arXiv:2512.14573v1 [math.AC] 16 Dec 2025

理想滤子之间的缺陷函数

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Abstract. 我们引入并研究与一对理想滤过相关的缺陷函数，它推广了理想的符号缺陷。假设其中一个滤过的 Rees代数是诺特的，且一个自然的分次模（用于衡量这两个滤过之间的相互作用）在其上有限生成，我们证明相应的缺陷函数渐近地是一个拟多项式。此外，当第一个滤过的 Rees代数是标准分次时，缺陷函数最终成为多项式。对于由单项式理想的饱和与普通幂产生的滤过，我们进一步分析拟多项式的结构。我们证明，在自然假设下，最终拟多项式的最高两个系数为常数。

1. Introduction

理想的符号幂与普通幂之间的比较，长期以来一直是交换代数和代数几何中的核心主题。符号幂体现几何消失条件，而普通幂反映代数重数；理解二者差异的性质，是研究爆破、Rees代数和渐近不变量的重要组成部分（参见[1, 13, 15, 20, 25]）。衡量这种差异的一个特别有用的量是符号缺陷，由 [17], 引入，用来记录给定理想 I 的模 $I^{(m)}/I^m$ 的最小生成元个数。这一不变量已在多种情形下得到研究，包括单项式理想、胖点概形的理想以及定义奇异簇的理想；它在研究包含问题和渐近现象方面也发挥着重要作用（参见 [11, 18, 30], 等）。

符号缺陷自然地融入更广阔的图景。交换代数中的许多构造，例如符号幂、普通幂、整闭包、饱和、冒理想滤子、紧闭包序列和混合幂序列，都会产生理想滤子。比较两个这样的滤子并衡量它们之间的差异是很自然的。受这一观点启发，我们为理想滤子对引入缺陷函数 。设环 R中的理想滤过为 $\mathcal{I} = \{I_n\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{J_n\}_{n\geqslant 0}$ ，其中该环是诺特

2020年 数学主题分类。13A02，13A15。关键词和短语。Hilbert函数、符号缺陷、齐次单项式理想、拟多项式。

局部环或域上的标准分次代数，我们定义

$$\operatorname{def}(\mathcal{I},\mathcal{J},n)=\mu\left(\frac{I_n+J_n}{J_n}\right),$$

which recovers symbolic defect when $\mathcal{I}$ is the symbolic power filtration and $\mathcal{J}$ is the ordinary power filtration (see Definition 3.1). This framework captures a wide range of natural examples and allows a unified treatment of several defect-type invariants.

A second major theme in commutative algebra is the study of *asymptotic behavior* of algebraic invariants. Prior work, including [9, 14, 26, 29, 34] and references therein, show that invariants such as Castelnuovo–Mumford regularity, Hilbert functions, and depth often stabilize or become eventually linear or quasi-polynomial. Recent work has extended these asymptotic phenomena to invariants such as the v-number [8, 16], Waldschmidt constants, and asymptotic resurgence [1, 10, 21]. In the context of symbolic defect, Drabkin and Guerrieri [11] proved that $\operatorname{sdef}(I,n)$ is eventually a quasi-polynomial provided the symbolic Rees algebra of I is Noetherian. Their proof relies on Hilbert–Serre theory applied to a finitely generated graded module over the symbolic Rees algebra.

我们的首个主要结果将 Drabkin–Guerrieri 定理推广到了任意滤过的情形。

定理 3.4. 设 $\mathcal{I}=\{I_i\}_{i\geqslant 0}$ 和 $\mathcal{J}=\{J_i\}_{i\geqslant 0}$ 是 R中的（齐次）理想滤子。假设 *Rees*代数 $\mathcal{R}(\mathcal{I})$是诺特的，且缺陷模

$$N=\bigoplus_{n\geqslant 0}\frac{I_n+J_n}{J_n}$$

是 $\mathcal{R}(\mathcal{I})$上的有限生成模。那么，$\operatorname{def}(\mathcal{I},\mathcal{J},n)$ 最终是关于 n 的拟多项式，其周期为 $\alpha=\operatorname{lcm}(\alpha_1,\alpha_2,\ldots,\alpha_s)$，其中 $\alpha_1,\alpha_2,\ldots,\alpha_s$ 是 $\mathcal{R}(\mathcal{I})$的生成元的次数。最后，如果 $\mathcal{R}(\mathcal{I})$ 作为 R上的代数是标准分次的，那么$\operatorname{def}(\mathcal{I},\mathcal{J},n)$ 最终是关于 n的多项式。

定理 3.4 可以通过将 N_n 替换为$\mathbf{F}(N_n)$来推广，其中$\mathbf{F}$是有限生成 R-模范畴上的一个函子，且与直和交换；同时考虑$\mu(\mathbf{F}(N_n))$ 而非 $\mu(N_n)$；见定理 3.9。特别地，推论 3.10 表明，对于任意有限生成的 R-模 M 以及任意固定的 $i\geqslant 0$, $\mu($, $\operatorname{Tor}_i^R(M,N_n))$ 渐近地是关于 n的拟多项式。

对于单项式理想，指数集和 Newton多面体的几何性质揭示了更多结构（参见 [4, 6, 12, 23, 30, 33]）。对于单项式理想的滤过，我们在定理 4.1 中给出了充分条件，以确保 $(\mathcal{I},\ \mathcal{J})$ 的缺陷函数增长与其整闭包的缺陷函数增长相当。这推广了 [30, 命题 4.5]，并拓展了这样一种观点：单项式滤过从其指数集的组合学中继承线性或准线性的渐近行为。

我们的第二个主要定理研究一个单项式理想 I 相对于另一个单项式理想 J 的饱和滤过与其普通幂滤过I之间的缺陷。对于这对特定的滤过，我们确定了描述该缺陷的最终拟多项式的最高两项系数。

