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含非Lipschitz Poisson跳跃噪声的二维Boussinesq方程的弱与强平均原理

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摘要

本文研究含非Lipschitz Poisson跳跃噪声的二维 Boussinesq方程的平均原理。具体而言，我们首先探讨涡量变量的适定性、正则性估计和紧性。随后，证明温度变量的遍历性。接着，在不同条件下证明，当时间尺度参数 ε 趋于零时，涡量变量依概率和 $2p$ 阶均值收敛到平均方程的解。最后，我们给出一个具体案例并开展数值模拟，以验证本文的主要结论。

关键词： Boussinesq方程、平均原理、泊松跳跃噪声、非Lipschitz。**2010年数学主题分类** 37N10, 70K65, 70K70。

1 引言

本文研究二维环面 $\mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$ 上的 Boussinesq方程

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u = \triangle u - \nabla p + \theta e_2, \\ \dfrac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta = \triangle \theta, \\ \nabla \cdot u = 0, \\ u(0, x, y) = u_0(x, y),\ \theta(0, x, y) = \theta_0(x, y), \end{cases} \tag{1.1}$$

空间中满足周期边值条件。其中 $u(t,\ x,\ y) = (u_1,\ u_2)$ 表示速度向量，$p = p(t,\ x,\ y)$ 是标量压力，变量 $\theta(t,\ x,\ y)$ 表示标量温度，向量 $e_2 = (0,\ 1)$。

由于热扩散率决定了热量在流体中的传递速率，尤其当流体的普朗特数（Pr）或刘易斯数（Le）显著小于 1 时，热扩散率远大于动量扩散率和溶质扩散率[1]。

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对（1.1）的第一个方程应用旋度算子，我们将考虑以下由泊松跳跃噪声驱动的双尺度随机Boussinesq方程

$$\begin{cases} \dfrac{\partial j^\varepsilon}{\partial t} + (u^\varepsilon \cdot \nabla) j^\varepsilon = \triangle j^\varepsilon + f(j^\varepsilon, \theta^\varepsilon) + \partial_x \theta^\varepsilon + \displaystyle\int_{|z|_H<1} \sigma_1(j^\varepsilon, \theta^\varepsilon, z)\dot{\tilde{\eta}}_1(t, dz), \\ \dfrac{\partial \theta^\varepsilon}{\partial t} + \dfrac{1}{\varepsilon}(u^\varepsilon \cdot \nabla)\theta^\varepsilon = \dfrac{1}{\varepsilon}\triangle \theta^\varepsilon + \displaystyle\int_{|z|_H<1} \sigma_2(j^\varepsilon, \theta^\varepsilon, z)\dot{\tilde{\eta}}_2^\varepsilon(t, dz), \\ j^\varepsilon(0) = j_0,\ \theta^\varepsilon(0) = \theta_0, \end{cases} \tag{1.2}$$

其中，$j^\varepsilon := \nabla^\perp \cdot u^\varepsilon = \partial_x u_2^\varepsilon - \partial_y u_1^\varepsilon$ 为涡量，小参数 $0 < \varepsilon \ll 1$ 描述 j^ε 与 θ^ε之间的时间尺度比。补偿泊松随机测度$\tilde{\eta}_1(t,\ z) = \eta_1(t,\ z) - \nu_1(z) \cdot t$和 $\tilde{\eta}_2^\varepsilon(t,\ z) = \eta_2(\frac{1}{\varepsilon}t,\ z) - \frac{1}{\varepsilon}\nu_2(z) \cdot t$定义在可测空间$(H,\ \mathscr{B}(H))$上，强度测度 $\nu_i(i = 1,\ 2)$定义在滤过概率空间$(\Omega\ \mathscr{F},\ \{\mathscr{F}_t\}_{t\geq 0},\ \mathbb{P})$上。

作为描述特定近似条件下流体流动和热输运规律的数学模型，Boussinesq方程广泛应用于流体力学 [2,3], 、环境水文学 [4,5], 、工程学科 [6] 等领域。

