真空存在时可压缩 NAVIER-STOKES-RIESZ 系统中稳态的渐近稳定性 ASYMPTOTIC STABILITY OF STEADY STATES FOR THE COMPRESSIBLE NAVIER-STOKES-RIESZ SYSTEM IN THE PRESENCE OF VACUUM

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摘要。我们研究了一维物理真空自由边界问题，该问题以具有吸引性 Riesz 势 $|x|^{-(2-s)}$ 的可压缩 Navier-Stokes-Riesz 系统，其中 $0 < s < 1/2$。对任意具有紧支撑的稳态密度轮廓，其径向质量密度在 $(2-s)$ 阶奇异，我们证明了其在加权 Sobolev 空间中具有相同质量的稳态解的 Lyapunov 型稳定性。具体而言，我们在拉格朗日坐标下建立了紧支撑稳态的收敛的收敛速率，其中权重刻画了解在真空自由边界附近的行为。证明中的困难和挑战不仅源于真空自由边界导致的退化性，也源于 Riesz 势的非局部特征。

Abstract. We study the one-dimensional physical vacuum free boundary problem for the compressible Navier-Stokes-Riesz system with attractive Riesz potential $|x|^{-(2-s)}$ with $0 < s < 1/2$. It is proved that for the adiabatic constant γ satisfying $2(1-s) < \gamma \leq 1 + 2s/3$, and additional condition that $3/8 < s < 1/2$, there exists a unique global-in-time strong solution. Specifically, we establish the Lyapunov-type stability of the compactly supported steady states with the same mass in weighted Sobolev spaces where the weights indicate the behavior of solutions near the vacuum free boundary. The difficulties and challenges in the proof are caused not only by the degeneracy due to the vacuum free boundary but also by the non-local feature of the Riesz potential.

1. 引言 INTRODUCTION

The motion of compressible viscous fluids with attractive Riesz-type interaction in $\mathbb{R}^n$ can be modeled by the following system of equations,

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \operatorname{div} \mathfrak{S} + \chi \rho \nabla W_k = 0, \tag{1.2}$$

对于 $t \geqslant 0$, $x \in \mathbb{R}^n$。其中，$\rho \geqslant 0$ 和 $\mathbf{u} \in \mathbb{R}^n$ 分别是流体的密度和速度，应力张量 $\mathfrak{S}$ 定义为

for $t \geqslant 0$, $x \in \mathbb{R}^n$. Here, $\rho \geqslant 0$ and $\mathbf{u} \in \mathbb{R}^n$ are the density and velocity of the fluid, respectively. The stress tensor $\mathfrak{S}$ is given by

$$\mathfrak{S} := p I_n - \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^\top - \frac{2}{3} (\operatorname{div} \mathbf{u}) I_n \right) - \lambda (\operatorname{div} \mathbf{u}) I_n,$$

其中 I_n 是 $n \times n$ 单位矩阵，$\mu > 0$ 是剪切黏度，$\lambda \geqslant 0$ 是体积黏度，而 p 是满足以下状态方程的压力，

where I_n is the $n \times n$ identity matrix, $\mu > 0$ is the shear viscosity, $\lambda \geqslant 0$ is the bulk viscosity, and p is the pressure that satisfies the equation of state,

$$p := K \rho^\gamma,$$

常数 $K > 0$ 和绝热指数 $\gamma \geqslant 1$。相互作用势 W_k 定义为 $W_k = W_k(x)$，其中 $k = s - n$ with $s \in (0, n/2)$ and W_k is given by

$$\nabla W_k(\mathbf{x}) := \begin{cases} \nabla (W_k * \rho)(x), & \text{for } 1 - n \leqslant k < 0, \\ \int_{\mathbb{R}^n} \nabla W_k(x - y)(\rho(y) - \rho(x))\, dy, & \text{for } -n < k \leqslant 1 - n. \end{cases} \tag{1.3}$$

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请注意，在更奇异的范围 $-n < k \le -1$ 内，假设密度 ρ 具有适当的正则性，我们需要减去一项以使积分收敛，或者也可以用柯西主值积分来定义该积分。这里，W_k 是梨曼核，其定义为

Notice that in the more singular range $-n < k \le -1$, we need to substract a monomial to make the integral converge, or also we can assume enough regularity on the density ρ. Alternatively, one could defined the integral using Cauchy principal value integrals. Here, W_k is the Riesz kernel given by

$$W_k(\mathbf{x}) := \begin{cases} \dfrac{|\mathbf{x}|^k}{k}, & \text{for } -n < k < 0, \\ \log|\mathbf{x}|, & \text{for } k = 0. \end{cases}$$

众所周知，W_k 由分数阶扩散过程控制，

As is well known, W_k is governed by a fractional diffusion process,

$$C_{n,s}(-\Delta)^s W_k = \delta_0,$$

其中

where

$$C_{n,s} := \begin{cases} \dfrac{\Gamma(n/2-s)}{(2s-n)\,\pi^{n/2}\,4^s\,\Gamma(s)}, & \text{for } k < \min\{0, 2-n\}, \\ 2\pi, & \text{for } k = 0 \text{ and } n = 2, \end{cases}$$

其中 Γ 为伽马函数。在 (1.2) 中，$\chi \in \mathbb{R}$ 表示排斥强度（当 $\chi < 0$ 时）或吸引强度（当 $\chi > 0$ 时）。系统 (1.1)-(1.2) 通常称为可压缩 Navier-Stokes-Riesz 系统，可用于描述多孔介质、等离子体物理、重力或星系动力学以及数学生物学中的群集模型，参见例如 [1, 6, 5, 27]。从现在起，我们固定 $K = \chi = 1$，因为我们只讨论吸引情形。

with Γ being the gamma function. In (1.2), $\chi \in \mathbb{R}$ represents the strength of repulsion if $\chi < 0$ or attraction if $\chi > 0$. The system (1.1)-(1.2) is usually called the compressible Navier-Stokes-Riesz system, and it can be used to describe porous media, plasma physics, gravitational or galactic dynamics, and swarming models in mathematical biology, see for instance [1, 6, 5, 27]. From now on, we will fix $K = \chi = 1$, since we are dealing only for the attractive case.

众所周知，该系统存在由两种相互竞争的行为所导致的静态解：流体粒子之间的排斥（压力）以及通过非局部自洽力产生的吸引。事实上，$\nabla \mathbf{u} = 0$，系统 (1.1)–(1.2) 可化为

It is well known that the system admits steady states resulting from the balance between two competing behaviors: the repulsion (pressure) between fluid particles and the attraction via a non-local self-consistent force. Indeed, by setting $\mathbf{u} = 0$, the system (1.1)-(1.2) is reduced to

$$\nabla(\bar{\rho}^\gamma) + \bar{\rho}\nabla\bar{\Psi}_k = 0, \tag{1.4}$$

其中 Ψ_k 满足将 ρ 替换为 $\bar{\rho}$ 后的 (1.3)。我们将 (1.4) 称为稳态扩散聚集方程。根据 [4, 8, 9, 2] 中的结果（见附录 A 中的总结），对于 $n < \gamma < \infty$（其中 $\gamma_c = 2(1-s/n)$ 以及

where Ψ_k satisfies (1.3) with ρ substituted by $\bar{\rho}$. We call (1.4) by the steady diffusion-aggregation equation. According to [4, 8, 9, 2] (see a summary in Appendix A), for $\gamma_c < \gamma < \gamma^*$, where $\gamma_c := 2(1-s/n)$ and

$$\gamma^* := \begin{cases} \frac{k-1}{k}, & \text{for } n \ge 1 \text{ and } 0 < s < \frac{1}{2}, \\ +\infty, & \text{for } n \ge 2 \text{ and } \frac{1}{2} \le s < \frac{n}{2}, \end{cases}$$

所有满足 (1.4) 的驻态 $\bar{\rho} \ge 0$（精确定义见附录 A）均关于其质心是径向对称递减，此外，在平移意义下，它是唯一的，并且可以画为 $\bar{\rho}(x) = \bar{\rho}(|x|)$。相关的非线性泛函 $\mathcal{F}$ 的唯一全局极小值，

all stationary states $\bar{\rho} \ge 0$ (see precise definition in Appendix A) satisfying (1.4), are radially decreasing about their center of mass, and unique up to translations. Moreover, they can be characterized as the unique global minimizer of the nonlinear functional $\mathcal{F}$ associated with (1.4),

$$\mathcal{F}[\rho] := \int_{\mathbb{R}^n} \frac{1}{\gamma-1}\rho^\gamma + \frac{1}{2}\rho(W_k * \rho)\,d\mathbf{x}, \tag{1.5}$$

在该类中

in the class

$$\mathcal{Y} := \left\{ \rho \in L^1_+(\mathbb{R}^n) \cap L^\gamma(\mathbb{R}^n),\ \|\rho\|_1 = 1,\ \int_{\mathbb{R}^n} x\rho(x)\,dx = 0 \right\}.$$

此外，它们是紧支撑的 $\operatorname{supp}(\rho) = B_R$，对于某个 $R > 0$，且 $\rho = \bar{\rho}_k$，$\bar{\rho}_k$ 对于 $k < \min\{0, 2-n\}$ 是严格递减的，对于 $k = 0$ 是常数。因此，$\gamma(p, 0)$ 是系统 (1.1)-(1.2) 的稳态解，并且 $(\bar{\rho}, 0)$ 也是 (1.6)-(1.7) 相关的下列能量泛函 $\mathcal{E}$ 的全局极小值，

Moreover, they are compactly supported $\operatorname{supp}(\rho) = B_R$ for some $R > 0$, and $\rho = \bar{\rho}_k$, with $\bar{\rho}_k$ for $k < \min\{0, 2-n\}$ strictly decreasing, and constant for $k = 0$. Therefore, $\gamma(\bar{\rho}, 0)$ is steady solution to the system (1.1)-(1.2), and $(\bar{\rho}, 0)$ is also a global minimizer of the following energy functional $\mathcal{E}$ associated with (1.6)-(1.7),

$$\mathcal{E}[\rho, \mathbf{u}] := \int_{\mathbb{R}^n} \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{\gamma-1}\rho^\gamma + \frac{1}{2}\rho W_k * \rho\,d\mathbf{x}.$$

Indeed, for any solution (ρ, u) to (1.1)-(1.2), we have

$$E[\rho, u] \geq E[\rho, 0] \equiv F[\rho] \geq F[\bar{\rho}].$$

事实上，对于(1.1)-(1.2)的任意解(ρ,u)，我们有

In this paper, our aim is to prove the global-in-time stability of the steady solution $(\bar{\rho}, 0)$ to the free boundary problem for the compressible Navier-Stokes-Riesz system (1.1)-(1.2) with $2(1-s) < \gamma \leq 2$. To be more specific, we consider

本文旨在通过考察可压缩 Navier-Stokes-Riesz 系统(1.1)-(1.2)的一维($n=1$)真空自由边界问题，证明稳态解$(\bar{\rho},0)$的全局时间稳定性，其中$2(1-s)<\gamma\leq 2$。具体地，我们考虑

$$\partial_t \rho + \partial_\xi (\rho u) = 0, \tag{1.6}$$

$$\rho(\partial_t u + u\partial_\xi u) + \partial_\xi p = \partial_{\xi\xi} u - \rho \partial_\xi V^k, \tag{1.7}$$

for $t > 0$ and $0 \leqslant \xi \leqslant a(t)$, where $k = 2s-1$, and $a(t)$ is a free boundary determined by

对于$t>0$和$0\leqslant\xi\leqslant a(t)$，其中$k=2s-1$，且$a(t)$是由下式确定的自由端点

$$\begin{cases} \dfrac{\mathrm{d}a(t)}{\mathrm{d}t} = u(t, a(t)), \\ a(0) = a_0, \end{cases} \tag{1.8}$$

for some fixed $a_0 > 0$. Here, we set $4\mu/3 + \lambda$ to be unity for convenience. At the origin, we impose a no-flow boundary condition,

对于某个固定的$a_0>0$。这里，为方便起见，我们令$4\mu/3+\lambda$为1。在原点处，我们施加无流边界条件，

$$u(t, 0) \equiv 0, \tag{1.9}$$

while at the free boundary, we impose the Neumann condition,

在自由端点处，我们施加诺伊曼条件，

$$\partial_\xi u(\tau, a(\tau)) = 0. \tag{1.10}$$

Furthermore, we will omit the dependence of V and W on k for the sake of simplicity.

为简洁起见，从现在起我们省略V和W对k的依赖。

The initial condition is given by

初始条件给定为

$$(\rho, u)(0, \xi) \equiv (\rho_0, u_0)(\xi), \tag{1.11}$$

with ρ_0 satisfying

其中ρ_0满足

$$\int_0^{a_0} \rho_0(\xi)\,\mathrm{d}\xi \equiv \int_0^{\bar{R}} \bar{\rho}(x)\,\mathrm{d}x, \quad \rho_0(a_0) = 0, \quad (\rho_0^{\gamma-1}(a_0))' = (\bar{\rho}^{\gamma-1}(\bar{R}))', \tag{1.12}$$

which implies that the initial mass of the fluid equals the mass of the steady state, the fluid continuously connects to the vacuum across the free boundary, and the sound speed $c = \sqrt{\rho^{\gamma-1}}$ coincides with that of the steady state to the first-order derivative at the vacuum boundary. Moreover, by the conservation of mass (1.6) and (1.10), we have $a'(t) = 0$. 还应注意，由于$\rho\in W^{1,\infty}$当$2(1-s)<\gamma\leq 2$时，

这表明，在初始时刻，流体的质量等于稳态的质量，流体在自由边界处与真空连续连接，声速$c=\sqrt{\rho^{\gamma-1}}$在真空边界处与稳态的声速具有相同的一阶导数。此外，由质量守恒(1.6)和(1.10)，我们还得到$a'(t)=0$。还应注意，由于$\bar{\rho}\in W^{1,\infty}$当$2(1-s)<\gamma\leq 2$时，

$$-\infty < (\rho_0^{\gamma-1}(a_0))' \equiv (\bar{\rho}^{\gamma-1}(\bar{R}))' < 0,$$

which implies that the vacuum boundary is physical for the compressible Euler equations according to the definition in Jang-Masmoudi's work [16, 20].

这意味着，根据 Jang-Masmoudi 的工作[16, 20]中的定义，该真空边界对于可压缩 Euler 方程而言是物理的。

Moreover, the free boundary problem (1.6)-(1.7) with boundary condition (1.9)-(1.10) and initial condition (1.11) admits a symmetric extension, that is, if we define $(\hat{\rho}, \hat{u})$ by

此外，自由边界问题(1.6)-(1.7)配合边界条件(1.9)-(1.10)及初始条件(1.11)具有对称延拓，即若定义$(\hat{\rho},\hat{u})$为

$$(\hat{\rho}, \hat{u})(\tau, \xi) \equiv \begin{cases} (\rho, u)(\tau, \xi), & \xi \geq 0, \\ (\rho, -u)(\tau, -\xi), & \xi < 0, \end{cases}$$

then, (ρ, u) is a solution to (1.6)-(1.7) over the free boundary $a(t)$ with boundary condition (1.9)-(1.10) and initial condition (1.11) if and only if $(\hat{\rho}, \hat{u})$ is the odd extension of (ρ, u) with respect to ξ and $(\hat{\rho}, \hat{u})$ is a solution to the free boundary problem (1.6)-(1.7) with boundary condition $\hat{u}(t, a(t)) = \hat{u}(t, -a(t)) = 0$ and initial condition (1.11). Since $\hat{u}$ is an odd extension of u, we have $\hat{u}(t, 0) = 0$. Without loss of

那么，(ρ,u)是自由边界$a(t)$上满足由(1.10)和(1.11)相应延拓而来的边界条件和初始条件的(1.6)-(1.7)的解，当且仅当$(\hat{\rho},\hat{u})$是(ρ,u)关于ξ的奇延拓，因此$\hat{u}(t,0)=0$。不失

generality, we may focus our attention on (ρ, u). For the sake of simplicity, we still denote the extended solution by (ρ, u).

由于 ρ 定义在 $\mathbb{R}$ 上，我们采用 ρ 的一个自然延拓，记为 $\tilde{\rho}$，

Since ρ is defined on $\mathbb{R}$, we use a natural extension of ρ denoted by $\tilde{\rho}$,

$$\tilde{\rho}(\xi) := \begin{cases} \rho(\xi), & -a(T) \le \xi \le a(T), \\ 0, & |\xi| > a(T). \end{cases}$$

因此，根据方程 (1.3)，$\tilde{p}$ 具有如下形式，

Therefore, according to (1.3), $\tilde{p}$ has the following form,

$$\tilde{p}(T,\xi) = \int_{-\infty}^{\xi} W''(\xi - \eta)(\tilde{\rho}(T,\eta) - \bar{\rho}(T,\xi))\, d\eta.$$

Our main result can be stated as follows: Assume (ρ_0, u_0, a_0) is a strong solution to the free boundary problem (1.6)-(1.7) with the adiabatic exponent γ satisfying $1 < \gamma < 1 + 2s/3$, and the initial data (ρ_0, u_0, a_0) is a small perturbation of the steady solutions $(\bar{\rho}, 0, \bar{a})$ in a suitable sense (see Theorem 2.1 and Theorem 2.2) with the same total mass, then for the adiabatic exponent γ satisfying $2(1-s) < \gamma < 1 + 2s/3$, under the additional condition that $3/8 < s < 1/2$, there is a unique global-in-time strong solution (ρ, u, a) to the free boundary problem of (1.6)-(1.7) with boundary conditions (1.9), (1.10) and initial conditions (1.11). Moreover, let $\eta(t,x)$ be defined by (2.1), (2.2), and (2.3). Then the steady solution $(\bar{\rho}, 0)$ is stable in the following sense

我们的结果可表述如下：假设初始数据 (ρ_0, u_0, a_0) 在适当意义下是稳态解 $(\bar{\rho}, 0, \bar{a})$ 的小扰动（见定理 2.1 和定理 2.2），且具有相同的总质量，则对于绝热指数 $2(1-s) < \gamma < 1 + 2s/3$，在附加条件 $3/8 < s < 1/2$ 下，自由边界问题 (1.6)-(1.7) 在边界条件 (1.9)，(1.10) 和初始条件 (1.11) 下存在唯一的全局强解 (ρ, u, a)。此外，设 $\eta(t,x)$ 由 (2.1)，(2.2) 和 (2.3) 定义，则稳态解 $(\bar{\rho}, 0)$ 在以下意义下是稳定的

$$\|(\rho(t,x) - \bar{\rho}(x), u(t,\eta(t,x)))\|_{L^\infty} \le C \|(\rho_0(x) - \bar{\rho}(x), u_0(\eta_0(x)))\|_{L^\infty}, \quad t \ge 0,$$

对某个常数 $C \ge 0$，C 该结论关于时间 t 一致成立。在某些适当加权的 L^2 范数下，也可以得到衰减率。这里，限制条件 $\gamma < 1 + 2s/3$ 是在适当函数空间中研究该问题的一项技术要求。详情请参见第 2.2 节。

uniformly in time. Decay rates can also be obtained in L^2 weighted norms. Here, the restriction $\gamma < 1 + 2s/3$ is a technical requirement for studying the problem in an appropriate functional space. For details, see Section 2.2.

The difficulties of the vacuum free boundary problem for the compressible Navier-Stokes-Riesz system are not only caused by the degeneracy due to the vacuum free boundary, but also caused by the non-local nature of the Riesz potential, which forces us to take into account of the effect of the solution over the whole domain even if we focus on the local behavior of the solution. This makes a huge difference compared to cases involving the Newtonian potential.

可压缩 Navier-Stokes-Riesz 系统的真空自由边界值问题之所以困难且具有挑战性，不仅是因为真空自由边界导致的退化，还因为 Riesz 势具有非局部性，即使我们关注解的局部行为，也必须考虑整个区域上解的影响。这与涉及牛顿势的情形有很大不同。

Last, we would like to mention some results related to our problem. The vacuum free boundary problem is a long-standing research topic in the compressible hydrodynamics due to its physical importance and mathematical difficulties. In the theory of inviscid flows, significant results on the local-in-time existence have been investigated. For the compressible Euler system, Jang-Masmoudi [16] made the breakthrough in establishing the unique local-in-time well-posedness of vacuum free boundary problem for the Euler-Poisson system with a moving physical vacuum boundary. Later in [17] they extended their method to three dimensional cases. A similar result was also obtained by Jang-Masmoudi using a different approach. If the self-gravitation is taking into consideration, Luo-Xin-Zeng [22] proved a general uniqueness theorem for the Euler-Poisson system with a moving physical vacuum boundary. Hadzic-Jang [13] even obtained nonlinear stability of some compactly supported expanding star-solutions of the mass-critical gravitational Euler-Poisson system, then they further constructed the global-in-time solutions without any symmetry assumptions in $\mathbb{R}^3$. In the theory of viscous flows, not

最后，我们想提一些相关研究，以结束本节。物理的真空自由边界问题是可压缩流体力学中一个由来已久的研究课题，其原因在于它具有重要的物理意义，同时也存在数学上的困难。在无黏流理论中，关于局部时间存在性的研究已取得显著成果。对于可压缩欧拉系统，Jang-Masmoudi [16] 率先在一维情形下建立了真空自由边界问题的局部适定性。随后，Constantin-Shkoller [3] 在 Sobolev 空间中获得了更光滑的局部解，且解一直延伸至移动边界。之后，他们在 [15] 将方法推广到三维情形。Jang-Masmoudi [20] 采用不同方法也得到了类似结果。若考虑自引力，Luo-Xin-Zeng [22] 证明了一个一般的唯一性定理，并为球对称运动建立了新的局部存在理论，其所用函数空间涉及的导数阶数低于以往构造的空间。Hadzic-Jang [13] 甚至证明了质量临界 Euler-Poisson 系统的一些紧支撑膨胀星解具有非线性稳定性，随后又在 [14] 构造了不作任何对称性假设的全局解。在黏性流理论中，不

仅有局部时间结果，也有全局时间结果。Jang [17] 建立了强解的局部适定性。后来，Jang [17] established the local-in-time well-posedness of strong solutions. Later, Jang [17] 研究了可压缩 Navier-Stokes 系统的真空自由边界问题，并构造了全局强解。Ou-Zeng [28]将研究推广到更符合实际的情形，其中黏性假设为依赖于密度，且存在外部引力。若考虑自引力，则存在一类重要的解，称为 Lane-Emden 解；它们具有球对称性且稳定，并且在所有可能的解中使能量最小（参见 [12]）。Luo-Xin-Zeng [24, 25]在 Lane-Emden 解方面取得了重要进展。他们研究了可压缩 Navier-Stokes-Poisson 系统的自由边界问题，并证明对于满足 $4/3 \leqslant \gamma \leqslant 2$ 的黏性气体星，Lane-Emden 解具有渐近非线性稳定性；此外，他们还证明了解在真空边界处一致成立的全局时间正则性。随后，Luo-Wang-Zeng [26] 通过研究白矮星和所有 $\gamma \geqslant 4/3$ 的多方体，推广了上述结果。黏性引起的额外耗散在获得这些全局时间结果方面发挥了重要作用。已有研究指出，阻尼引起的耗散同样能够保证全局时间结果。关于这方面的结果，请参阅 [23, 30, 31, 32]。关于不稳定性的结果，请参阅 [15, 18, 19]。

最近，Castillo-Chae-Chen-Cuan-Yu [7] 针对可压缩 Euler-Riesz 系统的柯西问题，获得了一个具有球对称性和有限能量的整体弱解，并且在某种意义下验证了稳态的非线性稳定性。他们的结果证明依赖于消失黏性方法。作为中间步骤，他们还研究了可压缩 Navier-Stokes-Riesz 系统的自由边界问题。为了解决自由边界问题，拉格朗日变换在他们的分析中也极为有用。由于该自由边界问题的解是 Euler-Riesz 系统（包含真空）的解的 transformation，他们可以将自由边界问题提升到非真空情形，这直接保证了拉格朗日变换

Very recently, Castillo-Chae-Chen-Cuan-Yu [7] obtained the Cauchy problem with spherical symmetry, finite energy for the Compressible Euler-Riesz system. Global existence of the compressible Euler-Riesz system, the nonlinear stability of steady states is also verified in a certain sense. The proof of their result relies on the vanishing viscosity method. As an intermediate step, they also studied the free boundary problem of the compressible Navier-Stokes-Riesz system. To solve the free boundary problem, the Lagrangian transformation is used. Moreover, they can lifted the free boundary problem to consider a non-vacuum case, which directly ensures that the Lagrangian transformation

$$(t,r) \mapsto (\mathcal{T} := t, x := \int_a^r \rho(t,y) y^{n-1} dy),$$

是一个光滑双射映射。事实上，我们有

is a smooth bijective mapping. Indeed, we have

$$\nabla_{(t,r)} x = (-\rho u r^{n-1}, \rho r^{n-1}), \quad \nabla_{t,r} \mathcal{T} = (1, 0), \tag{1.13}$$

这意味着根据反函数定理，拉格朗日变换是双射的。如果出现真空，则 (1.13) 不适用，因为 Lagrangian transformation is ill-defined by the inverse function theorem。我们的结果在以下两个方面具有重要意义：首先，通过考虑可压缩 Navier-Stokes-Riesz 系统的强解，我们成功证明了拉格朗日变换是一个双射映射；其次，我们获得了解的时间渐近行为，以及在某些特定加权 L^2 范数下的时间衰减估计，其中权函数还反映了解在接近真空自由边界处的行为。

This implies that the Lagrangian transformation is bijective by the inverse function theorem. Our result is important in the following two aspects: First, we succeed in justify that the Lagrangian transformation is a bijective mapping by considering the strong solution to the compressible Navier-Stokes-Riesz system; Second, we obtain a time asymptotic behavior of the solution and a time decay estimate in some particular weighted L^2 norms where the weights also indicate the behavior of the solution close to the vacuum free boundary.

本文的结构安排如下：第2节中，我们在拉格朗日变换下描述问题，随后给出主要结果。第3节中，我们为强解框架建立三项基本估计：(1) 低阶估计：利用波动方程技巧推导初始能量估计，并通过对近真空状态下空间导数进行黏性演化分析加以改进

The paper is structure is organized as follows: In Section 2, we describe the problem under the Lagrangian transformation and present the main results. In Section 3, we establish three fundamental estimates for the strong solution framework: (1) Lower-order estimates: Initial L^2 energy estimates derived via wave equation techniques, enhanced through viscous evolution analysis of spatial derivatives

(2) 点在刻画: 利用 Riesz 的表勒展开表示公式, 对扰动幅值进行引导分析 (leading-order analysis of perturbation amplitude using Riesz's series representation formula of Riesz); (3) 高阶正则性: 将低阶估计扩展至时间正则性的高阶估计, 并建立二阶空间导数的高阶估计。主要技术挑战在于处理 Riesz 势奇异的非局部结构, 我们借鉴低阶能量估计和高阶能量估计的结果, 通过分解方法应对这一问题。此外, 我们还建立了一些时间衰减估计。第4节中, 我们简要给出差分格式, 以证明强解的局部存在性与唯一性。随后运用第3节的结果证明定理 2.1 和定理 2.2。附录包含必要的背景材料: 可压缩 Navier-Stokes-Riesz 系统稳态的存在理论, 以及本文分析中广泛使用的 Hardy 型不等式。

(2) local analysis of perturbation amplitude using Riesz's series representation formula of Riesz; (3) Higher-order regularity: Extension of lower-order estimates with higher-order estimates in temporal regularity, as well as the higher-order estimates in second-order spatial derivatives. The principal technical challenge involves managing the Riesz potential's singular non-local structure, which is addressed through decompositions inspired by Riesz's series representation formula of Riesz. Additionally, some time-decay estimates are also established. In Section 4, we provide a compressible difference scheme formulation for demonstrating local existence and uniqueness of strong solutions, subsequently employing Section 3's results to prove **Theorem 2.1** and **Theorem 2.2**. Appendices include essential background material: existence theory for steady states of compressible Navier-Stokes-Riesz systems, and Hardy-type inequalities extensively utilized in our analysis.

