\mu_h$. Let Θ be an element of $\mathrm{Rel}(\mu_1,\dots,\mu_h)$, and let ν be a monomial in F. We claim that $[\Theta]_\nu$ is generated by Θ_{ij}^λ. Consider the degree-ν part of the equation $\Theta(\mu_1,\dots,\mu_h) = 0$. Let A be the set of i such that μ_i divides ν. If $|A| \le 1$, then no cancellation occurs in $[\Theta(\mu_1,\dots,\mu_h)]_\nu$, so we must have $[\Theta]_\nu = 0$. From now on, assume without loss of generality that $A = \{1,\dots,l\}$ with $l \ge 2$.

写作 $\tau_i = \nu/\mu_i \in R$，则 $[\Theta]_\nu$ 具有如下形式

$$[\Theta]_\nu = c_1\tau_1 w_1 + \cdots + c_l\tau_l w_l, \tag{A.26}$$

其中 $c_i \in k$。方程 $[\Theta(\mu_1,\dots,\mu_h)]_\nu = 0$ 随即推出

$$c_1\tau_1\mu_1 + \cdots + c_l\tau_l\mu_l = (c_1 + \cdots + c_l)\tau = 0, \tag{A.27}$$

因此 $c_1 + \cdots + c_l = 0$。由此可得。

$$[\Theta]_\nu = -(c_2 + \cdots + c_l)\tau_1 w_1 + c_2\tau_2 w_2 + \cdots + c_l\tau_l w_l \tag{A.28}$$

$$= c_2(\tau_2 w_2 - \tau_1 w_1) + \cdots + c_l(\tau_l w_l - \tau_1 w_1). \tag{A.29}$$

For $2 \le i \le l$, we have $\tau_i w_i - \tau_1 w_1 \in \mathrm{Rel}(\mu_1,\mu_i)$, since its evaluation at the pair (μ_1,μ_i) is $\tau_i\mu_i - \tau_1\mu_1 = \mu - \mu = 0$. Because the relations Θ_{1i}^λ ($\lambda \in \Lambda_{1i}$) generate $\mathrm{Rel}(\mu_1,\mu_i)$, we conclude that $[\Theta]_\nu$ is generated by the elements Θ_{ij}^λ in W_h.

□

现在，我们利用上述关于集合 S_{ij}^ν 的性质来证明布赫伯格判据。这是唯一用到假设 3.9 的步骤，即 F 上的单项式序具有序类型 ω。

Lemma A.10. *Assume Assumption 3.9. Let $G = \{g_1,\dots,g_h\}$ with $g_i \in F$ and $\mathrm{LT}(g_i) = \mu_i$. Let Θ^λ, $\lambda \in \Lambda$ be homogeneous elements of W_h (*)with respect to the grading induced by $\mu_1,\dots,\mu_h$ that generate $\mathrm{Rel}(\mu_1,\dots,\mu_h)$. If for each $\lambda \in \Lambda$, a possible remainder of $\Theta^\lambda(g_1,\dots,g_h)$ divided by G is zero, then G is a Gröbner basis.*

证明。利用由 $\mu_1,\dots,\mu_h$ 诱导的 W_h 的分次结构。在此证明中，我们需要考虑 W_h 中任意非零元素 Θ 的首项次数这一概念:

$$\deg(\Theta) = \min\nolimits_\prec\{\nu : [\Theta]_\nu \ne 0\}, \tag{A.30}$$

最低次单项式 ν in F，使得 Θ 具有一个次数为 ν 的分量。注意，由良序性，deg(Θ)唯一存在。

假设 G不是格拉斯曼簇¨obner，那么存在一个元素 f属于 $M = (g_1,\dots,g_h)$，使得 LT(f) 不能被任何一个 μ_i 整除。我们将固定 f，并由此导出矛盾。

由于 $f \in M$，存在 $P \in W_h$ 使得 $f = P(g_1,\dots,g_h)$。令 $\nu =$deg P。注意到 f 是 $[P]_\nu(\underline{\mu})$（其形式为 $c\nu, c \in k$）加上次数高于 ν 的项。因此，LM$(f) \succeq \nu$，且当且仅当 $[P]_\nu(\underline{\mu}) = 0$ 时，不等式严格成立。另一方面，注意到 LM(f) 不可能是 ν，因为根据由 $\underline{\mu}$ 诱导的分次的定义，ν 必须能被 μ_i 中的某一个整除，而 LT(f) 不能。因此，LM$(f) \succ \nu$，从而 $[P]_\nu(\underline{\mu}) = 0$。

我们已经得出结论，任何满足 $f = P(g_1,\dots,g_h)$ 的 $P \in W_h$ 都必须满足 deg $P \prec$LM(f)。根据假设 3.9，$\prec$ 的序型为 ω，因此 LM(f) 以下只有有限个单项式。由此，我们可以假设 P 的选取满足 $\nu =$ deg P 最大。

We claim that there is $P' \in W_h$ such that $f = P'(g_1, \dots, g_h)$ and $\deg P' \succ \nu$, which yields a contradiction. To prove the claim, recall that $[P]_\nu(\underline{\mu}) = 0$, so $[P]_\nu$ lies in the module of relations $\mathrm{Rel}(\mu_1, \dots, \mu_h)$. By our hypothesis, there exists elements $a^\lambda \in R$ such that

$$[P]_\nu = \sum_{\lambda \in \Lambda} a^\lambda \Theta^\lambda. \tag{A.31}$$

Since Θ^λ is homogeneous (say, of degree ν^λ), we may assume a^λ is homogeneous of degree ν/ν^λ by taking the degree-ν part of the right-hand side of (A.31). In other words, we may assume a^λ is of the form $c^\lambda \nu/\nu^\lambda$, where $c^\lambda \in k$. (Here, we have assumed without loss of generality that ν^λ divides ν for every λ in the sum.)

根据我们的假设以及除法表达式的定义要求，对于每个 λ，都存在$Q^\lambda \in W_h$，使得

$$Q^\lambda(g_1, \dots, g_h) = \Theta^\lambda(g_1, \dots, g_h) \tag{A.32}$$

以及 $\deg\ Q^\lambda \succeq \mathrm{LM}(\Theta^\lambda(g_1, \dots, g_h))$。$\Theta^\lambda$的齐次性（次数为 ν^λ）意味着

$$\Theta^\lambda(g_1, \dots, g_h) = \Theta^\lambda(\underline{\mu}) + (\text{terms strictly higher than } \nu^\lambda). \tag{A.33}$$

由于 $\Theta^\lambda \in \mathrm{Rel}(\underline{\mu})$，我们有 $\Theta^\lambda(\underline{\mu}) = 0$，因此 $\mathrm{LM}(\Theta^\lambda(g_1, \dots, g_h)) \succ \nu^\lambda$。由此，我们得出了一个重要结论

$$\deg Q^\lambda \succ \nu^\lambda = \deg \Theta^\lambda. \tag{A.34}$$

这使我们能够用严格更高阶的项替换 P 的“首项”，这正是我们想要的。具体来说，令

$$P' = P + \sum_{\lambda \in \Lambda} a^\lambda \Big(Q^\lambda - \Theta^\lambda \Big). \tag{A.35}$$