过去二十年来，随机 Boussinesq 方程的研究备受关注，因为这类方程为理解随机扰动下的流体动力学提供了重要模型。对这些随机系统的数学分析已在多个方向上取得显著进展。关于解的存在性与唯一性这一基础问题，研究者已进行了广泛探讨。Pu 和 Guo [7] 证明了具有部分黏性的随机二维Boussinesq方程的整体适定性。Du 和 Zhang [8] 获得了具有乘性噪声的逐路径解的局部与整体存在性，而 Alonso-Orán 等人 [9] 研究了存在输运噪声时的适定性。Sun 等人 [10],研究了三维情形，并证明了 Besov空间中温和解的整体存在性。Wu 和 Huang [11] 进一步研究了具有非线性噪声的分数阶Boussinesq方程的适定性。Lin 等人 [12] 证明了存在输运噪声和小粗糙初值时解的整体存在性。此外，黏性趋于零的极限情形也受到了特别关注。Yamazaki[13] 构造了零耗散随机 Boussinesq 系统的整体鞅解。Luo [14] 研究了具有输运噪声的随机二维无黏Boussinesq方程向确定性黏性系统的收敛性。

大偏差理论为理解随机Boussinesq系统中罕见事件发生的概率提供了洞见。Duan 和 Millet [15] 建立了随机影响下Boussinesq方程的大偏差原理。Sun 等人 [16] 研究了具有波动的动力边界条件的罕见事件。Zheng 和 Huang [17]随后将大偏差结果扩展到由Lévy噪声驱动的系统。此外，理解解在长时间尺度上的统计性质至关重要。Huang 等人 [18] 研究了具有退化噪声的二维随机分数阶 Boussinesq 方程的渐近性质。Dai 等人 [19]分析了具有记忆效应的部分耗散系统的渐近行为。

平均原理是处理多尺度系统的重要工具，在多个领域都有广泛应用，详见[20–22] 及其中的参考文献。随机微分方程（SDEs）的平均原理由 Khasminskii 首次提出于 [23, 24], ，此后该领域取得了大量研究进展。以往关于平均原理的研究主要集中于抽象系统。

关于抽象随机微分方程和随机偏微分方程中的平均原理的研究成果，参见 [25–32]及其中的参考文献。

此外，平均原理在物理模型中有着广泛的应用。Majda 等运用平均原理研究随机气候模型，从而从原始模型中提取有效约化方程，见 [33], 。Arnold 等通过平均原理研究 Lorenz-Maas 系统，并开展了有效数值模拟，见 [34]。Yeong 等将平均原理应用于研究多尺度混沌系统中带推力控制的粒子滤波器；详情见 [35] 。此外，Gao 在随机流体方程的平均方面取得了一系列杰出成果，例如见[36–38]。

然而，时至今日，关于多尺度 Boussinesq 方程的研究成果仍然十分有限。Zhang 等研究了湿大气动力学中快波的平均，见 [39]。Du 等考虑了弗劳德数较小时随机 Boussinesq 方程的平均原理，参见 [40]。

在已有研究中，通常要求快变量满足抽象的反应–扩散方程，这与我们的模型（1.2）不同。对流项的存在给冻结方程的处理以及确定平均方程中系数的性质带来了诸多困难。

本文的结构如下。在下一节中，我们介绍必要的记号，并阐述方程（1.2）中各个系数所满足的假设。第3节讨论非 Lipschitz 条件下方程（1.2）解的存在性与唯一性。此外，我们研究涡量变量的正则性估计，从而得到其关于 ε 的紧性。第4节专门考察温度变量对应冻结方程的遍历性。在第5节中，我们证明，在不同条件下，当 ε 趋于零时，涡量变量在概率意义和 2p 阶均值意义下都可以收敛到平均方程的解。最后一节给出一个说明性例子及其数值模拟。

2 记号与假设

2.1 记号

对于任意 $p \geq 1$，令 $L^p(\mathbb{T}^2)$ 表示定义在 $\mathbb{T}^2$ 上的 p 阶可积函数空间，其范数为

$$|f|_{L^p(\mathbb{T}^2)} = \left(\int_{\mathbb{T}^2} |f(x,y)|^p \, dxdy \right)^{\frac{1}{p}}.$$