2. 拉格朗日表述与主要结 2. LAGRANGIAN FORMULATION AND MAIN RESULTS

2.1. 拉格朗日表述。 为了研究高阶正则性, 我们采用拉格朗日变量表述该问题。令 x 为参考变量, 并定义拉格朗日变量 $\eta(t,x)$ 为

2.1. Lagrangian formulation. To study higher-order regularity, we adopt the Lagrangian variable. Let x be the reference variable and define the Lagrangian variable $\eta(t,x)$ by

$$\frac{d\eta(t,x)}{dt} = u(t,\eta(t,x)), \tag{2.1}$$

对于 $t>0$ 以及 $-R\le x\le R$, 满足初始条件

for $t>0$ and $-R\le x\le R$, with the initial condition

$$\eta(0,x) = \eta_0(x), \tag{2.2}$$

其中 $\eta_0(x)$ 是初始位置, 它将 $[-R,R]$ 映射到 $[-\bar a,\bar a]$ 并满足

where $\eta_0(x)$ is the initial position, which maps $[-R,R]$ to $[-\bar a,\bar a]$ satisfying

$$\int_{\eta_0(-x)}^{\eta_0(x)} \rho_0(y)\,dy = \int_{-x}^{x} \bar\rho(y)\,dy, \quad \eta_0(-x) = -\eta_0(x), \quad \eta_0(\pm R) = \pm\bar a, \tag{2.3}$$

从而

so that

$$\rho_0(\eta_0(x))\eta_0'(x) = \bar\rho(x). \tag{2.4}$$

事实上, η_0 可表示为

In fact, η_0 can be described by

$$\eta_0(x) = \varphi^{-1}(\psi(x)),$$

其中 φ 和 ψ 定义为

where φ and ψ are defined by

$$\psi(x) := \int_{-x}^{x} \bar\rho(y)\,dy, \quad -R\le x\le R,$$

$$\varphi(\xi) := \int_{-\xi}^{\xi} \rho_0(y)\,dy, \quad -a_0\le \xi\le a_0.$$

此外, (2.3) 中的第三个条件还意味着初始状态与稳态的速度在真空边界处具有相同的行为, 直至一阶导数。事实上, 考虑 $x=R$ 处的情形, 另一种情形 $x=-R$ 也可以用类似的方法讨论。利用 (2.4), 可得

In addition, the third condition in (2.3) also implies that the initial state and the steady state have the same behavior at the vacuum boundary up to their first-order derivatives. Indeed, let us consider the case at $x=R$, as the other case at $x=-R$ can be discussed in a similar way. By using (2.4), we have

$$\lim_{x\to R}\frac{(\rho_0^{\gamma-1}(\eta_0(x)))'}{(\bar\rho^{\gamma-1}(x))'} = \lim_{x\to R}\frac{(\eta_0(x)-\bar a)^{\gamma-1}\eta_0'(x)}{\bar\rho^{\gamma-1}(x)} = 1,$$

这表明 (1.12) 中的第三个条件成立。

which implies the third condition in (1.12).

我们分别将拉格朗日密度和速度定义为

We further define the Lagrangian density and velocity respectively by

$$\rho(t,x) := \rho(t,\eta(t,x)), \quad v(t,x) := u(t,\eta(t,x)).$$

对于 $t \geqslant 0$ 和 $-\bar{R} \leqslant x \leqslant \bar{R}$. Then (d, v) satisfies

$$\partial_t(d\partial_x\eta) = 0, \tag{2.5}$$

$$d\partial_t v + \frac{\partial_x(d^\gamma)}{\partial_x\eta} = \frac{1}{\partial_x\eta}\partial_x\left(\frac{\partial_x v}{\partial_x\eta}\right) - d\partial_x\psi, \tag{2.6}$$

for $t \geqslant 0$ and $-\bar{R} \leqslant x \leqslant \bar{R}$, where $\partial_x\psi$ is defined by

$$\partial_x\psi(t,x) := \int_{-\infty}^{\infty} W'(\eta(t,x) - \eta(t,y))(d(t,y) - d(t,x))\partial_y\tilde{\eta}(t,y)\,dy,$$

$$\tilde{\eta}(t,x) := \begin{cases} -\eta(t,\bar{R}) + \bar{R} + x, & x \leqslant -\bar{R}, \\ -\eta(t,-x), & -\bar{R} \leqslant x \leqslant 0, \\ 0, & x = 0, \\ \eta(t,x), & 0 \leqslant x \leqslant \bar{R}, \\ \eta(t,\bar{R}) - \bar{R} + x, & x > \bar{R}. \end{cases}$$

注意，拉格朗日密度可由 (2.5) 和 (2.4) 求得

Here, Lagrangian density can be derived from (2.5) and (2.4),

$$d(t,x) = \frac{\bar{\rho}(x)}{\partial_x\eta(t,x)}, \tag{2.7}$$

并且根据 (2.1) 有 $v = \partial_t\eta$，于是 (2.6) 可改写为

and due to (2.1) $v = \partial_t\eta$, then (2.6) can be rewritten as,

$$\bar{\rho}\partial_t v + \partial_x\left[\left(\frac{\bar{\rho}}{\partial_x\eta}\right)^\gamma - \bar{P}\right] + \Phi = \partial_x\left(\frac{\partial_x v}{\partial_x\eta}\right), \tag{2.8}$$

其中 Φ 的表达式为

where Φ is given by

$$\Phi(t,x) := \partial_x(\bar{P}(x)) + a\frac{\bar{\rho}(x)}{\partial_x\eta(t,x)}\int_{-\infty}^{\infty} W'(\eta(t,x) - \tilde{\eta}(t,y))(\bar{\rho}(y)\partial_{xx}\eta(t,x) - \bar{\rho}(x)\partial_y\tilde{\eta}(t,y))\,dy. \tag{2.9}$$

注意到，$\Phi(t,x) = 0$ 在 $\eta(t,x) = x$ 处趋于零，这正是定常方程 (1.4) 所预期的长时间渐近行为。

Notice that $\Phi(t,x) = 0$ vanishes for $\eta(t,x) = x$, which is the expected long time asymptotics due to the stationary equation (1.4).

下面推导 (2.8) 对应的边界条件。首先，由 (2.10) 和 (2.3) 可得

Let us derive the boundary conditions for (2.8). First, by (2.10) and (2.3),

$$\partial_x v(t, \pm\bar{R}) = 0. \tag{2.10}$$

然后，由于 η 满足微分方程 (2.1)，我们有

Then, since η satisfies the differential equation (2.1), we have

$$\eta(t,x) = \eta_0(x) + \int_0^t v(s,x)\,ds. \tag{2.11}$$

对 (2.11) 关于 x 求导，然后利用边界条件 (2.10) 和 (2.3)，可得

Taking ∂_x to (2.11), and then using the boundary conditions (2.10) and (2.3), we find

$$\partial_x\eta(t, \pm\bar{R}) = 1. \tag{2.12}$$

此外，除了初始条件 (2.2) 外，(1.11) 还意味着

In addition, besides the initial condition (2.2), (1.11) also implies

$$\partial_t\eta(0,x) \equiv u(0,\eta_0(x)) := \eta_1(x). \tag{2.13}$$

2.2. Main results. In this section, we give the definition of solutions to the problem (2.8) with (2.2), (2.13) and (2.12).

Definition 2.1. *We say that $\eta(t,x)$ is a solution to the problem (2.8) with initial data (2.2), (2.13) and boundary conditions (2.12) if η satisfies*

(1) $\eta(t,x) - x \in C([0,T];H^2(-R,R)) \cap C^1([0,T];H^1(-R,R))$, $\partial_x\eta(t,x) \geq c_1$ *for* $0 \leq t \leq T$ *and* $-R \leq x \leq R$ *with some constants* $c_1 > 0$;

(2) $\partial_{tx}\eta \in C([0,T];L^2(-R,R))$;

(3) $\bar{\rho}^{-\frac{1}{2}}\partial_t\eta \in C([0,T];L^2(-R,R))$, $\bar{\rho}^{-\frac{1}{2}}\partial_x\eta \in C([0,T];L^2(-R,R))$ *and* $\bar{\rho}^{-\frac{1}{2}}\partial_x\left(\frac{\partial_{tx}\eta}{\partial_x\eta}\right) \in C([0,T];L^2(-R,R))$;

(4) $\partial_x\eta(t,\ -R) = 1$ *hold in the sense of* $H^{-\frac{1}{2}}$*-trace for* $0 \leq t \leq T$;

(5) (2.8) *holds over* $[0,T] \times (-\bar{R},\bar{R})$ *a.e.*

Remark 2.1. *The condition (1) implies a positive lower bound for* $\partial_x\eta$*. Then, by the inverse function theorem, the transformation from the Lagrangian coordinates to the Eulerian coordinates is invertible, which means that the system (1.6)-(1.7) in Eulerian coordinates and the system (2.5)-(2.6) in Lagrangian coordinates are equivalent.*

Remark 2.2. *The condition (2) implies that* u *is Lipschitz continuous in* x*, i.e.,* $\partial_x u(t,\eta(t,x)) = \frac{\partial_{tx}\eta(t,x)}{\partial_x\eta(t,x)}$*. Therefore, by Picard-Lindelöf's theorem, the ordinary differential equation (2.1) admits a unique solution* η *which ensures the well-definiteness of Lagrangian variable.*

Remark 2.3. *The condition (3) gives the function spaces such that each term in (2.8) multiplied by* $\bar{\rho}^{-\frac{1}{2}}$ *belongs to* $C([0,T];L^2(-R,R))$*. Indeed,*

$$\left\|\bar{\rho}^{-\frac{1}{2}}\partial_x\left[\left(\frac{\bar{\rho}}{\partial_x\eta}\right)^\gamma\right]\right\|_{L^2(-R,R)} \leq C\left(\left\|\bar{\rho}^{\gamma-\frac{1}{2}}\partial_x\eta\right\|_{L^2(-R,R)} + \left\|\bar{\rho}^{-\frac{1}{2}}\partial_x\left(\bar{\rho}^\gamma\right)\right\|_{L^2(-R,R)}\right)$$
$$\leq C\left(\|\partial_x\eta\|_{L^\infty}^{\gamma-\frac{1}{2}}\left\|\bar{\rho}^{-\frac{1}{2}}\partial_x\bar{\rho}\right\|_{L^2(-R,R)} + 1\right),$$

这与 $\rho \in W^{1,\infty}$ 共同推出 $\bar{\rho}^{-\frac{1}{2}}\partial_x\left[\left(\frac{\bar{\rho}}{\partial_x\eta}\right)^\gamma - \bar{\rho}^\gamma\right] \in C([0,T];L^2(-R,R))$。此外，根据引理 B.4，

$$\left\|\bar{\rho}^{-\frac{1}{2}}\partial_x\left[\left(\frac{\bar{\rho}}{\partial_x\eta}\right)^\gamma - \bar{\rho}^\gamma\right](t)\right\|_{L^2} \leq C\left(\|\partial_x\eta(t)\|_{L^\infty}^{2\gamma-1}\left\|\bar{\rho}^{-\frac{1}{2}}\partial_x\bar{\rho}(t)\right\|_{L^2} + \left\|\bar{\rho}^{\gamma-\frac{1}{2}}\partial_x\eta(t)\right\|_{L^2}\right),$$

这意味着 $\bar{\rho}^{-\frac{1}{2}}\partial_x\left[\left(\frac{\bar{\rho}}{\partial_x\eta}\right)^\gamma - \bar{\rho}^\gamma\right] \in C([0,T];L^2(-R,R))$。因此，(2.8) 中的每一项都属于 $C([0,T];L^2(-R,R))$。结论上，条件 (5) 成立。

As a conclusion, the condition (5) follows.

Remark 2.4. *If* ρ *satisfies the conditions (1), (2) and (3) in* **Definition 2.1** hold, then

$$\|\partial_x v(t)\|_{H^1} \leq \|\partial_x\eta(t)\|_{L^\infty}\left\|\frac{\partial_{tx}\eta(t)}{\partial_x\eta(t)}\right\|_{H^1}$$
$$\leq \|\partial_x\eta(\|\left(\left\|\frac{1}{\partial_x\eta(t)}\right\|_{L^\infty}\|\partial_t\eta(t)\|_{W^{1,\infty}} + \left\|\partial_x\left(\frac{\partial_{tx}\eta(t)}{\partial_x\eta(t)}\right)\right\|_{L^2}\right),$$

这意味着 $\partial_x \eta \in C([0,T]; H^1(-R,R))$。因此，$\bar{R}$ 以 H^1 的意义存在 exists in the sense of H^1-trace. Since

$$\partial_x \eta(t,x) - \partial_x \eta_0(x) = \int_0^t \partial_x v(s,x)\,ds,$$

对于任意 x，由条件 (4) 和 $\partial_x \eta(t,\pm R) = \partial_x v(t,\pm R) = 0$ 我们有 $\partial_x \eta(t,\pm R) = 1$。因此，$\bar{R}$ 以 H^1 迹的意义存在 exists in the sense of H^1-trace.

Remark 2.5. 以定义 2.1 的意义存在，则除边界条件 (2.12) 外，我们还有 Besides the boundary conditions (2.12), we have

$$\eta(t,0) = 0. \tag{2.14}$$

事实上，令 $x \equiv 0$，则 (2.1) 变为关于 η 的常微分方程。回忆一下，在原点处，由于奇性，$u(t,0) = 0$ 当 $t \geq 0$ 时。且由于 u 的奇函数性质，$\eta(t,0) = 0$ 可以直接验证，$\eta(t,0) = 0$ 是该常微分方程的一个解。由于 $\partial_t \eta \in W^{1,\infty}$，根据皮卡-林德勒夫定理，$\eta(t,0) = 0$ 也是唯一解。 Indeed, let $x \equiv 0$, then (2.1) becomes an ordinary differential equation in η. Recall that $u(t,0) = 0$ for $t \geq 0$ due to the singularity at the origin and the odd symmetry of u. It can be directly verified that $\eta(t,0) = 0$ is a solution to this ordinary differential equation. Since $\partial_t \eta \in W^{1,\infty}$, by Picard-Lindelöf's theorem, $\eta(t,0) = 0$ is also the unique solution.

定义强解之后，我们可以构造由边界值问题 (1.6)–(1.11) 的唯一整体强解 (ρ, u, a)。事实上，由定义 2.1 中的条件 (1) 和反函数定理可知，η 定义了从固定参考区域 $[-R,R]$ 到自由边界区域 $[-a(t),a(t)]$ 的一个微分同胚，其中 Having the strong solution in hand, existing results on the free boundary value problem (1) in Definition 2.1 and the inverse function theorem η defines a diffeomorphism from the fixed reference domain $-R, R]$ to the free boundary domain $[-a(t), a(t$ with

$$a(t) := \eta(t,\bar{R}), \tag{2.15}$$

for $t \geq 0$. 此外，我们将映射 η 的逆映射记为 ξ。我们定义 (ρ, u) 为 We denoted the inverse map of η as ξ. We defined (ρ, u) as

$$\rho(\tau,\xi) := \frac{\bar{\rho}(x)}{\partial_x \eta(\tau,x)}, \quad u(\tau,\xi) := \partial_\tau \eta(\tau,x), \tag{2.16}$$

其中 $x = \eta(\tau,\xi)$ 适用于 $T \geq 0$ 以及 $-R \leq \xi \leq a(T)$。因此，我们给出在 with $x = \eta(\tau,\xi)$ for $T \geq 0$ and $-R \leq \xi \leq a(T)$. Accordingly, we give the (1.9), (1.10) 和 (1.11) 条件下，方程 (1.6) 和 (1.7) 的强解定义。 definition of strong solutions to (1.6) and (1.7) with (1.9), (1.10) and (1.11).

Definition 2.2. *(ρ, u, a) is called a strong solution to the compressible Navier-Stokes-Riesz system (1.6) and (1.7) with (1.9), (1.10) and boundary conditions (1.9), (1.10) in $T \times \bar{R}$ for some constant T, if*

1. *η is a strong solution to (2.8) with (2.2), (2.13) and (2.12);*
2. *(ρ, u, a) satisfies (2.15) and (2.16) a.e.*

现在陈述第一个主要定理。 Now we state the main theorem.

Theorem 2.1. *Let $\frac{1}{8} < s < 1/2$ 且 $2(\frac{1}{2} - s) < \gamma - 1 < 2s/3$. 假设 (2.3) 成立，且初值 η_0 是满足相容条件的奇函数，$\partial_x \eta_0(\pm\bar{R}) = 1$。此外，假设存在某个 $\varepsilon_0 > 0$ 使得 $\partial_x \eta_0 - 1$, $\partial_x u_0 \in L^\infty(-R,R) + \|\rho_0^{\frac{1}{2}} \partial_x u_0\|_{L^2(-R,R)} + \|\rho_0^{\frac{1}{2}} \partial_x v_0\|_{L^2(-R,R)} \leq \varepsilon_0$.*

Then, the problem (2.8) with (2.2), (2.13) and (2.12) admits a unique strong solution, satisfying (2.12) admits a unique strong solution such that

$$\|(\partial_x \eta - 1, \partial_x u)(t)\|_{L^\infty(-R,R)} + \|\rho^{\frac{1}{2}} \partial_x u(t)\|_{L^2(-R,R)} + \|\rho^{\frac{1}{2}} \partial_x v(t)\|_{L^2(-R,R)} \leq C\varepsilon_0, \quad t \geq 0, \tag{2.17}$$

其中某个常数 $C \geq 0$ 与时间 t 无关。此外，该解满足以下时间衰减估计，

for some constant C independent of time t, where

$$E(t) := \|(\partial_t \rho, \partial_t u)(t)\|^2_{L^2(-R,R)} + \|\rho^{-\frac{1}{2}} \partial_{xx} u(t)\|^2_{L^2(-R,R)} + \|\rho^{\frac{1}{2}} \partial_{xt} u(t)\|^2_{L^2(-R,R)} \tag{2.20}$$

Remark 2.6. *It is emphasized that (i) guarantees the existence of steady solutions in the sense of (2.4) under the condition $\gamma > 2(1-s)$. However, the condition $\gamma < 1+2s/3$ is necessary for the definition of strong solution, and more precisely, it is required to have the regularity $\bar{\rho}^{-\frac{1}{2}} \Phi \in C([0,T; L^2(-\bar{R},\bar{R}))$.*

注2.6. 我们强调，(i) 保证了稳态解的存在，并且能量估计（见第3节）也以类似地方式利用了这一条件。此外，条件 $\gamma < 1+2s/3$ 是定义强解所必需的；更准确地说，需要具有如下正则性 $\bar{\rho}^{-\frac{1}{2}} \Phi \in C([0,T; L^2(-\bar{R},\bar{R}))$。

The second main result concerns the original free boundary problem (1.6)-(1.11) in Eulerian coordinates.

第二个主要结果涉及欧拉坐标下的原始自由边界问题 (1.6)-(1.11)。

Theorem 2.2. *Let $1 < \gamma < 2$ and $2(1-s) < \gamma < 1+2s/3$. Assume that the initial condition (ρ_0, u_0) satisfies (1.12) and the compatibility condition (2.16) with $a_0 > 0$. Then (ρ, u, a) defined by (2.15) and (2.16) is the unique global strong solution to the system (1.6)-(1.7) with initial condition (1.11) and boundary conditions (1.9), (1.10).*

定理 2.2. 设 $1 < \gamma < 2$ 且 $2(1-s) < \gamma < 1+2s/3$。假设初始条件 (ρ_0, u_0) 满足 (1.12)，且相容条件 (2.16) 对 $a_0 > 0$ 成立。则由 (2.15) 和 (2.16) 定义的 (ρ, u, a) 是满足初始条件 (1.11) 以及边界条件 (1.9)、(1.10) 的系统 (1.6)-(1.7) 的唯一整体强解。

Remark 2.7. *Under the conditions of Theorem 2.1, the free boundary problem (1.6)-(1.7) admits a classic solution, by the discussion in Section 2.1, the classic solution is in the form of (2.16).*

注2.7. 在定理2.1的条件下，自由边界问题 (1.6)-(1.7) 存在经典解，则根据第2.1节的讨论，该经典解具有 (2.16) 的形式。

Remark 2.8. *From the construction of (2.16), the density $\rho(t,x)$ is positive, is compactly supported in $(a(t,-R), a(t,R))$ for any $t \geq 0$, and continuously connected to the vacuum near the free boundary.*

注2.8. 由 (2.16) 的构造可知，强解 $\rho(t,x)$ 在自由边界区域上非负，并且在自由边界附近与真空连续连接。

Remark 2.9. *Recall that the sound speed of the flow is $c = \rho^{\frac{\gamma-1}{2}}$. Thus, by the fact that $\rho \in W^{1,\infty}$, we see that the sound speed c is $\frac{\gamma-1}{2}$-Hölder continuous near the vacuum. Indeed, note that*

$$|\rho_1^{\frac{\gamma-1}{2}}(x) - \rho_2^{\frac{\gamma-1}{2}}(x)| \leq \frac{1}{\frac{\gamma-1}{2}} |\rho_1^{\frac{\gamma-1}{2}-1}(x) + \rho_2^{\frac{\gamma-1}{2}-1}(x)| \cdot |\rho_1(x) - \rho_2(x)|.$$

This together with the fact that $\rho \in W^{1,\infty}$ implies that the sound speed c is $\frac{\gamma-1}{2}$-Hölder continuous near the vacuum.

注2.9. 按照我们的构造，流动的声速在真空附近是 $\frac{\gamma-1}{2}$-Hölder 连续的。事实上，注意到

$$|\rho_1^{\frac{\gamma-1}{2}}(x) - \rho_2^{\frac{\gamma-1}{2}}(x)| \leq \frac{1}{\frac{\gamma-1}{2}} |\rho_1^{\frac{\gamma-1}{2}-1}(x) + \rho_2^{\frac{\gamma-1}{2}-1}(x)| \cdot |\rho_1(x) - \rho_2(x)|.$$

这与 $\rho \in W^{1,\infty}$ 这一事实共同意味着，声速 $c \equiv \sqrt{\gamma}\rho^{\frac{\gamma-1}{2}}$ 在真空附近具有 $\frac{\gamma-1}{2}$-Hölder 连续性。

Let us illustrate the main ideas and the structure of the proof.

下面说明证明的主要思路和结构。

To tackle the free boundary problem (1.6)-(1.7) with (1.9)-(1.11), we reformulate the system in Lagrangian coordinates (2.1) and (2.2), where the free boundary problem is reformulated as a fixed initial boundary value problem (2.8) with constraints (2.12)-(2.13) on a fixed domain through the Lagrangian particle trajectory framework (2.1) and (2.2). Notably, we derive novel estimates for the Riesz potential that address its inherent non-local structure. Unlike the Newtonian potential case analyzed in [24], where localization was feasible, we develop two direct treatments of the Riesz potential in (3.10), including a refined version that leverages the energy structure of the linear operator $\mathcal{L}$ in (3.7) for the basic energy estimate, which also reveals the competition between the fluid pressure and the self-consistent attractie force in the energy (1.5) (see **Remark 3.1**). The other is a strategic decomposition (**Lemma 3.1**, **Lemma 3.2**, **Lemma 3.8**) for the higher spatial regularity. Our approach also

在拉格朗日粒子轨迹框架 (2.1) 和 (2.2) 下，自由边界问题 (1.6)-(1.7) 及其条件 (1.9)-(1.11) 被重新表述为固定区域上的初边值问题 (2.8)，并满足约束条件 (2.12)-(2.13)。值得注意的是，我们针对 Riesz 势固有的非局部结构推导了新的估计。与 [24] 中分析的牛顿势情形不同，在后者中可以进行局部化处理，我们针对 Riesz 势的非局部性提出了两种直接处理方法。其一是为基本能量估计建立线性算子 $\mathcal{L}$（见 (3.7)）的估计（见 (3.10)），这也揭示了能量 (1.5) 中流体压力与自洽吸引力之间的竞争（见 **Remark 3.1**）。其二是为高阶空间正则性采用策略性分解（引理 3.1、引理 3.2、引理 3.8）。我们的方法还

involves several steps. We first establish a uniform lower bound for the determinant of the Jacobian matrix associated with the Lagrangian coordinate transformation, specifically $\partial_x \eta$. Building on the methodology in [24], we investigate the representation formula of $\partial_x \eta$ in further details and we advance their analytical framework. Our key innovations lie in establishing a bootstrap pointwise estimate (**Lemma** 3.4), achieved through systematic applications of weighted Sobolev space techniques with carefully constructed test functions.

涉及一个关键方面：我们建立与拉格朗日坐标变换相关的雅可比矩阵行列式的正下界和上界，具体而言为$\partial_x \eta$。在[24]中方法的基础上，我们进一步深入研究$\partial_x \eta$的表示公式，并推进了其分析框架。我们的关键创新在于建立自举逐点估计(引理 3.4)，这是通过系统地运用加权 Sobolev 空间技术并精心构造测试函数实现的。

Notations. Throughout this paper, C (或C_ε, $C_{s,\gamma}$) denotes a positive constant which may vary from line to line. $L^p(-R,R)$ ($1 \le p < \infty$) denotes the p-th integrable Lebesgue space with norm defined by

符号说明。本文中，C (或C_ε, $C_{s,\gamma}$) 表示一般的正常数。$L^p(-R,R)$ ($1 \le p < \infty$) 表示第 p 阶可积 Lebesgue 空间，其范数定义为

$$\|f\|_{L^p} := \left(\frac{1}{2R} \int_{-R}^{R} |f|^p \, dx \right)^{\frac{1}{p}}.$$

L^∞ denotes the essentially bounded functional space with norm defined by

L^∞ 表示本质有界可测函数空间，其范数定义为

$$\|f\|_{L^\infty} := \inf\{C \ge 0 : |f| \le C \text{ in } (-R,R) \text{ a.e.}\}.$$

$H^\alpha(-R,R)$ ($\alpha \in \mathbb{N}$ or $\alpha \in (0,1)$) denotes the Sobolev space with norm defined by

$H^\alpha(-R,R)$ ($\alpha \in \mathbb{N}$ 或 $\alpha \in (0,1)$) 表示 Sobolev 空间，其范数定义为

$$\|f\|_{H^\alpha} := \|f\|_{L^2} + \sum_{n=1}^{\alpha} \|\partial_x^n f\|_{L^2}.$$

In addition, we use the following definition for the fractional Sobolev spaces $H^\alpha(-R,R)$ $\alpha \in (0,1)$:

此外，我们采用以下定义来表示分数阶 Sobolev 空间$H^\alpha(-R,R)$ $\alpha \in (0,1)$ 的范数：

$$\|f\|_{H^\alpha} := \|f\|_{L^2} + \frac{1}{2R} \int_{-R}^{R} \int_{-R}^{R} \frac{|f(x) - f(y)|^2}{|x-y|^{1+2\alpha}} \, dy \, dx.$$

We also introduce the concept of the Cauchy principal value given a function f over $[a,b]$ with a singularity at $a \le c \le b$, we define its Cauchy principal value over $[a,b]$ as

我们还引入柯西主值的概念，给定定义在 $[a,b]$ 上的函数 f，其奇点为 $a \le c \le b$，则定义其在 $[a,b]$ 上的柯西主值为

$$\text{p.v.} \int_a^b f(x)\,dx := \lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^b f(x)\,dx \right).$$

3. A PRIORI ESTIMATES

3. 先验估计

In the rest of this work, we set R to be unity for notational simplicity. Let us denote the perturbation of η around x by

在本文余下部分，为简化记号，令 R 等于 1，将 η 在 x 附近的扰动记为

$$w(t,x) := \eta(t,x) - x.$$

for $t \ge 0$ and $-1 \le x \le 1$. Then by (2.8), w satisfies

对于 $t \ge 0$ 且 $-1 \le x \le 1$，由 (2.8)，w 满足

$$\partial_{tt} w + \partial_x \left[\left(\frac{\bar{\rho}}{1+\partial_x w} \right)^\gamma - \bar{\rho}^\gamma \right] + \Phi = \partial_x \left(\frac{\partial_{tx} w}{1+\partial_x w} \right), \tag{3.1}$$

for $t \gg 0$ and $-1 \le x \le 1$. Moreover, (2.12), (2.14), (2.22) and (2.23) imply that

对于 $t \gg 0$ 且 $-1 \le x \le 1$，此外，(2.12)，(2.14)，(2.22) 和 (2.23) 表明(2.13) imply that

$$\partial_x w(t,\pm 1) = 0, \quad w(t,0) = 0, \tag{3.2}$$

$$(w, \partial_t w)(0,x) = (w_0, w_1)(x). \tag{3.3}$$

Here, Φ has the form given in (2.9):

这里，Φ 的形式如 (2.9) 所示：

$$\Phi(t,x) = \partial_x(\bar{P}(x)) + a\,\frac{\bar{\rho}(x)}{\partial_x \Gamma(t,x)} \int_{-\infty}^{\infty} W'(w(t,x) - w(t,y) + x - y)\,\delta(t,x,y)\,dy,$$

其中$\bar{\rho}(x,y) \equiv \rho(y)\partial_x w(t,x) - \rho(x)\partial_y w(t,y) + \rho(y) - \rho(x)$。我们利用 (3.1) 和 (3.2) 进一步定义 $\bar{\rho}\partial_t w_1$ 和 $\bar{\rho}\partial_t w_2$，
with $\bar{\rho}(x,y) \equiv \rho(y)\partial_x w(t,x) - \rho(x)\partial_y w(t,y) + \rho(y) - \rho(x)$. We further define $\bar{\rho}\partial_t w_1$ and $\bar{\rho}\partial_t w_2$ by using (3.1) and (3.2),

$$\bar{\rho}(x)\partial_t w_1(x) := -\partial_x\left[\left(\frac{\bar{\rho}(x)}{1+\partial_x \tilde{w}(x)}\right)^{\gamma} - \bar{\rho}^{\gamma}(x)\right] = \Phi_1(0,x) + \partial_x\left(\frac{\partial_x w_1(x)}{1+\partial_x \tilde{w}_0(x)}\right),$$

此外，我们还将 w 记为 w 在 $\mathbb{R}$ 上的延拓，
In addition, we also denote w the extension of w over $\mathbb{R}$.

$$\tilde{w}(t,x) := \tilde{f}(t,x) - x,$$

for $t \geq 0$ and $x \in \mathbb{R}$. 此处，我们定义初始扰动
We introduce the initial perturbation

$$E_0 := \sup_{-1\leq x\leq 1} |\partial_x w_0| + \sup_{-1\leq x\leq 1} |\partial_x w_1|.$$

我们作出先验假设，存在一个足够大的常数 $A > 0$，使得
We make the *a priori* assumption that there exists a suitably large constant $A > 0$ such that

$$\sup_{0\leq t\leq T,\,-1\leq x\leq 1} |\partial_x w(t,x)| \leq AE_0, \qquad \sup_{0\leq t\leq T,\,-1\leq x\leq 1} |\partial_{tx} w(t,x)| \leq AE_0, \tag{3.4}$$

对于 $T > 0$。关于 A 的定义，参见 (3.43)，其将在后文引理 3.4 的证明中给出，并注意到 A 与 T 无关。
for $T > 0$. For the definition of A we refer to (3.43) as introduced in the proof of Lemma 3.4, and note that A is independent of T.