于是我们有 $P'(g_1, \dots, g_h) = P(g_1, \dots, g_h)$。此外，$-\sum_{\lambda \in \Lambda} a^\lambda \Theta^\lambda$ 项恰好抵消了 P 的 ν 次部分，而 $+\sum_{\lambda \in \Lambda} a^\lambda Q^\lambda$ 项只涉及高于 ν 的次数，因为

$$\deg(a^\lambda Q^\lambda) \succ \deg(a^\lambda \nu^\lambda) = \nu. \tag{A.36}$$

由此可得 $\deg\ P' \succ \deg\ P$，因此 P' 按照命题中的要求构造完成。至此，我们完成了引理 A.10 的证明。□

Combining Lemma A.9 and Lemma A.10, we complete the proof of the “if” part of the Buchberger criterion (Proposition 3.11). The “only if” part is in fact elementary: if $G = \{g_1, \dots, g_h\}$ is a Gröbner basis for M, then $S_{ij}^\nu(g_i, g_j)$ is in M, so its remainder in every division expression by G is zero by Lemma A.2. Hence the proof of every assertion in §3 is complete.

B. 附录：动机性考量

我们将部分结果推广到代数叠的动机格罗滕迪克环中（有关介绍，请参见 [20]）。为实现这一点，我们需要对模空间进行几何方面的考察。然而，并不需要运用太多复杂的代数几何工具，因为为了在动机的格罗滕迪克环中进行研究，我们只需关注约化结构。

This section contains two main topics. The first is a proof of Theorem B.1 (or Theorem 1.7), which occupies §B.1∼ §B.4. As noted in Remark 2.6, the proof of Proposition 2.3 does not naïvely generalize to the motivic setting. In order to prove Theorem B.1, we will need to develop new machinaries. The second is an exposition of Gröbner strata, which we present in §B.5. Abundant results on Gröbner strata already exist in literature, for example, see [44]. The results in previous chapters are easily generalizable to the geometric setting. Furthermore, Lemma 4.4 and Lemma 4.5 can be easily upgraded to their motivic versions. So we actually get motivic decompositions of $[\mathrm{Quot}_{d,n}^d(R)]$ with respect to Gröbner strata. We refer the readers to §B.5.2 for the precise statements.

定理 B.1（对定理 1.7 的略微修改）。设 k 为任意一个域， R 为形如 $k[[x_1,...,x_m]]/I$ 的完备局部环。对于任意满足 $0 \le r \le \min\{n,d\}$ 的 n、d、r，以下恒等式 $\widehat{\ }$ 在 $K_0(\mathrm{Var}_k)$ 中成立：

$$[\mathrm{Coh}_n^r(R)] = \frac{[\mathrm{Quot}_{d,n}^r(R)][\mathrm{GL}_{d-r}]}{\mathbb{L}^{d(n-d)+(d-r)^2}[\mathrm{GL}_d]}. \tag{B.1}$$

$\widehat{\ }$通过设定 $R=\mathcal{O}_{X,p}$ 并代入，即可由公式 (B.1) 恢复得到 (1.21)。

$$[\mathrm{Gr}(d,r)] = [\mathrm{GL}_d]/\mathbb{L}^{r(d-r)}[\mathrm{GL}_r][\mathrm{GL}_{d-r}]. \tag{B.2}$$

实际上，我们只会使用最重要的特殊情形 $r=d$。

B.0.1. *Notation and assumptions.* In order to keep notation clean, we will again use $\mathrm{Coh}_n(R)$, $\mathrm{Coh}_n^r(R)$, $\mathrm{Quot}_{d,n}(R)$, $\mathrm{Quot}_{d,n}^r(R)$, $C_n(R)$, and $C_n^r(R)$ to denote the induced reduced structures of the original schemes and stacks, see §B.1. In the whole chapter, the symbol S is reserved for denoting a reduced affine k-scheme, which will only be used in the situation where we consider affine scheme points of various schemes and stacks.

B.0.2. *Outline of the proof for Theorem B.1.* In §B.2.2, we will first observe a group action $\mathrm{GL}[F] \curvearrowright \mathrm{Quot}_{d,n}^r(R)$ and introduce an important group scheme $\mathscr{G}$ over $\mathrm{Quot}_{d,n}^r(R)$, which is the "stablizer bundle" of the $\mathrm{GL}[F]$-action. In §B.3, we will construct a Zariski GL_n-torsor $\mathfrak{Q}$ over $\mathrm{Quot}_{d,n}^r(R)$ that sits in the following diagram

$$\begin{array}{ccc} & \mathfrak{Q} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\varpi} \\ \mathrm{Quot}_{d,n}^r(R) & & C_n^r(R) \end{array} \tag{B.3}$$

我们将证明 ϖ 存在 Zariski 局部截面，即存在 Zariski 覆盖 $U \to C_n^r(R)$，以及$\sigma\colon U \to \mathfrak{Q}$，使得 $\varpi\sigma = \mathrm{Id}_U$。接着我们证明，$\mathrm{GL}[F]_U$是 $\mathfrak{Q}_U$上的 fppf 托orsor $\mathscr{G}_U$，其中$\mathscr{G}_U = \sigma^*\pi^*\mathscr{G}$ 是 U上的群概形。然后我们在引理 B.6 中证明， U 存在一个 Zariski 分层，在该分层上 $\mathscr{G}_U$ 是特殊的（也就是说，任意 fppf 托orsor 都是 Zariski 局部平凡的）。最后，我们利用引理 B.2 和引理 B.3 中对 GL[F] 和 $\mathscr{G}$ 的显式描述，建立公式 (B.1)。

B.1. 若干几何对象的函子描述。为了便于后续参考，我们总结了所讨论的各种几何对象的点函子。提醒读者，$C_n(R)$ 是交换簇（§2.1）。固定 V_n 为一个维数为 n 、定义在 k 上的向量空间。以下，设 X 为一个约化的 k-概形。

$$\mathrm{Coh}_n(R)(X) = \left\{\begin{array}{l}\text{groupoid of coherent sheaves } \mathcal{F} \text{ over } X_R,\\ \text{such that } \mathcal{F} \text{ is flat of rank } n \text{ over } X.\end{array}\right\}. \tag{B.4}$$

$$\mathrm{Coh}_n^r(R)(X) = \left\{\begin{array}{l}\text{groupoid of coherent sheaves } \mathcal{F} \text{ over } X_R,\\ \text{such that } \mathcal{F} \text{ is flat of rank } n \text{ over } X, \text{ and}\\ \mathcal{F}/\mathfrak{m}\mathcal{F} \text{ is flat of rank } r \text{ over } X.\end{array}\right\}. \tag{B.5}$$

$$\mathrm{Quot}_{d,n}(R)(X) = \left\{N \subseteq \mathcal{O}_{X_R}^d, \text{ such that } \mathcal{O}_{X_R}^d/N \in \mathrm{Coh}_n(R)(X).\right\}. \tag{B.6}$$

$$\mathrm{Quot}_{d,n}^r(R)(X) = \left\{N \subseteq \mathcal{O}_{X_R}^d, \text{ such that } \mathcal{O}_{X_R}^d/N \in \mathrm{Coh}_n^r(R)(X).\right\}. \tag{B.7}$$

$$C_n(R)(X) = \left\{\begin{array}{l}\text{equivalent classes of pairs } (\mathcal{F},\iota) \text{ where } \mathcal{F} \in \mathrm{Coh}_n(R)(X),\\ \text{and } \iota : \mathcal{O}_X \otimes V_n \simeq \mathcal{F} \text{ is an isomorphism of } \mathcal{O}_X\text{-modules.}\end{array}\right\}. \tag{B.8}$$

$$C_n^r(R)(X) = \left\{\begin{array}{l}\text{equivalent classes of pairs } (\mathcal{F},\iota) \text{ where } \mathcal{F} \in \mathrm{Coh}_n^r(R)(X),\\ \text{and } \iota : \mathcal{O}_X \otimes V_n \simeq \mathcal{F} \text{ is an isomorphism of } \mathcal{O}_X\text{-modules.}\end{array}\right\}. \tag{B.9}$$