特别地，令

$$H := \left\{ f \in L^2(\mathbb{T}^2); \int_{\mathbb{T}^2} f dxdy = 0 \right\}$$

其内积为（·, ·）。

对于任意 $\alpha > 0$，令 H^α 为 $\mathbb{T}^2$ 上具有 α阶广义导数且取值于 H的函数所构成的 Sobolev 空间，并赋予其范数

$$|u|_{H^\alpha} = |u|_H + \left| (-\triangle)^{\frac{\alpha}{2}} u \right|_H.$$

此外，$\langle\cdot,\ \cdot\rangle$ 表示 H^1 与 H^{-1}的对偶配对。

$\mathscr{B}_b(H)$ 表示有界博雷尔函数 $\varphi(x)$：$H\to\mathbb{R}$ 的空间，其赋范为

$$|\varphi|_0=\sup_{x\in H}|\varphi(x)|.$$

$C_0^k(\mathbb{T}^2)$ 表示直至 k 阶导数均有界的 k 阶连续可微函数空间。

2.2 假设

我们给出方程 (1.2) 中系数所满足的假设。

假设1. f: $H\times H\to H$，对任意 u_1、u_2、v_1、$v_2\in H$，存在一个凹连续非递减函数 $\kappa(\cdot)$ 以及某个 $p\geq 1$，使得

$$|f(u_1,v_1)-f(u_2,v_2)|_H^{2p}\leq\kappa\left(|u_1-u_2|_H^{2p}+|v_1-v_2|_H^{2p}\right),$$

其中 $\kappa(\cdot)$：$\mathbb{R}^+\to\mathbb{R}^+,\kappa(0)=0$，且 $\int_{0^+}\frac{du}{\kappa(u)}=\infty$。可以选取一些常数$C_1,C_2>0$，使得 $\kappa(u)\leqslant C_1u+C_2,\ \forall u\geqslant 0$。因此，我们有如下增长条件：

$$|f(u,v)|_H^{2p}\leq C_1\left(1+|u|_H^{2p}+|v|_H^{2p}\right).$$

假设2. σ_1：$H\times H\times H\to H$，对于某个 $p\geq 1$，且对于任意 $u_1,u_2,v_1,v_2\in H$，有

$$\int_{|z|_H<1}|\sigma_1(u_1,v_1,z)-\sigma_1(u_2,v_2,z)|_H^{2p}\,\nu_1(dz)\leq\kappa\left(|u_1-u_2|_H^{2p}+|v_1-v_2|_H^{2p}\right),$$

以及增长条件：

$$\int_{|z|_H<1}|\sigma_1(u,v,z)|_H^{2p}\,\nu_1(dz)\leq C_1\left(1+|u|_H^{2p}+|v|_H^{2p}\right).$$

假设3. σ_2：$H\times H\times H\to H$，对于某个 $p\geq 1$，对于任意 $u_1,u_2,v_1,v_2\in H$，存在 $L_{\sigma_2}>0$ 使得

$$\int_{|z|_H<1}|\sigma_2(u_1,v_1,z)-\sigma_2(u_2,v_2,z)|_H^{2p}\,\nu_2(dz)\leq L_{\sigma_2}|v_1-v_2|_H^{2p}+\kappa\left(|u_1-u_2|_H^{2p}\right).$$

因此，我们有如下增长条件：

$$\int_{|z|_H<1}|\sigma_2(u,v)|^{2p}\,\nu_2(dz)\leq L_{\sigma_2}|v|_H^{2p}+C_1\left(1+|u|_H^{2p}\right).$$

假设4. $\lambda_p:=2p\lambda-M_p\left(\underbrace{(p-1)_p}+\underbrace{\frac{L_{\sigma_2}}{p}}+\underline{L_{\sigma_2}}\right)>0$，其中 λ是 $-\triangle$ 在 H 中的最小特征值，且 $M_p:=2p\cdot(2p-1)\cdot 2^{2p-3}$。

注2.1. 1. 对于任意 $x,h\in H$ 和 $p\geq 1$，我们有

$$\left||x+h|_H^{2p}-|x|_H^{2p}-2p\,|x|_H^{2(p-1)}(x,h)\right|\leq M_p\left(|x|_H^{2(p-1)}|h|_H^2+|h|_H^{2p}\right),\qquad(2.1)$$