3.1. **低阶估计。**在本小节将给出两个结果，分别表述于引理 3.1 和引理 3.2 中，涉及 (2.8) 的基本能量结构。此外，能量的时间衰减率将在引理 3.3 中给出。最后一部分，我们将在引理 3.4 中给出逐点估计。
3.1. **Lower-order estimates.** In this subsection, we will show two results, formulated in **Lemma 3.1** and **Lemma 3.2**, concerning the basic energy structure for (2.8). Moreover a time-decay rate of the energy will be shown in **Lemma 3.3**. In the last part, we will provide a pointwise estimate in **Lemma 3.4**.

3.1.1. **基本能量估计。**首先，我们给出 w 的时间导数和空间导数加权范数的估计。
3.1.1. *Basic energy estimate.* Here we give an estimate for the weighted norm of both time and spatial derivative of w.

引理 3.1. *对于 $\gamma > 2(1-s)/(1+s)$，存在一个 $E_0 > 0$ 使得在满足先验假设 (3.4) 的条件下，*
Lemma 3.1. *For $\gamma > 2(1-s)/(1+s)$, there exists $E_0 > 0$ such that under the a priori assumption* (3.4)*, it holds that*

$$\|\bar{\rho}^{\frac{1}{2}}\partial_t w(t)\|_{L^2}^2 + \|\bar{\rho}^{\gamma}\partial_x w(t)\|_{L^2}^2 + \int_0^t \|\partial_{tx} w(s)\|_{L^2}^2\,ds \leq C_{s,\gamma}\left(\|\bar{\rho}^{\frac{1}{2}}w_1\|_{L^2}^2 + \|\bar{\rho}^{\gamma}\partial_x w_0\|_{L^2}^2\right), \quad 0 \leq t \leq T, \tag{3.5}$$

对于某个常数 $C_{s,\gamma} > 0$。
for some constant $C_{s,\gamma} > 0$.

*证明。*将 (3.1) 乘以 $\partial_t w$，然后对空间变量积分，并利用分部积分和 (3.2)，可得
Proof. Multiplying (3.1) by $\partial_t w$, then integrating the product with respect to spatial variable, using the integration by parts and (3.2), we have

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}\bar{\rho}|\partial_t w|^2\,dx + \underbrace{\frac{d}{dt}\int_{-1}^{1}\frac{1}{\gamma-1}\bar{\rho}^{\gamma}\left[\left(\frac{1}{1+\partial_x w}\right)^{\gamma} + (\gamma-1)\partial_x w - 1\right]dx}_{(3.6)_1}$$

$$+ \underbrace{\frac{1}{2}\frac{d}{dt}\int_{-1}^{1} w\cdot\mathcal{L}w\,dx}_{(3.6)_2} + \underbrace{\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}\int_{-1}^{1} R_{0,2}(t,x,y)\bar{\rho}(y)\bar{\rho}(x)\,dy\,dx}_{(3.6)_3}$$

$$+ \underbrace{\int_{-1}^{1}\frac{|\partial_{tx} w|^2}{1+\partial_x w}\,dx}_{(3.6)_4} \equiv 0, \tag{3.6}$$

其中 $\mathcal{L}$ 是一个线性算子，定义为

$$\mathcal{L}w(t,x) := \mathcal{L}_1 w(t,x) + \mathcal{L}_2 w(t,x), \tag{3.7}$$

$$\mathcal{L}_1 w(t,x) := \bar{\rho}^2(x) \int_{-\infty}^{\infty} \frac{x-y}{|x-y|^{3-2s}} (\partial_x \tilde{w}(t,x) - \partial_y \tilde{w}(t,y))\, dy,$$

$$\mathcal{L}_2 w(t,x) := -\bar{\rho}(x) \int_{-\infty}^{\infty} \frac{2(s-1)}{|x-y|^{3-2s}} (\bar{\rho}(x) - \bar{\rho}(y))(\tilde{w}(t,x) - \tilde{w}(t,y))\, dy,$$

且非线性项 $R_{0,2}$ 的形式如下，

$$R_{0,2}(t,x,y) := \frac{1}{2s-1} \frac{1}{|\eta(t,x) - \eta(t,y)|^{1-2s}} - \frac{1}{2s-1} \frac{1}{|x-y|^{1-2s}}$$

$$- \frac{x-y}{|x-y|^{3-2s}} (w(t,x) - w(t,y)) - \frac{s-1}{|x-y|^{3-2s}} (w(t,x) - w(t,y))^2. \tag{3.8}$$

对于(3.6)，根据先验假设，(3.4)中的估计 **a priori assumption**，我们有

$$\left(\frac{\gamma}{2} - E_0\right) \int_{-1}^{1} \bar{\rho}^{\gamma-2} |\partial_x w|^2 \, dx \leq (3.6)_1 \leq \gamma \int_{-1}^{1} \bar{\rho}^{\gamma-2} |\partial_x w|^2 \, dx, \tag{3.9}$$

其中我们用到了以下初等代数不等式，

$$\left(\frac{\gamma}{2} - E_0\right) z^2 \leq \frac{1}{\gamma - 1} \left(\left(1 + z\right)^{\gamma-1} - 1 \right) \pm z - \frac{1}{\gamma - 1} \leq \gamma z^2,$$

对于充分小的 E_0 有 $|z| \leq E_0$ 充分小，这给出了(3.4)中对 E_0 的限制 for $|z| \leq E_0$ sufficiently small, that gives a restriction for ϵ_0 in (3.4).

对于(3.6)$_2$，我们断言

$$|(3.6)_2| \leq 2(1-s) \int_{-1}^{1} \bar{\rho} |\partial_x w|^2 \, dx. \tag{3.10}$$

事实上，回顾 (3.7)，一方面，通过直接计算，我们得到 Indeed, recall (3.7), on the one hand, by direct computation, we find

$$\int_{-1}^{1} w \cdot \mathcal{L}w \, dx = -2(1-s) \iint_{-1<x<y<1} \frac{|w(x) - w(y)|^2}{|x-y|^{3-2s}} \bar{\rho}(x) \bar{\rho}(y) \, dx \, dy. \tag{3.11}$$

另一方面，由于 ρ 支撑紧，且 ρ 是(1.4)的解，因此 $\bar{\rho}$ is the solution to (1.4), then

$$\bar{\rho}'(x) = \int_{x}^{\infty} \int_{-\infty}^{\infty} \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)(\bar{\rho}(z) - \bar{\rho}(y)) \, dz \, dy,$$

对于任意 $x \in [0,1]$，由于对任意的 $\varepsilon > 0$，Since for arbitrary $\varepsilon > 0$,

$$\left| \int_{y-\varepsilon}^{y+\varepsilon} \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)(\bar{\rho}(z) - \bar{\rho}(y)) \, dz \right| \leq \bar{\rho}(y) \|\bar{\rho}\|_{W^{1,\infty}} \int_{y-\varepsilon}^{y+\varepsilon} \frac{1}{|y-z|^{2-2s}} \, dz$$

$$\leq \frac{1}{s} \bar{\rho}(y) \|\bar{\rho}\|_{W^{1,\infty}} \varepsilon^{2s},$$

我们有 we have

$$\lim_{\varepsilon \to 0} \int_{y-\varepsilon}^{y+\varepsilon} \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)(\bar{\rho}(z) - \bar{\rho}(y)) \, dz = 0. \tag{3.12}$$

注意到 $(y-z)/|y-z|^{3-2s}$ 关于 y 是奇对称，因此我们推得 Noting $(y-z)/|y-z|^{3-2s}$ is odd symmetric about y, we deduce that

$$\left(\int_{-\infty}^{y-\varepsilon} + \int_{y+\varepsilon}^{\infty} \right) \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)^2 \, dz = 0,$$

结合 (3.12), 可得 With (3.12) implies

$$\bar{\rho}^{\gamma}(x) = \int_x^{\infty} \text{p.v.} \int_{-\infty}^{\infty} \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)\bar{\rho}(z)\,\mathrm{d}z\,\mathrm{d}y.$$

为简化符号，下文积分中将省略柯西主值 p.v.。因此可得 For notational simplicity, we will omit p.v. in the integrals hereafter. Therefore, it follows that

$$\begin{aligned}\int_{-1}^{1} \bar{\rho}^{\gamma}|\partial_x w|^2\,\mathrm{d}x &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)\bar{\rho}(z)|\partial_x \tilde{w}(x)|^2\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{y}\int_{-\infty}^{\infty} \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)\bar{\rho}(z)|\partial_x \tilde{w}(x)|^2\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

其中最后一行用到了富比尼定理；由对称性，对 y 和 z 的积分可交换。由于 where the last line is due to Fubini's theorem; by the symmetry of integration in y and z, we also have

$$\int_{-1}^{1} \bar{\rho}^{\gamma}|\partial_x w|^2\,\mathrm{d}x = \int_{-\infty}^{\infty}\int_{-\infty}^{z}\int_{-\infty}^{\infty} \frac{z-y}{|z-y|^{3-2s}} \bar{\rho}(z)\bar{\rho}(y)|\partial_x \tilde{w}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}z,$$

then

$$\begin{aligned}\int_{-1}^{1} \bar{\rho}^{\gamma}|\partial_x w|^2\,\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{z}^{y} \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)\bar{\rho}(z)|\partial_x \tilde{w}(x)|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \iint_{-\infty<z<y<\infty} \frac{y-z}{|y-z|^{3-2s}} \bar{\rho}(y)\bar{\rho}(z)\int_{z}^{y} |\partial_x \tilde{w}(x)|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad (3.13)\end{aligned}$$

由微积分基本定理，对于 $y \geq z$，有 By the fundamental theorem of calculus, for $y \geq z$, we have

$$|\tilde{w}(y) - \tilde{w}(z)| \leq \left(\int_z^y |\partial_x \tilde{w}(x)|^2\,\mathrm{d}x\right)^{1/2} (y-z)^{1/2};$$

结合 (3.13) 可得 which together with (3.13) implies

$$\int_{-1}^{1} \bar{\rho}^{\gamma}|\partial_x w|^2\,\mathrm{d}x \geq \iint_{-1<z<y<1} \bar{\rho}(y)\bar{\rho}(z) \frac{|w(y)-w(z)|^2}{|y-z|^{3-2s}}\,\mathrm{d}y\,\mathrm{d}z. \qquad (3.14)$$

对于 (3.6), 我们首先证明 $R_{0,2}$ 的一个逐点估计 For (3.6), we first prove a pointwise estimate for $R_{0,2}$,

$$|R_{0,2}(t,x,y)| \leq 4(1-s)(3-2s)A\varepsilon_0 \frac{|w(x)-w(y)|^2}{|x-y|^{3-2s}}, \qquad (3.15)$$

于是立即得到 then immediately we have

$$|(3.6)_3| \leq 4(1-s)(3-2s)A\varepsilon_0 \int_{-1}^{1}\int_{-1}^{1} \frac{|w(x)-w(y)|^2}{|x-y|^{3-2s}} \bar{\rho}(x)\bar{\rho}(y)\,\mathrm{d}y\,\mathrm{d}x.$$

接下来，利用 (3.14) 可得 Next, using (3.14) we find that

$$|(3.6)_3| \leq 4(1-s)(3-2s)A\varepsilon_0 \int_{-1}^{1} \bar{\rho}^{\gamma}|\partial_x w|^2\,\mathrm{d}x. \qquad (3.16)$$

接下来证明 (3.15)。我们计算 To prove (3.15), we compute

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{1}{2s-1}\frac{1}{|a(w(t,x)-w(t,y))+(1-a)(x-y)|^{1-2s}}\right) \\ &\quad = \frac{x-y+a(w(t,x)-w(t,y))}{|x-y+a(w(t,x)-w(t,y))|^{3-2s}}(w(t,x)-w(t,y)), \qquad (3.17)\end{aligned}$$

and

$$\frac{\mathrm{d}}{d\beta}\left(\frac{x-y+\beta(w(t,x)-w(t,y))}{|x-y+\beta(w(t,x)-w(t,y))|^{3-2s}}\right)$$
$$=(s-1)\frac{w(t,x)-w(t,y)}{|x-y+\beta(w(t,x)-w(t,y))|^{3-2s}}$$
$$+\delta_0(x-y+\beta(w(t,x)-w(t,y)))\frac{w(t,x)-w(t,y)}{|x-y+\beta(w(t,x)-w(t,y))|^{2-2s}}. \tag{3.18}$$

Applying (3.8), (3.17) and (3.18), we have

$$|R_{0,2}(t,x,y)|$$
$$=\left|\int_0^1\frac{x-y+\alpha(w(t,x)-w(t,y))}{|x-y+\alpha(w(t,x)-w(t,y))|^{3-2s}}(w(t,x)-w(t,y))\,\mathrm{d}\alpha\right.$$
$$\left.-\frac{x-y}{|x-y|^{3-2s}}(w(t,x)-w(t,y))-\frac{2s-2}{2}\frac{1}{|x-y|^{3-2s}}(w(t,x)-w(t,y))\right|$$
$$=\frac{|w(t,x)-w(t,y)|^2}{|x-y|^{3-2s}}\left|\int_0^1\int_0^\alpha(s-1)\frac{1}{|1+\beta\frac{w(t,x)-w(t,y)}{x-y}|^{3-2s}}\,\mathrm{d}\beta\mathrm{d}\alpha-(s-1)\right|$$
$$=2(1-s)\frac{|w(t,x)-w(t,y)|^2}{|x-y|^{3-2s}}\left|\int_0^1\frac{1-\beta}{|1+\beta\frac{w(t,x)-w(t,y)}{x-y}|^{3-2s}}\,\mathrm{d}\beta-\frac{1}{2}\right|,$$

where we used the *a priori* **assumption** (3.4), and the second equality holds since $\int_0^1\int_0^\alpha\mathrm{d}\beta\mathrm{d}\alpha=\frac{1}{2}$. Moreover, by ***a priori* assumption** (3.4) again, then we have

$$\left|\frac{1}{|1+\beta\frac{w(t,x)-w(t,y)}{x-y}|^{3-2s}}-1\right|\le 2(3-2s)A\epsilon_0,$$

where we used the following elementary algebraic inequality,

$$\left|\frac{1}{|1+z|^{3-2s}}-1\right|\le 2(3-2s)\epsilon,$$

for $|z|<\epsilon$ with ϵ sufficiently small. Therefore

$$|R_{0,2}(t,x,y)|\le 4(1-s)(3-2s)A\epsilon_0\frac{|w(x)-w(y)|^2}{|x-y|^{3-2s}}.$$

For (3.6), by the *a priori* **assumption** (3.4), we have

$$\|\nabla^{\frac{1}{4}}R_{0,2}\|_{L^2}^2\le\frac{1}{1+A\epsilon_0}\int_0^t\int_{\mathbb{T}^1}\int_{-1}^1|\theta_x w|^2\,dx\,ds, \tag{3.19}$$

Therefore, combining (3.9), (3.10), (3.15), (3.19), (3.5) and choosing ϵ_0 sufficiently small we obtain (3.5) which proves **Lemma 3.1**. □

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Remark 3.1. *In fact, the estimate that terms (3.6a) and (3.6b) exceeds the interaction between the fluid pressure and the self-consistent attractive force. Let us illustrate it from a variational point of view.*

注 3.1. 事实上，两个项(3.6a)和(3.6b)之间的估计揭示了流体压力与自洽吸引力之间的竞争。下面从变分观点对此加以说明。

Recall from (1.5) that, in the one-dimensional case,

$$\mathcal{F}[\rho] = \int_{\mathbb{R}} f\Big(\frac{1}{\gamma-1}\rho^{\gamma}\Big)\,dx + \frac{1}{2}\int_{\mathbb{R}} \rho\,(W**\rho)\,dx,$$

then it can be rewritten in the Lagrangian coordinates and use the mass conservation (2.7) to get

$$\begin{aligned}\mathcal{F}[\rho] &\equiv \frac{1}{\gamma-1}\int_{-1}^{1} \bar{\rho}^{\gamma}(\partial_x \eta)^{1-\gamma}\,dx \\ &\quad + \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} W(\eta(t,x)-\eta(t,y))\bar{\rho}(x)\bar{\rho}(y)\,dx\,dy =: \mathcal{H}[\eta].\end{aligned}$$

We set $\eta = x + \varepsilon w$. Formally, computing,

$$(\partial_x \eta)^{1-\gamma} = (1+\varepsilon\partial_x w)^{1-\gamma} = 1 + \varepsilon(1-\gamma)\partial_x w + \frac{1}{2}\varepsilon^2(1-\gamma)(-\gamma)(\partial_x w)^2 + O(1)\varepsilon^3,$$

and

$$\begin{aligned}W(\eta(t,x)-\eta(t,y)) &\equiv W(x-y+\varepsilon(w(x)-w(y))) \\ &\equiv W(x-y) + W'(x-y)\varepsilon[w(x)-w(y)] \\ &\quad + \frac{1}{2}W''(x-y)\varepsilon^2[w(x)-w(y)]^2 + O(1)\varepsilon^3.\end{aligned}$$

Since the unique global minimizer of the non-linear energy functional $\mathcal{F}$ is attained at $\bar{\rho}$, we obtain

$$\begin{aligned}&\mathcal{H}[x+\varepsilon w] \equiv \mathcal{H}(x) \\ &\quad + \varepsilon\Big\{-\int_{-1}^{1}\bar{\rho}^{\gamma}\partial_x w\,dx + \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} W'(x-y)[w(x)-w(y)]\bar{\rho}(x)\bar{\rho}(y)\,dy\,dx\Big\} \\ &\quad + \frac{1}{2}\varepsilon^2\Big\{\gamma\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_x w|^2\,dx - 2(1-s)\iint_{x<y}\frac{|w(x)-w(y)|^2}{|x-y|^{3-2s}}\bar{\rho}(x)\bar{\rho}(y)\,dy\,dx\Big\} + o(1)\varepsilon^3.\end{aligned}$$

For the free energy, εw above, using the stationary equation (1.4), we have

$$\begin{aligned}&-\int_{-1}^{1}\bar{\rho}^{\gamma}\partial_x w\,dx + \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} W'(x-y)[w(x)-w(y)]\bar{\rho}(x)\bar{\rho}(y)\,dy\,dx \\ &\quad \equiv \int_{-1}^{1}\partial_x(\bar{\rho}^{\gamma})\cdot w\,dx + \int_{-1}^{1}\int_{-1}^{1} W'(x-y)w(x)\bar{\rho}(x)(\bar{\rho}(y)-\bar{\rho}(x))\,dy\,dx \equiv 0.\end{aligned}$$

Therefore, since $\mathcal{H}[x+\varepsilon w]$ attains its unique minimum at $\varepsilon = 0$, it holds at least formally that

$$\gamma\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_x w|^2\,dx \geq 2(1-s)\iint_{-1\leq x<y\leq 1}\frac{|w(x)-w(y)|^2}{|x-y|^{3-2s}}\bar{\rho}(x)\bar{\rho}(y)\,dy\,dx. \tag{3.20}$$

Since $\gamma > 2(1-s)$, both (3.14) and (3.20) explain that the fluid pressure dominates the self-consistent attractive force.

In the following, we will provide the estimate for the spatial derivative of w near the vacuum.

下面，我们将进行引理3.1中关于真空附近 w 的空间导数估计。

Lemma 3.2. *For any $\gamma > 1$, there exists an $\varepsilon_0 > 0$ such that under the a priori assumption* (3.4)*, the following holds*

引理 3.2. *对于 $\gamma > 1$，存在一个 $\varepsilon_0 > 0$ 使得在先验假设* (3.4) *下，以下结论成立*

$$\|\partial_x w(t)\|_{L^2}^2 + \int_0^t \|\bar{\rho}^{\frac{\gamma-3}{2}} \partial_x w(s)\|_{L^2}^2 \, ds \leq C_{\gamma,\bar{\rho}}(\|\partial_x w_0\|_{L^2}^2 + \|\bar{\rho}^{\frac{\gamma-1}{2}} w_0\|_{L^2}^2 + \|\bar{\rho}^{\frac{\gamma-3}{2}} \partial_x w_0\|_{L^2}^2), \quad 0 \leq t \leq T, \tag{3.21}$$

for some constant $C_{\gamma,\bar{\rho}} > 0$.

对于某个常数 $C_{\gamma,\bar{\rho}} > 0$。

Proof. We may rewrite (3.1) as

证明. 将(3.1)改写为

$$-\partial_x \left(\frac{\partial_x w}{1+\partial_x w} \right) + \partial_x \left[\left(\frac{\bar{\rho}}{1+\partial_x w} \right)^\gamma - \bar{\rho}^\gamma \right] = -\bar{\rho}\partial_t w - \bar{\rho} w. \tag{3.22}$$

Then multiplying (3.22) by w, integrating the product with respect to spatial variable, using the integration by parts and boundary conditions (3.2),

然后将(3.22)乘以 w 对空间变量积分，并利用分部积分和边界条件(3.2)，

$$\begin{aligned}
&\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^1 \partial_x w - \ln(1+\partial_x w) \, dx}_{(3.23)_1} + \underbrace{\int_{-1}^1 \bar{\rho}^\gamma \partial_x w \left[1 - \left(\frac{1}{1+\partial_x w} \right)^{\gamma-1} \right] dx}_{(3.23)_2} \\
&= \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^1 -\bar{\rho} w \partial_t w \, dx}_{(3.23)_3} + \underbrace{\int_{-1}^1 \bar{\rho} |\partial_x w|^2 \, dx}_{\cdots (3.23)_4 \cdots} + \underbrace{\int_{-1}^1 -w \cdot \mathcal{L} w \, dx}_{(3.23)_5} \\
&\quad + \frac{1}{2} \underbrace{\int_{-1}^1 \int_{-1}^1 -\bar{\rho}(x)\bar{\rho}(y)(w(t,x) - w(t,y)) R_{1,1}(t,x,y) \, dy \, dx}_{(3.23)_6},
\end{aligned} \tag{3.23}$$

where the nonlinear term $R_{1,1}$ has the form

其中，非线性项 $R_{1,1}$ 的形式为

$$R_{1,1}(t,x,y) := \frac{n(t,x) - n(t,y)}{|x(t,x) - x(t,y)|^{3-2s}} - \frac{x - y}{|x - y|^{3-2s}} = 2(s-1) \frac{w(t,x) - w(t,y)}{|x - y|^{3-2s}}. \tag{3.24}$$

For $(3.23)_1$, by the *a priori assumption* (3.4), we have

对于 $(3.23)_1$，由先验假设(3.4)，可得

$$\frac{1}{4} \int_{-1}^1 |\partial_x w|^2 \, dx \leq (3.23)_1 \leq \int_{-1}^1 |\partial_x w|^2 \, dx, \tag{3.25}$$

where we used the following elementary algebraic inequality

其中用到了以下初等代数不等式

$$\frac{1}{4} z^2 \leq z - \ln(1+z) \leq z^2,$$

for z sufficiently small.

当 z 足够小时，对于 $|z| < \varepsilon_0$ small。

For $(3.23)_2$, by the *a priori assumption* (3.4), we have

对于 $(3.23)_2$，由先验假设(3.4)，可得

$$(\gamma - 1)\varepsilon_0 \int_{-1}^1 \bar{\rho}^\gamma |\partial_x w|^2 \, dx \leq (3.23)_2 \leq \frac{\gamma}{2} \int_{-1}^1 \bar{\rho}^\gamma |\partial_x w|^2 \, dx, \tag{3.26}$$

where we used the following elementary algebraic inequality

其中我们使用了如下初等代数不等式

$$(\gamma - 1)z^2 \leq z \left[1 - \left(\frac{1}{1+z} \right)^{\gamma-1} \right] \leq 2\gamma z^2,$$

对于 $|z| \leqslant \epsilon$，其中 ϵ 充分小。

For $|z| \leqslant \epsilon$ with ϵ sufficiently small.

对于(3.23)，由积分基本定理并利用 $w(t,0) = 0$，可得

For (3.23), by the fundamental theorem of calculus using that $w(t,0) = 0$, we have

$$|I_{3,1}| \leq \epsilon \int_{-1}^{1} |\partial_x w|^2 \, dx + C\epsilon \int_{-1}^{1} |\partial_x w|^2 \, dx, \tag{3.27}$$

其中用到了如下事实：$\rho \in L^\infty$，当 $2(1-s) \leqslant 1$，即 $s \geqslant \frac{1}{2}$ 时 $\frac{1}{|\cdot|^{2(1-s)}} \in L^1$。

where we have used the fact that $\rho \in L^\infty$, $\frac{1}{|\cdot|^{2(1-s)}} \in L^1$ for $2(1-s) \leqslant 1$, i.e. $s \geqslant \frac{1}{2}$.

对于(3.24)，由微积分基本定理并利用 $\partial_t w(t,0) = 0$，可得

For (3.24), by the fundamental theorem of calculus using that $\partial_t w(t,0) = 0$, we have

$$|I_{3,4}| \leq C \int_{-1}^{1} |\partial_x w|^2 \, dx. \tag{3.28}$$

对于(3.25)，我们有与(3.14)相同的估计，

For (3.25), we have same estimate as (3.14),

$$|I_{3,5}| \leq 2(1-s) \int_{-1}^{1} |\partial_x w|^2 \, dx. \tag{3.29}$$

对于(3.26)，我们断言

For (3.26), we claim that

$$|R_{1,1}(t,x,y)| \leq 2(1-s)(3-2s) \frac{A^2 \epsilon_0^2}{|x-y|^{2-2s}}, \tag{3.30}$$

于是我们有

then we have

$$|I_{3,6}| \leq A^2 \epsilon_0^2 (1-s)(3-2s) \int_{-1}^{1} \int_{-1}^{1} \frac{|w(t,x) - w(t,y)|}{|x-y|^{3-2s}} \rho(x) \rho(y) \, dy \, dx.$$

采用与(3.14)相同的估计，我们可以将(3.26)界定为

Using the same estimate as (3.14), we can bound (3.26) as

$$|I_{3,6}| \leq A^2 \epsilon_0^2 (1-s)(3-2s) \int_{-1}^{1} |\partial_x w|^2 \, dx. \tag{3.31}$$

还需证明(3.30)。我们计算得

It remains to prove (3.30). We compute

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{\alpha(w(t,x) - w(t,y)) + (1-\alpha)(x-y)}{|\alpha(w(t,x) - w(t,y)) + (1-\alpha)(x-y)|^{3-2s}} \\ &= 2(s-1) \frac{w(t,x) - w(t,y)}{|x - y + \alpha(w(t,x) - w(t,y))|^{3-2s}} \\ &\quad + \delta_0(x - y + \alpha(w(t,x) - w(t,y))) \frac{w(t,x) - w(t,y)}{|x - y + \alpha(w(t,x) - w(t,y))|^{3-2s}}. \end{aligned} \tag{3.32}$$

对(3.24)应用微积分基本定理，并考虑(3.32)，我们有

Applying the fundamental theorem of calculus to (3.24), taking (3.32) into account, we have

$$|R_{1,1}(t,x,y)| = 2(1-s) \frac{|w(t,x) - w(t,y)|}{|x-y|^{3-2s}} \left| \int_0^1 1 - \frac{1}{|1 + \alpha \frac{w(t,x) - w(t,y)}{x-y}|^{3-2s}} \, d\alpha \right|,$$

其中我们采用先验假设(3.4)。因此当 ϵ_0 足够小时，(3.32)右端的第一项为零，因为 $1 - \frac{x-y}{|x-y|} = 0$。此外，再次由先验假设(3.4)，可得

where we used the *a priori* assumption (3.4). Hence, the first term on the right hand side in (3.32) vanishes since $1 - \frac{x-y}{|x-y|} = 0$. Moreover, by ***a prior* assumption** (3.4) again, we have

$$\left| 1 - \frac{1}{|1 + \alpha \frac{w(t,x) - w(t,y)}{x-y}|^{3-2s}} \right| \leq 2(3-2s) A \epsilon_0,$$

其中用到了以下初等代数不等式，
where we used the following elementary algebraic inequality,

$$\left|\frac{1}{1+z}-1\right| \le 2|z|, \quad \text{if } |z| \le \frac{1}{2}.$$

对于$|z| \le \frac{1}{2}$时显然成立。因此
which is obviously true. Therefore

$$|R_{1,1}(t,x,y)| \le 2(1-s)(3-2s)\frac{A_1\epsilon_0^2}{|x-y|^{2-2s}}.$$

因此，结合(3.25)、(3.26)、(3.27)、(3.28)、(3.29)、(3.30)、(3.31)的结果3.2，并注意到$2(1-s)(3-2s) \le 2(1+s)$，再对结果关于时间t积分，取ϵ_0充分小，可得(3.21)，从而证明了引理3.2。
Thus, combining the results (3.25), (3.26), (3.27), (3.28), (3.29), (3.30), (3.31) with (3.32) and noting that $2(1-s)(3-2s) \le 2(1+s)$, then integrating the resulting result with respect to time t, by choosing and ϵ_0 sufficiently small, we obtain (3.21) which proves **Lemma 3.2**. □

得到基本能量估计后，我们进一步推导出关于w的时间导数和空间导数的时间加权估计。
Having obtained basic energy estimates, we further derive time-weighted estimates for the time and spatial derivatives of w.