注意到

$$C_n^r(R) = C_n(R) \times_{\mathrm{Coh}_n(R)} \mathrm{Coh}_n^r(R), \tag{B.10}$$

and $\mathrm{Coh}_n^r(R) = [C_n^r(R)/\mathrm{GL}_n]$. Furthermore, when n is fixed, there is a large Artin local quotient R' of R such that for all d and r, $\mathrm{Coh}_n(R) = \mathrm{Coh}_n(R')$, $\mathrm{Coh}_n^r(R) = \mathrm{Coh}_n^r(R')$, $\mathrm{Quot}_{d,n}(R) = \mathrm{Quot}_{d,n}(R')$, $\mathrm{Quot}_{d,n}(R) = \mathrm{Quot}_{d,n}(R')$, $\mathrm{Quot}_{d,n}^r(R) = \mathrm{Quot}_{d,n}^r(R')$, $C_n(R) = C_n(R')$ and $C_n^r(R) = C_n^r(R')$.

B.2. **Geometry of** $\mathrm{Quot}_{d,n}^r(R)$**.** By the last few sentences of §B.1, we may and will replace R by a large Artin quotient. Write $\dim_k R = b+1$ and $F = R^d = \bigoplus_{i=1}^d Ru_i$. If a submodule $N \subseteq F \otimes S$ lies in $\mathrm{Quot}_{d,n}^r(R)(S)$, then the quotient $(F\otimes S)/N$ is a flat S-module of rank n, and $(\mathfrak{m}F \otimes S) \cap N$ is a flat S-module of rank $bd - n + r$, so we obtain a locally closed immersion

$$\mathrm{Quot}_{d,n}^r(R) \hookrightarrow \mathrm{Gr}(F, bd + d - n) \tag{B.11}$$

将子模 $N \in F \otimes S$ 映射到 $F \otimes S$ 中对应的模。事实上，$\mathrm{Quot}_{d,n}^r(R)$ 可以闭嵌入到 $\mathrm{Gr}(F, bd+d-n)$ 的局部闭子簇中，该子簇参数化了与 $\mathfrak{m}F$ 的交恰好具有维数 $bd - n + r$ 的子空间。注意，当 $d = r$ 时，(B.11) 是一个闭嵌入。令

$$0 \to \mathscr{N} \to \underline{F} \to \mathscr{Q} \to 0 \tag{B.12}$$

be the restriction of the tautological sequence of universal bundles over $\mathrm{Gr}(F, bd+d-n)$ to $\mathrm{Quot}_{d,n}^r(R)$, i.e., over a point $x \in \mathrm{Quot}_{d,n}^r(R)(S)$ corresponding to an $N \subseteq F \otimes S$, the fiber $\mathscr{N}_x$ is N, while the fiber of $\mathscr{Q}_x$ is $(F\otimes S)/N$. Furthermore, (B.12) is not only a sequence of $\mathrm{Quot}_{d,n}^r(R)$-bundles, but also a sequence of coherent $\mathrm{Quot}_{d,n}^r(R) \times \operatorname{Spec} R$ modules. There is another related construction:

$$\mathrm{Quot}_{d,n}^r(R) \to \mathrm{Quot}_{d,r}^r(R) = \mathrm{Gr}(F/\mathfrak{m}F, d - r) \tag{B.13}$$

将 $N \subseteq F \otimes S$ 发送至 $N/(\mathfrak{m}F \otimes S)$。不难看出，这个态射是满射，且其纤维逐点同构，但我们不会用到这一性质。将 $\mathrm{Gr}(F/\mathfrak{m}F, d-r)$ 上泛丛的典范正合列拉回到 $\mathrm{Quot}_{d,n}^r(R)$ 上，所得结果记为

$$0 \to \overline{\mathscr{N}} \to \underline{F}/\mathfrak{m}\underline{F} \to \overline{\mathscr{Q}} \to 0 \tag{B.14}$$

显然，$\overline{\mathscr{N}} = \mathscr{N}/\mathfrak{m}\underline{F}$ 和 $\overline{\mathscr{Q}} = \mathscr{Q}/\mathfrak{m}\mathscr{Q}$。

B.2.1. *Group action.* The most important feature of $\mathrm{Quot}_{d,n}^r(R)$ is that it is equipped with an algebraic group action. For a finite R-module M, let $\mathrm{GL}[M]$ be the group $\mathrm{Aut}_R(M)$, but viewed as a linear algebraic group over k. Then $\mathrm{GL}[F]$ acts on $\mathrm{Quot}_{d,n}^r(R)$, in the sense that any $g \in \mathrm{GL}[F](S)$ acts on $\mathrm{Quot}_{d,n}^r(R)(S)$ by sending a submodule $N \subseteq F \otimes S$ to $g \cdot N$.

存在一个投影 $\mathrm{GL}[F] \to \mathrm{GL}[F/\mathfrak{m}F]$，将 g 映射到 (g 模 $\mathfrak{m}$)。令 $U(\mathfrak{m}F)$ 为其核。$\mathrm{GL}[F/\mathfrak{m}F]$ 在 $\mathrm{Gr}(F/\mathfrak{m}F, d-r)$ 上的典范作用，与 $\mathrm{GL}[F]$ 通过 (B.13) 在 $\mathrm{Quot}_{d,n}^r(R)$ 上的作用相容。$\mathrm{GL}[F]$ 的结构很容易理解：

引理 B.2. 按照上述记号，正合列

$$1 \to U[\mathfrak{m}F] \to \mathrm{GL}[F] \to \mathrm{GL}[F/\mathfrak{m}F] \to 1, \tag{B.15}$$

存在典范分裂。此外，$\mathrm{GL}[F/\mathfrak{m}F] \simeq \mathrm{GL}_d$ 和 $U[\mathfrak{m}F]$ 是一个维数为 bd^2 的 k-分裂幺幂群（如果一个幺幂群是 k-分裂的，则它是由 $\mathbb{G}_a$ 在 k上的连续扩张构成的）。