定理 4.2。 设 $R = \mathbb{K}[x_1, \ldots, x_r]$ 是一个标准分次多项式环。设 I 和 J 是单项式理想。考虑 R中理想的两个滤过 $\mathcal{I} = \{I^n : J^\infty\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{I^n\}_{n\geqslant 0}$ 。假设 *Rees*代数 $\mathcal{R}(\mathcal{I})$是诺特的，且缺陷模

$$N = \bigoplus_{n\geqslant 0} \frac{I_n + J_n}{J_n}$$

是 $\mathcal{R}(\mathcal{I})$上的有限生成模。进一步假设 $(\mathcal{I}, \mathcal{J})$ 的拟多项式缺陷函数（由定理 *3.4* 得出）具有如下形式

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = a_c(n)n^c + a_{c-1}(n)n^{c-1} + \ \textit{lower order terms},$$

其中 $n \gg 0$，而 c 是次数。那么，以下结论成立：

(1) 系数 $a_c(n)$是常数。*(2)* 若 I 由次数相同的单项式生成，且 高度$(I) \geqslant 2$，则 $a_{c-1}(n)$ 的非零值为常数。

论文结构。 第 2 节汇集关于滤过、Rees代数、希尔伯特函数和符号缺陷的预备知识。第 3 节证明定理 3.4，并确立缺陷函数的拟多项式行为。第 4 节转向单项式滤过：我们比较缺陷函数及其整闭包类似物，并证明定理 4.2，该定理涉及单项式理想的饱和缺陷的首项系数。

2. Preliminaries

In this section, we introduce various notations and auxiliary results that will be used. Throught the paper, let $\mathbb{K}$ be a field and let R be either a Noetherian local ring with maximal ideal $\mathfrak{m}$ or a standard graded $\mathbb{K}$-algebra with maximal homogeneous ideal $\mathfrak{m}$.

2.1. Filtration of Ideals.

We shall recall the definition of a filtration of ideals, and discuss various rings and algebras that are associated to filtrations.

Definition 2.1. A collection $\mathcal{I} = \{I_i\}_{i\in\mathbb{N}}$ of ideals in R is called a *filtration* if the following conditions hold:

(1) 它是一个理想的分次族，即 $I_0 = R$，并且对每个 $m,\ n \in \mathbb{N}$, $I_m I_n \subseteq I_{m+n}$；以及
(2) 对任意 $n \in \mathbb{N}$，都有 $I_{n+1} \subseteq I_n$.

给定 R 中理想的一个滤子 $\mathcal{I} = \{I_i\}_{i\in\mathbb{N}}$，其*Rees*代数 定义为

$$\mathcal{R}(\mathcal{I}) = \bigoplus_{n=0}^{\infty} I_n t^n \subseteq R[t],$$

and its *special fiber ring* is given by

$$\mathcal{F}(\mathcal{I}) = \mathcal{R}(\mathcal{I}) \otimes_R \frac{R}{\mathfrak{m}}.$$

A filtration $\mathcal{I} = \{I_i\}_{i\in\mathbb{N}}$ in a graded ring R is called *homogeneous* if every I_i is a homogeneous ideal. When R is a polynomial ring, we call $\mathcal{I}$ a *monomial filtration* if each I_i is a monomial ideal.

2.2. **Hilbert Functions and Asymptotic Behaviour.** We now define a quasi-polynomial and recall the Hilbert-Serre theorem in the theory of Hilbert function.

Definition 2.2. A *quasi-polynomial* function in $\mathbb{N}$ is a function $f : \mathbb{N} \to \mathbb{Q}$ such that for some $p \in \mathbb{N}$, there exist polynomials $f_0, f_1, \ldots, f_{n-1}$ with rational coefficients having the property that for all $m \in \mathbb{N}$, $f(m) = f_r(m)$ whenever $m \equiv r \pmod{p}$. The value of p is called the *period* of f.

Theorem 2.3 ([32, 4.4.1])**.** *If a numerical function $\phi : \mathbb{N} \to \mathbb{Z}$ has the generating function*

$$\sum_{n=0}^{\infty} \phi(n)t^n = \frac{q(t)}{\prod_{i=1}^{s}(1 - t^{d_i})}$$

for some $d_1, d_2, \ldots, d_s \in \mathbb{N}$, and some polynomial $q(t) \in \mathbb{Z}[t]$, then $\phi(n) = Q(n)$ for sufficiently large n, where $Q(n)$ is a quasi-polynomial with period $p = \operatorname{lcm}(d_1, d_2, \ldots, d_s)$.

The following theorem is a well-known result in the theory of Hilbert-function of graded modules over a graded ring R.

Theorem 2.4 (Hilbert-Serre)**.** *Let $M = \bigoplus_{n\in\mathbb{Z}} M_i$ be a finitely generated graded module of dimension d over a graded ring $R = \bigoplus_{i\in\mathbb{Z}} R_i$. Then, there exists a quasi-polynomial Q_M of degree $d-1$ such that the Hilbert function of M satisfies $H_M(n) = Q_M(n)$ for $n \gg 0$. Moreover, the period of $Q_M(n)$ divides $\operatorname{lcm}(\deg r_1, \ldots, \deg r_d)$ where $r_1, \ldots, r_d$ is a homogeneous system of parameters of M.*

When the ring R is a *standard* graded ring, the following theorem shows that the Hilbert-function is a polynomial for large values of n.