引理 3.3. *对于$2(1+s) \le \gamma$，存在一个$\epsilon_0 > 0$使得在先验假设(3.4)下，以下结论成立：*
Lemma 3.3. *For $2(1+s) \le \gamma$, there exists an $\epsilon_0 > 0$ such that under the a priori assumption* (3.4) *the following holds*

$$\begin{aligned}
&(1+t)\|\sqrt{\bar{\rho}}\partial_t w(t)\|_{L^2}^2 + (1+t)\|\bar{\rho}^{\frac{\gamma}{2}}\partial_x w(t)\|_{L^2}^2 + \int_0^t (1+s)\|\partial_{tx} w(s)\|_{L^2}^2 \,\mathrm{d}s \\
&\quad \le C_{s,\gamma}(\|\partial_x w_0\|_{L^2}^2 + \|\sqrt{\bar{\rho}}w_0\|_{L^2}^2 + \|\bar{\rho}^{\frac{\gamma}{2}}\partial_x w_0\|_{L^2}^2), \quad 0 \le t \le T, \qquad (3.33)
\end{aligned}$$

对于某个常数$C_{s,\gamma} > 0$。
for some constant $C_{s,\gamma} > 0$.

证明. 回顾引理3.1，我们有
Proof. Recalling Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}|\partial_t w|^2 \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\frac{1}{\gamma-1}\bar{\rho}^{\gamma}\left[\left(\frac{1}{1+\partial_x w}\right)^{\gamma-1} + (\gamma-1)\partial_x w - 1\right]\mathrm{d}x \\
&\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} w \cdot \mathcal{L}w \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\int_{-1}^{1} p(x)p(y)R_{0,2}(t,x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + \int_{-1}^{1}\frac{|\partial_{tx} w|^2}{1+\partial_x w}\,\mathrm{d}x = 0,
\end{aligned}$$

将上述方程乘以$1+t$，可得
Multiplying the above equation by $1+t$ we find that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)\int_{-1}^{1}|\partial_t w|^2 \,\mathrm{d}x\right] - \frac{1}{2}\underbrace{\int_{-1}^{1}|\partial_t w|^2 \,\mathrm{d}x}_{(3.34)_1} \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\left\{(1+t)\int_{-1}^{1}\frac{1}{\gamma-1}\bar{\rho}^{\gamma}\left[\left(\frac{1}{1+\partial_x w}\right)^{\gamma-1} + (\gamma-1)\partial_x w - 1\right]\mathrm{d}x\right\}}_{(3.34)_2} \\
&\quad - \underbrace{\int_{-1}^{1}\frac{1}{\gamma-1}\bar{\rho}^{\gamma}\left[\left(\frac{1}{1+\partial_x w}\right)^{\gamma-1} + (\gamma-1)\partial_x w - 1\right]\mathrm{d}x}_{(3.34)_3}
\end{aligned}$$

$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[(1+t)\int_{-1}^{1} w\cdot LL_w w\,\mathrm{d}x\right]}_{(3.34)_4}=\underbrace{\frac{1}{2}\int_{-1}^{1} w\cdot LL_w w\,\mathrm{d}x}_{(3.34)_5}$$

$$+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left((1+t)\int_{-1}^{1}\int_{-1}^{1}\bar{p}(x)\bar{p}(y)R_{0,2}(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\right)}_{(3.34)_6}$$

$$-\underbrace{\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\bar{p}(x)\bar{p}(y)R_{0,2}(t,x,y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.34)_7}+\underbrace{(1+t)\int_{-1}^{1}\frac{|\partial_x w|^2}{1+\partial_x w}\,\mathrm{d}x}_{(3.34)_8}=0. \tag{3.34}$$

For $(3.34)_1$, by the fundamental theorem of calculus and using that $\partial_t w(t,0)=0$, we have

$$|(3.34)_1|\le C\int_{-1}^{1}|\partial_x w|^2\,\mathrm{d}x. \tag{3.35}$$

For $(3.34)_2$ and $(3.34)_3$, by the *a priori* assumption (3.4),

$$\left(\frac{\gamma}{2}-A\varepsilon_0\right)(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma-1}|\partial_x w|^2\,\mathrm{d}x\le(3.34)_2\le\gamma(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma-1}|\partial_x w|^2\,\mathrm{d}x, \tag{3.36}$$

where we used the following elementary algebraic inequality,

$$\left(\frac{\gamma}{2}-\varepsilon\right)z^2\le\frac{1}{\gamma-1}\left(\frac{1}{1+z}\right)^{\gamma-1}+z-\frac{1}{\gamma-1}\le\gamma z^2,$$

for $|z|\le\varepsilon$ with ε sufficiently small. We find also that

$$|(3.34)_3|\le C\int_{-1}^{1}\bar{\rho}^{\gamma-1}|\partial_x w|^2\,\mathrm{d}x. \tag{3.37}$$

For $(3.34)_4$ and $(3.34)_5$, using (3.11) and (3.14) we obtain

$$|(3.34)_4|\le 2(1-s)(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma-1}|\partial_x w|^2\,\mathrm{d}x, \tag{3.38}$$

$$|(3.34)_5|\le 2(1-s)\int_{-1}^{1}\bar{\rho}^{\gamma-1}|\partial_x w|^2\,\mathrm{d}x. \tag{3.39}$$

For $(3.34)_6$ and $(3.34)_7$, 类似于引理 3.1 中对 $(3.6)_1$ 和 $(3.6)_3$ 的估计, in **Lemma 3.1**,

$$|(3.34)_6|\le 8(1-s)(3-2s)A\varepsilon_0(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma-1}|\partial_x w|^2\,\mathrm{d}x, \tag{3.40}$$

and

$$|(3.34)_7|\le 8(1-s)(3-2s)A\varepsilon_0\int_{-1}^{1}\bar{\rho}^{\gamma-1}|\partial_x w|^2\,\mathrm{d}x. \tag{3.41}$$

For $(3.34)_8$, 以及先验假设由的估计 (3.4), *a priori* **assumption**,

$$\frac{1}{1+A\varepsilon_0}(1+t)\int_{-1}^{1}|\partial_x w|^2\,\mathrm{d}x\le(3.34)_8\le\frac{1}{1-A\varepsilon_0}(1+t)\int_{-1}^{1}|\partial_x w|^2\,\mathrm{d}x. \tag{3.42}$$

Therefore combining the results of (3.35), (3.36), (3.37), (3.38), (3.39), (3.40), (3.41), (3.42) and (3.34), (3.22) and choosing ϵ_0 sufficiently small, we obtain (3.33) which prove **Lemma 3.3**. □

3.1.2. *Lagrangian coordinates in Lagrangian domain.* Let us give the pointwise estimate on $w(t,x)$ which is the key point of this paper. This estimate holds for arbitrary time and it is independent of the lifespan T of the solution. Thus it will be used to extend the solution globally in time. Before stating the estimate, we first introduce the Lagrangian coordinates in the Lagrangian domain.

Lemma 3.4. *For $\gamma \geq 2(1-s)$, $s \in \mathbb{N}$ and $A \geq 1$ large enough such that*

$$\begin{cases} (1-10^{-N})\eta > c_2(1-s), \\ A \geq \dfrac{C_{s,\gamma}}{1-10^{-N}-c_2(1-s)/\eta}, \end{cases} \tag{3.43}$$

for a fixed constant $C_{s,\gamma} > 0$ (which will be determined later), under the bootstrap assumption (3.4), it holds that

$$\|\partial_x w(t)\|_{L^\infty} \leq (1-10^{-N-1})A\epsilon_0, \quad 0 \leq t \leq T. \tag{3.44}$$

Proof. Rewriting the equation (2.8) as

$$\partial_t(\ln \partial_x \tau) = F(t,x) + \partial_x\left[\left(\frac{\bar{\rho}}{\partial_x \eta}\right)^\gamma - 1\right] + \Phi. \tag{3.45}$$

From the initial condition (2.2) and boundary condition (2.12), by fundamental theorem of calculus, we have

$$\partial_x \tau(t,x) = \partial_x \tau_0(x) \exp\left[-\int_0^t \int_x^1 \partial_x(\ln \partial_x \tau)\, dy\, d\tau\right],$$

因此, 由(3.45)可得,

$$\partial_x \tau(t,x) = \partial_x \tau_0(x) \exp\left(F(t,x) - \int_0^t \int_x^1 G(\tau,y)\, dy\, d\tau\right), \tag{3.46}$$

其中, 令 where we set

$$F := \int_0^t \left(\frac{\bar{\rho}}{\partial_x \eta}\right)^\gamma d\tau,$$
$$G := \rho u_t - \partial_x(F^\gamma) + \Phi.$$

再次利用(3.46), 注意到 Using again (3.46), we note that

$$\partial_x F = \left(\frac{\bar{\rho}}{\partial_x \eta}\right)^\gamma = \left(\frac{\bar{\rho}}{\partial_x \eta_0 J}\right)^\gamma \exp(-\gamma F) \exp\left(\gamma \int_0^t \int_x^1 G(\tau,y)\, dy\, d\tau\right).$$

将上述结果乘以 $\gamma \exp(\gamma F)$, 得到 Multiplying the above result by $\gamma \exp(\gamma F)$ gives

$$\partial_x \exp(\gamma F) = \gamma \left(\frac{\bar{\rho}}{\partial_x \eta_0 J}\right)^\gamma \exp\left(\gamma \int_0^t \int_x^1 G(\tau,y)\, dy\, d\tau\right).$$

对 $[0,t]$ 进行时间积分, 并注意到 $F(0,x) = 0$, 可得 Taking integration in time over $[0,t]$ and noting $F(0,x) = 0$ it follows that

$$\exp(F) = \left[1 + \gamma \left(\frac{\bar{\rho}}{\partial_x \eta_0 J}\right)^\gamma \int_0^t \exp\left(\gamma \int_0^s \int_x^1 G(\tau,y)\, dy\, d\tau\right) ds\right]^{\frac{1}{\gamma}}$$

回顾 (3.46), we have

$$\begin{aligned}\partial_x\eta(t,x) &= \partial_x\eta_0(x)\exp\left(-\int_0^t\int_x^1 G(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau\right)\\ &\quad\times\left[1+\gamma\left(\frac{\bar\rho(x)}{\partial_x\eta_0(x)}\right)^\gamma\int_0^t\exp\left(\gamma\int_0^s\int_x^1 G(\tau,y)\,\mathrm{d}y\,\mathrm{d}\tau\right)\mathrm{d}s\right]^{1/\gamma}\\ &= \partial_x\eta_0(x)\Bigg[\exp\left(-\gamma\bar\rho^\gamma(x)t-\gamma H_1(t,x)-\gamma H_1(0,x)-\gamma\int_0^t H_2(s,x)\,\mathrm{d}s\right)\\ &\quad+\gamma\left(\frac{\bar\rho(x)}{\partial_x\eta_0(x)}\right)^\gamma\int_0^t\exp\Bigg(-\gamma\bar\rho^\gamma(x)(t-s)-\gamma H_1(t,x)+\gamma H_1(s,x)\\ &\qquad\qquad-\gamma\int_s^t H_2(\tau,x)\,\mathrm{d}\tau\Bigg)\mathrm{d}s\Bigg]^{1/\gamma},\end{aligned}\tag{3.47}$$

其中我们denote

$$\begin{aligned}H_1(t,x) &:= \int_x^1\bar\rho(y)\partial_x\eta(t,y)\,\mathrm{d}y,\\ H_2(t,x) &:= \int_x^1\Phi_1(t,x)\,\mathrm{d}x.\end{aligned}$$

接下来，我们给出H_1和H_2的estimates for H_1 and H_2.

对于H_1，应用霍尔德不等式和引理 3.1, Lemma 3.1, we have

$$\|H_1(t,x)\|_{L^\infty}\le\left(\int_x^1\bar\rho(y)\,\mathrm{d}y\right)^{1/2}\left(\int_x^1\bar\rho(y)|\partial_x\eta(t,y)|^2\,\mathrm{d}y\right)^{1/2}\le C_{s,E_0},\tag{3.48}$$

对于H_2，回顾 (2.9)和(1.4)，我们有 we have

$$\begin{aligned}H_2(t,x) &= -\int_x^1\int_{-\infty}^{\infty}\bar\rho(y)\frac{y-z}{|y-z|^{3-2s}}(\bar\rho(z)-\rho(y))\,\mathrm{d}z\,\mathrm{d}y\\ &\quad+\frac{1}{2}\int_x^1\int_{-\infty}^{\infty}\frac{\bar\rho(y)}{|\eta(t,y)-\eta(t,z)|^{3-2s}}(\bar\rho(z)\partial_y T(t,y)-\bar\rho(y)\partial_z T(t,z))\,\mathrm{d}z\,\mathrm{d}y\\ &= \int_x^1\int_{-\infty}^{x}\bar\rho(y)\bar\rho(z)R_{1,0}(t,y,z)\,\mathrm{d}z\,\mathrm{d}y,\end{aligned}$$

其中$R_{1,0}$具有如下形式 where $R_{1,0}$ has the form

$$R_{1,0}(t,x,y)\equiv\frac{\eta(t,x)-\eta(t,y)}{|\eta(t,x)-\eta(t,y)|^{3-2s}}-\frac{x-y}{|x-y|^{3-2s}}.\tag{3.49}$$

现在我们来估计$R_{1,0}$. Now let us estimate $R_{1,0}$, we compute

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}a}\frac{a(\eta(t,x)-\eta(t,y))+(1-a)(x-y)}{|a(\eta(t,x)-\eta(t,y))+(1-a)(x-y)|^{3-2s}}\\ &\equiv 2(s-1)\frac{w(t,x)-w(t,y)}{|x-y+a(w(t,x)-w(t,y))|^{3-2s}}\\ &\quad+\delta_0(x-y+a(w(t,x)-w(t,y)))\frac{w(t,x)-w(t,y)}{|x-y+a(w(t,x)-w(t,y))|^{2-2s}}.\end{aligned}\tag{3.50}$$

Applying the fundamental theorem of calculus to (3.49), taking (3.50) into account, we obtain

$$|R_{1,0}(t,x,y)| \equiv 2(1-s)\frac{|w(t,x)-w(t,y)|}{|x-y|^{3-2s}}\int_0^1 \frac{1}{1+\alpha\frac{w(t,x)-w(t,y)}{x-y}|^{3-2s}}\,\mathrm{d}\alpha,$$

where we use the ***a prior* assumption** (3.4) such that $|\frac{w(t,x)-w(t,y)}{x-y}| \leqslant A\epsilon_0 \ll 1$ (当ϵ_0足够小时) if ϵ_0 is sufficiently small. Moreover, by the ***a prior*** **assumption** (3.4) again, we have

$$\frac{1}{|1+\alpha\frac{w(t,x)-w(t,y)}{x-y}|^{3-2s}} \leqslant \frac{1}{(1-A\epsilon_0)^{3-2s}},$$

因此

$$|R_{1,0}(t,x,y)| \leqslant \frac{2(1-s)A\epsilon_0}{(1-A\epsilon_0)^{3-2s}}\frac{1}{|x-y|^{2-2s}}. \tag{3.51}$$

Applying this above estimate, and noticing (1.4), by (3.51), we have

$$\begin{aligned}|H_2(t,x)| &\leq 2(1-s)A\epsilon_0\int_x^1\int_0^x \bar{\rho}(y)\bar{\rho}(z)\frac{1}{|y-z|^{2-2s}}\,\mathrm{d}z\,\mathrm{d}y\\ &\leq \frac{2(1-s)A\epsilon_0}{(1-A\epsilon_0)^{3-2s}}\bar{\rho}^{\gamma}(x),\end{aligned} \tag{3.52}$$

where we have used the fact that,

$$W'(y-z) = \frac{1}{|y-z|^{2-2s}}$$

for $y \geqslant z$.

Combining (3.48), (3.51), (3.52), (3.47) with (3.47), we obtain the lower and upper bounds for $\partial_x\eta$ by (3.47),

$$L(t,x,\epsilon_0) \leqslant \partial_x\eta(t,x) \leqslant U(t,x,\epsilon_0),$$

where U and L respectively have the form

$$\begin{aligned}U(t,x,\epsilon) :=& (1+\epsilon)(1+C_{s,\gamma}\epsilon)\left[g_1(t,x,\epsilon)+\frac{1+C_{s,\gamma}\epsilon}{(1-\epsilon)^{\gamma}}\frac{1}{1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}}\left(1-g_1(t,x,\epsilon)\right)\right]^{1/\gamma},\\ L(t,x,\epsilon) :=& (1-\epsilon)(1-C_{s,\gamma}\epsilon)\left[g_2(t,x,\epsilon)+\frac{1-C_{s,\gamma}\epsilon}{(1+\epsilon)^{\gamma}}\frac{1}{1+\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}}\left(1-g_2(t,x,\epsilon)\right)\right]^{1/\gamma},\end{aligned}$$

with

$$g_1(t,x,\epsilon) = e^{-\gamma\bar{\rho}^{\gamma-1}(x)\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]t} \quad\text{and}\quad g_2(t,x,\epsilon) = e^{-\gamma\bar{\rho}^{\gamma-1}(x)\left[1+\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]t}.$$

为了估计 U_ϵ，我们来计算 In order to estimate U_ϵ, let us compute

$$\partial_t U_\epsilon(t,x,\varepsilon) = (1+C_{s,\gamma}\varepsilon)\left[g_1(t,x,\varepsilon) + \frac{1+C_{s,\gamma}\varepsilon}{2(1-s)A\epsilon}(1-g_1(t,x,\varepsilon))\right]^{1/\gamma}$$

$$+C_{s,\gamma}(1+\varepsilon)\left[g_1(t,x,\varepsilon) + \frac{1+C_{s,\gamma}\varepsilon}{2(1-s)A\epsilon}(1-g_1(t,x,\varepsilon))\right]^{1/\gamma}$$

$$+\frac{(1+\varepsilon)(1+C_{s,\gamma}\varepsilon)}{\gamma}\left[g_1(t,x,\varepsilon) + \frac{1+C_{s,\gamma}\varepsilon}{2(1-s)A\epsilon}(1-g_1(t,x,\varepsilon))\right]^{(1-\gamma)/\gamma}$$

$$\times\left[\left(2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}} - \frac{2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}(1+C_{s,\gamma}\varepsilon)}{(1+\varepsilon)^\gamma\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]}\right)\nabla\bar{P}^\gamma(x)\,tg_1(t,x,\varepsilon)\right.$$

$$\left.+\left(\frac{C_{s,\gamma}}{(1+\varepsilon)^\gamma\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]} + \frac{\gamma(1+C_{s,\gamma}\varepsilon)}{(1+\varepsilon)^{\gamma+1}\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]}\right.\right.$$

$$\left.\left.+\frac{2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}(1+C_{s,\gamma}\varepsilon)}{(1+\varepsilon)^\gamma\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]^2}\right)(1-g_1(t,x,\varepsilon))\right], \quad (3.53)$$

and

$$\partial_\varepsilon U_\epsilon(t,x,\varepsilon) = 2C_{s,\gamma}\left[g_1(t,x,\varepsilon) + \frac{1+C_{s,\gamma}\varepsilon}{2(1-s)A\epsilon}(1-g_1(t,x,\varepsilon))\right]^{1/\gamma}$$

$$+2(1+C_{s,\gamma}\varepsilon)\frac{1}{\gamma}\left[g_1(t,x,\varepsilon) + \frac{1+C_{s,\gamma}\varepsilon}{2(1-s)A\epsilon}(1-g_1(t,x,\varepsilon))\right]^{(1-\gamma)/\gamma}$$

$$\times\left[\left(2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}} - \frac{2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}(1+C_{s,\gamma}\varepsilon)}{(1+\varepsilon)^\gamma\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]}\right)\nabla\bar{P}^\gamma(x)\,tg_1(t,x,\varepsilon)\right.$$

$$\left.+\left(\frac{C_{s,\gamma}}{(1+\varepsilon)^\gamma\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]} + \frac{\gamma(1+C_{s,\gamma}\varepsilon)}{(1+\varepsilon)^{\gamma+1}\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]}\right.\right.$$

$$\left.\left.+\frac{2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}(1+C_{s,\gamma}\varepsilon)}{(1+\varepsilon)^\gamma\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]^2}\right)(1-g_1(t,x,\varepsilon))\right]$$

$$+(1+\varepsilon)(1+C_{s,\gamma}\varepsilon)\frac{1-\gamma}{\gamma^2}\left[g_1(t,x,\varepsilon) + \frac{1+C_{s,\gamma}\varepsilon}{2(1-s)A\epsilon}(1-g_1(t,x,\varepsilon))\right]^{(1-2\gamma)/\gamma}$$

$$\times\left[\left(2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}} - \frac{2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}(1+C_{s,\gamma}\varepsilon)}{(1+\varepsilon)^\gamma\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]}\right)\nabla\bar{P}^\gamma(x)\,tg_1(t,x,\varepsilon)\right.$$

$$\left.+\left(\frac{C_{s,\gamma}}{(1+\varepsilon)^\gamma\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]} + \frac{\gamma(1+C_{s,\gamma}\varepsilon)}{(1+\varepsilon)^{\gamma+1}\left[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}\right]}\right.\right.$$

$$+\frac{2(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]^2}\Bigg)(1-g_1(t,x,\epsilon))\Bigg]^2$$

$$+(1+\epsilon)(1+C_{s,\gamma}\epsilon)\frac{1}{\gamma}\Big[g_1(t,x,\epsilon)+\frac{1+C_{s,\gamma}\epsilon}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]}(1-g_1(t,x,\epsilon))\Big]^{(1-\gamma)/\gamma}$$

$$\times\Bigg[\Bigg(4(1-s)^2A^2\frac{(1+2(1-s)A\epsilon)^2}{(1-A\epsilon)^{8-4s}}-\frac{4(1-s)^2A^2\frac{(1+2(1-s)A\epsilon)^2}{(1-A\epsilon)^{8-4s}}(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]}\Bigg)\gamma^2\bar{\rho}^{2\gamma}(x)t^2g_1(t,x,\epsilon)$$

$$+\Bigg(-\frac{4(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}C_{s,\gamma}}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]}-\frac{4(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma+1}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]}$$

$$-\frac{8(1-s)^2A^2\frac{(1+2(1-s)A\epsilon)^2}{(1-A\epsilon)^{8-4s}}(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]^2}+4(1-s)(3-2s)A^2\frac{1+(1-s)A\epsilon}{(1-A\epsilon)^{5-2s}}$$

$$+\frac{4(1-s)(3-2s)A^2\frac{1+(1-s)A\epsilon}{(1-A\epsilon)^{5-2s}}(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]}\Bigg)\gamma\bar{\rho}^{\gamma}(x)tg_1(t,x,\epsilon)$$

$$+\Bigg(\frac{2\gamma C_{s,\gamma}}{(1-\epsilon)^{\gamma+1}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]}+\frac{\gamma(\gamma+1)(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma+2}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]}+\frac{4(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}C_{s,\gamma}}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]^2}$$

$$+\frac{4(1-s)A\frac{1+2(1-s)A\epsilon}{(1-A\epsilon)^{4-2s}}\gamma(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma+1}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]^2}+\frac{8(1-s)^2A^2\frac{(1+2(1-s)A\epsilon)^2}{(1-A\epsilon)^{8-4s}}(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]^3}$$

$$+\frac{4(1-s)(3-2s)A^2\frac{1+(1-s)A\epsilon}{(1-A\epsilon)^{5-2s}}(1+C_{s,\gamma}\epsilon)}{(1-\epsilon)^{\gamma}[1-\frac{2(1-s)A\epsilon}{(1-A\epsilon)^{3-2s}}]^2}\Bigg)(1-g_1(t,x,\epsilon))\Bigg]. \quad (3.54)$$

对于固定的 $t\in[0,T]$ 和 $x\in[-1,1]$，在 $\epsilon=0$ 附近对 U 关于 ϵ 应用泰勒定理，我们有

Applying Taylor's theorem to $U(t,x,\epsilon)$ with respect to ϵ for fixed $t\in[0,T]$ and $x\in[-1,1]$, we have

$$U(t,x,\epsilon)=U(t,x,0)+\partial_\epsilon U(t,x,0)\epsilon+\frac{\partial_{\epsilon\epsilon}U(t,x,\epsilon')}{2}\epsilon^2,$$

对于某个 $\epsilon'\in(0,\epsilon)$。利用 (3.53) 和 (3.54)，我们得到以下与时间无关的一致估计：

for some $\epsilon'\in(0,\epsilon)$. Using (3.53) and (3.54), we obtain the following uniform estimates independent of time,

$$\partial_\epsilon U(t,x,0)=1+C_{s,\gamma}+\frac{C_{s,\gamma}+\gamma+2(1-s)A}{\gamma}(1-e^{-\gamma\bar{\rho}^{\gamma}(x)t})$$
$$\leq\frac{2(1-s)}{\gamma}A+\frac{\gamma+1}{\gamma}(1+C_{s,\gamma}),$$

and

$$\partial_{\epsilon\epsilon}U(t,x,\epsilon')\leq 2C_{s,\gamma}\left(1+\frac{1+C_{s,\gamma}\epsilon'}{(1-\epsilon')^{\gamma}[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}]}\right)^{1/\gamma}$$

$$
\begin{aligned}
&+2(1+C_{s,\gamma})+2C_{s,\gamma}\frac{1}{\gamma}\left(1+\frac{(1+C_{s,\gamma}\epsilon')(t-s)M}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}\right)^{(1-\gamma)/\gamma}(t-s)\\
&\times\left(2(t-s)A\frac{1+2(1-s)A\epsilon'}{(1-A\epsilon')^{4-2s}}+\frac{C_{s,\gamma}}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}+\frac{\gamma(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^{\gamma+1}\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}\right.\\
&\left.+\frac{2(t-s)A\frac{1+2(1-s)A\epsilon'}{(1-A\epsilon')^{4-2s}}(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]^2}+\frac{2(t-s)A\frac{1+2(1-s)A\epsilon'}{(1-A\epsilon')^{4-2s}}(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]^2}\right)\\
&+(1+\epsilon')(1+C_{s,\gamma}\epsilon')\frac{1-\gamma}{\gamma^2}\left(1+\frac{(1+C_{s,\gamma}\epsilon')(t-s)M}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}\right)^{(1-2\gamma)/\gamma}(t-s)^2\\
&\times\left(2(t-s)A\frac{1+2(1-s)A\epsilon'}{(1-A\epsilon')^{4-2s}}+\frac{C_{s,\gamma}}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}+\frac{\gamma(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^{\gamma+1}\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}\right.\\
&\left.+\frac{2(t-s)A\frac{1+2(1-s)A\epsilon'}{(1-A\epsilon')^{4-2s}}(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]^2}+\frac{2(t-s)A\frac{1+2(1-s)A\epsilon'}{(1-A\epsilon')^{4-2s}}(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]^2}\right)^2\\
&+(1+\epsilon')(1+C_{s,\gamma}\epsilon')\frac{1}{\gamma}\left(1+\frac{(1+C_{s,\gamma}\epsilon')(t-s)M}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}\right)^{(1-\gamma)/\gamma}(t-s)^2\\
&\times\left(4(t-s)^2A^2\frac{(1+2(1-s)A\epsilon')^2}{(1-A\epsilon')^{8-4s}}+\frac{2\gamma C_{s,\gamma}}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}+\frac{\gamma(\gamma+1)(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^{\gamma+2}\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]}\right.\\
&+\frac{4(t-s)A\frac{1+2(1-s)A\epsilon'}{(1-A\epsilon')^{4-2s}}C_{s,\gamma}}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]^2}+\frac{8(t-s)^2A^2\frac{(1+2(1-s)A\epsilon')^2}{(1-A\epsilon')^{8-4s}}(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]^3}\\
&+4(1-s)(3-2s)A^2\frac{1+(1-s)A\epsilon'}{(1-A\epsilon')^{5-2s}}+\frac{4(1-s)(3-2s)A^2\frac{1+(1-s)A\epsilon'}{(1-A\epsilon')^{5-2s}}(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]^2}\\
&\left.+\frac{4(1-s)(3-2s)A^2\frac{1+(1-s)A\epsilon'}{(1-A\epsilon')^{5-2s}}(1+C_{s,\gamma}\epsilon')}{(1-\epsilon')^\gamma\left[1-\frac{2(1-s)A\epsilon'}{(1-A\epsilon')^{3-2s}}\right]^2}\right)\\
&\le 4C_{s,\gamma}+(1+C_{s,\gamma})\frac{1}{\gamma}2^{(1-\gamma)/\gamma}(2(1-s)A(1+2e^{-1})+C_{s,\gamma}+\gamma)\\
&\quad+\frac{1-\gamma}{\gamma^2}2^{(1-2\gamma)/\gamma}(2(1-s)A(1+2e^{-1})+C_{s,\gamma}+\gamma)^2\\
&\quad+\frac{2^{1/\gamma}}{\gamma}(4(1-s^2)A^2e^{-2}+2\gamma C_{s,\gamma}+\gamma(\gamma+1)+4(1-s)AC_{s,\gamma}+8(1-s)^2A^2\\
&\quad+4(1-s)(3-2s)A^2e^{-1}+4(1-s)(3-2s)A^2e^{-1}+4(1-s)(3-2s)A^2)\\
&:=C'_{s,\gamma}.
\end{aligned}
$$

因此我们推导出

Therefore we deduce that

$$
U(t,x,E_0)\le 1+\left(\frac{2(1-s)}{\gamma}A+\frac{\gamma+1}{\gamma}(1+C'_{s,\gamma})+\frac{1}{2}C'_{s,\gamma}E_0\right)E_0.
$$

可压缩

由于 $\gamma > 2(1-s)$, 令 $A \geq \frac{C_0}{2(1-s)/\gamma - 1}$ for some $N \in \mathbb{N}$ such that $(1+N)\gamma > 2(1-s)$, 则 then

$$U(t,x,E_0) \leq 1 + (1 + 2^{N-1})AE_0,$$

当 E_0 sufficiently small. Let us emphasize that the upper bound is independent of t and x, therefore we have

$$\partial_x^{\gamma} U(t,x) - 1 \leq (1 + 2^{N-1})AE_0, \tag{3.55}$$

对所有 $t \in [0,T]$ 以及 $x \in [-1,1]$.