具体细节可参见 [41]。

2. 当 $p=1$, $\lambda_p=2\lambda-2L_{\sigma_2}$ 是一个共同的耗散条件时。

3 适定性、正则性估计与紧性

本节讨论方程 (1.2) 的适定性、正则性估计以及涡量变量 j^ε 的紧性。此外，我们将 (1.2) 的解定义如下：

定义 3.1. 一个 $H\times H$ 值的 càdlàg $\mathscr{F}_t$-适应过程 $(j^\varepsilon,\theta^\varepsilon)$ 称为方程 (1.2) 的解，若满足以下条件

1. $(j^\varepsilon,\theta^\varepsilon)\in L^\infty([0,T];H\times H)\cap L^2([0,T];H^1\times H^1)$, $\mathbb{P}$-a.s.;
2. 以下等式 $\mathbb{P}$-几乎处处成立：

$$j^\varepsilon+\int_0^t(u^\varepsilon\cdot\nabla)\,j^\varepsilon ds=j_0+\int_0^t\triangle j^\varepsilon ds+\int_0^t f(j^\varepsilon,\theta^\varepsilon)ds+\int_0^t\int_{|z|_H<1}\sigma_1(j^\varepsilon,\theta^\varepsilon,z)\dot{\tilde{\eta}}_1(ds,dz),$$
$$\theta^\varepsilon+\frac{1}{\varepsilon}\int_0^t(u^\varepsilon\cdot\nabla)\,\theta^\varepsilon ds=\theta_0+\frac{1}{\varepsilon}\int_0^t\triangle\theta^\varepsilon ds+\int_0^t\int_{|z|_H<1}\sigma_2(j^\varepsilon,\theta^\varepsilon,z)\dot{\tilde{\eta}}_2^\varepsilon(ds,dz).$$

现在，我们给出以下结果：

定理 命题 3.2. 假设假设 *1–4* 成立，则对任意 $(j_0,\theta_0)\in H\times H$，

1. (1.2) 存在唯一解 $(j^\varepsilon,\theta^\varepsilon)$；

2. 对任意 $T>0$，且对某个 $p\geq 1$，有

$$\mathbb{E}\sup_{t\in[0,T]}|j^\varepsilon|_H^{2p}+\mathbb{E}\int_0^T|j^\varepsilon|_H^{2(p-1)}\,|\nabla j^\varepsilon|_H^2\,ds\leqslant C_T\left(1+|j_0|_H^{2p}+|\theta_0|_H^{2p}\right),\tag{3.1}$$

and

$$\sup_{t\in[0,T]}\mathbb{E}\,|\theta^\varepsilon|_H^{2p}+\mathbb{E}\int_0^T|\theta^\varepsilon|_H^{2(p-1)}\,|\nabla\theta^\varepsilon|_H^2\,ds\leqslant C_T\left(1+|j_0|_H^{2p}+|\theta_0|_H^{2p}\right).\tag{3.2}$$

证明。 关于 Boussinesq 方程的适定性已有大量研究成果。本文将适定性推广到非 Lipschitz 噪声情形。详情见附录。

下面验证 (3.1) 和 (3.2)。注意到

$$((u\cdot\nabla)v,w)=-\left((u\cdot\nabla)w,v\right),\ and\ ((u\cdot\nabla)v,v)=0,\quad for\ any\ u,v,w\in H^1.$$

根据带 Lévy 噪声的伊藤公式（参见 [42]），有

$$|\theta^\varepsilon|_H^{2p}=|\theta_0|_H^{2p}+\frac{2p}{\varepsilon}\int_0^t|\theta^\varepsilon|_H^{2(p-1)}\,\langle\triangle\theta^\varepsilon,\theta^\varepsilon\rangle\,ds+2p\int_0^t\int_{|z|_H<1}|\theta^\varepsilon|_H^{2(p-1)}\,(\theta^\varepsilon,\sigma_2(j^\varepsilon,\theta^\varepsilon,z))\,\tilde{\eta}_2^\varepsilon(ds,dz)$$
$$+\int_0^t\int_{|z|_H<1}\left[|\theta^\varepsilon+\sigma_2(j^\varepsilon,\theta^\varepsilon,z)|_H^{2p}-|\theta^\varepsilon|_H^{2p}-2p\,|\theta^\varepsilon|_H^{2(p-1)}\,(\theta^\varepsilon,\sigma_2(j^\varepsilon,\theta^\varepsilon,z))\right]\eta_2(\frac{1}{\varepsilon}ds,dz),$$