证明。显然，$\mathrm{GL}[F/\mathfrak{m}F] \simeq \mathrm{GL}_d$。态射 $\pi : \mathrm{GL}[F] \to \mathrm{GL}[F/\mathfrak{m}F]$ 存在一个由结构态射 $k \hookrightarrow R$ 诱导的截面 $\mathrm{GL}[F/\mathfrak{m}F] \hookrightarrow \mathrm{GL}[F]$。注意，$U[\mathfrak{m}F](S)$ 由形如 g 的对象组成。

$$g(u_i) \in u_i + \mathfrak{m}F \otimes S,\ 1 \le i \le d. \tag{B.16}$$

因此，$U[\mathfrak{m}F]$ 在抽象意义上同构于仿射空间 $\mathbb{A}^{bd^2}$。它是幺幂的，因为对于任意 $g \in U[\mathfrak{m}F]$（S），都有 $(g - \mathrm{Id})^b = 0$。通过对 b（R的 k维数）应用归纳法，可以得出 $U[\mathfrak{m}F]$ 是 k-分裂的这一事实：当 $b = 1$时，该断言显然成立；否则，存在一个元素 $0 \neq v \in R$ 使得

$v\mathfrak{m}=0$。子群 $U[vF]\subseteq U[\mathfrak{m}F]$（其 S-点由满足 $g(u_i)\in u_i+vF\otimes S$ 的元素 g 构成）包含于 $U[\mathfrak{m}F]$ 的中心中，并且同构于 $\mathbb{G}_a^{d^2}$。令 $F'=(R/v)^d$。我们有一个正合列。

$$1\to U[vF]\to U[\mathfrak{m}F]\to U[\mathfrak{m}F']\to 1. \tag{B.17}$$

根据归纳假设，$U[\mathfrak{m}F']$ 是 k-分裂的。因此，$U[\mathfrak{m}F]$ 是 k-分裂的。 t. □

B.2.2. 商概形$^r_{d,n}(R)$上的群概形。§B.2.1 中的构造可以整体化。它们是商概形$^r_{d,n}(R)$上更为重要的群概形。定义如下：

- $\mathrm{GL}[\underline{F}]:=\mathrm{Aut}_R(\underline{F})$、$\mathrm{GL}[\underline{F}/\mathfrak{m}\underline{F}]:=\mathrm{Aut}_R(\underline{F}/\mathfrak{m}\underline{F})$ 和 $\mathrm{GL}[\overline{\mathscr{N}}]:=\mathrm{Aut}_R(\overline{\mathscr{N}})$。
- $U(\mathfrak{m}\underline{F}):=\ker(\mathrm{GL}[\underline{F}]\to\mathrm{GL}[\underline{F}/\mathfrak{m}\underline{F}])$.
- $\mathscr{G}$ 是 $\mathrm{GL}[\underline{F}]$ 中稳定 $\mathscr{N}$ 且在 $\mathscr{Q}$ 上约化为恒等元的子群概形。
- $\mathrm{Fil}_1\mathscr{G}:=\ker(\mathscr{G}\to\mathrm{GL}[\underline{F}/\mathfrak{m}\underline{F}])=U(\mathfrak{m}\underline{F})\cap\mathscr{G}$，即 $\mathrm{Fil}_1\mathscr{G}$ 是 $\mathrm{GL}[\underline{F}]$ 的子群概形，在整个 $\underline{F}/\mathfrak{m}\underline{F}$ 上约化为恒等映射。
- $\mathrm{Fil}_2\mathscr{G}$ 是 $\mathrm{GL}[\underline{F}]$ 的子群概形，它在 $\overline{\mathscr{N}}$ 和 $\overline{\mathscr{Q}}$ 上的约化都为恒等映射。

Note that $\mathrm{GL}[\underline{F}]$ *resp.* $\mathrm{GL}[\underline{F}/\mathfrak{m}\underline{F}]$ *resp.* $U(\mathfrak{m}\underline{F})$ is just a trivial $\mathrm{GL}[F]$ *resp.* $\mathrm{GL}[F/\mathfrak{m}F]$ *resp.* $U[\mathfrak{m}F]$ bundle over $\mathrm{Quot}^r_{d,n}(R)$, and the split exact sequence (B.15) admits an obvious global analogue. We will call $\mathscr{G}$ the *stabilizer bundle*, in the sense that the fiber of $\mathscr{G}$ at an S-point is the subgroup of $\mathrm{GL}[F](S)$ that acts trivially on the corresponding module. It will play an important role in studying the action of $\mathrm{GL}[F]$ on $\mathrm{Quot}^r_{d,n}(R)$. The group scheme $\mathscr{G}$ admits a three step filtration of normal subgroups:

$$\underline{1}\triangleleft\mathrm{Fil}_1\mathscr{G}\triangleleft\mathrm{Fil}_2\mathscr{G}\triangleleft\mathrm{Fil}_3\mathscr{G}=\mathscr{G}. \tag{B.18}$$

注意，在特殊情形 $d=r$ 中，我们有 $\mathscr{G}=\mathrm{Fil}_2\mathscr{G}$。

引理 B.3. 以下结论成立：

(a) $\mathrm{Fil}_1\mathscr{G}$ *is a unipotent group scheme of rank* $d(bd-n+r)$.
(b) $\mathrm{gr}_2\mathscr{G}$ *is a subgroup scheme of* $\mathrm{GL}[\underline{F}/\mathfrak{m}\underline{F}]$ *which is Zariski locally isomorphic to* $\mathbb{G}_a^{r(d-r)}$.[9]
(c) $\mathrm{Fil}_2\mathscr{G}$ *is a unipotent group scheme of rank* $bd^2-d(n-d)-(d-r)^2$.
(d) $\mathrm{gr}_3\mathscr{G}\simeq\mathrm{GL}[\overline{\mathcal{N}}]$. *In particular, it is Zariski locally isomorphic to* GL_{d-r}.

Proof.

(a) 我们首先描述 $\mathrm{Fil}_1\mathscr{G}$ 的点函子。$x\in$商概形$^r_{d,n}(R)(S)$ 对应于子模 $N\subseteq F\otimes S$。那么 $\mathrm{Fil}_1\mathscr{G}$ 在 x 上的纤维是

$$\mathrm{Fil}_1\mathscr{G}|_x=\{g\in\mathrm{GL}[F](S):\ g(u_i)\in u_i+N\cap(\mathfrak{m}F\otimes S),1\le i\le d\}. \tag{B.19}$$

这已经意味着 $\mathrm{Fil}_1\mathscr{G}$ 是幺幂的。此外，这还意味着 $\mathrm{Fil}_1\mathscr{G}$ 具有一个底层向量丛结构，该结构同构于 $(\mathscr{N}\cap\mathfrak{m}\underline{F})^{\oplus d}$。这一断言由 $\mathrm{rk}(\mathscr{N}\cap\mathfrak{m}\underline{F})=bd-n+r$ 这一事实得出。

(b) 令 $\mathscr{H}_2$ 为保持滤过 (B.14) 且在 $\overline{\mathscr{Q}}$ 和 $\overline{\mathscr{N}}$ 上均约化为恒等映射的 $\mathrm{GL}[\underline{F}/\mathfrak{m}\underline{F}]$ 的子群概形。则 $\mathrm{gr}_2\mathscr{G}\subseteq\mathscr{H}_2$。(B.15) 的典范分裂导出一个态射

$$\mathscr{H}_2\hookrightarrow\mathrm{GL}[\underline{F}/\mathfrak{m}\underline{F}]\to\mathrm{GL}[\underline{F}] \tag{B.20}$$