Similarly, 类似地，我们可以按照估计 U 时的相同步骤来估计 Γ, 从而得到 estimating U to obtain

$$\partial_x^{\gamma} \Gamma(t,x) - 1 \geq -(1 + 2^{N-1})AE_0, \tag{3.56}$$

对所有 $t \in [0,T]$ 以及 $x \in [-1,1]$.

Therefore (3.55) and (3.56) prove (3.44) in **Lemma 3.4.** □

3.2. Higher-order estimates. In this subsection, we will give two auxiliary results as preparation for **Lemma 3.1**, **Lemma 3.2** and **Lemma 3.3** in which higher regularity in time variable, see **Lemma 3.5**, **Lemma 3.6** and **Lemma 3.7**. An energy estimate for higher regularity in spatial variable will also be proved in **Lemma 3.8**.

3.2. 高阶估计。 本小节将给出两个有引理 3.1、引理 3.2 和引理 3.3 类似的结果，涉及时间变量的更高正则性，见引理 3.5、引理 3.6 和引理 3.7。引理 3.8 还将证明关于空间变量更高正则性的能量估计。

First, let us give an estimate for the time derivative and mixed time-space derivative of w.

首先，我们给出 w 的三阶时间导数和时空混合导数加权范数的估计。

Lemma 3.5. *For $\gamma > 2(1-s)$, there exists $\varepsilon_0 > 0$, such that if the initial data satisfies the a priori assumption* (3.4)*, it holds that*

引理 3.5. 对于 $\gamma > 2(1-s)$，存在 $\varepsilon_0 > 0$，使得若初值满足先验假设 (3.4)，则有

$$\begin{aligned} &\|\sqrt{\rho_0}\partial_{tt}w(t)\|_{L^2}^2 + \|\sqrt{\rho_0^{\gamma}}\partial_{tx}w(t)\|_{L^2}^2 + \int_0^t \|\partial_{ttx}w(s)\|_{L^2}^2 \mathrm{d}s \\ &\quad \leq C_{s,\gamma}(\|\sqrt{\rho_0}\partial_t w_1\|_{L^2}^2 + \|\sqrt{\rho_0^{\gamma}}\partial_x w_1\|_{L^2}^2), \quad 0 \leq t \leq T, \end{aligned} \tag{3.57}$$

where $C_{s,\gamma}$ is a constant 其中 $C_{s,\gamma}$ 为某个常数 > 0。.

Proof. Differentiating (3.1) with respect to t and multiplying the resulting equation by $\partial_{tt}w$, then integrating by parts and using (3.2), we have

证明. 对 (3.1) 关于 t 求导，并将所得结果乘以 $\partial_{tt}w$，然后对乘积关于空间变量积分，利用分部积分和边界条件 (3.2)，得到

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1} \rho_0 |\partial_{tt}w|^2 \mathrm{d}x + \frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_{-1}^{1} \bar{\rho}^{\gamma}\left(\frac{1}{1+\partial_x w}\right)^{\gamma+1} |\partial_{tx}w|^2 \mathrm{d}x}_{(3.58)_1} \\ &\quad + \frac{\gamma(\gamma+1)}{2}\underbrace{\int_{-1}^{1} \bar{\rho}^{\gamma}\left(\frac{1}{1+\partial_x w}\right)^{\gamma+2} (\partial_{tx}w)^3 \mathrm{d}x}_{(3.58)_2} \\ &\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_{-1}^{1} \partial_{tw} \cdot \mathcal{L}\partial_{tw}\mathrm{d}x}_{(3.58)_3} \\ &\quad + \frac{1-s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\int_{-1}^{1}\int_{-1}^{1} \bar{\rho}(x)\bar{\rho}(y)|\partial_t w(t,x) - \partial_t w(t,y)|^2 R_3(t,x,y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.58)_4} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-s)(3-2s)}{2}\underbrace{\int_{-1}^{1}\int_{-1}^{1}\bar{\rho}(x)\bar{\rho}(y)(\partial_t w(t,x)-\partial_t w(t,y))^3 R_4(t,x,y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.58)_5}\\
&+\frac{(1-s)(3-2s)}{2}\underbrace{\int_{-1}^{1}\int_{-1}^{1}\bar{\rho}(x)\bar{\rho}(y)\frac{(\partial_t w(t,x)-\partial_t w(t,y))^3}{|x-y|^{4-2s}}\,\mathrm{d}y\,\mathrm{d}x}_{(3.58)_6}\\
&+\underbrace{\int_{-1}^{1}\frac{|\partial_{tx}w|^2}{1+\partial_x w}\,\mathrm{d}x}_{(3.58)_7}=\underbrace{\int_{-1}^{1}\left(\frac{\partial_{tx}w}{1+\partial_x w}\right)^2\partial_{ttx}w\,\mathrm{d}x}_{(3.58)_8}.
\end{aligned}
\tag{3.58}
$$

其中 R_3 和 R_4 表示
where R_3 and R_4 denote

$$R_3(t,x,y):=-\frac{1}{|\eta(t,x)-\eta(t,y)|^{3-2s}}+\frac{1}{|x-y|^{3-2s}},\tag{3.59}$$

$$R_4(t,x,y):=-\frac{1}{|\eta(t,x)-\eta(t,y)|^{4-2s}}+\frac{1}{|x-y|^{4-2s}}.\tag{3.60}$$

对于 $(3.58)_1$ 和 $(3.58)_7$，由先验假设，根据估计式 (3.14)，可得
For $(3.58)_1$ and $(3.58)_7$, by the *a priori* **assumption**, the estimate (3.14), we have

$$\left(\frac{1}{1+A\epsilon_0}\right)^{\gamma+1}\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x\le(3.58)_1\le\left(\frac{1}{1-A\epsilon_0}\right)^{\gamma+1}\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x\tag{3.61}$$

and

$$\frac{1}{1+A\epsilon_0}\int_{-1}^{1}|\partial_{tx}w|^2\,\mathrm{d}x\le(3.58)_7\le\frac{1}{1-A\epsilon_0}\int_{-1}^{1}|\partial_{tx}w|^2\,\mathrm{d}x.\tag{3.62}$$

对于 $(3.58)_2$，由先验假设，根据估计式 (3.14)，可得
For $(3.58)_2$, by the *a priori* **assumption**, we have

$$|(3.58)_2|\le\frac{\gamma(\gamma+1)A\epsilon_0}{2}\left(\frac{1}{1-A\epsilon_0}\right)^{\gamma+2}\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x.\tag{3.63}$$

对于 $(3.58)_3$，采用与证明估计式 (3.14) 类似的方法，可得
For $(3.58)_3$, using a similar method to proving the estimate (3.14), we have

$$|(3.58)_3|\le 2(1-s)\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x.\tag{3.64}$$

对于 $(3.58)_4$，我们断言
For $(3.58)_4$, we claim that

$$|R_3(t,x,y)|\le\frac{(3-2s)A\epsilon_0(1+A\epsilon_0)}{(1-A\epsilon_0)^{5-2s}}\frac{1}{|x-y|^{3-2s}}.\tag{3.65}$$

于是可得
Then we have

$$|(3.58)_4|\le\frac{(3-2s)A\epsilon_0(1+A\epsilon_0)}{(1-A\epsilon_0)^{5-2s}}\int_{-1}^{1}\int_{-1}^{1}\frac{|\partial_t w(t,x)-\partial_t w(t,y)|^2}{|x-y|^{3-2s}}\bar{\rho}(x)\bar{\rho}(y)\,\mathrm{d}y\,\mathrm{d}x.$$

采用与处理 $(3.58)_4$ 相同的做法，我们可以将 $(3.58)_4$ 有界估计为 $(3.58)_4$ 的形式，即
Using the same method to $(3.58)_4$, we can estimate $(3.58)_4$ as

$$|(3.58)_4|\le 2(3-2s)A\epsilon_0\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x.\tag{3.66}$$

接下来还需证明(3.65)。我们计算得

$$\frac{\mathrm{d}}{\mathrm{d}a}\frac{1}{|a(\eta(t,x)-\eta(t,y))+(1-a)(x-y)|^{3-2s}}$$
$$=(2s-3)\frac{x-y+a(w(t,x)-w(t,y))}{|x-y+a(w(t,x)-w(t,y))|^{5-2s}}\cdot(w(t,x)-w(t,y)). \tag{3.67}$$

应用微积分基本定理，并考虑到(3.67)，可得

Applying the fundamental theorem of calculus to (3.59) and taking (3.67) into account, we have

$$|R_3(t,x,y)|=(3-2s)\frac{|w(t,x)-w(t,y)|}{|x-y|^{5-2s}}\left|\int_0^1\frac{1+a\frac{\eta(t,x)-\eta(t,y)}{x-y}}{|1+a\frac{\eta(t,x)-\eta(t,y)}{x-y}|^{5-2s}}\,\mathrm{d}a\right|.$$

设(3.40)中的ϵ_0充分小，则有

Let ϵ_0 in the Lemma 3.1 be sufficiently small, then we have

$$\frac{1+\frac{\eta(t,x)-\eta(t,y)}{x-y}}{|1+\frac{\eta(t,x)-\eta(t,y)}{x-y}|^{5-2s}}\leqslant\frac{1+A\epsilon_0}{(1-A\epsilon_0)^{5-2s}},$$

因此

therefore

$$|R_3(t,x,y)|\leqslant\frac{(3-2s)A\epsilon_0(1+A\epsilon_0)}{(1-A\epsilon_0)^{5-2s}}\frac{1}{|x-y|^{3-2s}}.$$

对于(3.58)$_5$，我们给出一个逐点估计，

For $(3.58)_5$, we also give a pointwise estimate,

$$|R_4(t,x,y)|\leqslant\frac{(4-2s)A\epsilon_0(1+A\epsilon_0)}{(1-A\epsilon_0)^{6-2s}}\frac{1}{|x-y|^{4-2s}}, \tag{3.68}$$

于是我们得到

Then we have

$$|(3.58)_5|\leqslant\frac{(4-2s)A\epsilon_0(1+A\epsilon_0)}{(1-A\epsilon_0)^{6-2s}}\int_{-1}^{1}\int_{-1}^{1}\frac{|\partial_t w(t,x)-\partial_t w(t,y)|^2}{|x-y|^{4-2s}}\bar{\rho}(x)\bar{\rho}(y)\,\mathrm{d}y\,\mathrm{d}x$$
$$\leqslant\frac{(4-2s)A^2\epsilon_0^2(1+A\epsilon_0)}{(1-A\epsilon_0)^{6-2s}}\int_{-1}^{1}\int_{-1}^{1}\frac{|\partial_t w(t,x)-\partial_t w(t,y)|^2}{|x-y|^{3-2s}}\bar{\rho}(x)\bar{\rho}(y)\,\mathrm{d}y\,\mathrm{d}x.$$

采用处理(3.58)$_3$时的相同估计，我们可以将(3.58)$_5$控制为

Using the same estimate as dealing with $(3.58)_3$, we can bound $(3.58)_5$ as

$$|(3.58)_5|\leqslant\frac{(4-2s)A^2\epsilon_0^2(1+A\epsilon_0)}{(1-A\epsilon_0)^{6-2s}}\int_{-1}^{1}\int_{-1}^{1}\rho|\partial_t w|^2\,\mathrm{d}y\,\mathrm{d}x. \tag{3.69}$$

还需证明(3.68)。我们计算得

It remains to prove (3.68). We compute

$$\frac{\mathrm{d}}{\mathrm{d}a}\frac{1}{|a(\eta(t,x)-\eta(t,y))+(1-a)(x-y)|^{4-2s}}$$
$$=(2s-4)\frac{x-y+a(w(t,x)-w(t,y))}{|x-y+a(w(t,x)-w(t,y))|^{6-2s}}\cdot(w(t,x)-w(t,y)). \tag{3.70}$$

应用微积分基本定理，并考虑到(3.70)，可得

Applying the fundamental theorem of calculus to (3.60) and taking (3.70) into account, we have

$$|R_4(t,x,y)|=(4-2s)\frac{|w(t,x)-w(t,y)|}{|x-y|^{6-2s}}\left|\int_0^1\frac{1+a\frac{\eta(t,x)-\eta(t,y)}{x-y}}{|1+a\frac{\eta(t,x)-\eta(t,y)}{x-y}|^{6-2s}}\,\mathrm{d}a\right|.$$

设(3.40)中的ϵ_0充分小，则有

Let ϵ_0 in the Lemma 3.1 be sufficiently small, then we have

$$\frac{1+\frac{\eta(t,x)-\eta(t,y)}{x-y}}{|1+\frac{\eta(t,x)-\eta(t,y)}{x-y}|^{6-2s}}\leqslant\frac{1+A\epsilon_0}{(1-A\epsilon_0)^{6-2s}}$$

and consequently

从而

$$|R_4(t,x,y)| \le \frac{(4-2s)A\varepsilon_0(1+A\varepsilon_0)}{(1-A\varepsilon_0)^{6-2s}} \frac{1}{|x-y|^{4-2s}}.$$

For $(3.58)_6$, by the estimate in (3.4), we have

对于$(3.58)_6$，由先验假设中的估计 (3.4)，我们有

$$|(3.58)_6| \le A\varepsilon_0 \int_{-1}^{1}\int_{-1}^{1} \frac{|\partial_t w(t,x)-\partial_t w(t,y)|^2}{|x-y|^{3-2s}} \bar{\rho}(x)\bar{\rho}(y)\,dy\,dx.$$

Using the same argument as in $(3.58)_3$, we can bound $(3.58)_6$ as

采用处理 $(3.58)_3$ 时的相同论证，我们可以将 $(3.58)_6$ 有界估计为

$$|(3.58)_6| \le A\varepsilon_0 \Big[\int_{-1}^{1} \bar{\rho}^{\gamma}|\partial_{tx}w|^2\,dx\Big]. \tag{3.71}$$

For $(3.58)_8$, by the estimate in (3.4) and Hölder's inequality, we have

对于 $(3.58)_8$，由先验假设中的估计 (3.4) 和 Hölder 不等式，我们有

$$|(3.58)_8| \le \epsilon\Big[\int_{-1}^{1}|\partial_{tx}w|^2\,dx + C_\epsilon A^2\varepsilon_0^2\Big[\int_{-1}^{1}|\partial_{tx}w|^2\,dx. \tag{3.72}$$

Combining the estimates (3.61), (3.62), (3.64), (3.66), (3.69), (3.71), (3.72) and (3.73), and choosing ϵ and ε_0 sufficiently small, we obtain (3.57) which proves the **Lemma 3.5**. □

结合 (3.61)、(3.62)、(3.64)、(3.66)、(3.69)、(3.71)、(3.72) 和 (3.73) 的结果，并注意到 $2(1-s)>\gamma$ 且 $1-s<1$，以及 ε_0 充分小，由引理 3.1 可知，$2c_*$ 和

$$\frac{1}{(1+A\epsilon_0)^\gamma}\int_{-1}^{1}\bar{\rho}^\gamma|\partial_{tx}w|^2\,\mathrm{d}x \le (3.74)_2 \le \frac{1}{(1-A\epsilon_0)^\gamma}\int_{-1}^{1}\bar{\rho}^\gamma|\partial_{tx}w|^2\,\mathrm{d}x \tag{3.76}$$

and

$$|(3.74)_6| \le \frac{A\epsilon_0}{|1+A\epsilon_0|^2}\int_{-1}^{1}|\partial_{tx}w|^2\,\mathrm{d}x. \tag{3.77}$$

For $(3.74)_3$, using the same estimates as in (3.58) in the proof of **Lemma 3.5**, we have

$$|(3.74)_3| \le 2(1-s)\int_{-1}^{1}\bar{\rho}^\gamma|\partial_{tx}w|^2\,\mathrm{d}x. \tag{3.78}$$

For $(3.74)_4$, recalling (3.65), we have

$$|(3.74)_4| \le \frac{(3-2s)A\epsilon_0(1+A\epsilon_0)}{(1-A\epsilon_0)^{5-2s}}\int_{-1}^{1}\int_{-1}^{1}\frac{|\partial_t w(t,x)-\partial_t w(t,y)|^2}{|x-y|^{3-2s}}\bar{\rho}(x)\bar{\rho}(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Using the same estimate as dealing with $(3.58)_3$ in **Lemma 3.5**, we can bound $(3.74)_4$ as follows

$$|(3.74)_4| \le \frac{(3-2s)A\epsilon_0(1+A\epsilon_0)}{(1-A\epsilon_0)^{5-2s}}\int_{-1}^{1}\bar{\rho}|\partial_{tx}w|^2\,\mathrm{d}x. \tag{3.79}$$

For $(3.74)_5$, by Hölder inequality and Poincaré inequality, we have

$$|(3.74)_5| \le \varepsilon\int_{-1}^{1}|\partial_{tx}w|^2\,\mathrm{d}x + C_\varepsilon\int_{-1}^{1}\bar{\rho}|\partial_{tt}w|^2\,\mathrm{d}x, \tag{3.80}$$

where we have used the fact that $\bar{\rho}_x \le 2$.

Combining (3.75), (3.76), (3.78), (3.79), (3.80), (3.77), (3.74), and choosing $\varepsilon = \frac{1}{2}(1-s)$, by **Lemma 3.1** and choosing ϵ_0 sufficiently small, we obtain (3.73) which proves **Lemma 3.6**. □

Having established basic energy estimates, we further derive a time-weighted estimate for the second-order time derivative and mixed time-space derivative of w.

Lemma 3.7 *For $\frac{1}{2} < s < 1$, there exists an $\epsilon_0 > 0$ such that under the a priori assumption* (3.4)*, it holds that*

$$\begin{aligned}&(1+t)\|\bar{\rho}^{\frac{1}{2}}\partial_{tt}w(t)\|_{L^2}^2 + (1+t)\|\bar{\rho}^{\frac{\gamma}{2}}\partial_{tx}w(t)\|_{L^2}^2 + \int_0^t(1+s)\|\partial_{tx}w(s)\|_{L^2}^2\,\mathrm{d}s\\ &\le C_{s,\gamma}\sum_{i=0}^{1}\left(\|\partial_t^i\partial_x w_0\|_{L^2}^2 + \|\bar{\rho}_0^{\frac{1}{2}}\partial_t^i w_1\|_{L^2}^2 + \|\bar{\rho}_0^{\frac{\gamma}{2}}\partial_x^i w_0\|_{L^2}^2\right), \quad 0 \le t \le T,\end{aligned} \tag{3.81}$$

for some constants $C_{s,\gamma} > 0$.

证明. 回顾引理3.5可知，我们有

Proof. Recall Lemma 3.5, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}|\partial_t w|^2\,\mathrm{d}x+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\bar{\rho}^{\gamma}\left(\frac{1}{1+\partial_x w}\right)^{\gamma+1}|\partial_{tx}w|^2\,\mathrm{d}x\\
&\quad+\frac{\gamma(\gamma+1)}{2}\int_{-1}^{1}\bar{\rho}^{\gamma}\left(\frac{1}{1+\partial_x w}\right)^{\gamma+2}|\partial_{tx}w|^2\,\mathrm{d}x\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\partial_t w\cdot\mathcal{L}\partial_t w\,\mathrm{d}x\\
&\quad+\frac{1-s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\int_{-1}^{1}\bar{\rho}(x)\bar{\rho}(y)|\partial_t w(t,x)-\partial_t w(t,y)|^2R_3(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad+\frac{(1-s)(3-2s)}{2}\int_{-1}^{1}\int_{-1}^{1}(\partial_t w(t,x)-\partial_t w(t,y))^3R_4(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad+\frac{(1-s)(3-2s)}{2}\int_{-1}^{1}\int_{-1}^{1}\frac{(\partial_t w(t,x)-\partial_t w(t,y))^3}{|x-y|^{4-2s}}\,\mathrm{d}y\,\mathrm{d}x\\
&\quad+\int_{-1}^{1}\frac{|\partial_{tx}w|^2}{1+\partial_x w}\,\mathrm{d}x=\int_{-1}^{1}\left(\frac{\partial_{tx}w}{1+\partial_x w}\right)^2\partial_{tx}w\,\mathrm{d}x.
\end{aligned}
$$

将上述方程乘以 $1+t$，得到

Multiplying the above equation by $1+t$ we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)\int_{-1}^{1}|\partial_t w|^2\,\mathrm{d}x\right]-\underbrace{\frac{1}{2}\int_{-1}^{1}|\partial_t w|^2\,\mathrm{d}x}_{(3.82)_1}\\
&\quad+\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\underbrace{(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma}\left(\frac{1}{1+\partial_x w}\right)^{\gamma+1}|\partial_{tx}w|^2\,\mathrm{d}x}_{(3.82)_2}\right]\\
&\quad=\underbrace{\frac{\gamma}{2}\int_{-1}^{1}\bar{\rho}^{\gamma}\left(\frac{1}{1+\partial_x w}\right)^{\gamma+1}|\partial_{tx}w|^2\,\mathrm{d}x}_{(3.82)_3}\\
&\quad+\underbrace{\frac{\gamma(\gamma+1)}{2}(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma}\left(\frac{1}{1+\partial_x w}\right)^{\gamma+2}|\partial_{tx}w|^2\,\mathrm{d}x}_{(3.82)_4}\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\underbrace{(1+t)\int_{-1}^{1}\partial_t w\cdot\mathcal{L}\partial_t w\,\mathrm{d}x}_{(3.82)_5}\right]-\underbrace{\frac{1}{2}\int_{-1}^{1}\partial_t w\cdot\mathcal{L}\partial_t w\,\mathrm{d}x}_{(3.82)_6}\\
&\quad+\frac{1-s}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\underbrace{(1+t)\int_{-1}^{1}\int_{-1}^{1}\bar{\rho}(x)\bar{\rho}(y)|\partial_t w(t,x)-\partial_t w(t,y)|^2R_3(t,x,y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.82)_7}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1-s}{2}\underbrace{\int_{-1}^{1}\int_{-1}^{1}\bar{\rho}(x)\bar{\rho}(y)|\partial_t w(t,x)-\partial_t w(t,y)|^2 R_3(t,x,y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.82)_8}\\
&+\underbrace{\frac{(1-s)(3-2s)}{2}(1+t)\int_{-1}^{1}\int_{-1}^{1}(\partial_t w(t,x)-\partial_t w(t,y))^3 R_4(t,x,y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.82)_9}\\
&+\underbrace{\frac{(1-s)(3-2s)}{2}(1+t)\int_{-1}^{1}\int_{-1}^{1}\frac{(\partial_t w(t,x)-\partial_t w(t,y))^3}{|x-y|^{4-2s}}\,\mathrm{d}y\,\mathrm{d}x}_{(3.82)_{10}}\\
&+\underbrace{(1+t)\int_{-1}^{1}\frac{|\partial_{tx}w|^2}{1+\partial_x w}\,\mathrm{d}x}_{(3.82)_{11}}\equiv\underbrace{(1+t)\int_{-1}^{1}\left(\frac{\partial_{tx}w}{1+\partial_x w}\right)^2\partial_{tx}w\,\mathrm{d}x}_{(3.82)_{12}}.
\end{aligned}
\tag{3.82}
$$

对于$(3.82)_1$，根据Poincaré不等式，我们得到
For $(3.82)_1$, by Poincaré inequality, we find

$$
|(3.82)_1|\le C\int_{-1}^{1}|\partial_{tx}w|^2\,\mathrm{d}x. \tag{3.83}
$$

对于$(3.82)_2$, $(3.82)_3$, $(3.82)_4$, 和$(3.82)_{11}$，由(3.4)中的先验假设估计可得
For $(3.82)_2$, $(3.82)_3$, $(3.82)_4$, and $(3.82)_{11}$, by the smallness assumption (3.4) of ***a priori* assumption**,

$$
\begin{aligned}
&\frac{1}{(1+A\epsilon_0)^{\gamma+2}}(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x\le(3.82)_2,(3.82)_3\\
&\qquad\le\frac{1}{(1-A\epsilon_0)^{\gamma+2}}(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x,
\end{aligned}
\tag{3.84}
$$

$$
|(3.82)_4|\le\frac{1}{(1-A\epsilon_0)^{\gamma+1}}\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x, \tag{3.85}
$$

and

$$
\frac{1}{1+A\epsilon_0}(1+t)\int_{-1}^{1}|\partial_{tx}w|^2\,\mathrm{d}x\le(3.82)_{11}\le\frac{1}{1-A\epsilon_0}(1+t)\int_{-1}^{1}|\partial_{tx}w|^2\,\mathrm{d}x. \tag{3.86}
$$

对于$(3.82)_5$和$(3.82)_6$，采用处理引理3.5中(3.58)$_3$时的相同估计，我们有
For $(3.82)_5$ and $(3.82)_6$, using the same estimate as dealing with $(3.58)_3$ in **Lemma 3.5**, we have

$$
|(3.82)_5|\le(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x \tag{3.87}
$$

and

$$
|(3.82)_6|\le\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x. \tag{3.88}
$$

对于$(3.82)_7$，利用(3.65)并采用处理引理3.5中(3.58)$_3$时相同的估计，我们得到
For $(3.82)_7$, using (3.65) and applying the same estimate as dealing with $(3.58)_3$ in **Lemma 3.5**, we have

$$
|(3.82)_7|\le\frac{(3-2s)A\epsilon_0(1+A\epsilon_0)}{(1-A\epsilon_0)^{5-2s}}(1+t)\int_{-1}^{1}\bar{\rho}^{\gamma}|\partial_{tx}w|^2\,\mathrm{d}x. \tag{3.89}
$$

For I_{32}, using (3.68), applying the same estimate as dealing with I_{31} in **Lemma 3.5**, we find that

$$|I_{32}| \le \frac{(4-2s)A_{E_0}(1+A_{E_0})}{(1-A_{E_0})^{6-2s}} \int_{-1}^{1} \bar{p}^{\theta} |\partial_{tx} w|^2 \, dx. \tag{3.90}$$

对于 I_{32}，利用 (3.68)，我们有

$$|I_{32}| \le \frac{(4-2s)A_{E_0}(1+A_{E_0})}{(1-A_{E_0})^{6-2s}} \int_{-1}^{1} \bar{p}^{\theta} |\partial_{tx} w|^2 \, dx. \tag{3.90}$$

For I_{33}, using (3.68), we have

$$|I_{33}| \le A_{E_0}^2 (1+t) \int_{-1}^{1} \int_{-1}^{1} \frac{|\partial_t w(t,x) - \partial_t w(t,y)|^2}{|x-y|^{3-2s}} \bar{p}(x) \bar{p}(y) \, dy \, dx. \tag{3.91}$$

对于 I_{33}，由先验假设中的估计 (3.4)，可得

$$|I_{33}| \le A_{E_0}^2 (1+t) \int_{-1}^{1} \int_{-1}^{1} \frac{|\partial_t w(t,x) - \partial_t w(t,y)|^2}{|x-y|^{3-2s}} \, dy \, dx.$$

利用引理 3.5 中与 (3.58) 相同的估计，我们得到

$$|I_{33}| \le A_{E_0} (1+t) \int_{-1}^{1} \bar{p}^{\theta} |\partial_{tx} w|^2 \, dx. \tag{3.92}$$

For I_{34}, using the estimate (3.4) and Hölder's inequality, we have

$$|I_{34}| \le \frac{1}{2(1-A_{E_0})^2} (1+t) \int_{-1}^{1} |\partial_{tx} w|^2 \, dx + \frac{A_{E_0}^2 z^2}{2(1-A_{E_0})^2} (1+t) \int_{-1}^{1} |\partial_{tx} w|^2 \, dx. \tag{3.93}$$

综合 (3.83)、(3.84)、(3.85)、(3.87)、(3.88)、(3.89)、(3.90)、(3.91)、(3.92)、(3.86)、(3.93) 的结果，并注意到 $y \ge 2(1-s)$，然后对所得结果关于时间 t 进行积分，取足够小的 ε_0，即可得到 (3.81)，从而证明引理 3.7。 □

Combining the estimates (3.83), (3.84), (3.85), (3.87), (3.88), (3.89), (3.90), (3.91), (3.92), (3.86), (3.93), and noticing that $y \ge 2(1-s)$, then integrating the resultant with respect to time t, by choosing ε_0 sufficiently small, we obtain (3.81) which proves **Lemma 3.7**.