定理 2.5 ([28, 定理 13.2。])。 设 R 是域 $\mathbb{K}$ 上的标准分次代数，且 M 是维数为 d 的有限生成分次 R-模。存在一个次数为 $d-1$ 的多项式 $\phi_M(X)$，使得当 $n \gg 0$ 时，有 维数$_{\mathbb{K}}(M_n) = \phi_M(n)$。

3. Defect functions of pairs and asymptotic behaviour

本节中，我们引入理想的两组滤过 $(\mathcal{I},\ \mathcal{J})$ 的缺陷函数这一概念（见定义 3.1）。这个不变量衡量两组滤过之间"包含关系的差异"。我们证明，当 $\mathcal{I}$ 的 Rees代数是诺特的，且相应的 缺陷模 在该 Rees代数上有限生成时，缺陷函数最终是一个拟多项式（在标准分次情形下是多项式；见定理 3.4）。

Recall that throughout R will denote either a Noetherian local ring or a standard graded algebra over a field.

Definition 3.1. Let $\mathcal{I} = \{I_i\}_{i\geqslant 0}$ and $\mathcal{J} = \{J_i\}_{i\geqslant 0}$ be (homogeneous) filtrations of ideals in R. The *defect function* of the pair $(\mathcal{I}, \mathcal{J})$ is defined by

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = \mu\left(\frac{I_n + J_n}{J_n}\right).$$

The module $N = \bigoplus_{n\geqslant 0} \dfrac{I_n + J_n}{J_n}$ is called the *defect module* of $(\mathcal{I}, \mathcal{J})$.

The following example illustrates the notion of symbolic function in Definition 3.1.

Example 3.2. Let $R = \mathbb{K}[x_1, x_2, x_3]$, and let $P = (x_2, x_3)$ and $I = (x_1x_2, x_2x_3, x_3x_1)$ be ideals in R. Consider the following filtrations of ideals:

$$\mathcal{I} = \{I^n : P^\infty\}_{n\in\mathbb{N}} \text{ and } \mathcal{J} = \left\{I^{(n)}\right\}_{n\in\mathbb{N}}.$$

Clearly, $I^n : P^\infty = (x_1, x_2)^n \cap (x_1, x_3)^n$. We claim that

$$(x_1, x_2)^n \cap (x_1, x_3)^n = (x_1^{n-i}x_2^i x_3^i \mid i = 0, 1, 2, \ldots, n).$$

Indeed, the containment $(x_1^{n-i}x_2^i x_3^i \mid i = 0, 1, 2, \ldots, n) \subseteq (x_1, x_2)^n \cap (x_1, x_3)^n$ is a straightforward computation. On the other hand, suppose $m \in (x_1, x_2)^n \cap (x_1, x_3)^n$ is a monomial in R. Then, m is divisible by $x_1^{n-i_1}x_2^{i_1}$ and $x_1^{n-i_2}x_3^{i_2}$ for some $i_1, i_2 \leqslant n$. Set $j = \min\{i_1, i_2\}$ then $x_1^{n-j}x_2^j$ and $x_1^{n-j}x_3^j$ divide m. Therefore, $m \in (x_1^{n-i}x_2^i x_3^i \mid i = 0, 1, 2, \ldots, n)$.
For $n = 1$, the only generator of $\frac{I:P^\infty}{I}$ is given by (x_1) and hence $\mu\left(\dfrac{I : P^\infty}{I}\right) = 1$. Now, by [7, Lemma 4.1], for $n \geqslant 2$, we get

$$I^{(n)} = I^n + (D(n)),$$

where $D(n) = \{x_1^{a_1}x_2^{a_2}x_3^{a_3} \mid a_{i_1} + a_{i_2} \geqslant n \text{ for } \{i_1, i_2\} \subseteq \{1, 2, 3\} \text{ and } a_1 + a_2 + a_3 \leqslant 2n - 1\}$. This implies that

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = \mu\left(\frac{I^n : P^\infty}{I^{(n)}}\right) = \left\lceil \frac{n}{2} \right\rceil.$$

注记 3.3. 设 $I, J \subseteq R$ 是理想。

(1) 考虑 $\mathcal{I} = \{I^{(n)}\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{I^n\}_{n\geqslant 0}$。那么，$(\mathcal{I}, \mathcal{J})$ 的缺陷函数为

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = \mu\left(\frac{I^{(n)}}{I^n}\right),$$

which is a well-studied under the nomenclature of *symbolic defect* of I and is often denoted by $\operatorname{sdef}(I, n)$.

(2) Consider $\mathcal{I} = \{\overline{I^n}\}_{n\geqslant 0}$ and $\mathcal{J} = \{I^n\}_{n\geqslant 0}$. Then, the defect function of $(\mathcal{I}, \mathcal{J})$ is

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = \mu\left(\frac{\overline{I^n}}{I^n}\right),$$

which is closely related to the symbolic defect of I, but has not been much investigated.

(3) Let $\mathcal{I} = (I^{(n)} : J^\infty)_{n\geqslant 0}$ and $\mathcal{J} = \{I^n\}_{n\geqslant 0}$. Then, the defect function of $(\mathcal{I}, \mathcal{J})$ is

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = \mu\left(\frac{(I^n : J^\infty)}{I^n}\right).$$

We now prove our first main result regarding the asymptotic behaviour of the defect function. We show that under certain conditions on the Rees algebra of $\mathcal{I}$ and the defect module between $\mathcal{I}$ and $\mathcal{J}$, the defect function is eventually a quasi-polynomial (respectively, a polynomial in the standard graded case).