其像位于 $\mathrm{Fil}_2\mathscr{G}$ 中。这意味着 $\mathrm{gr}_2\mathscr{G}=\mathscr{H}_2$。令 $\mathrm{Spec}\,S$ 为 $\mathrm{Quot}^r_{d,\ n}(R)$ 的一个 Zariksi 仿射开集，在其上 $\overline{\mathscr{Q}}$ 和 $\overline{\mathscr{N}}$ 是平凡化的。于是我们可以选取 (B.14) 的一个截面，并写作 $(\underline{F}/\mathfrak{m}\underline{F})_S=\overline{\mathscr{N}}_S\oplus\overline{\mathscr{Q}}_S$。那么 S-点 $\mathscr{H}_2|_S$ 可以写成矩阵。

$$\begin{bmatrix}\mathrm{Id}_{\overline{\mathscr{N}}_S} & B\\ 0 & \mathrm{Id}_{\overline{\mathscr{Q}}_S}\end{bmatrix},\ B\in\mathcal{H}om(\overline{\mathscr{Q}},\overline{\mathscr{N}})(S). \tag{B.21}$$

这尤其意味着 $\mathrm{gr}_2\mathscr{G}|_S=\mathbb{G}_{a\,S}^{r(d-r)}$。

(c) 这由 (a) 和 (b) 得出。

[9]我们是指，存在一个 Zariksi 覆盖 U of $\mathrm{Quot}^r_{d\ ,\ n}(R)$，使得 $\mathrm{gr}_2\mathscr{G}_U\simeq\mathbb{G}_{a\,U}^{r(d-r)}$，。我们对 (d) 也采用相同的约定。

(d) 设 $\mathscr{H}$ 为保持滤过 (B.14) 并且在 $\overline{\mathscr{Q}}$ 上约化为恒等映射的 $\mathrm{GL}[\underline{F}/\mathfrak{m}\underline{F}]$ 子群概形。我们有 $\mathscr{G}/\mathrm{Fil}_1\mathscr{G} \subseteq \mathscr{H}$。与 (b) 中一样，(B.15) 的典范分裂再次诱导出一个态射 $\mathscr{H} \to \mathrm{GL}[\underline{F}]$，其像包含于 $\mathscr{G}$。这意味着 $\mathscr{G}/\mathrm{Fil}_1\mathscr{G} = \mathscr{H}$ 和 $\mathrm{gr}_3\mathscr{G} = \mathscr{H}/\mathscr{H}_2$。

设 $\mathrm{Spec}\, S$ 为 $\mathrm{Quot}^r_{d,n}(R)$ 的一个 Zariksi 仿射开集，在该开集上 $\overline{\mathscr{Q}}$ 和 $\overline{\mathscr{N}}$ 均已平凡化。选取与 (b) 中相同的 (B.14) 的分裂。则 $\mathscr{H}|_S$ 的 S-点可以写成矩阵形式。

$$\begin{bmatrix} A & B \\ 0 & \mathrm{Id}_{\overline{\mathscr{Q}}_S} \end{bmatrix},\ A \in \mathrm{GL}[\overline{\mathscr{N}}](S),\ B \in \mathcal{H}om(\overline{\mathscr{Q}}, \overline{\mathscr{N}})(S). \tag{B.22}$$

如果对其取商（B.21），那么 $\mathrm{gr}_3\mathscr{G} \simeq \mathrm{GL}[\overline{\mathscr{N}}]$ 的断言显然成立。 □

B.3. 联系 $\mathrm{Quot}^r_{d,n}(R)$ 和 $C^r_n(R)$。我们首先将 (B.3) 中的 $\mathfrak{Q}$ 构造为对应于丛 $\mathscr{Q}$ 的 GL_n-主丛。更准确地说，令 V_n 为一个固定的 k-向量空间，其维数为 n，并令 $\underline{V_n}$ 为 $\mathrm{Quot}^r_{d,n}(R)$ 上对应的平凡丛，那么

$$\mathfrak{Q} = \mathcal{I}som_{\mathrm{Quot}^r_{d,n}(R)}(\underline{V_n}, \mathscr{Q}). \tag{B.23}$$

因此，我们得到一个投影 $\pi\colon \mathfrak{Q} \to$ 商概形$^r_{d,n}(R)$。另一方面，取 (B.9) 中的 $X = \mathfrak{Q}$ 和 $\mathcal{F} = \pi^*\mathscr{Q}$，由于 $\pi^*\mathfrak{Q} \to \mathfrak{Q}$ 是一个平凡挠子（例如，存在一个来自对角线的整体截面），$\mathcal{F} \to \mathfrak{Q}$ 是一个平凡丛，并且是 $\mathrm{Coh}^r_n(R)(X)$ 中的一个对象。令

$$\iota : \mathcal{O}_X \otimes_k V_n \simeq \mathcal{F} \tag{B.24}$$

为由求值映射诱导的 $\mathcal{O}_X$-模同构

$$\underline{V_n} \times_{\mathrm{Quot}^r_{d,n}(R)} \mathfrak{Q} \to \mathscr{Q}. \tag{B.25}$$

这对 $(\mathcal{F}, \iota)$ 导出了一个态射 $\varpi\colon \mathfrak{Q} \to C^r_n(R)$。

On the level of S-points, ϖ can be easily understood as follows: let x be an S-point of $\mathfrak{Q}$ corresponding to a module $N \subseteq F \otimes S$ with an S-isomorphism $\iota : V_n \otimes S \simeq (F \otimes S)/N$, then $\varpi(x) = ((F \otimes S)/N, \iota) \in C^r_n(R)(S)$. It is easily seen that ϖ is surjective, and is invariant under $\mathrm{GL}[F]$-action. Indeed, let $g \in \mathrm{GL}[F](S)$, and $x = (N, \iota)$ as above, we have $\varpi g(x) = ((F\otimes S)/gN, g\iota) \sim ((F\otimes S)/N, \iota)$. This implies that $\varpi g(x) = \varpi(x)$.

B.3.1. 局部截面。存在一个 Zariski 仿射覆盖 $U = \mathrm{Spec} D \to C^r_n(R)$，使得 $C^r_n(R)(U)$ 的典范点对应于一对（等价类）。

$$((F \otimes D)/N^{\mathrm{taut}}, \iota^{\mathrm{taut}}). \tag{B.26}$$

Fixing a pair (B.26) in the equivalence class, we get a U-point σ of $\mathfrak{Q}$ such that $\varpi\sigma = \mathrm{Id}_U$. Therefore, σ is a section of ϖ over U. Note that such sections are well defined up to an action of $\mathrm{GL}[F](U)$. Let $\mathfrak{Q}_U = \mathfrak{Q} \times_{C^r_n(R)} U$. Since ϖ is invariant under $\mathrm{GL}[F]$, the section σ induces a morphism of U-schemes

$$p : \mathrm{GL}[F]_U = U \times \mathrm{GL}[F] \xrightarrow{\sigma\times\mathrm{Id}} \mathfrak{Q} \times \mathrm{GL}[F] \dot{\to} \mathfrak{Q}. \tag{B.27}$$