In the following, we will prove a higher order estimate in spatial variable.

引理 3.8. 对于 $\gamma \in (1, 4/3]$，存在常数 $C > 0$ 使得

Lemma 3.8. *For $\gamma \in (1, 4/3]$, there exists a constant C such that*

$$\|a^{2\gamma-1} \partial_{xxx} w(t)\|_{L^2}^2 + \int_0^t \|a^{3\gamma-1} \partial_{xxx} w(s)\|_{L^2}^2 \, ds \le C \big(\|a^{\frac{1}{2}} w_1\|_{L^2}^2 + \|\partial_{xx} w_0\|_{L^2}^2 + \|a^{\gamma-2} \partial_{xx} w_1\|_{L^2}^2 + \|a^{2\gamma-2} \partial_{xxx} w_0\|_{L^2}^2 \big), \quad 0 \le t \le T, \tag{3.94}$$

对于某个常数 $C_{\gamma, T} > 0$。

for some constant $C_{\gamma,T} > 0$.

证明. 令

Proof. Let us set

$$Q := \ln \partial_x w, \quad \tilde{Q} := \ln \partial_x \bar{w}. \tag{3.95}$$

那么，在先验假设 (3.4) 下，对于任意 $a > 0$，我们有

Then, we have that for arbitrary $a > 0$, under the *a priori* assumption (3.4),

$$\frac{1}{2} \bar{p}^{a} \partial_x w \le \bar{p}^{a} e^{Q} \le \bar{p}^{a} \partial_x w,$$

这里用到了以下初等代数不等式

where we used the following elementary algebraic inequality

$$\frac{1}{2} z \le \ln(1+z) \le z$$

当 $|z| \ll \varepsilon$ 且 ε 足够小时。此外，我们有

for $|z| \ll \varepsilon$ with ε sufficiently small. Moreover, we have

$$\frac{1}{1+A_{E_0}} \bar{p}^{a} \partial_{xx} w \le \bar{p}^{a} \partial_x Q \le \frac{1}{1-A_{E_0}} \bar{p}^{a} \partial_{xx} w,$$

现在让我们考虑 $\mathcal{G}$ 的方程。由方程 (2.8) 可得

Now let us consider the equation for $\mathcal{G}$. From the equation (2.8) we have

$$\partial_{tx}\mathcal{G}+\gamma\left(\frac{\bar{\rho}}{\partial_x\eta}\right)^{\gamma}\partial_x\mathcal{G}=\bar{\rho}\partial_{tt}\eta+\bar{\rho}(W'*\bar{\rho})\left[\left(\frac{1}{\partial_x\eta}\right)^{\gamma}-1\right]+\Phi. \tag{3.96}$$

将 (3.96) 乘以 $\bar{\rho}^{3\gamma-1}\partial_x\mathcal{G}$，然后对空间变量积分，得到

Multiplying (3.96) by $\bar{\rho}^{3\gamma-1}\partial_x\mathcal{G}$ and integrating the resulting identity with respect to the spatial variable, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1}^{1}\bar{\rho}^{3\gamma-1}|\partial_x\mathcal{G}|^2\,\mathrm{d}x+\underbrace{\int_{-1}^{1}\frac{\gamma\bar{\rho}^{3\gamma-1}}{(\partial_x\eta)^{\gamma}}|\partial_x\mathcal{G}|^2\,\mathrm{d}x}_{(3.97)_1}\\&=\underbrace{\int_{-1}^{1}\bar{\rho}^{3\gamma-1}\partial_x\mathcal{G}\cdot\partial_{tx}w\,\mathrm{d}x}_{(3.97)_2}+\underbrace{\int_{-1}^{1}\bar{\rho}^{3\gamma-1}(W'*\bar{\rho})\partial_x\mathcal{G}\cdot\left[\left(\frac{1}{\partial_x\eta}\right)^{\gamma}-1\right]\mathrm{d}x}_{(3.97)_3}+\underbrace{\int_{-1}^{1}\bar{\rho}^{2\gamma-1}\partial_x\mathcal{G}\cdot\Phi\,\mathrm{d}x}_{(3.97)_4}.\end{aligned} \tag{3.97}$$

下文中，我们逐项估计 (3.97)$_1$–(3.97)$_4$，其中 $c>0$ 取充分小。

In what follows, we estimate (3.97)$_1$–(3.97)$_4$, where $c>0$ is chosen to be sufficiently small.

对于 (3.97)$_1$，利用 (3.4)，可得

For (3.97)$_1$, using (3.4), we have that

$$\frac{\gamma}{(1+A\varepsilon)^{\gamma}}\int_{-1}^{1}\bar{\rho}^{3\gamma-1}|\partial_x\mathcal{G}|^2\,\mathrm{d}x\le(3.97)_1\le\frac{\gamma}{(1-A\varepsilon)^{\gamma}}\int_{-1}^{1}\bar{\rho}^{3\gamma-1}|\partial_x\mathcal{G}|^2\,\mathrm{d}x. \tag{3.98}$$

对于 (3.97)$_2$，由 Young 不等式可得

For (3.97)$_2$, by Young's inequality, we have

$$(3.97)_2\le\varepsilon\int_{-1}^{1}\bar{\rho}^{3\gamma-1}|\partial_x\mathcal{G}|^2\,\mathrm{d}x+C_\varepsilon\int_{-1}^{1}\bar{\rho}^{\gamma+1}|\partial_{tt}w|^2\,\mathrm{d}x. \tag{3.99}$$

对于 (3.97)$_3$，由 (3.4) 和杨氏不等式可得

For (3.97)$_3$, it follows from (3.4) and Young's inequality that

$$(3.97)_3\le\varepsilon\int_{-1}^{1}\bar{\rho}^{3\gamma-1}|\partial_x\mathcal{G}|^2\,\mathrm{d}x+C_\varepsilon\int_{-1}^{1}\bar{\rho}^{\gamma+1}|\partial_x w|^2\,\mathrm{d}x, \tag{3.100}$$

这里我们使用了以下初等代数不等式

Here we used the following elementary algebraic inequality

$$\left|1-\left(\frac{1}{1+z}\right)^{\gamma}\right|\le 2\gamma|z|.$$

For $(3.97)_4$, we introduce the expression of $\mathcal{L}$ in (2.9) and by splitting it into four terms, we deduce

对于$(3.97)_4$，我们代入(2.9)中的$\mathcal{L}$表达式，并将其分为四项，从而得出

$$
\begin{aligned}
(3.97)_4 = & \underbrace{\int_{-1}^{1} \bar{\rho}^{2\gamma-1}\partial_x \mathcal{G}\cdot \mathcal{L}w\,\mathrm{d}x}_{(3.101)_1} \\
& + \underbrace{\int_{-1}^{1} \bar{\rho}^{2\gamma}(x)\partial_x \mathcal{G}(x)\int_{-1}^{1}(\bar{\rho}(y)-\bar{\rho}(x))R_{1,1}(x,y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.101)_2} \\
& + \underbrace{\int_{-1}^{1} \bar{\rho}^{2\gamma+1}(x)\partial_x \mathcal{G}(x)\int_{-1}^{1}(\partial_x w(x)-\partial_y w(y))R_{1,0}(x,y)\frac{1}{1+\partial_x w}\,\mathrm{d}y\,\mathrm{d}x}_{(3.101)_3} \\
& + \underbrace{\int_{-1}^{1} \bar{\rho}^{2\gamma+1}(x)\partial_x \mathcal{G}(x)\int_{-1}^{1}(\partial_x w(x)-\partial_y w(y))\frac{x-y}{|x-y|^{3-2s}}\frac{-\partial_x w(x)}{1+\partial_x w(x)}\,\mathrm{d}y\,\mathrm{d}x}_{(3.101)_4}.
\end{aligned}
\tag{3.101}
$$

For $(3.101)_1$, we need to use the expression of $\mathcal{L}$ in (3.7), and $\mathcal{G}$ in (3.8),

对于$(3.101)_1$，我们需要利用(3.7)中$\mathcal{L}$的表达式来估计两项，

$$
(3.101)_1 = \underbrace{\int_{-\infty}^{\infty} \bar{\rho}^{2\gamma+1}(x)\partial_x \tilde{\mathcal{G}}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\partial_x \tilde{w}(x)-\partial_y \tilde{w}(y))\,\mathrm{d}y\,\mathrm{d}x}_{(3.102)_1}
$$

$$
= \underbrace{-\int_{-\infty}^{\infty} \bar{\rho}^{2\gamma}(x)\partial_x \tilde{\mathcal{G}}(x)\int_{-\infty}^{\infty}\frac{-2(1-s)}{|x-y|^{3-2s}}(\bar{\rho}(x)-\bar{\rho}(y))(\tilde{w}(x)-\tilde{w}(y))\,\mathrm{d}y\,\mathrm{d}x}_{(3.102)_2}. \tag{3.102}
$$

For $(3.102)_1$, we further split it into three parts,

对于$(3.102)_1$，我们进一步将其分成三部分，

$$
(3.102)_1 = \underbrace{\int_{-\infty}^{\infty} \bar{\rho}^{\frac{\gamma+3}{2}}(x)\partial_x \tilde{\mathcal{G}}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\bar{\rho}^{\frac{3\gamma-1}{2}}(x)\partial_x \tilde{w}(x)-\bar{\rho}^{\frac{3\gamma-1}{2}}(y)\partial_y \tilde{w}(y))\,\mathrm{d}y\,\mathrm{d}x}_{(3.103)_1}
$$

$$
\underbrace{-\int_{-\infty}^{\infty} \bar{\rho}^{\frac{\gamma+1}{2}}(x)\partial_x \tilde{\mathcal{G}}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\bar{\rho}^{\frac{3\gamma-1}{2}}(x)-\bar{\rho}^{\frac{3\gamma-1}{2}}(y))(\bar{\rho}(x)-\bar{\rho}(y))\partial_y \tilde{w}(y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.103)_2}
$$

$$
\underbrace{-\int_{-\infty}^{\infty} \bar{\rho}^{\frac{\gamma+1}{2}}(x)\partial_x \tilde{\mathcal{G}}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\bar{\rho}^{\frac{3\gamma-1}{2}}(x)-\bar{\rho}^{\frac{3\gamma-1}{2}}(y))\bar{\rho}(y)\partial_y \tilde{w}(y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.103)_3}.
\tag{3.103}
$$

For $(3.103)_1$, by the Hölder inequality, Lemma B.3, and the definition of H^{-s}, we obtain that

$$|(3.103)_1| \le \left(\int_{-1}^{1}\int_{-1}^{1} \frac{1}{|x-y|^{1-2s}}\bar{\rho}^{\gamma+3}(x)|\partial_{xx}w|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \times \left(\int_{-1}^{1}\int_{-1}^{1} \frac{|\bar{\rho}^{\frac{3\gamma-1}{2}}(x)\partial_x w(x) - \bar{\rho}^{\frac{3\gamma-1}{2}}(y)\partial_y w(y)|^2}{|x-y|^{3-2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} \lesssim \|\bar{\rho}^{\frac{\gamma+3}{2}}\partial_{xx}G\|_{L^2}\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x w\|_{H^{1-s}}.$$

由 Gagliardo-Nirenberg 不等式可得

$$\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x w\|_{H^{1-s}} \lesssim \|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x w\|_{L^2}^{s}\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x w\|_{H^1}^{1-s} \le C\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x w\|_{L^2} + C\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x w\|_{L^2}^{\frac{1}{1-s}} \le C(\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x w\|_{L^2} + \|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x G\|_{L^2})^{\frac{1}{1-s}},$$

其中我们用到了 $s\in(0,\frac{1}{2})$ 的事实. 由 Hardy 型不等式推论 **Corollary B.1**, 可得

$$\|\bar{\rho}^{\frac{3\gamma-3}{2}}\partial_x w\|_{L^2} \lesssim \|\bar{\rho}^{\frac{3\gamma-3}{2}+1}\partial_x w\|_{L^2} + \|\bar{\rho}^{\frac{3\gamma-3}{2}+1}\partial_x G\|_{L^2}.$$

因此, 由杨氏不等式可推得

$$|(3.103)_1| \le C\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x G\|_{L^2}^2 + C\|\bar{\rho}^{\frac{3\gamma-3}{2}}\partial_x w\|_{L^2}^2. \tag{3.104}$$

对于 $(3.103)_2$, 应用分部积分可得

$$\begin{aligned}(3.103)_2 \equiv & \underbrace{\int_{-\infty}^{\infty}\partial_x(\bar{\rho}^{\frac{\gamma+1}{2}}(x))\cdot\tilde{Q}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\bar{\rho}^{\frac{3\gamma-1}{2}}(x)-\bar{\rho}^{\frac{3\gamma-1}{2}}(y))(\bar{\rho}(x)-\bar{\rho}(y))\partial_y\tilde{w}(y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.105)_1} \\ & + \underbrace{\int_{-\infty}^{\infty}\bar{\rho}^{\frac{\gamma+1}{2}}(x)\tilde{Q}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\bar{\rho}^{\frac{3\gamma-1}{2}}(x)-\bar{\rho}^{\frac{3\gamma-1}{2}}(y))(\bar{\rho}(x)-\bar{\rho}(y))\partial_y\tilde{w}(y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.105)_2} \\ & + \underbrace{\int_{-\infty}^{\infty}\bar{\rho}^{\frac{\gamma+1}{2}}(x)\tilde{Q}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}\partial_x(\bar{\rho}^{\frac{3\gamma-1}{2}}(x))\cdot(\bar{\rho}(x)-\bar{\rho}(y))\partial_y\tilde{w}(y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.105)_3} \\ & + \underbrace{\int_{-\infty}^{\infty}\bar{\rho}^{\frac{\gamma+1}{2}}(x)\tilde{Q}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\bar{\rho}^{\frac{3\gamma-1}{2}}(x)-\bar{\rho}^{\frac{3\gamma-1}{2}}(y))\partial_x(\bar{\rho}(x))\cdot\partial_y\tilde{w}(y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.105)_4}.\end{aligned} \tag{3.105}$$

对于 $(3.105)_1$, 我们进一步将其拆分为两项,

$$\begin{aligned}(3.105)_1 \equiv & \underbrace{\int_{-\infty}^{\infty}\frac{\gamma+1}{2}\bar{\rho}^{\frac{\gamma-1}{2}}(x)\partial_x\bar{\rho}(x)\cdot\tilde{Q}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\bar{\rho}^{\frac{3\gamma-1}{2}}(x)-\bar{\rho}^{\frac{3\gamma-1}{2}}(y))\partial_y\tilde{w}(y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.106)_1} \\ & - \underbrace{\int_{-\infty}^{\infty}\frac{\gamma+1}{2}\bar{\rho}^{\frac{\gamma-1}{2}}(x)\partial_x\bar{\rho}(x)\cdot\tilde{Q}(x)\int_{-\infty}^{\infty}\frac{x-y}{|x-y|^{3-2s}}(\bar{\rho}^{\frac{3\gamma-1}{2}}(x)-\bar{\rho}^{\frac{3\gamma-1}{2}}(y))\bar{\rho}(y)\partial_y\tilde{w}(y)\,\mathrm{d}y\,\mathrm{d}x}_{(3.106)_2}.\end{aligned} \tag{3.106}$$

For $(3.106)_1$, by Hölder's inequality, Lemma B.3, and the fact that $\bar{\rho}^{-1+\delta}(y)\,\mathrm{d}y\,\mathrm{d}x$ is a finite measure on $[-1,1]^2$ since $\delta > -2s$, we have

$$|(3.106)_1| \le \left(\int_{-1}^1\int_{-1}^1 \frac{\bar{\rho}^{-1+\delta}(y)}{|x-y|^{1-2s}}\bar{\rho}^{\gamma+1}(x)|\mathcal{G}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12}$$
$$\times\left(\int_{-1}^1\int_{-1}^1 \frac{1}{|x-y|^{1-2s}}\bar{\rho}^{1-\delta}(y)|\partial_y\tilde{w}(y)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12}$$
$$\le \|\bar{\rho}^{\frac{\gamma+2s+\delta}{2}}\mathcal{G}\|_{L^2}\|\bar{\rho}^{\frac{1-\delta}{2}}\partial_x w\|_{L^2}.$$

根据 $\delta > -2s$ 的选取，我们得到

$$\|\bar{\rho}^{\frac{\gamma+2s+\delta}{2}}\mathcal{G}\|_{L^2}\|\bar{\rho}^{\frac{1-\delta}{2}}\partial_x w\|_{L^2}.$$

此外，根据推论 B.1 以及 $\delta \le 4-3\gamma$ 的选取，我们有

$$\|\bar{\rho}^{\frac{1-\delta}{2}}\partial_y w\|_{L^2} \le C(\|\bar{\rho}^{\frac{3-\delta}{2}}\partial_x w\|_{L^2}+\|\bar{\rho}^{\frac{3-\delta}{2}}\partial_{xx} w\|_{L^2}) \le C(\|\bar{\rho}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}+\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x\mathcal{G}\|_{L^2}).$$

因此，由杨氏不等式可得

$$|(3.106)_1| \le \varepsilon\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x\mathcal{G}\|_{L^2}^2 + C_\varepsilon\|\bar{\rho}^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2,$$

其中 $\varepsilon > 0$ 充分小。

类似地，利用霍尔德不等式和引理 B.3，并选取一个固定常数 $\delta \in (-2s, 4-3\gamma]$，我们有

$$|(3.106)_2| \le \left(\int_{-1}^1\int_{-1}^1 \frac{1}{|x-y|^{1-2s}}\bar{\rho}^{\gamma-\delta}(x)|\mathcal{G}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12}$$
$$\times\left(\int_{-1}^1\int_{-1}^1 \frac{\bar{\rho}^{-1+\delta}(x)}{|x-y|^{1-2s}}\bar{\rho}^{2}(y)|\partial_y w(y)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12},$$

$$|(3.105)_2| \le \left(\int_{-1}^1\int_{-1}^1 \frac{\bar{\rho}^{-1+\delta}(y)}{|x-y|^{1-2s}}\bar{\rho}^{\gamma+1}(x)|\mathcal{G}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12}$$
$$\times\left(\int_{-1}^1 \frac{1}{|x-y|^{1-2s}}\bar{\rho}^{1-\delta}(y)|\partial_y w(y)|^2\,\mathrm{d}x\right)^{\frac12},$$

$$|(3.105)_3| \le \left(\int_{-1}^1\int_{-1}^1 \frac{\bar{\rho}^{-1+\delta}(y)}{|x-y|^{1-2s}}\bar{\rho}^{2\gamma-1}(x)|\mathcal{G}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12}$$
$$\times\left(\int_{-1}^1\int_{-1}^1 \frac{1}{|x-y|^{1-2s}}\bar{\rho}^{1-\delta}(y)|\partial_y w(y)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12},$$

and

$$|(3.105)_4| \le \left(\int_{-1}^1\int_{-1}^1 \frac{\bar{\rho}^{-1+\delta}(y)}{|x-y|^{1-2s}}\bar{\rho}^{\frac{\gamma+1}{2}}(x)|\mathcal{G}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12}$$
$$\times\left(\int_{-1}^1\int_{-1}^1 \frac{1}{|x-y|^{1-2s}}\bar{\rho}^{1-\delta}(y)|\partial_y w(y)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12},$$

Then, following the same approach as estimating (3.106)$_1$ by using Lemma B.3, (3.106)$_2$, (3.105)$_1$, (3.105)$_3$, (3.105)$_4$ and Corollary B.1, and Young's inequality, we have

$$(3.106)_2,\ (3.105)_2,\ (3.105)_3,\ (3.105)_4 \le \varepsilon\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x\tilde{\mathcal{G}}\|_{L^2}^2 + C_\varepsilon\|\bar{\rho}^{\frac{3}{2}}\partial_x\tilde{w}\|_{L^2}^2.$$

因此，以下结论成立：

Therefore, the following holds

$$(3.103)_2 \le \varepsilon\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x\tilde{\mathcal{G}}\|_{L^2}^2 + C_\varepsilon\|\bar{\rho}^{\frac{3}{2}}\partial_x\tilde{w}\|_{L^2}^2. \tag{3.107}$$

对于(3.103)$_3$，由霍尔德不等式和杨氏不等式可得

For $(3.103)_3$, by Hölder inequality and Young's inequality, we obtain

$$(3.103)_3 \le \left(\int_{-1}^{1}\int_{-1}^{1}\frac{1}{|x-y|^{1-2s}}\bar{\rho}^{\gamma+2-\delta}(x)|\partial_x\tilde{\mathcal{G}}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}} \times \left(\int_{-1}^{1}\int_{-1}^{1}\frac{1}{|x-y|^{1-2s}}\bar{\rho}^{\delta-2}(y)|\partial_x\tilde{w}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}};$$

其中$\delta \in (2s, 4s-1)$。接着，按照利用引理B.3估计(3.106)$_1$的相同方法，使用推论B.1，和杨氏不等式，可得

where $\delta \in (2s, 4s-1)$. Then, following the same approach as estimating $(3.106)_1$ by using Lemma B.3, Corollary B.1, and Young's inequality, we find

$$(3.103)_3 \le \varepsilon\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x\tilde{\mathcal{G}}\|_{L^2}^2 + C_\varepsilon\|\bar{\rho}^{\frac{3}{2}}\partial_x\tilde{w}\|_{L^2}^2. \tag{3.108}$$

当$\varepsilon > 0$充分小时

for $\varepsilon > 0$ sufficiently small.

结合(3.104)、(3.107)和(3.108)的结果，可得

Combining the results (3.104), (3.107) and (3.108), we have

$$(3.102)_1 \le \varepsilon\|\bar{\rho}^{\frac{3\gamma-1}{2}}\partial_x\tilde{\mathcal{G}}\|_{L^2}^2 + C_\varepsilon\|\bar{\rho}^{\frac{3}{2}}\partial_x\tilde{w}\|_{L^2}^2. \tag{3.109}$$

对于$(3.102)_2$，我们进一步将其拆分为两项，

For $(3.102)_2$, we further split it into two terms,

$$(3.102)_2 \le \underbrace{\int_{-\infty}^{\infty}\bar{\rho}^{2\gamma-\frac{1}{2}}(x)|\partial_x\tilde{\mathcal{G}}|(x)\int_{-\infty}^{\infty}\frac{2(1-s)}{|x-y|^{3-2s}}|\bar{\rho}(x)-\bar{\rho}(y)|^{\frac{3}{2}}|\tilde{w}(x)-\tilde{w}(y)|\,\mathrm{d}y\,\mathrm{d}x}_{(3.110)_1}$$
$$+ \underbrace{\int_{-\infty}^{\infty}\bar{\rho}^{2\gamma-\frac{1}{2}}(x)|\partial_x\tilde{\mathcal{G}}|(x)\int_{-\infty}^{\infty}\frac{2(1-s)}{|x-y|^{3-2s}}|\bar{\rho}(x)-\bar{\rho}(y)|^{\frac{1}{2}}|\bar{\rho}(y)||\tilde{w}(x)-\tilde{w}(y)|\,\mathrm{d}y\,\mathrm{d}x}_{(3.110)_2}. \tag{3.110}$$

对于$(3.110)_1$，根据霍尔德不等式和(3.114)，我们推得

For $(3.110)_1$, by Hölder inequality and (3.114), we deduce that

$$|(3.110)_1| \le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\rho}^{-\frac{3-4s}{3-2s}}(y)\bar{\rho}^{4\gamma-1-\frac{3-4s}{3-2s}}(x)|\partial_x\tilde{\mathcal{G}}(x)|^2\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\times \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|\tilde{w}(x)-\tilde{w}(y)|^2}{|x-y|^{3-4s}}\bar{\rho}^{\frac{3-4s}{3-2s}}(x)\bar{\rho}^{\frac{3-4s}{3-2s}}(y)\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\le C\left(\int_{-1}^{1}\bar{\rho}^{4\gamma-1-\frac{3-4s}{3-2s}}(x)|\partial_x\tilde{\mathcal{G}}(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\times \left(\int_{-1}^{1}\int_{-1}^{1}\frac{|w(x)-w(y)|^2}{|x-y|^{3-2s}}\rho(x)\rho(y)\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{3-4s}{2(3-2s)}}\|w\|_{L^\infty}^{\frac{2}{2(1-2s)}},$$

由牛顿-莱布尼茨公式和霍尔德不等式，重选取一个固定常数$\kappa \in (0, 4-3\gamma)$，可得

By Newton-Leibniz formula and Hölder's inequality, for a fixed constant $\kappa \in (0, 4-3\gamma)$, we have

$$\|w\|_{L^\infty} \le \|\partial_x w\|_{L^1} \le \|\bar{\rho}^{-\frac{1-\kappa}{2}}\|_{L^2}\|\bar{\rho}^{\frac{1-\kappa}{2}}\partial_x w\|_{L^2} = C\|\bar{\rho}^{\frac{1-\kappa}{2}}\partial_x w\|_{L^2}.$$

由推论 B.C，可得

$$\|w\|_{L^\infty} \le C\left(\|\bar\rho^{\frac{3-\delta'}{2}}\partial_x w\|_{L^2} + \|\bar\rho^{\frac{3-\delta'}{2}}\partial_x \mathcal{G}\|_{L^2}\right).$$

Therefore, by (3.14), Young's inequality and the choice of δ' which that $\delta' < 3\gamma$, we have

$$(3.110)_1 \le \epsilon\|\bar\rho^{\frac{3\gamma-1}{2}}\partial_x \mathcal{G}\|_{L^2}^2 + C_\epsilon\|\bar\rho^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2.$$

对于 $(3.110)_2$，由 Hölder 不等式，(3.14) 和杨氏不等式，我们得到

$$\begin{aligned}(3.110)_2 &\le \left(\int_{-1}^1\int_{-1}^1 \frac{1}{|x-y|^{1-2s}}\bar\rho^{4-2\gamma}(x)|\partial_x\tilde{\mathcal{G}}(x)|^2\,dy\,dx\right)^{\frac12}\\ &\quad\times\left(\int_{-1}^1\int_{-1}^1 \frac{|\tilde w(x)-\tilde w(y)|^2}{|x-y|^{3-2s}}\bar\rho^{2\gamma}(x)\bar\rho^{2\gamma}(y)\,dy\,dx\right)^{\frac12},\\ &\le \varepsilon\|\bar\rho^{\frac{3\gamma-1}{2}}\partial_x\mathcal{G}\|_{L^2}^2 + C_\varepsilon\|\bar\rho^{\frac{\gamma}{2}}\partial_x w\|.\end{aligned}$$

Therefore we obtain that

$$(3.102)_1 \le \epsilon\|\bar\rho^{\frac{3\gamma-1}{2}}\partial_x\mathcal{G}\|_{L^2}^2 + C_\epsilon\|\bar\rho^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2. \tag{3.111}$$

结合 (3.109) 和 (3.111) 的结果，我们得到

$$(3.101)_1 \le \epsilon\|\bar\rho^{\frac{3\gamma-1}{2}}\partial_x\mathcal{G}\|_{L^2}^2 + C_\epsilon\|\bar\rho^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2. \tag{3.112}$$