Theorem 3.4. *Let $\mathcal{I} = \{I_i\}_{i\geqslant 0}$ and $\mathcal{J} = \{J_i\}_{i\geqslant 0}$ be (homogeneous) filtrations of ideals in R. Assume that the Rees algebra $\mathcal{R}(\mathcal{I})$ is a Noetherian and the defect module*

$$N = \bigoplus_{n\geqslant 0} \frac{I_n + J_n}{J_n}$$

is a finitely generated module over $\mathcal{R}(\mathcal{I})$. Then, $\operatorname{def}(\mathcal{I}, \mathcal{J}, n)$ is eventually a quasi-polynomial in n with period $\alpha = \operatorname{lcm}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, where $\alpha_1, \alpha_2, \ldots, \alpha_s$ are the degrees of the generators of $\mathcal{R}(\mathcal{I})$. Finally, if $\mathcal{R}(\mathcal{I})$ is standard graded as an algebra over R, then $\operatorname{def}(\mathcal{I}, \mathcal{J}, n)$ is eventually a polynomial in n.

Proof. We claim that, for any $n \in \mathbb{N}$, $\mu\left(\dfrac{I_n + J_n}{J_n}\right) = \dim_{\mathbb{K}}\left(\dfrac{I_n + J_n}{\mathfrak{m}I_n + J_n}\right)$. Indeed, consider the following isomorphism of R-modules:

$$\frac{R}{\mathfrak{m}} \otimes_R \left(\frac{I_n + J_n}{J_n}\right) \cong \frac{\left(\frac{I_n+J_n}{J_n}\right)}{\mathfrak{m}\left(\frac{I_n+J_n}{J_n}\right)} \cong \frac{I_n + J_n}{\mathfrak{m}(I_n + J_n) + J_n} = \frac{I_n + J_n}{\mathfrak{m}I_n + J_n}.$$

因此，根据中山引理，我们得到所需的等式

$$\mu\left(\frac{I_n + J_n}{J_n}\right) = \dim_{\mathbb{K}}\left(\frac{I_n + J_n}{\mathfrak{m}I_n + J_n}\right).$$

令 $M=\bigoplus_{n\in\mathbb{N}} M_n=\bigoplus_{n\in\mathbb{N}} \frac{I_n+J_n}{\mathfrak{m}I_n+J_n}$。于是， M是特殊纤维锥 $\mathcal{F}(\mathcal{I})=\frac{\mathcal{R}(\mathcal{I})}{\mathfrak{m}\mathcal{R}(\mathcal{I})}$上的有限生成分次模，其中 $(\mathcal{F}(\mathcal{I}))_0=\mathbb{K}$； M在$\infty\mathcal{F}(\mathcal{I})$上的模结构继承自 N在 $\mathcal{R}(\mathcal{I})$上的模结构。因此，根据定理 2.4， $h_M(z)=\sum \mathbf{def}(\mathcal{I},\ \mathcal{J},\ n)z^n$最终是$_{n=0}$一个拟多项式。

Furthermore, if the Rees algebra $\mathcal{R}(\mathcal{I})$ is standard graded, then so is $\mathcal{F}(\mathcal{I})=\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. Therefore, $h_M(z)=\sum_{n=0}^{\infty}\operatorname{def}(\mathcal{I},\mathcal{J},n)z^n$ is asymptotically a polynomial in n with period $\alpha=\operatorname{lcm}(\alpha_1,\alpha_2,\ldots,\alpha_s)$, by Theorem 2.5. □

The following example illustrates Theorem 3.4.

Example 3.5 ([27, Example 4.12])**.** Let $I=(x_1x_2,x_2x_3,x_3x_1,x_1x_4)\subseteq\mathbb{K}[x_1,\ldots,x_4]$. Then, for the filtrations $\mathcal{I}=\{I^{(n)}\}_{n\geqslant 0}$ and $\mathcal{J}=\{I^n\}_{n\geqslant 0}$, their defect function is

$$\operatorname{def}(\mathcal{I},\mathcal{J},n)=\begin{cases}2k^2-1, & n=2k\\ 2k^2+2k, & n=2k+1\end{cases}$$

The following corollaries are direct consequences of Theorem 3.4.

Corollary 3.6. *Let a,b be positive integers. For the filtrations $\mathcal{I}=\{I^{an}\}_{n\geqslant 0}$ and $\mathcal{J}=\{I^{(bn)}\}_{n\geqslant 0}$ where I is a (homogeneous) ideal in R, the defect function*

$$\operatorname{def}(\mathcal{I},\mathcal{J},n)=\mu\left(\frac{I^{an}+I^{(bn)}}{I^{(bn)}}\right)$$

is eventually a polynomial in n.

Proof. The assertion follows from Theorem 3.4, noting that the Rees algebra $\mathcal{R}(\mathcal{I})$ is a standard graded algebra over R. □

Corollary 3.7. *Let I, J be (homogeneous) ideals in R. For the filtrations $\mathcal{I}=\{(I^n:J^\infty)\}_{n\geqslant 0}$ and $\mathcal{J}=\{I^n\}_{n\geqslant 0}$, if the Rees algebra $\mathcal{R}(\mathcal{I})$ is Noetherian, then the defect function*

$$\operatorname{def}(\mathcal{I},\mathcal{J},n)=\mu\left(\frac{(I^n:J^\infty)}{I^n}\right)$$

is eventually a quasi-polynomial in n.