引理 B.4. 设 $\mathscr{G}$ 为 $\mathrm{Quot}^r_{d,n}(R)$（§B.2.2）上的稳定子丛，并设 $\mathscr{G}_U := \sigma^*\pi^*\mathscr{G}$ 为 U 上的拉回群概形。那么，p 将 $\mathfrak{Q}$ 实现为 $\mathrm{GL}[F]_U$ 对 fppf 子群 $\mathscr{G}_U$ 的 fppf 商。特别地，我们有：

(a) p是一个 fppf 覆盖。(b) $\mathrm{GL}[F]_U$是 $\mathfrak{Q}_U$ 上的一个 fppf $\mathscr{G}_U$-挠子，在覆盖 p 下是平凡的。

Proof. First we check that $\mathfrak{Q}_U$ is the fppf quotient. Let S be an fppf affine over U and x be an S-point of σU, corresponding to a module $N \subseteq F \otimes S$ with an S-isomorphism $\iota : V_n \otimes S \simeq (F \otimes S)/N$. For any $g \in \mathrm{GL}[F](S)$, we have $p(x, g) = ((F \otimes S)/gN, g\iota)$. Since $p(x, g) = p(x, g')$ if and only if $g'g^{-1}$ is in the stabilizer of x, i.e., $g'g^{-1}$ is in the fiber of $\mathscr{G}_S$ over x, we obtain the fppf quotient assertion. Once this is established, (a) and (b) are just formal properties of fppf quotient, see [19, Proposition 4.31]. □

B.4. **The motivic formula.** We say that a unipotent group scheme over a base Y is Y-*split*, or simply *split*, if it is a successive extension of $\mathbb{G}_{a,Y}$. We say a group scheme $\mathcal{G}$ over Y is *special*, if any fppf $\mathcal{G}$-torsor is Zariski locally trivial. It is well-known that a split unipotent group scheme is special, and an extension of special group schemes is special.

引理 B.5. 记号同§B.2.2。存在一个 Zariski 分层

$$\mathrm{Quot}^r_{d,n}(R) = \bigsqcup_{\beta\in\mathbf{B}} Z_\beta, \tag{B.28}$$

使得对于每个 β，$\mathrm{Fil}_2\mathscr{G}_{Z_\beta}$ 是一个分裂的 Z_β-幺零群概形。

Proof. Note that it suffices to show that $\mathrm{Fil}_1\mathscr{G}_{Z_\beta}$ is split for some Zariski stratification $\{Z_\beta\}$, thanks to Lemma B.3(b). Take a nonempty affine open $\operatorname{Spec} S \subseteq \mathrm{Quot}^r_{d,n}(R)$ and view it as an S-point of $\mathrm{Quot}^r_{d,n}(R)$. It then corresponds to a submodule $N \subseteq F\otimes S$. Shrinking S, we can assume that $N' = N\cap(\mathfrak{m}F\otimes S)$ is free. The group scheme $\mathrm{Fil}_1\mathscr{G}_S$ can be described as the functor

$$T/S \to \{g\in U[\mathfrak{m}F](T) | g(u_i)\in u_i + N'\otimes T, 1\le i\le d\}. \tag{B.29}$$

Let $\{\omega_i\}_{i=1}^{bd+d-n}$ be a basis of N'. There is a nonzero element $a\in N'$ such that $\mathfrak{m}a = 0$. Indeed, we know that for any element $b\in N'$, $\mathfrak{m}^n b = 0$ for sufficiently large n. So multiply b by some suitable element $u\in\mathfrak{m}$, we have $ub\neq 0$ but $\mathfrak{m}ub = 0$. Let $a = ub$ and express $a = \sum_i s_i\omega_i$. There is at least one s_i, say s_1, which is nonzero. Since S is reduced (recall that we have assumed everything is reduced in §B.0.1), the localization $S[s_1^{-1}]$ is nonzero. Therefore, $\{a\}\cup\{\omega_i\}_{i=2}^{bd+d-n}$ is a basis of $N'[s_1^{-1}]$. Therefore, after shrinking S, we can assume that N' admits a basis $\{\omega_i\}_{i=1}^{bd+d-n}$ such that $\mathfrak{m}\omega_1 = 0$.

将 N'替换为 N'/ω_1，并执行相同的技巧。根据归纳法，我们可以看到，在将 S充分缩小若干次之后，存在 N'的一组基 $\{\omega_i\}_{i=1}^{bd+d-n}$，使得 $\mathfrak{m}\omega_j \subseteq$张成 $\{\omega_1,\dots,\omega_{j-1}\}$。定义一个幺幂群概形的中心列。

$$0\triangleleft\mathscr{U}_{1,S}\triangleleft\mathscr{U}_{2,S}\triangleleft\dots\triangleleft\mathscr{U}_{bd-n+r,S} = \mathrm{Fil}_1\mathscr{G}_S \tag{B.30}$$

通过将点函子定义为

$$\mathscr{U}_{j,S} : T/S \to \{g\in U[\mathfrak{m}F](T) | g(u_i)\in u_i + \operatorname{span}_S\{\omega_1,\dots,\omega_j\}\otimes_S T, 1\le i\le d\}. \tag{B.31}$$

由构造可容易地得出 $\mathscr{U}_{j,S}/\mathscr{U}_{j,S}\simeq\mathbb{G}_{a,S}^d$。因此，$\mathrm{Fil}_1\mathscr{G}_S$ 是 S-分裂的。

将 $\mathrm{Quot}^r_{d,n}(R)$ 替换为闭子概形 $\mathrm{Quot}^r_{d,n}(R)\times_{\operatorname{Spec}}S$，并重复上述过程。由于 $\mathrm{Quot}^r_{d,n}(R)$ 是有限型的 k，因此该过程将在有限步内终止，并得到所需的分层。□

引理 B.6. 记号同上。存在一个 Zariski 分层

$$U = \bigsqcup_{\gamma\in\mathbf{C}} U_\gamma, \tag{B.32}$$

使得 $\mathscr{G}_{U_\gamma}$ 是一个特殊的 U_γ-群概形。

证明。由引理 B.5 和引理 B.3(d)，我们可知存在一个分层 $\{U_\gamma\}$，使得 $\mathrm{Fil}_2\mathscr{G}_{U_\gamma}$ 是分裂的，且 $\mathrm{gr}_3\mathscr{G}_{U_\gamma}$ 同构于 $\mathrm{GL}_{d-r,\,U_\gamma}$。众所周知，$\mathrm{GL}_{d-r,\,U_\gamma}$ 是特殊的。因此，$\mathscr{G}_{U_\gamma}$ 作为特殊群概形的扩张，也是特殊的。□

定理 B.1 的证明。根据引理 B.6，在 U 上存在一个分层 (B.32)，使得 $\mathscr{G}_U$ 在每个分层上的限制都是特殊的。由引理 B.4、$\mathscr{G}_{U_\gamma}$ 的特殊性以及引理 B.3 可知，

$$[U_\gamma][\mathrm{GL}[F]] = [\mathfrak{Q}_{U_\gamma}][\mathrm{GL}_{d-r}]\mathbb{L}^{bd^2-d(n-d)-(d-r)^2}. \tag{B.33}$$