对于 $(3.101)_2$，由 (3.30)，我们有

$$\begin{aligned}(3.101)_2 &\le 2(1-s)(3-2s)A\epsilon_0\\ &\quad\times\int_{-1}^1\bar\rho^{\frac{\gamma+3}{2}}(x)|\partial_x\mathcal{G}(x)|\int_{-1}^1\frac{|w(x)-w(y)|}{|x-y|^{2-2s}}\,dy\,dx.\end{aligned}$$

然后，采用与估计 (3.102) 类似的方法，我们将 $(3.101)_2$ 估计为

$$(3.101)_2 \le A\epsilon_0(\|\bar\rho^{\frac{3\gamma-1}{2}}\partial_x\mathcal{G}\|_{L^2}^2 + \|\bar\rho^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2). \tag{3.113}$$

对于 $(3.101)_3$ 和 $(3.101)_4$，由 (3.51)，我们可以证明

$$\begin{aligned}&|(3.101)_3|, |(3.101)_4|\\ &\le \frac{2(1-s)A\epsilon_0}{(1-A\epsilon_0)^{3-2s}}\underbrace{\int_{-1}^1\bar\rho^{\frac{\gamma+3}{2}}(x)|\partial_x\mathcal{G}(x)|\int_{-1}^1\frac{|\bar\rho^{\frac{3\gamma-1}{2}}(y)\partial_y w(y)-\bar\rho^{\frac{3\gamma-1}{2}}(x)\partial_x w(x)|}{|x-y|^{2-2s}}\,dy\,dx}_{(3.114)_1}\\ &\quad+\frac{2(1-s)A\epsilon_0}{(1-A\epsilon_0)^{3-2s}}\underbrace{\int_{-1}^1\bar\rho^{\frac{\gamma+3}{2}}(x)|\partial_x\mathcal{G}(x)|\int_{-1}^1\frac{|\bar\rho^{\frac{3\gamma-1}{2}}(x)-\bar\rho^{\frac{3\gamma-1}{2}}(y)|\,|\partial_y w(y)|}{|x-y|^{2-2s}}\,dy\,dx}_{(3.114)_2}.\end{aligned} \tag{3.114}$$

对于 $(3.114)_1$ 和 $(3.114)_2$，按照使用霍尔德不等式估计 (3.106)$_1$ 的相同方法，结合引理 B.3、推论 B.1，并应用杨氏不等式，我们得

$$|(3.114)_1|, |(3.114)_1| \le \epsilon\|\bar\rho^{\frac{3\gamma-1}{2}}\partial_x\mathcal{G}\|_{L^2}^2 + C_\epsilon\|\bar\rho^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2.$$

Therefore, we obtain

$$|(3.101)_3|, |(3.101)_4| \le A\epsilon_0(\|\bar\rho^{\frac{\gamma+3}{2}}\partial_x\mathcal{G}\|_{L^2}^2 + \|\bar\rho^{\frac{\gamma}{2}}\partial_x w\|_{L^2}^2). \tag{3.115}$$

Combining (3.112), (3.113) and (3.115) we get, 结合(3.112)、(3.113)和(3.115)的结果，我们得到

$$|(3.97)_4| \le \varepsilon \|\bar{\rho}^{\frac{3\gamma-1}{2}} \partial_x \mathcal{G}\|_{L^2}^2 + C_\varepsilon \|\bar{\rho}^{\frac{\gamma}{2}} \partial_x w\|_{L^2}^2. \tag{3.116}$$

Collecting the results (3.109), (3.110), (3.111), (3.116), and taking ε sufficiently small, we obtain (3.94) which proves **Lemma 3.8**. 结合(3.98)-(3.100)的结果，取ε足够小，我们得到(3.94)，即证明了引理3.8。

□

4. Proof of main theorems 主要定理的证明

4.1. Proof of Theorem 2.1. The proof consists of two parts. First, we prove the local existence and uniqueness of strong solutions on some finite time interval. For this purpose, recall that, from (2.8), we have **定理2.1的证明。**证明分为两部分。首先，我们证明在某个有限时间区间上的强解的局部存在性与唯一性。为此，回顾(2.8)可知，

$$\partial_t \eta = v, \tag{4.1}$$

$$\bar{\rho}\partial_t v + \partial_x \left[\left(\frac{\bar{\rho}}{\partial_x \eta} \right)^\gamma - \bar{\rho}^\gamma \right] + \Phi \equiv \partial_x \left(\frac{\partial_x v}{\partial_x \eta} \right), \tag{4.2}$$

where Φ is given by 其中Φ由下式给出

$$\begin{aligned} \Phi(t,x) &:= \partial_x(\bar{P}(x)) \\ &+ \frac{\bar{\rho}(x)}{\partial_x \eta(t,x)} \int_{-\infty}^{\infty} W'(\eta(t,x) - \bar{\eta}(t,y))(\bar{\rho}(y)\partial_x \eta(t,x) - \bar{\rho}(x)\partial_y \bar{\eta}(t,y))\,\mathrm{d}y. \end{aligned}$$

To derive the desired scheme, as before, we choose $N \in \mathbb{N}$ and $h = 1/N$. We decompose the domain $[-1,1]$ by grid points $x_k = kh$, $k = -N, \cdots, 0, \cdots, N$. Then we approximate (4.1) and (4.2) by the following ordinary differential equations, 有限差分格式定义如下：设N为正整数，且$h = 1/N$。我们利用网格点$x_k = kh$, $k = -N, \cdots, 0, \cdots, N$将区域$[-1,1]$离散化。然后，我们用以下常微分方程近似式(4.1)和(4.2)

$$\frac{\mathrm{d}\eta_k}{\mathrm{d}t} = v_k, \tag{4.3}$$

$$\begin{aligned} \bar{\rho}_k \frac{\mathrm{d}v_k}{\mathrm{d}t} &+ \frac{1}{h}\left\{ \bar{\rho}_{k+1}\left[\left(\frac{h}{\eta_{k+1} - \eta_k}\right)^\gamma - 1\right] - \bar{\rho}_k \left[\left(\frac{h}{\eta_k - \eta_{k-1}}\right)^\gamma - 1\right] \right\} + \Phi_k \\ &\equiv \frac{1}{h}\left(\frac{v_{k+1} - v_k}{\eta_{k+1} - \eta_k} - \frac{v_k - v_{k-1}}{\eta_k - \eta_{k-1}} \right), \end{aligned} \tag{4.4}$$

where 其中

$$\begin{aligned} \bar{\rho}_k &:= \bar{\rho}(x_k), \\ \bar{\eta}_k &:= \sum_{i \ge k+1} \sum_{j \le i} W(x_i - x_j) h^2, \\ \Phi_k &:= -\bar{\rho}_k \sum_{j \ne k} h W'(x_k - x_j)(\bar{\rho}_j - \bar{\rho}_k) \\ &\quad + \frac{h\bar{\rho}_k}{\eta_{k+1} - \eta_k} \sum_{j \ne k} h W'(\eta_k - \bar{\eta}_j)\left(\bar{\rho}_j \frac{v_{k+1} - v_k}{h} - \bar{\rho}_k \frac{v_{j+1} - v_j}{h} \right), \end{aligned}$$

$$\eta_{-N} := -1, \quad \eta_k := \eta_{k-1} + h,$$

$$\bar{\eta}_k := \begin{cases} \eta_{-N} + 1 + kh, & k \le -N-1, \\ \eta_k, & -N \le k \le N, \\ \eta_N - 1 + kh, & k \ge N+1. \end{cases}$$

Notice that the formulas for $\rho^N(x)$ and $p^N(x)$ are symmetric with respect to k and $-k$, thus we only need to consider $k \geq 0$ in the following.

注意，(2.9) 和 (4.4) 中对 $\rho^N(x)$ 和 $p^N(x)$ 的近似公式经过选取，使其与 (2.12) 和 (2.13) 中的稳态方程一致。此外，这将使我们能够得到能量估计的离散版本。

Moreover, this will allow us to obtain a discrete version of the energy estimate. From (2.2) and (2.13), this system is supplemented by the following initial conditions,

此外，这将使我们能够得到能量估计的离散版本。由 (2.2) 和 (2.13) 可知，该系统还满足以下初始条件，

$$\eta_k(0) = \eta_0(x_k), \quad v_k(0) = \eta_1(x_k), \tag{4.5}$$

and, in order to ensure (2.14), we further impose

并且，为满足边界条件 (2.12) 和 (2.14)，我们进一步施加

$$\eta_0(t) \equiv 0, \quad \frac{\eta_N(t) - \eta_{N-1}(t)}{h} \equiv 1. \tag{4.6}$$

Then, we use (4.6) to determine η_N and the terms v_N and v_{N-1}. It follows that

然后，我们利用 (4.6) 来确定 η_N 和 v_N 中的项 v_N 与 v_{N-1}。由此可得

$$\eta_N = \eta_{N-1} + h, \quad v_N = v_{N-1}. \tag{4.7}$$

In addition, we define the approximation of E in (2.20),

此外，我们定义 (2.20) 中 E 的近似，

$$\begin{aligned} E_N(t) := & \max_{-N+1 \leq k \leq N} \left| \left(\frac{\eta_k(t) - \eta_{k-1}(t)}{h} - 1, \frac{v_k(t) - v_{k-1}(t)}{h} \right) \right|^2 \\ & + \sum_{k=-N+1}^{N-1} h \frac{\gamma - 1}{\rho_k} \left| \frac{1}{h} \left(\frac{\eta_{k+1}(t) - \eta_k(t)}{h} - \frac{\eta_k(t) - \eta_{k-1}(t)}{h} \right) \right|^2 + \sum_{k=-N}^{N} h \frac{1}{\rho_k} \left| \frac{dv_k(t)}{dt} \right|^2. \end{aligned}$$

Then following the argument in [24], we can prove the existence of η_{N-1} and obtain the following lemma.

然后沿用 [24] 中的论证，我们可以证明 η_{N-1} 的存在性，并得到以下引理。

Lemma 4.1. *Let $0 < s < 1/2$ and $\gamma > 2(1-s)$. Suppose $E(0) < \infty$ and $\eta_0(\pm 1) = \dots = \eta_0(\pm(N-1)) = 0$. Then there exist $N \in \mathbb{N}$ and $\epsilon_0 > 0$ such that if $E(0) < \epsilon_0$ and $N > N_0$, then the problem admits a unique solution $(\eta_1, \cdots, \eta_{N-1})$ on $[0, T^*]$ for some constant $T^* > 0$ independent of N satisfying*

引理 4.1. *设 $0 < s < 1/2$ 且 $\gamma > 2(1-s)$。假设 $E(0) < \infty$ 且 $\eta_0(\pm 1) = \dots = \eta_0(\pm(N-1)) = 0$。则存在 $N_0 \in \mathbb{N}$ 和 $\epsilon_0 > 0$，使得，若 $E(0) < \epsilon_0$ 且 $N > N_0$，则该问题在某个与 N 无关、并满足 $T^* > 0$ 的常数 T^* 上存在唯一解 $(\eta_1, \cdots, \eta_{N-1})$，其中该常数与 N 无关，并满足*

$$E_N(t) \leq C E_N(0) \leq 2C E(0), \quad 0 \leq t \leq T^*,$$

for some constant $C > 0$ independent of N.

其中存在常数 $C > 0$，与 N 无关。

Proof. It follows from the ODE theory that the ODE system (4.3) and (4.4) with initial data (4.5) has a solution in a time interval $[0, T_N)$. Let $T_N > 0$ be the maximum existence time. Then for $k = -N, \cdots, 0, \cdots, N$, and $t \in [0, T_N)$,

证明. 由常微分方程理论可知，初始数据 (4.5) 对应的常微分方程组 (4.3) 和 (4.4) 在某个时间区间 $[0, T_N)$ 上存在解。令 $T_N > 0$ 为最大存在时间。那么对于 $k = -N, \cdots, 0, \cdots, N$ 且 $t \in [0, T_N)$，

$$\begin{aligned} \left| \frac{\eta_k(t) - \eta_{k-1}(t)}{h} - 1 \right| &\leq \left| \frac{\eta_0(x_k) - \eta_0(x_{k-1})}{h} - 1 \right| + \int_0^t \left| \frac{v_k(s) - v_{k-1}(s)}{h} \right| ds \\ &\leq \|\eta_0' - 1\|_{L^\infty} + t \sup_{s \in [0,t]} \sqrt{E_N(s)}. \end{aligned}$$

This implies, due to $\eta_0(t) = 0$, that for $k = -N, \cdots, 0, \cdots, N$ 且 $t \in [0, T_N)$,

这意味着，由于 $\eta_0(t) = 0$，对于 $k = -N, \cdots, 0, \cdots, N$ 且 $t \in [0, T_N)$，

$$\left| \frac{\eta_k(t) - \eta_{k-1}(t)}{h} - 1 \right| \leq \max_{-N \leq k \leq N} \left| \frac{\eta_k(t) - \eta_{k-1}(t)}{h} - 1 \right| \leq \|\eta_0' - 1\|_{L^\infty} + t \sup_{s \in [0,t]} \sqrt{E_N(s)}.$$

Hence, for $k = -N, \cdots, 0, \cdots, N$ and $t \in [0, T_N)$,

因此，对于 $k = -N, \cdots, 0, \cdots, N$ 且 $t \in [0, T_N)$，

$$\left| \frac{\eta_k(t) - \eta_{k-1}(t)}{h} - 1 \right| \leq 2\epsilon, \quad \left| \frac{\eta_k(t)}{x_k} - 1 \right| \leq 2\epsilon, \tag{4.8}$$

provided that

前提是

$$\|\eta_0' - 1\|_{L^\infty} \leq \epsilon, \quad T \sup_{s \in [0,T]} \sqrt{E_N(s)} \leq \epsilon, \tag{4.9}$$

for some constant $\epsilon > 0$.

对于某个常数 $\epsilon > 0$。

Similar as the proof in [24], we prove that for sufficiently large $N \in \mathbb{N}$, there exists a constant $\bar{C}$, independent of N such that if T satisfies (4.9), then

与[24]中的证明类似，我们证明：当 $N \in \mathbb{N}$ 足够大时，存在一个与 N 无关的常数 $\bar{C}$，使得若 T 满足(4.9)，则

$$E_N(t) \le \bar{C} E_N(0) + \bar{C} \int_0^t E_N(s) + E_N^2(s)\, \mathrm{d}s, \tag{4.10}$$

for $0 \le t \le T$.

当 $0 \le t \le T$.

First, taking the time derivative on both sides of (4.4), we obtain

首先，对(4.4)两边求时间导数，得到

$$\begin{aligned}
\bar{\rho}_k \frac{\mathrm{d}^2 v_k}{\mathrm{d}t^2} &= \frac{\gamma}{h}\left[-\theta_{k+1}\left(\frac{h}{\eta_{k+1}-\eta_k}\right)^{\gamma+1} \frac{v_{k+1}-v_k}{h} \right. \\
&\quad \left. + \theta_k \left(\frac{h}{\eta_k - \eta_{k-1}}\right)^{\gamma+1} \frac{v_k - v_{k-1}}{h} \right] + \frac{\mathrm{d}\Phi_k}{\mathrm{d}t} \\
&= \frac{1}{h}\left[-\left(\frac{v_{k+1}-v_k}{\eta_{k+1}-\eta_k}\right)^2 + \frac{\frac{\mathrm{d}v_{k+1}}{\mathrm{d}t} - \frac{\mathrm{d}v_k}{\mathrm{d}t}}{\eta_{k+1}-\eta_k} - \left(\frac{v_k - v_{k-1}}{\eta_k - \eta_{k-1}}\right)^2 + \frac{\frac{\mathrm{d}v_k}{\mathrm{d}t} - \frac{\mathrm{d}v_{k-1}}{\mathrm{d}t}}{\eta_k - \eta_{k-1}} \right].
\end{aligned} \tag{4.11}$$

Multiplying (4.11) by $h\frac{\mathrm{d}v_k}{\mathrm{d}t}$ and taking the summation of the product from $k = -N+1$ to $k = N-1$, since the rest of the terms are similar to the case in [24], we only discuss the third term on the left-hand side of (4.11).

将(4.11)乘以 $h\frac{\mathrm{d}v_k}{\mathrm{d}t}$，并对乘积从 $k = -N+1$ 到 $k = N-1$ 求和，由于大多数项与[24]中的情形类似，我们只讨论(4.11)左端的第三项。

For the third term, we have

对于第三项，有

$$\begin{aligned}
\sum_{k=-N+1}^{N-1} h \frac{\mathrm{d}v_k}{\mathrm{d}t}\frac{\mathrm{d}\Phi_k}{\mathrm{d}t} &= (s-1)\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\sum_{-N \le j < k \le N} h^2 \bar{\rho}_j \bar{\rho}_k \frac{1}{|x_j - x_k|^{1-2s}} \left(\frac{v_j - v_k}{x_j - x_k}\right)^2}_{(4.12)_1} \\
&\quad + \frac{1-s}{2}\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\sum_{k=-N}^{N} \sum_{j \ne k} h^2 \bar{\rho}_j \bar{\rho}_k \left(-\frac{1}{|\eta_j - \eta_k|^{3-2s}} + \frac{1}{|x_j - x_k|^{3-2s}} \right)(v_j - v_k)^2}_{(4.12)_2} \\
&\quad + \frac{(1-s)(3-2s)}{2} \underbrace{\sum_{k=-N}^{N} \sum_{j \ne k} h^2 \bar{\rho}_j \bar{\rho}_k \left(-\frac{\eta_j - \eta_k}{|\eta_j - \eta_k|^{5-2s}} + \frac{x_j - x_k}{|x_j - x_k|^{5-2s}} \right)(v_j - v_k)^3}_{(4.12)_3}.
\end{aligned} \tag{4.12}$$

In addition, we deduce from (4.8) that

此外，由(4.8)可推得

$$(1 - 2E)(x_k - x_{k-1}) \le \eta_k - \eta_{k-1} \le (1 + 2E)(x_k - x_{k-1}),$$

which further implies that

这进一步意味着

$$(1 - 2E)(x_k - x_j) \le \eta_k - \eta_j \le (1 + 2E)(x_k - x_j), \quad k > j.$$

Therefore, for $k \ne j$, j, we have

因此，对于 $k \ne j$，j，有

$$\left| \frac{1}{|\eta_j - \eta_k|^{3-2s}} - \frac{1}{|x_j - x_k|^{3-2s}} \right| \le 2(3-2s)E \frac{1}{|x_j - x_k|^{3-2s}},$$

where we have used the following elementary algebraic inequalities,

其中我们使用了以下初等代数不等式，

$$\left| \frac{1}{1+z|^{3-2s}} - 1 \right| \le 2(3-2s)E.$$

然后，我们对 $(4.12)_2$ 进行如下估计，
Then, we estimate $(4.12)_2$ as follows,

$$|(4.12)_2| \le 4(3-2s)\epsilon \sum_{-N\le j<k\le N} h^2 \bar\rho_j \bar\rho_k \frac{1}{|x_j - x_k|^{1-2s}} \left(\frac{v_j - v_k}{x_j - x_k}\right)^2. \tag{4.13}$$

对于 $(4.12)_3$，同理可得，
For $(4.12)_3$, similarly, we obtain

$$\left| \frac{1}{|v_j - v_k|^{4-2s}} + \frac{1}{|x_j - x_k|^{4-2s}} \right| \le 2(4-2s)\epsilon \frac{1}{|x_j - x_k|^{4-2s}},$$

其中用到了代数不等式，
where we have used the algebraic inequality,

$$\left| 1 - \frac{1}{|1+z|^{4-2s}} \right| \le 2(4-2s)\epsilon.$$

此外，由 E_N 的定义可推得，
Moreover, by the definition of E_N that,

$$-E_N(t)(x_k - x_{k-1}) \le v_k - v_{k-1} \le E_N(t)(x_k - x_{k-1}),$$

这进一步意味着
which further implies that

$$-E_N(t)(x_k - x_j) \le v_k - v_j \le E_N(t)(x_k - x_j).$$

因此，我们对 $(4.12)_3$ 作如下估计，
Therefore, we estimate $(4.12)_3$ as follows,

$$|(4.12)_3| \le 4(4-2s)\epsilon E_N(t) \sum_{-N\le j<k\le N} h^2 \bar\rho_j \bar\rho_k \frac{1}{|x_j - x_k|^{1-2s}} \left(\frac{v_j - v_k}{x_j - x_k}\right)^2. \tag{4.14}$$

接下来考虑 $(4.12)_1$，根据 $E_N(t)$ 的定义，我们有
Next we consider $(4.12)_1$. By the definition of $E_N(t)$, we have

$$|v_k(t) - v_{k-1}(t)|^2 \le E_N(t),$$

这意味着
which implies that

$$|v_j(t) - v_k(t)|^2 \le (k-j) E_N(t).$$

因此，我们可以将 $(4.12)_1$ 估计为
Therefore, we can estimate $(4.12)_1$ as

$$\sum_{-N\le j<k\le N} h^2 \bar\rho_j \bar\rho_k \frac{1}{|x_j - x_k|^{1-2s}} \left(\frac{v_j - v_k}{x_j - x_k}\right)^2 \le \sum_{-N\le j<k\le N} h \bar\rho_j \bar\rho_k \frac{E_N(t)}{|x_j - x_k|^{1-2s}} \le C E_N(t). \tag{4.15}$$

再次，根据柯西不等式，
Indeed, by Cauchy's inequality,

$$|v_i - v_j| = \left| \sum_{k=j}^{i-1} \frac{v_{k+1} - v_k}{h} h \right| \le \left[\sum_{k=j}^{i-1} \left(\frac{v_{k+1} - v_k}{h} \right)^2 h \right]^{\frac12} (x_i - x_j)^{\frac12} \le E_N^{\frac12}(t)(x_i - x_j)^{\frac12},$$

这意味着 (4.15) 因此，将 (4.13) 和 (4.14) 应用于 (4.11)，我们可以得到
which implies (4.15). Therefore, by applying (4.13) and (4.14) to (4.11), we can obtain

$$\sum_{k=-N}^{N} h \bar\rho_k \left| \frac{\mathrm{d}v_k}{\mathrm{d}t} \right|^2 \le \bar C E_N(0) + \bar C \int_0^t E_N(s) + E_N^2(s)\,\mathrm{d}s. \tag{4.16}$$

Second, we denote $\mathcal{G}_k = \ln \rho_k$. Now, we rewrite (4.4) as follows,

$$\frac{\mathrm{d}\mathcal{G}_{k+1}}{\mathrm{d}t} - \frac{\mathrm{d}\mathcal{G}_k}{\mathrm{d}t} = \bar{\theta}_{k+1}\frac{1}{h}\left[\left(\frac{h}{\eta_{k+1}-\eta_k}\right)^{\gamma} - \left(\frac{h}{\eta_k-\eta_{k-1}}\right)^{\gamma}\right]$$
$$= \bar{\rho}_k^{\gamma}\frac{\mathrm{d}v_k}{\mathrm{d}t} + \frac{\bar{\theta}_{k+1}-\bar{\theta}_k}{h}\left[\left(\frac{h}{\eta_{k+1}-\eta_k}\right)^{\gamma} - 1\right] + \Phi_k. \tag{4.17}$$

将(4.17)乘以$p_k\sim\frac{1}{h}(\mathcal{G}_{k+1}-\mathcal{G}_k)$，并对从$k=-N+1$到$k=N-1$的乘积求和，由于大多数项与[24]中的情形类似，我们只讨论右端的第三项。

Multiplying (4.17) by $p_k\sim\frac{1}{h}(\mathcal{G}_{k+1}-\mathcal{G}_k)$, and taking the summation of the product from $k=-N+1$ to $k=N-1$, since most terms are similar to the case in [24], we only discuss the third term on the right-hand side.

对于右侧的第三项，通过直接计算可得

For the third term on the right-hand side, by direct computation, we have

$$\sum_{k=-N}^{N} h^2\bar{\rho}_k^{2\gamma-1}\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\Phi_k$$
$$= \underbrace{\sum_{k=-N}^{N} h^2\bar{\rho}_k^{2\gamma+1}\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\sum_{j\neq k}\frac{x_k-x_j}{|x_k-x_j|^{3-2s}}\left(\frac{\eta_{k+1}-\eta_k}{h}-\frac{\eta_{j+1}-\eta_j}{h}\right)}_{(4.18)_1}$$
$$+ \underbrace{\sum_{k=-N}^{N} h^2\bar{\rho}_k^{2\gamma+1}\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\sum_{j\neq k}\frac{x_k-x_j}{|x_k-x_j|^{3-2s}}(\bar{\rho}_k-\bar{\rho}_j)(\eta_k-\eta_j)}_{(4.18)_2}$$
$$+ \underbrace{\sum_{k=-N}^{N} h^2\bar{\rho}_k^{2\gamma+1}\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\times\sum_{j\neq k}\left(\frac{\eta_k-\eta_j}{|\eta_k-\eta_j|^{3-2s}}-\frac{x_k-x_j}{|x_k-x_j|^{3-2s}}\right)\left(\frac{\eta_{k+1}-\eta_k}{h}-\frac{\eta_{j+1}-\eta_j}{h}\right)}_{(4.18)_3}$$
$$+ \underbrace{\sum_{k=-N}^{N} h^2\bar{\rho}_k^{2\gamma+1}\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\times\sum_{j\neq k}\frac{1}{|\eta_k-\eta_j|^{3-2s}}\left(\left(\frac{h}{\eta_{k+1}-\eta_k}\right)^{\gamma}-1\right)\left(\frac{\eta_{k+1}-\eta_k}{h}-\frac{\eta_{j+1}-\eta_j}{h}\right)}_{(4.18)_4}$$
$$+ \underbrace{\sum_{k=-N}^{N} h^2\bar{\rho}_k^{2\gamma}\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\times\sum_{j\neq k}\left(\frac{\eta_k-\eta_j}{|\eta_k-\eta_j|^{3-2s}}-\frac{x_k-x_j}{|x_k-x_j|^{3-2s}}-\frac{2(s-1)}{|x_k-x_j|^{3-2s}}(\eta_k-\eta_j)\right)(\bar{\rho}_j-\bar{\rho}_k)}_{(4.18)_5}. \tag{4.18}$$

接着，对于(4.18)，由(4.8)可推得

$$(1-2\bar{E}(x_k-x_{k-1}) \le \eta_k-\eta_{k-1} \le (1+2\bar{E})(x_k-x_{k-1}),$$

这进一步意味着

$$(1-2\bar{E})(x_k-x_j) \le \eta_k-\eta_j, \eta_{k+1}-\eta_{j+1} \le (1+2\bar{E})(x_k-x_j), \quad k \ge j.$$

Therefore we obtain

$$|(4.18)_1| \le \sum_{k=-N}^{N} h^2 \bar{p}_k^{2\gamma+1} \left|\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\right| \sum_{j\ne k} \frac{4\epsilon}{|x_k-x_j|^{2\theta}} \le \bar{C}(1+E_N(t)), \quad (4.19)$$

这里用到了以下事实the fact that

$$\left|\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\right| \le \frac{1}{1-2\bar{E}} \left|\frac{1}{h}\left(\frac{\eta_{k+1}-\eta_k}{h}-\frac{\eta_k-\eta_{k-1}}{h}\right)\right|,$$

由于(4.8)成立，且存在$\theta \in (0,1)$使得since (4.8) holds, and there exists $\theta \in (0,1)$ such that

$$\ln x - \ln y = \frac{1}{\theta x+(1-\theta)y}(x-y),$$

根据拉格朗日中值定理，类似地，我们可以证明by the Lagrange mean value theorem. Similarly, we can show that

$$|(4.18)_2, (4.18)_3, (4.18)_4, (4.18)_5| \le \sum_{k=-N}^{N} h^2 \bar{p}_k^{2\gamma} \left|\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\right| \sum_{j\ne k} |x_k-x_j|^{2\theta} \le \bar{C}(1+E_N(t)). \quad (4.20)$$

结合(4.19)和(4.20)，可得From (4.19) and (4.20), we have

$$\sum_{k=-N}^{N} h^2 \bar{p}_k^{2\gamma-1} \left|\frac{\mathcal{G}_{k+1}-\mathcal{G}_k}{h}\right|^2 \le \bar{C}E_N(t). \quad (4.21)$$

因此，我们可以得到Therefore, we can obtain

$$\sum_{k=-N}^{N} h\bar{p}_k^{2\gamma-1} \left|\frac{1}{4}\left(\frac{\eta_{k+1}-\eta_k}{h}+\frac{\eta_k-\eta_{k-1}}{h}\right)\right|^2 \le \bar{C}E_N(0) + \bar{C}\int_0^t E_N(s)+E_N^2(s)\,ds. \quad (4.22)$$

于是，由(4.16)和(4.22)推出(4.10)。令$T^* := \sup\{t : E_N(t) \le 2\bar{C}E_N(0)\}$，其中$\bar{C} > 1$是(4.10)中的常数(4.10需要考虑$\bar{C}<1$的情形)。否则，$\sqrt{2\bar{C}E_N(0)}$小于$E_N(0)$，Then, (4.16) and (4.22) imply (4.10). Set $T^* := \sup\{t : E_N(t) \le 2\bar{C}E_N(0)\}$, where $\bar{C} > 1$ is the constant in (4.10) (we consider $\bar{C}<1$ case). Otherwise, T^* is $+\infty$ if $\sqrt{2\bar{C}E_N(0)}$ is smaller than $E_N(0)$, otherwise the result is already obtained since $E_N(0) \to E(0)$ as $N \to \infty$. By the definition of T^*, we have

$$T^* \sup_{s\le T^*} \sqrt{E_N(s)} \le T^* \sqrt{2\bar{C}E_N(0)} \le \epsilon,$$

这意味着T^*满足(4.9)。因此，由(4.10)可得which yields that T^* satisfies (4.9). Therefore, it follows from (4.10) that

$$2\bar{C}E_N(0) = E_N(T^*) \le \bar{C}E_N(0) + \bar{C}\int_0^{T^*} E_N(s)+E_N^2(s)\,ds$$
$$\le \bar{C}E_N(0) + 2\bar{C}^2 E_N(0)(1+2\bar{C}E_N(0))T^*,$$

这意味着which implies

$$T^* \ge \frac{1}{2\bar{C}(1+2\bar{C}E_N(0))} \ge \frac{1}{2\bar{C}(1+4\bar{C}E(0))}.$$

通过取$C = 2\bar{C}$，证明该引理。Taking $C = 2\bar{C}$, we prove the lemma. □

To obtain the approximation sequence of η, we define η_h by

$$\eta_h(t,x) := \eta_{k-1}(t) + \frac{\eta_k(t) - \eta_{k-1}(t)}{h}(x - x_{k-1}),$$

for $x_{k-1} \leqslant x \leqslant x_k$. Then by using Lemma 4.1, we can obtain all the needed estimates as in [24, Lemma A.6]. In a similar way, we can prove the convergence of $\{\eta_h\}_h$ as h goes to zero and obtain the unique strong solution in the sense of **Definition 2.1**.