Proof. The assertion follows from Theorem 3.4, under the assumption that the Rees algebra $\mathcal{R}(\mathcal{I})$ is Noetherian. □

定理 3.4 还得出了一个已知结果：局部环中的理想或域上的标准分次代数中的齐次理想的符号缺陷最终呈现拟多项式行为。

推论 3.8 ([11, 定理 2.4])。 设 I 是 R 中的一个（齐次）理想，使得 I 的符号 *Rees* 代数 $\mathcal{R}_s(I) = \bigoplus_{n\geqslant 0} I^{(n)}t^n$ 是诺特环，并设 $\mathcal{R}_s(I)$ 作为 R-代数的极小生成元的次数为 d_1、.....、d_s。那么，符号缺陷

$$\operatorname{sdef}(I,n) = \mu\left(\frac{I^{(n)}}{I^n}\right)$$

最终是关于 n 的拟多项式，其拟周期为 $d =$最小公倍数$(d_1, \ldots, d_s)$。

证明。 由假设可知，Rees代数 $\mathcal{R}(\mathcal{I})$是诺特的，因此该结论可由关于滤过 $\mathcal{I} = \{I^{(n)}\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{I^n\}_{n\geqslant 0}$的定理 3.4 得出。 □

定理 3.4 适用于更一般的情形：将缺陷模$N = \bigoplus_{n\geqslant 0} \dfrac{I_n + J_n}{J_n}$ 的分次分量 N_n 替换为函子 $\mathbf{F}$作用于$\overline{(N_n)}$ 所得的结果。

定理 3.9。 设 $\mathcal{I} = \{I_i\}_{i\geqslant 0}$ 和 $\mathcal{J} = \{J_i\}_{i\geqslant 0}$ 是 R中的理想滤子。假设 *Rees*代数 $\mathcal{R}(\mathcal{I})$ 是诺特的，且缺陷模 $N = \bigoplus_{n\geqslant 0} \dfrac{I_n + J_n}{J_n}$ 是 $\mathcal{R}(\mathcal{I})$上的有限生成模。设$\mathbf{F}$: R-模$^{\mathrm{fg}} \longrightarrow R$-模^{fg} 是有限生成 R-模范畴上的函子，并且与直和交换，即$\mathbf{F}\left(\bigoplus_{\lambda\in\Lambda} M_\lambda\right) = \bigoplus_{\lambda\in\Lambda} \mathbf{F}(M_\lambda)$。那么， $\mu(\mathbf{F}(N_n))$ 最终是关于 n 的拟多项式，其周期为 $\alpha =$最小公倍数$(\alpha_1, \alpha_2, \ldots, \alpha_s)$，其中 $\alpha_1, \alpha_2, \ldots, \alpha_s$ 是 $\mathcal{R}(\mathcal{I})$ 的生成元的次数。此外，若 $\mathcal{R}(\mathcal{I})$ 作为 R 上的代数是标准分次的，则 $\mu(\mathbf{F}(N_n))$ 最终是关于 n的多项式。

Proof. The proof goes in the same line of arguments as that of Theorem 3.4. Particularly, let $N_n = \dfrac{I_n + J_n}{I_n}$. Similar to what was done in Theorem 3.4, by considering

$$\mathbf{F}(N_n) \otimes_R \frac{R}{\mathfrak{m}} \cong \frac{\mathbf{F}(N_n)}{\mathfrak{m}\mathbf{F}(N_n)},$$

and applying Nakayama's lemma, we get

$$\mu(\mathbf{F}(N_n)) = \dim_{\mathbb{K}}\left(\frac{\mathbf{F}(N_n)}{\mathfrak{m}\mathbf{F}(N_n)}\right).$$

Since N is a finitely generated module over $\mathcal{R}(\mathcal{I})$, $\bigoplus_{n\geqslant 0}\left(\dfrac{\mathbf{F}(N_n)}{\mathfrak{m}\mathbf{F}(N_n)}\right)$ is a finitely generated graded module over the special fiber cone $\mathcal{F}(\mathcal{I})$. The conclusion now follows from Theorems 2.4 and 2.5. □

Theorem 3.9 gives the following corollaries.

推论 3.10. 设 $\mathcal{I} = \{I_i\}_{i\geqslant 0}$ 和 $\mathcal{J} = \{J_i\}_{i\geqslant 0}$ 是 R中的理想滤子。假设 *Rees*代数 $\mathcal{R}(\mathcal{I})$ 是诺特的，且缺陷模 $N = \bigoplus_{n\geqslant 0} \dfrac{I_n + J_n}{J_n}$ 是 $\mathcal{R}(\mathcal{I})$上的有限生成模。那么，下列结论成立：

(1) 对于环 R的任意乘法封闭集 S， $\mu\left(S^{-1}(N_n)\right)$ 最终是 n中的一个准多项式。

(2) For R-modules $M_1, M_2, \ldots, M_k$ and $i_1, i_2, \ldots, i_k \in \mathbb{N}$,

$$\mu\Big(\mathrm{Tor}_{i_1}^R\Big(M_1,\ \mathrm{Tor}_{i_2}^R\big(M_2,\ \ldots\ \mathrm{Tor}_{i_k}^R(M_k, N_n)\ldots\big)\Big)\Big)$$

is eventually a quasi-polynomial in n.

Furthermore, in these situations, the period of the quasi-polynomial is $\alpha = \mathrm{lcm}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, where the numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$ are the degrees of the generators of $\mathcal{R}(\mathcal{I})$. In addition, if $\mathcal{R}(\mathcal{I})$ is standard graded as an algebra over R then these quasi-polynomials are in fact polynomials.

Proof. (1) The assertion follows directly from Theorem 3.9, noting that

$$S^{-1}\left(\bigoplus_{n\geqslant 0}\frac{I_n + J_n}{J_n}\right) = \bigoplus_{n\geqslant 0} S^{-1}\left(\frac{I_n + J_n}{J_n}\right).$$

(2) The conclusion is also a direct consequence of Theorem 3.9, due to the fact that

$$\mathrm{Tor}_i^R\left(M, \bigoplus_{n\geqslant 0}\frac{I_n + J_n}{J_n}\right) \cong \bigoplus_{n\geqslant 0}\mathrm{Tor}_i^R\left(M, \frac{I_n + J_n}{J_n}\right)$$

for any R-module M and any $i \in \mathbb{N}$. □

注记 3.11. 我们不知道推论 3.10 的类似结论是否也适用于 Ext 模。此类结论是否存在，取决于 Ext 与直和之间的交换性，这方面已有大量研究（例如参见 [2] 和 [3]）。