由于各种 $\mathfrak{Q}_{U_\gamma}$ 也构成了 $\mathfrak{Q}_U$的 Zariski 分层，而 U 是 $C_n^r(R)$的 Zariski 覆盖，因此我们得到

$$[C_n^r(R)][\mathrm{GL}[F]] = [\mathfrak{Q}][\mathrm{GL}_{d-r}]\mathbb{L}^{bd^2-d(n-d)-(d-r)^2}. \tag{B.34}$$

由于 $[\mathfrak{Q}] = [\text{商概形}^r_{d,\ n}(R)][\mathrm{GL}_n]$, $[C^r_n(R)] = [\mathrm{Coh}^r_n(R)][\mathrm{GL}_n]$ 以及 $[\mathrm{GL}[F]] = [\mathrm{GL}_d]\mathbb{L}^{bd^2}$，由引理 B.2，我们得到公式 (B.1)。□

B.5. **Gröbner strata.** As noted at the beginning, the theory of Gröbner strata in the Hilbert scheme of points is well developed. Our case is slightly modified in the sense that we work with (1) power series rings, and (2) Gröbner basis for modules rather than ideals. Since we only classify submodules of $F := R^d$ with finite codimension, one shall expect that the Gröbner theory in our case bears no much difference from the Gröbner theory in the usual polynomial ring case. Moreover, one can even reduce the theory of Gröbner basis for modules to the theory of Gröbner basis for ideals, in the sense that submodules $N \subseteq F$ of rank r and codimension n can be regarded, via square-zero extension, as ideals in the square-zero deformation $R[[u_1, ..., u_r]]/(u_i u_j)$. Therefore we can realize $\mathrm{Quot}^d_{d,n}(R)$ as certain subscheme of the Hilbert scheme of points $\mathrm{Hilb}_n(R[[u_1, ..., u_d]]/(u_i u_j))$. Ideally, the Gröbner strata on the latter (the Hilbert scheme) shall restrict to Gröbner strata on $\mathrm{Quot}^d_{d,n}(R)$. These observations and intuitions suggest that the theory of Gröbner strata in our case is not too different from the usual case.

设 R 为域 k 上的一个幂级数环的单项式子环。在本节中，我们建立格拉斯曼簇 ¨obner 分层理论，该理论恰好足以用于对商概形 $^d_{d,n}(R)$ 进行分层。所有处理方式均与 [44], 类似且平行，因此我们将尽可能简洁地加以阐述。

B.5.1. *Gröbner basis for submodules of $F \otimes S$.* Let S be a finite type algebra over k. Then the ring $R \otimes S$ is nothing other than the subring of a power series ring $S[[\underline{x}]]$ generated by certain monomials. A submodule $M \subseteq F \otimes S$ is called **monic**, if $\mathrm{LT}(M)$ is a monomial submodule. (This notion is exclusive to Gröbner theory over a ring, since over a field, any submodule is monic.) A Gröbner basis of a submodule M is a finite subset G of M such $\mathrm{LT}(M)$ agrees with the submodule in $F \otimes S$ generated by $\mathrm{LT}(g)$ for $g \in G$. Since $R \otimes S$ is Noetherian, every submodule M admits a Gröbner basis. Similar to §3, a Gröbner basis $G = \{g_1, \ldots, g_h\}$ is **reduced** if

(a) $\mathrm{LT}(g_1) \ldots, \mathrm{LT}(g_h)$ 是单项式，且彼此不可整除;

(b) 对于任意 i、j，g_i 的任何非首项均不能被 $\mathrm{LT}(g_j)$ 整除。

与经典情形类似（参见 [5, 定理 2.11] 和 [59, 定理 4]），不难看出，一个子模 M 存在约化的格罗布纳基，当且仅当 M 是首一的。我们还注意到，布赫伯格判据在这种情况下仍然成立。

B.5.2. *Gröbner subfunctors and the induced stratification.* Following §4, let $\{M_\alpha : \alpha \in \Xi\}$ denote the set of finite-codimensional monomial submodules of $\mathfrak{m}F$, and let $\Delta(\alpha)$ and $C(\alpha)$ be the set of monomials not contained in M_α and the set of corners, respectively. Let $n(\alpha) := \dim_k \mathfrak{m}F/M_\alpha$. Though §4 is specifically written for the cusp, the notation above applies to any monomial subring R of a power series ring over k.

我们首先定义 $\mathrm{Hilb}_{d,n}(R) = \text{商概形}^d_{d,n+d}(R)$ 的以下两个子函子。设 $\preceq$ 为 R 上的一个单项式序，对于每个满足 $n(\alpha) = n$ 的 $\alpha \in \Xi$，我们通过其在仿射 k-代数上的点来定义函子 $\mathrm{Hilb}^\Delta(\alpha)$ 和 $\mathrm{Hilb}(\alpha)$

$$\mathrm{Hilb}^\Delta(\alpha)(S) = \{\phi : F \otimes S \twoheadrightarrow Q, \text{ such that } Q \text{ is free over } S \text{ with basis } \phi(\Delta(\alpha))\} \,/\sim \tag{B.35}$$

$$\mathrm{Hilb}(\alpha)(S) = \left\{ \begin{array}{l} \phi : F \otimes S \twoheadrightarrow Q, \text{ such that } \ker\phi \text{ has a reduced Gröbner basis of form} \\ g_\mu = \mu + \displaystyle\sum_{\nu \in \Delta(\alpha), \mu \prec \nu} s_{\mu,\nu}\nu, \text{ where } \mu \in C(\alpha). \end{array} \right\} \Big/ \sim \tag{B.36}$$

Note that there is a bijection between Ξ and the set of standard sets in the sense of [44]. Therefore the two subfunctors defined here are the direct analogues of the functors Hilb^Δ and $\mathrm{Hilb}^{\prec\Delta}$ defined in *loc.cit.* It is easy to check that $\mathrm{Hilb}^\Delta(\alpha)$ and $\mathrm{Hilb}(\alpha)$ are sheaves in the big Zariski site over k. Note that the definition of $\mathrm{Hilb}^\Delta(\alpha)$ does not require the monomial order.