对于 $x_{k-1} \leqslant x \leqslant x_k$，利用引理 4.1，可得到与 [24, 引理 A.5] 中相同的所有必要估计，然后按照 [24, Lemma A.6] 中完全相同的紧性论证，我们可以证明当 h 趋于零时，$\{\eta_h\}_h$ 收敛，并得到定义 2.1 意义下的唯一强解。

Lemma 4.2. *Let $1/2 < \gamma < 1$ and assume $E(0) < \infty$. Suppose that $\eta(0) = 1$ and $\partial_x \eta_0 \neq 0$. Then there exists a time $T^* > 0$ such that the problem admits a unique solution η on $[0, T^*]$ satisfying*

$$E(t) \leqslant CE(0), \quad 0 \leqslant t \leqslant T^*, \tag{4.23}$$

where $C > 0$ is a constant.

引理 4.2. *设 $1/2 < \gamma < 1$ 且 $E(0) < \infty$。假设 $\eta(0) = 1$ 且 $\partial_x \eta_0 \neq 0$。则存在一个 $T^* > 0$，使得问题在某个常数 T^* 时间内存在唯一解 η，其中 T^* 为某个常数。*

Secondly, to obtain the uniform-in-time boundedness of the following energy functional in (2.20),

其次，为了通过标准的延拓论证，从局部存在性和唯一性定理推出强解的整体适定性，我们需要推导下述能量泛函 (2.20) 关于时间的一致有界性，

$$E(t) = \|\partial_x \eta(t)\|_{L^\infty} + \|\rho^{\gamma-1} - 1\|_{L^\infty} + \|\rho^{\frac{2\gamma-1}{2}} \partial_{xx}\eta(t)\|_{L^2}^2 + \|\rho^{\frac{1}{2}} \partial_t u(t)\|_{L^2}^2 + \|\partial_t \eta(t)\|_{L^2}^2 + \|\partial_{tx}\eta(t)\|_{L^2}^2 + \|\partial_x u(t)\|_{L^2}^2 + \|\partial_{xx} u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2.$$

The uniform bounds of $\|\partial_x \eta(t)\|_{L^\infty}$ can be proved by using **Lemma 3.4** and the standard continuity argument. The estimates of $\|\partial_{xx}\eta(t)\|_{L^2}$ and $\|\partial_t \eta(t)\|_{L^2}$ are given by **Lemma 3.5**. The uniform boundedness of $\|\partial_{tx}\eta(t)\|_{L^2}$ is given by **Lemma 3.6**. Let us consider $\|\partial_{tx}\eta(t)\|_{L^\infty}$ and $\|\partial_{xx} u(t)\|_{L^2}$ and $\|\partial_t u(t)\|_{L^2}$ are uniformly bounded, while the uniform bound of $\|\partial_{tx}\eta(t)\|_{L^2}$ is given by **Lemma 3.6**. 的一致有界性由引理 3.4、引理 3.5 和引理 3.6 给出。剩下的工作是证明 $\|\partial_{tx}\eta(t)\|_{L^\infty}$ 的一致有界性。为此，我们将使用 Gagliardo-Nirenberg 不等式，

To prove the uniform boundedness of $\|\partial_{tx}\eta\|_{L^\infty}$. By Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\partial_{tx}\eta(t)\|_{L^\infty} &\leq C\|\partial_x \eta(t)\|_{L^\infty} \left\| \frac{\partial_{tx}\eta(t)}{\partial_x \eta(t)} \right\|_{L^2}^{\frac{1}{2}} \left\| \frac{\partial_{tx}\eta(t)}{\partial_x \eta(t)} \right\|_{H^1}^{\frac{1}{2}} \\ &\leq C\|\partial_x \eta(t)\|_{L^\infty} \left(\left\| \frac{\partial_{tx}\eta(t)}{\partial_x \eta(t)} \right\|_{L^2} + \left\| \left(\frac{\partial_{tx}\eta(t)}{\partial_x \eta(t)} \right)_x \right\|_{L^2} \right).\end{aligned}$$

Recall (3.95), we find

回顾 (3.95)，可得

$$\|\partial_{tx}\eta(t)\|_{L^\infty} \leq C\|\partial_x \eta(t)\|_{L^\infty} \left(\left\| \frac{\partial_t \mathcal{G}(t)}{\partial_x \eta(t)} \right\|_{L^2} + \|\partial_{tx}\eta(t)\|_{L^2} \right)$$

$$\leq C\|\partial_x \eta(t)\|_{L^\infty} \left(\left\| \frac{1}{\partial_x \eta(t)} \right\|_{L^\infty} \|\partial_t \mathcal{G}(t)\|_{L^2} + \|\partial_{tx}\eta(t)\|_{L^2} \right).$$

Since the uniform bound of $\|\partial_x \eta(t)\|_{L^\infty}$ and $\|\partial_x \eta(t)\|_{L^\infty}$ and $\|\partial_x \eta(t)\|_{L^\infty}$ are uniformly bounded, while the uniform bound of $\|\partial_{tx}\eta(t)\|_{L^2}$ is given by **Lemma 3.6**. Let us consider $\|\partial_t \mathcal{G}(t)\|_{L^2}$. Recall (3.96), we can obtain

由于 $\|\partial_x \eta(t)\|_{L^\infty}$ 的一致有界性表明 $\|\partial_x \eta(t)\|_{L^\infty}$ 和 $\|\partial_x \eta(t)\|_{L^\infty}$ 一致有界，而 $\|\partial_{tx}\eta(t)\|_{L^2}$ 的一致有界性由引理 3.6 给出。考虑 $\|\partial_t \mathcal{G}(t)\|_{L^2}$。回顾 (3.96)，可得

$$\|\partial_t \mathcal{G}(t)\|_{L^2} \leq C(1 + \|\rho^{\frac{2\gamma-1}{2}} \partial_{xx}\eta(t)\|_{L^2} + \|\rho^{\frac{1}{2}} \partial_t u(t)\|_{L^2} + \|\Phi(t)\|_{L^2})$$

$$\leq C(1 + \|\rho^{\frac{2\gamma-1}{2}} \partial_{xx}\eta(t)\|_{L^2} + \|\rho^{\frac{1}{2}} \partial_t u(t)\|_{L^2} + \|\Phi(t)\|_{L^2}),$$

for some constant $C > 0$. As $\|\rho^{\frac{2\gamma-1}{2}} \partial_{xx}\eta(t)\|_{L^2}$ and $\|\rho^{\frac{1}{2}} \partial_t u(t)\|_{L^2}$ are uniformly bounded, it suffices to estimate $\|\Phi(t)\|_{L^2}$. Since

对于某个常数 $C > 0$，由于 $\|\rho^{\frac{2\gamma-1}{2}} \partial_{xx}\eta(t)\|_{L^2}$ 和 $\|\rho^{\frac{1}{2}} \partial_t u(t)\|_{L^2}$ 一致有界，只需估计 $\|\Phi(t)\|_{L^2}$。因为 $\|\quad(\quad)\|_{L^2}$。Since

$$\|\Phi(t)\|_{L^2} \leq \|\rho^{-\frac{1}{2}}\|_{L^\infty} \|\rho^{\frac{1}{2}} \Phi(t)\|_{L^2},$$

and

$$\|\bar{\rho}^{-\frac{1}{2}} \Phi(t)\|_{L^2} \leq C(\|\partial_x \eta(t)\|_{L^\infty}^2 + \|\bar{\rho}^{\frac{2\gamma-1}{2}} \partial_{xx}\eta(t)\|_{L^2}^2),$$

We have $\|\partial_{tx}\eta(t)\|_{L^2}$ 由引理 B.6 一致有界，因此 $\|\partial_{tx}\eta(t)\|_{L^2}$ 一致有界。由此可得 $\|\partial_{tx}\eta(t)\|_{L^2}$ 一致有界。This finishes the proof of (2.17) in Theorem 2.1. 这就完成了定理 2.1 中 (2.17) 的证明。

The time decay estimates (2.18) and (2.19) are direct consequences of Lemma 3.3 and Lemma 3.7. 定理 2.1 中的时间衰减估计 (2.18) 和 (2.19) 是引理 3.3 和引理 3.7 的直接结果。

4.2. Proof of Theorem 2.2. 该结果由 (ρ, u) 的构造 (2.16) 得出。The construction (2.16) of (ρ, u).

Appendix A. Stationary states

In this section, we recall some important results on the existence and uniqueness modulo translations, corresponding to (1.4). This appendix is a summary of the main results in [8, 4, 9, 2]. Consider the equation 在本节中，我们回顾一些关于 (1.4) 对应的模平移存在性与唯一性的重要定理。本附录总结了 [8, 4, 9, 2] 中的主要结果。考虑方程

$$\nabla(\bar\rho^\gamma) + \bar\rho\nabla\Psi_k[\bar\rho] = 0, \quad \mathbf{x}\in\mathbb{R}^n. \tag{A.1}$$

Remark A.1. 我们回顾临界扩散指数 $\gamma = 1 - k/n$。若 $\gamma \ge \gamma_c$ 则称为公平竞争区间；若 $\gamma \ge \gamma_c$，则称为扩散主导区间；$0 < \gamma < \gamma_c$ 则称为吸引主导区间。*Recall the critical diffusion exponent $\gamma = 1 - k/n$. Then γ is called the fair competition regime; $\gamma > \gamma_c$ is called the diffusion-dominated regime; $\gamma < \gamma_c$ is called the attraction-dominated regime.*

我们根据 [8, 4, 9, 2] 给出 (A.1) 的解的定义。Definition of solutions to (A.1) according to [8, 4, 9, 2].

Definition A.1. 给定 $\gamma > 1 - 1/n$ 且 $k \in (\max\{-2, 1-n\}, 2)$，若 $\bar\rho \in L^1_+(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$，$\bar\rho^{\gamma-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$，$\nabla\Psi_k[\bar\rho] \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ 且满足 (A.1)，其意义定义为 $\mathbb{R}^n$ 上的分布。若 $n \ge 3$ 且 $k \in [2-n, 1-n]$，则还要求 $\bar\rho \in C^{0,\alpha}(\mathbb{R}^n)$ 对某个 $\alpha \in (1-k-n, 1)$。*Let $\gamma > 1 - 1/n$ and $k \in (\max\{-2, 1-n\}, 2)$. We call $\bar\rho$ a solution to (A.1) if $\bar\rho \in L^1_+(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\bar\rho^{\gamma-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\nabla\Psi_k[\bar\rho] \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\bar\rho$ satisfies (A.1) in the sense of distributions in $\mathbb{R}^n$. If $-n < k \le 1-n$, we further require $\bar\rho \in C^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (1-k-n, 1)$.*

为此，我们列出扩散主导区间的结果。回顾如下。For our purpose, we list the results for the diffusion-dominated regime. We recall that

$$\gamma^* := \begin{cases} \frac{k}{k-1}, & \text{for } n = 1 \text{ and } 0 < s < \frac{1}{2}, \\ +\infty, & \text{for } n \ge 2 \text{ and } \frac{1}{2} \le s < \frac{n}{2}. \end{cases}$$

Theorem A.1. 设 $n \ge 1$ 且 $k \in (\max\{-2, 1-n\}, 0)$。方程 (A.1) 的所有解都是径向对称非增的。令 F 由 (1.5) 定义。若 $\gamma > \gamma_c$，则 F 在 Y 上存在全局极小化子 $\bar\rho$。此外，所有全局极小化子 $\bar\rho \in Y$ 均为径向对称非增、紧支撑、一致有界，并且在其支撑内属于 C^∞。而且，根据定义 A.1，当 $\gamma_c < \gamma \le \gamma^*$ 时，所有 F 的全局极小化子都是方程 (A.1) 的解。最后，若 $\gamma_c < \gamma < 2$，则有 $\bar\rho \in W^{1,\infty}(\mathbb{R}^n)$。*All solutions of (A.1) are radially symmetric non-increasing. Let F be defined in (1.5). If $\gamma > \gamma_c$, then there exists a global minimiser $\bar\rho$ of F on Y. Further, all global minimisers $\bar\rho \in Y$ are radially symmetric, non-increasing, compactly supported, uniformly bounded and C^∞ inside their support. Moreover, all global minimisers of F are solutions of (A.1), according to Definition A.1, when $\gamma_c < \gamma \le \gamma^*$. Finally, if $\gamma_c < \gamma < 2$, we have $\bar\rho \in W^{1,\infty}(\mathbb{R}^n)$.*

定理 A.2. 设 $n \ge 1$，$k \in (-1, 0)$，且 $\gamma > \gamma_c$。方程 (A.1) 的所有解都是能量 F 在 Y 上的全局极小化子，并且在 Y 中的方程 (A.1) 的解是唯一的。**Theorem A.2.** *Let $n \ge 1$, $k \in (-1, 0)$, and $\gamma > \gamma_c$. All solutions of (A.1) are global minimisers of the energy F in Y. Further, solutions of (A.1) in $\mathcal{Y}$ are unique.*

Appendix B. Some important inequalities

在本小节中，我们列出证明中广泛使用的一些不等式。In this subsection, we list some inequalities extensively used in our proof.

第一个是 Hardy 型不等式。The first one is a Hardy type inequality.

引理 B.1 (Hardy 型不等式 I)。设 $k > -1$ 为给定实数，且 f 为满足以下条件的函数。**Lemma B.1 (Hardy type inequality I).** *Let $k > -1$ be a given real number, and f be a function satisfying*

$$\int_0^{\frac{1}{2}} r^k (|f|^2 + |\partial_r f|^2)\, dx < \infty,$$

则有 *it holds that*

$$\int_0^{\frac{1}{2}} x^{k-2}|f|^2\,\mathrm{d}x \le C\int_0^{\frac{1}{2}} x^k(|f|^2+|\partial_x f|^2)\,\mathrm{d}x.$$

作为引理 B.1 的推论，我们还注意到以下结果。 *As a consequence of Lemma B.1, we also have the following result.*

引理 B.2 (Hardy 型不等式 II) 设 $k \ge 1$ 为给定实数，且 f 为满足以下条件的函数 ***Lemma B.2** (Hardy-type inequality II). Let k be a given real number and f be a function satisfying*

$$\int_{\frac{1}{2}}^1 (1-x)^k(|f|^2+|\partial_x f|^2)\,\mathrm{d}x < \infty,$$

则有 *it holds that*

$$\int_{\frac{1}{2}}^1 (1-x)^{k-2}|f|^2\,\mathrm{d}x \le C\int_{\frac{1}{2}}^1 (1-x)^k(|f|^2+|\partial_x f|^2)\,\mathrm{d}x.$$

引理 B.1 和引理 B.2 的证明见 [21]。 *The proofs of Lemma B.1 and Lemma B.2 can be found in [21].*

根据定理 A.1，对于 $m \le m_* \le 2$，$\rho \in W^{1,\infty}(\mathbb{R})$，因此当 ϵ 充分小时，我们有 *By Theorem A.1, for $m \le m_* \le 2$, $\rho \in W^{1,\infty}(\mathbb{R})$, thus there exists sufficiently small, we have*

$$\begin{aligned} &c_1(x+\bar{R}) \le \bar{\rho}(x) \le c_2(x+\bar{R}), && -\bar{R} \le x \le -\bar{R}+\epsilon,\\ &c_1(\bar{R}-x) \le \bar{\rho}(x) \le c_2(\bar{R}-x), && \bar{R}-\epsilon \le x \le \bar{R}, \end{aligned}$$

对于某个常数 $c_1, c_2 > 0$。因此 *for some constant $c_1, c_2 > 0$. Therefore*

$$\bar{\rho}(x) \approx 1-|x| \text{ as } x \text{ is close to } \pm 1. \tag{B.1}$$

通过应用引理 B.1 和引理 B.2，便可直接得到以下结果。 *Then the following result follows directly by applying* **Lemma B.1** and **Lemma B.2**.

推论 B.1 设 $k \ge 1$ 为给定的实数，ρ 为定理 A.1 中对应于 $N=1$ 且 $m \le m_* \le 2$ 在 $[-R,R]$ 上紧支撑的稳态解，其中常数 $R > 0$。若 f 满足 ***Corollary B.1.** Let k be a given real number, ρ be the steady solution in Theorem A.1 corresponding to $N=1$ and $m \le m_* \le 2$ supported in $[-R,R]$ for some constant $R > 0$. If f satisfies*

$$\int_{-\bar{R}}^{\bar{R}} \bar{\rho}^k(|f|^2+|\partial_x f|^2)\,\mathrm{d}x < \infty,$$

则有 *then it holds that*

$$\int_{-\bar{R}}^{\bar{R}} \bar{\rho}^{k-2}|f|^2\,\mathrm{d}x \le C\int_{-\bar{R}}^{\bar{R}} \bar{\rho}^k(|f|^2+|\partial_x f|^2)\,\mathrm{d}x.$$

此外，我们的证明还需要以下结果。 *In addition, we need the following results in our proof.*

引理 B.3 对于 $\alpha < 1$，存在一个常数 $C > 0$ 使得 ***Lemma B.3.** For $\alpha < 1$, there exists a constant C such that*

$$\int_{-1}^1 \frac{1}{\bar{\rho}^{\alpha}(y)|x-y|^{1-2s}}\,\mathrm{d}y \le C\bar{\rho}^{2s-\alpha}(x). \tag{B.2}$$

此外，若 $1+2s > \alpha+s$ 和 $\beta > 0$，则 *Furthermore, if $1+2s > \alpha+s$ and $\beta > 0$, then*

$$\int_{-1}^1\int_{-1}^1 \frac{1}{\bar{\rho}^{\alpha}(x)\bar{\rho}^{\beta}(y)|x-y|^{1-2s}}\,\mathrm{d}x\,\mathrm{d}y \le C. \tag{B.3}$$

Proof. To prove (B.2), by (B.1), we have

$$\int_{-1}^{1} \frac{1}{\bar\rho^{\alpha}(y)|x-y|^{1-2s}}\,dy \le C\left(\int_{-1}^{x} \frac{1}{(1+y)^{\alpha}(x-y)^{1-2s}}\,dy + \int_{x}^{1} \frac{1}{(1-y)^{\alpha}(y-x)^{1-2s}}\,dy\right).$$

我们来计算
Let us compute

$$I_1(x) := \int_{-1}^{x} \frac{1}{(1+y)^{\alpha}(x-y)^{1-2s}}\,dy,$$

$$I_2(x) := \int_{x}^{1} \frac{1}{(1-y)^{\alpha}(y-x)^{1-2s}}\,dy.$$

对于 I_1，通过变量替换 $t = \frac{x-y}{1+x}$，则
For I_1, by using the change of variable $t = \frac{x-y}{1+x}$, then

$$I_1 = (1+x)^{2s-\alpha}\int_0^1 t^{-\alpha}(1-t)^{-1+2s}\,dt.$$

回顾贝塔函数的定义，
We recall the definition of Beta function,

$$B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}\,dt,$$

then

$$I_1(x) = (1+x)^{2s-\alpha}B(1-\alpha, 2s). \tag{B.4}$$

类似地，我们有
Similarly, we have

$$I_2(x) = (1-x)^{2s-\alpha}B(1-\alpha, 2s). \tag{B.5}$$

因此由 (B.4)，(B.5) 和 (B.1)，可得 (B.2)。
Therefore by (B.4), (B.5) and (B.1), we get (B.2).

现在，我们利用 (B.2) 证明 (B.3)，得到
Now, we use (B.2) to prove (B.3), and we find that

$$\int_{-1}^{1}\int_{-1}^{1} \frac{1}{\bar\rho^{\alpha}(x)\bar\rho^{\beta}(y)|x-y|^{1-2s}}\,dx\,dy \le \int_{-1}^{1} \bar\rho^{2s-\alpha-\beta}(y)\,dy.$$

因此由 (B.1) 可得 (B.3)。
Therefore by (B.1) we get (B.3).

□

正如注 2.3 中所述，我们给出关于 $\|\bar\rho^{-\frac{1}{2}}\Phi(t)\|_{L^2}$ 的以下估计。
As we mentioned in Remark 2.3, we give the following estimates for $\|\bar\rho^{-\frac{1}{2}}\Phi(t)\|_{L^2}$.

Lemma B.1. *For $\gamma \le 1 + 2s/3$, $s \ge \frac{1}{2}$, there exists a constant C such that*

$$\|\bar\rho^{-\frac{1}{2}}\Phi(t)\|_{L^2} \le C\left(\|\partial_x\eta(t)\|_{H^{\gamma-1}}^2 + \|\eta(t)\|_{H^{\frac{2\gamma-1}{2}}}^2\right). \tag{B.6}$$

Proof. 可得
It holds that

$$\begin{aligned}
\|\bar\rho^{-\frac{1}{2}}\Phi(t)\|_{L^2}^2 \le{}& \int_{-1}^{1}\frac{1}{\bar\rho(x)}\left(\int_{-1}^{1}\frac{\bar\rho(y)|\partial_x\eta(t,x)-\partial_y\eta(t,y)|}{|x-y|^{1-2s}}\,dy\right)^2 dx \\
&+ \int_{-1}^{1}\frac{1}{\bar\rho(x)}\left(\int_{-1}^{1}\frac{|\bar\rho(x)-\bar\rho(y)|\cdot|\partial_y\eta(t,y)|}{|x-y|^{1-2s}}\,dy\right)^2 dx \\
&+ \int_{-1}^{1}\frac{1}{\bar\rho(x)}\left(\int_{-1}^{1}\bar\rho(y)|\partial_x\eta(t,x)-\partial_y\eta(t,y)|\cdot|R_{1,\alpha}(t,x,y)|\,dy\right)^2 dx \\
&+ \int_{-1}^{1}\frac{1}{\bar\rho(x)}\left(\int_{-1}^{1}|\bar\rho(x)-\bar\rho(y)|\cdot|\partial_y\eta(t,y)R_{1,\alpha}(t,x,y)|\,dy\right)^2 dx.
\end{aligned}$$

By the estimate (3.5), (3.51), D 并取 $\delta \in (0, 2s-1, 4-3\gamma]$, we have

$$\begin{aligned}
\|\Phi^{-1}(t)\|_{L^2} \le & \int_{-1}^{1} \frac{1}{\bar{\rho}(x)} \left(\int_{-1}^{1} \frac{|\tilde{\rho}(y)| |\partial_x \pi(t,x) - \partial_y \pi(t,y)|}{|x-y|^{1+2s}} \, dy \right)^2 dx \\
& + \int_{-1}^{1} \frac{1}{\bar{\rho}(x)} \left(\int_{-1}^{1} \frac{|\bar{\rho}(x) - \bar{\rho}(y)| \cdot |\partial_y \pi(t,y)|}{|x-y|^{1+2s}} \, dy \right)^2 dx \\
\le & \underbrace{\int_{-1}^{1} \frac{1}{\bar{\rho}(x)} \left(\int_{-1}^{1} \frac{|\bar{\rho}^{\frac{-1+\delta}{2}}(y)| |\bar{\rho}^{\frac{3-\delta}{2}}(x) \partial_x \pi(t,x) - \bar{\rho}^{\frac{3-\delta}{2}}(y) \partial_y \pi(t,y)|}{|x-y|^{1+2s}} \, dy \right)^2 dx}_{I_1} \\
& + \underbrace{\int_{-1}^{1} \frac{1}{\bar{\rho}(x)} \left(\int_{-1}^{1} \frac{|\bar{\rho}^{\frac{-1+\delta}{2}}(y)| |\bar{\rho}^{\frac{3-\delta}{2}}(x) - \bar{\rho}^{\frac{3-\delta}{2}}(y)| \cdot |\partial_y \pi(t,y)|}{|x-y|^{1+2s}} \, dy \right)^2 dx}_{I_2} \\
& + \underbrace{\int_{-1}^{1} \frac{1}{\bar{\rho}(x)} \left(\int_{-1}^{1} \frac{|\bar{\rho}(x) - \bar{\rho}(y)| \cdot |\partial_y \pi(t,y)|}{|x-y|^{1+2s}} \, dy \right)^2 dx}_{I_3}.
\end{aligned}$$

For I_1, by Holder's inequality, Lemma (B.3) and the choice of δ such that δ 选择, 2可得 we have

$$\begin{aligned}
I_1 \le & \int_{-1}^{1} \int_{-1}^{1} \frac{1}{\bar{\rho}(x) \bar{\rho}^{1-\delta}(y) |x-y|^{1+2s}} \, dy \, dx \\
& \times \int_{-1}^{1} \int_{-1}^{1} \frac{|\bar{\rho}^{\frac{3-\delta}{2}}(x) \partial_x \pi(t,x) - \bar{\rho}^{\frac{3-\delta}{2}}(y) \partial_y \pi(t,y)|^2}{|x-y|^{1+2s}} \, dy \, dx \\
\le & C \|\bar{\rho}^{\frac{3-\delta}{2}} \partial_x \pi(t)\|_{\dot{H}^s}^2.
\end{aligned}$$

By Gagliardo-Nirenberg's inequality, 根据 Gagliardo-Nirenberg 不等式,

$$\|\bar{\rho}^{\frac{3-\delta}{2}} \partial_x \pi(t)\|_{\dot{H}^s} \le C \|\bar{\rho}^{\frac{3-\delta}{2}} \partial_x \pi(t)\|_{L^2}^{1-s} \|\bar{\rho}^{\frac{3-\delta}{2}} \partial_x \pi(t)\|_{\dot{H}^1}^{s}.$$

Then, by the choice of δ such that 然后, 根据 C 满足 δ 的取值选择, 可得 we have

$$I_1 \le C \|\partial_x \pi(t)\|_{L^2}^{2-s} \left(\|\rho^{\frac{3}{2}} \partial_x \pi(t)\|_{L^2}^{2} + \|\rho^{\frac{3}{2}} D_x \pi(t)\|_{L^2}^{2} \right). \tag{B.7}$$

For I_2 and I_3, by 根据霍尔德 Holder's inequality, we have

$$\begin{aligned}
I_2 \le & \|\partial_x \pi(t)\|_{L^\infty}^2 \int_{-1}^{1} \int_{-1}^{1} \frac{1}{\bar{\rho}(x) \bar{\rho}^{1-\delta}(y) |x-y|^{1+2s}} \, dy \, dx \\
& \times \int_{-1}^{1} \int_{-1}^{1} \frac{|\rho^{\frac{3}{2}}(x) - \rho^{\frac{3}{2}}(y)|^2}{|x-y|^{1+2s}} \, dy \, dx,
\end{aligned}$$

and

$$I_3 \le \|\partial_x \pi(t)\|_{L^\infty}^2 \int_{-1}^{1} \int_{-1}^{1} \frac{1}{\bar{\rho}(x) |x-y|^{1+2s}} \, dy \, dx \int_{-1}^{1} \int_{-1}^{1} \frac{|\rho(x) - \rho(y)|^2}{|x-y|^{1+2s}} \, dy \, dx.$$

Then following the estimates of I_1 by Lemma B.3 and Gagliardo-Nirenberg's inequality, 摘下来, 根据引理 B.3 和 Gagliardo-Nirenberg 不等式对 I_i 的估计,

$$I_2 + I_3 \le C \|\partial_x \pi(t)\|_{L^\infty}^2 \tag{B.8}$$

Therefore, (B.7) and (B.8) 因此, 由 (B.7) 和 (B.8) 可得 we have (B.6). □

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