4. Monomial ideals and leading coefficients of defect functions

本节研究多项式环 R 中一对单项式滤过（即每个理想都是单项式理想的滤过）的缺陷函数。我们的首要目标是比较缺陷函数与由整闭包滤过产生的缺陷的增长情况；见定理 4.1。这推广了 [30, 命题 4.5] 中关于符号缺陷的相应结果。随后，我们证明第二个主要定理：对于滤过 $\mathcal{I} = \{I^n :\ J^\infty\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{I^n\}_{n\geqslant 0}$，描述缺陷的最终拟多项式的首项系数为常数；并且在等次生成理想 I 满足较弱的高度条件时， n 的次高次幂的系数也为常数；见定理 4.2。

定理 4.1. 设 $R = \mathbb{K}[x_1, \ldots, x_r]$ 是域 $\mathbb{K}$ 上的标准分次多项式环。设 $\mathcal{I} = \{I_n\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{J_n\}_{n\geqslant 0}$ 是 R 中的单项式滤过，且对所有 $n \geqslant 0$， $I_n \subseteq J_n$ 。

(1) Set $T_n = \dfrac{\mathfrak{m}\overline{I}_n \cap \overline{J}_n}{\mathfrak{m}\overline{J}_n}$. If $\lim\limits_{n\to\infty}\dfrac{\mu(\overline{I}_n)}{\dim_{\mathbb{K}}(T_n) - \mu(\overline{J}_n)} \neq 1$, then $\lim\limits_{n\to\infty}\dfrac{\mathrm{def}(\mathcal{I}, \mathcal{J}, n)}{\mathrm{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n)} < \infty$.

(2) Set $S_n = \dfrac{I_n \cap \overline{J}_n}{J_n}$. *If* $\lim\limits_{n\to\infty} \dfrac{\mu(S_n)}{\mathrm{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n)} < \infty$, *then* $0 < \lim\limits_{n\to\infty} \dfrac{\mathrm{def}(\mathcal{I}, \mathcal{J}, n)}{\mathrm{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n)} < \infty$.

证明。当 $n \gg 0$ 时，$\mathbf{def}(\mathcal{I}, \mathcal{J}, n) = 0$ 或 $\mathbf{def}(\mathcal{I}, \mathcal{J}, n) = \overline{0}$，结论立即成立。因此，我们可以假设对 $n \gg 0$，$\mathbf{def}(\mathcal{I}, \mathcal{J}, n)$ 和 $\mathbf{def}(\mathcal{I}, \mathcal{J}, n)$ 均非零。

To prove (1), consider the following short exact sequence :

$$0 \longrightarrow \overline{J}_n \longrightarrow \overline{I}_n \longrightarrow \frac{\overline{I}_n + \overline{J}_n}{\overline{J}_n} \longrightarrow 0.$$

Tensoring with $\mathbb{K}$ and applying Nakayama's lemma, we get

$$\mu(\overline{J}_n) + \ \mathrm{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n) = \dim_{\mathbb{K}} T_n + \mu(\overline{I}_n).$$

That is, $\mathrm{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n) = \dim_{\mathbb{K}} T_n + \mu(\overline{I}_n) - \mu(\overline{J}_n)$.

Let $P(n) = \dim_{\mathbb{K}} T_n - \mu(\overline{J}_n)$ and $F_n = \mathrm{def}(\mathcal{I}, \mathcal{J}, n)/\mathrm{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n)$. Clearly,

$$F_n \leqslant \frac{\mu(I_n)}{\mu(\overline{I}_n) + P(n)}.$$

It follows from [22, Corollary 4.4] that the rates of growth of $\mu(I_n)$ and $\mu(\overline{I}_n)$ are both $\ell(\mathcal{I}) - 1$, where $\ell(\mathcal{I})$ is the analytic spread of $\mathcal{I}$. Therefore, if $\deg(P(n)) > \ell(\mathcal{I}) - 1$ then $F_n \to 0$ as $n \to \infty$. On the other hand, if $\deg(P(n)) \leqslant \ell(\mathcal{I}) - 1$, then F_n is bounded above by a finite number. In both cases, we have $\lim_{n\to\infty} F_n$ is finite.

To prove (2), we consider the following commutative diagram :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & J_n & \longrightarrow & \overline{J_n} & \xrightarrow{\phi} & \dfrac{\overline{J}_n}{J_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_n & \longrightarrow & \overline{I_n} & \longrightarrow & \dfrac{\overline{I}_n}{J_n} & \longrightarrow & 0.
\end{array}$$

Using the Snake Lemma, we get the following exact sequence:

$$0 \longrightarrow \frac{\overline{J}_n \cap I_n}{J_n} \longrightarrow \frac{I_n}{J_n} \longrightarrow \frac{\overline{I}_n}{\overline{J}_n} \longrightarrow C_n \longrightarrow 0$$

其中 C_n 表示映射 $\dfrac{I_n}{J_n} \longrightarrow \dfrac{\overline{I}_n}{\overline{J}_n}$ 的余核。将其与 $\mathbb{K}$ 张量化并应用中山引理，可得

$$\mathrm{def}(\mathcal{I}, \mathcal{J}, n) \leqslant \mathrm{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n) + \mu\left(S_n\right).$$

由此可得 $F_n \leqslant \frac{\mu(S_n)}{\overline{\overline{\quad}}}$ 定义为$(\mathcal{I}, \mathcal{J}, n)^{+1}$ 因此,

$$0 < \lim_{n\to\infty} \frac{\operatorname{def}(\mathcal{I}, \mathcal{J}, n)}{\operatorname{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n)} < \infty \text{ whenever } \lim_{n\to\infty} \frac{\mu(S_n)}{\operatorname{def}(\overline{\mathcal{I}}, \overline{\mathcal{J}}, n)} < \infty. \quad \square$$