命题 B.7。以下结论成立:

(a) $\mathrm{Hilb}^{\Delta}(\alpha)$ 由 $\mathrm{Hilb}_{d,n}(R)$ 的开子概形表示，并且它们构成一个开覆盖

$$\mathrm{Hilb}_{d,n}(R) = \bigcup_{\substack{\alpha\in\Xi \\ n(\alpha)=n}} \mathrm{Hilb}^{\Delta}(\alpha). \tag{B.37}$$

(b) 希尔伯特概形(α) 由希尔伯特概形$^{\Delta}(\alpha)$ 的一个闭子概形表示。

(c) $\mathrm{Hilb}_{d,\ n}(R)$ 存在 Zariski 分层

$$\mathrm{Hilb}_{d,n}(R) = \bigsqcup_{\substack{\alpha\in\Xi \\ n(\alpha)=n}} \mathrm{Hilb}(\alpha). \tag{B.38}$$

因此，$[$希尔伯特概形$_{d,\ n}(R)]$ $= \sum_{n}$ $_{\alpha\in\Xi}$ $_{()=n}$ $[$希尔伯特概形$(\alpha)]$ 属于 K_0(簇$_k$)。

证明。

(a) 根据定义显然可知，$\mathrm{Hilb}_{d,\ n}(R)$ 是子函子 $\mathrm{Hilb}(\Delta(\alpha))$ $\alpha\ \in\Xi, n(\alpha)\ = n$ 的并。现在我们来证明其开性。

应用 (B.11)，取 $r = d$ 和 $n_{(\mathrm{B.11})} = n + d$，我们得到一个局部闭浸入。

$$\mathrm{Hilb}_{d,n}(R) \hookrightarrow \mathrm{Gr} := \mathrm{Gr}(F, bd - n). \tag{B.39}$$

It is a closed embedding since $d = r$, as is observed before. Consider the subspace $\mathrm{span}_k\,\Delta(\alpha) \subseteq \mathfrak{m}F$. Let $\mathcal{V}(\alpha) \subseteq \mathrm{Gr}(k)$ be the subset parametrizing subspaces that have trivial intersection with $\mathrm{span}_k\,\Delta(\alpha)$. Since $\dim \mathrm{span}_k\,\Delta(\alpha) = n$, the subspaces of $\mathfrak{m}F$ of dimension $bd - n$ (i.e., of the complementary dimension to $\mathrm{span}_k\,\Delta(\alpha)$) have generically trivial intersection with $\mathrm{span}_k\,\Delta(\alpha)$. Therefore, $\mathcal{V}(\alpha)$ is open. Now we consider $\mathcal{V}(\alpha)$ as an open subscheme of Gr. Pulling $\mathcal{V}(\alpha)$ back to $\mathrm{Hilb}_{d,n}(R)$ exactly gives $\mathrm{Hilb}^{\Delta}(\alpha)$ which is, in particular, an open subscheme of $\mathrm{Hilb}_{d,n}(R)$.

(b) 证明几乎与 [44, 定理 1] 完全相同。设 $\phi \in$ 希尔伯特概形$^{\Delta}(\alpha)(S)$。对于每个 $\mu \in C(\alpha)$，都存在一个唯一的如下形式的多项式

$$f_\mu = \mu + \sum_{\nu\in\Delta(\alpha)} d_{\mu,\nu}\nu \in \ker\phi. \tag{B.40}$$

Let $J \subseteq S$ be the ideal generated by $d_{\mu,\nu}$ where μ runs over $C(\alpha)$ and ν runs over elements of Δ such that $\nu \prec \mu$. A morphism $\psi : S \to S'$ in the category of k-algebras factors through $S \to S/J$ if and only if $\ker(\phi\otimes\psi)$ is monic with Gröbner basis $\{\psi(f_\mu)\}$. The same argument as in *loc.cit* shows that $\mathrm{Hilb}(\alpha)$ is represented by a closed subscheme of $\mathrm{Hilb}^{\Delta}(\alpha)$.

(c) 这一断言直接由 (a) 和 (b) 得出，因为希尔伯特概形$_{d,n}(R)$ 的每个域值点都恰好属于一个分层希尔伯特概形(α)。

□

警告。我们的分层概念并不要求任何一个分层的闭包是若干分层的不交并。由于我们计算的是动机而非交理论，我们的 Zariski 分层概念已足够。正如 [44]，中所指出的，Gr¨obner 分层的一般理论并不能保证命题B.7(c)在这种更强的意义下构成一个分层。

B.5.3. 计数结果的动机版本。设 $R = k[[T^2, T^3]]$，并假定 §4 中的设定。定义一个 Zariski 子层 $\mathrm{Hilb}^0(\alpha) \subseteq \mathrm{Hilb}(\alpha)$ 为

$$\mathrm{Hilb}^0(\alpha)(S) = \left\{ \begin{array}{l} \phi : F\otimes S \twoheadrightarrow Q, \text{ such that } \ker\phi \text{ has a reduced Gröbner basis of form} \\ g_\mu = \mu + \displaystyle\sum_{\mu\in\Delta(\alpha),\mu\prec\nu} s_{\mu,\nu}\nu, \text{ where } \mu \in C(\alpha), \text{ such that } s_{\mu,\nu} = 0 \text{ if } \mu = \mu_i^0. \end{array} \right\} \Big/ \sim \tag{B.41}$$

这是引理 4.4 中定义的集合 $\mathrm{Hilb}^0(\alpha)$ 的动机版本。回顾一下，我们还在猜想 6.1 中为纯-K 首项数据 α 定义了一个簇 $V(\alpha)$。下面是引理 4.4 和引理 4.5 的动机升级版：

定理 B.8. $\mathrm{Hilb}^0(\alpha)$ 可由 $\mathrm{Hilb}(\alpha)$ 的一个闭子概形表示。此外，我们在 $K_0(\mathrm{Var}_k)$ 中有以下恒等式：

$$[\mathrm{Hilb}(\alpha)] = [\mathrm{Hilb}^0(\alpha)] \cdot \mathbb{L}^{\sum_i \left|\Delta(\alpha)_{\succ \mu_i^0(\alpha)}\right|}. \tag{B.42}$$

$$[\mathrm{Hilb}^0(\alpha)] = [V(\alpha|_{K_\alpha})] \cdot \mathbb{L}^{\sum_{i\in K_\alpha}\left|C(\alpha|_{J_\alpha})_{\succ \mu_i^0}\right|}. \tag{B.43}$$

Proof. To show that $\mathrm{Hilb}^0(\alpha)$ is a closed subscheme of $\mathrm{Hilb}(\alpha)$, we simpy replace the ideal J in the proof of Lemma B.7(2) by an ideal J_0 generated by J together with $d_{\mu,\nu}$ where $\mu = \mu_i^0$. A morphism $\psi : S \to S'$ in the category of k-algebras factors through $S \to S/J_0$ if and only if $\ker(\phi \otimes \psi)$ is monic with Gröbner basis $\{\psi(f_\mu)\}$, such that $\psi(f_\mu) = \psi(\mu)$ if $\mu = \mu_i^0$. It then follows that $\mathrm{Hilb}^0(\alpha)$ is represented by a closed subscheme of $\mathrm{Hilb}(\Delta(\alpha))$, hence a closed subscheme of $\mathrm{Hilb}(\alpha)$. For the two formulas, one can simply upgrade the proofs of Lemma 4.4 and Lemma 4.5. Indeed, one first establishes a stronger structural result

$$\mathrm{Hilb}(\alpha) \simeq \mathrm{Hilb}^0(\alpha) \times \mathbb{A}^{\sum_i \left|\Delta(\alpha)_{\succ \mu_i^0(\alpha)}\right|}. \tag{B.44}$$

$$\mathrm{Hilb}^0(\alpha) \simeq V(\alpha|_{K_\alpha}) \times \mathbb{A}^{\sum_{i\in K_\alpha}\left|C(\alpha|_{J_\alpha})_{\succ \mu_i^0}\right|}. \tag{B.45}$$

通过在上述引理的证明中使用 S-点（而不是我们之前使用的 k-点）来进行处理。不难看出，这两个引理的证明在本质上具有函子性，并且同样适用于 S-点。具体细节留给读者。□

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