现在我们证明第二个主要结果，它表明， n 在 $\mathcal{I} = I^n : J^\infty$和 $\mathcal{J} = \{I^n\}_{n\geqslant 0}$ 之间的缺陷函数中，首项系数和次高次幂的系数是常数。该结果的证明受到 [31, 定理 1.2] 的证明启发。在开始证明之前，回顾一下：对于满足 $I_0 = R$ 和 $I \subseteq I_1$的齐次理想的乘法滤过 $\mathcal{F} = \{I_n\}_{n\geqslant 0}$， $L^{\mathcal{F}} = \bigoplus R/I_{n+1}$ 具有 $R[It]$-模$_{n\geqslant 0}$结构（细节见 [31, 2.3] ）。这一模结构如下产生：利用 $\mathcal{R}(\mathcal{F}) = \bigoplus I_n t^n$ 是分次 $R[It]$-模这一事实以及短正合列$_{n\geqslant 0}$

$$0 \longrightarrow \mathcal{R}(\mathcal{F}) \longrightarrow R[t] \longrightarrow L^{\mathcal{F}}(-1) \longrightarrow 0,$$

我们得到 $L^{\mathcal{F}}(-1)$ 上的 $R[It]$-模结构，进而得到 $L^{\mathcal{F}}$ 的 $R[It]$-模结构。

定理 4.2。 设 $R = \mathbb{K}[x_1, \ldots, x_r]$ 是一个标准分次多项式环。设 I和 J为单项式理想。考虑 R 中理想的两个滤过 $\mathcal{I} = \{I^n : J^\infty\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{I^n\}_{n\geqslant 0}$ 。假设 ***Rees***代数 $\mathcal{R}(\mathcal{I})$ 是诺特的，且缺陷模

$$N = \bigoplus_{n\geqslant 0} \frac{I_n + J_n}{J_n}$$

is a finitely generated module over $\mathcal{R}(\mathcal{I})$. *Suppose further that the quasi-polynomial (by Theorem 3.4) defect function of* $(\mathcal{I}, \mathcal{J})$ *is of the form*

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = a_c(n)n^c + a_{c-1}(n)n^{c-1} + \text{ lower order terms},$$

for $n \gg 0$, *where* c *is the degree. Then, the following statements hold:*

(1) The coefficient $a_c(n)$ *is constant.*

(2) If I *is generated by monomials in the same degree and* $\operatorname{ht}(I) \geqslant 2$, *then the non-zero values of* $a_{c-1}(n)$ *are constant.*

Proof. (1) The assertion is a direct application of [24, Theorem 2.4] on the $\mathcal{R}(\mathcal{I})$-module $N = \bigoplus_{n\geqslant 0} \frac{I_n + J_n}{J_n}$, where $I_n = I^n : J^\infty$.

(2) Suppose that the quasi-polynomial of $\operatorname{def}(\mathcal{I}, \mathcal{J}, n)$ has degree c and period α (it is worthwhile to note that, by Theorem 2.4, the polynomials appearing in the asymptotic quasi-polynomial of $\operatorname{def}(\mathcal{I}, \mathcal{J})$ have the same degree c). Let $L^{\mathcal{I}} = \bigoplus_{n\geqslant 0} \frac{R}{I^{n+1} : J^\infty}$ and $L^{\mathcal{J}} = \bigoplus_{n\geqslant 0} \frac{R}{I^{n+1}}$。然后，对于 $W = \bigoplus_{n\geqslant 0} \frac{I^{n+1} : J^\infty}{I^{n+1}}$，我们有以下 $R[It]$-模的短正合列

$$0 \longrightarrow W \longrightarrow L^{\mathcal{J}} \longrightarrow L^{\mathcal{I}} \to 0. \tag{4.1}$$

利用以下事实： I 是由同次数单项式生成的单项式理想，存在一个齐次的 I-表面元 u，它对于滤子 $\mathcal{I}$ 也是表面的（详见[31, *1.2* 的证明。]）。令 $p = ut$，则根据 [31]，映射 $L^{\mathcal{I}}(-1) \xrightarrow{\cdot p} L^{\mathcal{I}}$的核集中在小于或等于 $n_0 \in \mathbb{N}$的次数中，使得对于所有 $n \geqslant n_0$，都有 $(I^{n+1} : u) = I^n$。这给出了以下短正合列：

$$0 \longrightarrow \left(\frac{W}{pW}\right)_{n\geqslant n_0+1} \longrightarrow \left(\frac{L^{\mathcal{J}}}{pL^{\mathcal{J}}}\right)_{n\geqslant n_0+1} \longrightarrow \left(\frac{L^{\mathcal{I}}}{pL^{\mathcal{I}}}\right)_{n\geqslant n_0+1} \longrightarrow 0. \tag{4.2}$$

Consider the standard graded $\mathbb{K}$ algebra $S = R/(u)$, the S-ideals $I' = I/(u)$ and $J' = \dfrac{J+(u)}{(u)}$, and the Noetherian S-filtration $\mathcal{I}' = \left\{\dfrac{I_n+(u)}{(u)}\right\}_{n\geqslant 0}$. Note that, $\operatorname{grade}(I) = \operatorname{ht}(I) \geqslant 2$ which gives $\operatorname{grade}(I') \geqslant 1$. Let $\mathcal{J}' = \left\{\dfrac{I^n+(u)}{(u)}\right\}_{n\geqslant 0}$. Using [24], we have $\dim_{\mathbb{K}}\left((I^n : J^\infty)/I^n\right)$ and $\dim_{\mathbb{K}}\left(\dfrac{(I^n : J^\infty)+(u)}{I^n+(u)}\right)$ are constants for $n \gg 0$, we denote these constants by r and s respectively. Furthermore, by Theorem 2.3, $\operatorname{def}(\mathcal{I}', \mathcal{J}', n)$ is eventually a quasi-polynomial of period α' and degree l, i.e.,

$$\operatorname{def}(\mathcal{I}', \mathcal{J}', n) = b_l(n)n^l + \text{ lower order terms.}$$

By (1), $b_l(n)$ is a constant b for all values of n.

For $n \geqslant n_0 + 1$, consider the multiplication map:

$$f : \frac{I^{n-1} : J^\infty}{I^{n-1}} \longrightarrow \frac{I^n : J^\infty}{I^n},$$

defined by $f(x + I^{n-1}) = ux + I^n$. We first compute the kernel of the map f. An element $x + I^{n-1} \in \operatorname{Ker}(f)$ if and only if $ux \in I^n$ and hence, we have:

$$\operatorname{Ker}(f) = \frac{(I^{n-1} : J^\infty) \cap (I^n : u)}{I^{n-1}}.$$

Since u is an I-superficial element and $n \geqslant n_0 + 1$, we have $I^n : u = I^{n-1}$. Therefore, $\operatorname{Ker}(f) = 0$. Note that the image of the map f is given by:

$$\operatorname{Im}(f) = \frac{u(I^{n-1} : J^\infty) + I^n}{I^n}.$$

Consequently, $\operatorname{Coker}(f) = \dfrac{I^n : J^\infty}{u(I^{n-1} : J^\infty) + I^n} \cong \dfrac{(I^n : J^\infty)+(u)}{I^n+(u)}$ (from (4.1) and (4.2)). Combining the above, we get the following short exact sequence for all $n \gg 0$:

$$0 \longrightarrow \frac{I^{n-1} : J^\infty}{I^{n-1}} \longrightarrow \frac{I^n : J^\infty}{I^n} \longrightarrow \frac{(I')^n : (J')^\infty}{(I')^n} \longrightarrow 0.$$

因此， $r \geqslant s$。如果 $r > s$，那么 $c = 0$，无需证明。

现在，假设 $r = s$。我们有 $l = c-1$。最后，由于 $\mathbf{def}(\mathcal{I}, \mathcal{J}, n)$–$\mathbf{def}(\mathcal{I}, \mathcal{J}, n-1) \leqslant \mathbf{def}(\mathcal{I}', \mathcal{J}', n)$，我们得到以下表达式：

$$ac + a_{c-1}(n) - a_{c-1}(n-1) \leqslant b \text{ for all } n \gg 0,$$

其中 a 是拟多项式 $\mathbf{def}(\mathcal{I},\ \mathcal{J},\ n)$ 关于$n \gg 0$的常数最高次系数。因此，对于 $n \gg 0$，我们得到 $a_{c-1}(n) \leqslant \gamma n + \lambda$，其中某些常数为 γ 和 λ（见 [5,定理 4.1.2.]）。假设 $a_{c-1}(n)$非零，则 $a_{c-1}(n)$是关于 n 的周期函数。由此可得 $\gamma = 0$，因此当 n充分大时，$a_{c-1}(n)$ 为常数。□

例 3.5 表明，定理 4.2 中 $a_{c-1}(n)$ 的表达式对于某些 $n \gg 0$ 的取值可能为零。然而，下面的例子通过展示 $a_{c-1}(n)$ 的非零取值为常数，说明了定理 4.2。

例 4.3 ([27, 例 4.11])。令 $I = (x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5,\ x_5x_1) \subseteq \mathbb{K}[x_1,\ \ldots\ldots,\ x_5]$。那么，对于滤子 $\mathcal{I} = \{I^{(n)}\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{I^n\}_{n\geqslant 0}$，它们的缺陷函数为

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = \begin{cases} \frac{15}{2}k^3 - \frac{15}{2}k^2 + 1 & \text{if } n = 3k, \\ \frac{15}{2}k^3 - \frac{5}{2}k & \text{if } n = 3k+1, \\ \frac{15}{2}k^3 + \frac{15}{2}k^2 & \text{if } n = 3k+2. \end{cases}$$

We end the paper with an example which shows that I is equigenerated in Theorem 4.2 is necessary for the non-zero values of $a_{c-1}(n)$ to be constant.

例 4.4 ([7, 例 6.12])。设 $I = (xyz,\ x^2z,\ y^3x,\ z^4y) \subseteq \mathbb{K}[x,\ y,\ z]$。那么，对于滤过 $\mathcal{I} = \{I^{(n)}\}_{n\geqslant 0}$ 和 $\mathcal{J} = \{I^n\}_{n\geqslant 0}$，它们的缺陷函数为

$$\operatorname{def}(\mathcal{I}, \mathcal{J}, n) = \begin{cases} 0, & n = 1 \\ \frac{11}{5}n - 2, & n \equiv 0 \pmod 5 \\ \frac{11}{5}n - \frac{6}{5}, & n \equiv 1 \pmod 5 \\ \frac{11}{5}n - \frac{7}{5}, & n \equiv 2 \pmod 5 \\ \frac{11}{5}n - \frac{3}{5}, & n \equiv 3 \pmod 5 \\ \frac{11}{5}n - \frac{4}{5}, & n \equiv 4 \pmod 5 \end{cases}$$

致谢。 第一作者获得印度理工学院克勒格布尔分校的创业资助。第二作者获得西蒙斯基金会的部分资助。第三作者感谢印度政府通过总理研究奖学金支持其开展这项工作。本项目的部分工作是在第三作者访问第二作者所在的杜兰大学期间完成的。作者感谢杜兰大学的热情接待。作者感谢使用计算机代数系统 Macaulay2 [19] 测试计算结果。

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