因此

$$\mathbb{E}\,|\theta^\varepsilon|_H^{2p}=|\theta_0|_H^{2p}+\frac{2p}{\varepsilon}\mathbb{E}\int_0^t|\theta^\varepsilon|_H^{2(p-1)}\,\langle\triangle\theta^\varepsilon,\theta^\varepsilon\rangle\,ds+\frac{1}{\varepsilon}\mathbb{E}\int_0^t\int_{|z|_H<1}\left[|\theta^\varepsilon+\sigma_2(j^\varepsilon,\theta^\varepsilon,z)|_H^{2p}\right.$$

$$- |\theta^\varepsilon|_H^{2p} - 2p\,|\theta^\varepsilon|_H^{2(p-1)}\,(\theta^\varepsilon, \sigma_2(j^\varepsilon, \theta^\varepsilon, z))\Big]\,\nu_2(dz)ds.$$

根据假设3、4和（2.1），我们有

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\,|\theta^\varepsilon|_H^{2p} + \frac{2p\gamma}{\varepsilon}\mathbb{E}\,|\theta^\varepsilon|_H^{2(p-1)}\,|\nabla\theta^\varepsilon|_H^2 \leq & -\frac{2p(1-\gamma)}{\varepsilon}\mathbb{E}\,|\theta^\varepsilon|_H^{2(p-1)}\,|\nabla\theta^\varepsilon|_H^2 \\
& + \frac{M_p}{\varepsilon}\mathbb{E}\int_{|z|_H<1}\left(|\theta^\varepsilon|_H^{2(p-1)}\,|\sigma_2(j^\varepsilon,\theta^\varepsilon,z)|_H^2 + |\sigma_2(j^\varepsilon,\theta^\varepsilon,z)|_H^{2p}\right)\nu_2(dz) \\
\leq & -\frac{2p(1-\gamma)\lambda}{\varepsilon}\mathbb{E}\,|\theta^\varepsilon|_H^{2p} + \frac{M_p}{\varepsilon}\mathbb{E}\int_{|z|_H<1}\left(\frac{p-1}{p}\,|\theta^\varepsilon|_H^{2p} + \frac{1}{p}\,|\sigma_2(j^\varepsilon,\theta^\varepsilon,z)|_H^{2p}\right)\nu_2(dz) \\
& + \frac{M_p L_{\sigma_2}}{\varepsilon}\mathbb{E}\,|\theta^\varepsilon|_H^{2p} + \frac{M_p C_1}{\varepsilon}\left(1 + \mathbb{E}\,|j^\varepsilon|_H^{2p}\right) \\
\leq & -\frac{\lambda_{p,\gamma}}{\varepsilon}\mathbb{E}\,|\theta^\varepsilon|_H^{2p} + \frac{C}{\varepsilon}\left(1 + \mathbb{E}\,|j^\varepsilon|_H^{2p}\right), \qquad (3.3)
\end{aligned}$$

其中 $\lambda_{p,\ \gamma} := 2p(1-\gamma)\lambda - M_p\left(\underline{\frac{(p-1)}{p}} + \underline{\frac{L_{\sigma_2}}{p}} + L_{\sigma_2}\right) > 0$，当 γ 足够小时。利用格朗沃尔不等式，我们有

$$\begin{aligned}
\mathbb{E}\,|\theta^\varepsilon|_H^{2p} + \frac{2p\gamma}{\varepsilon}\mathbb{E}\int_0^t |\theta^\varepsilon|_H^{2(p-1)}\,|\nabla\theta^\varepsilon|_H^2\,ds \leq & |\theta_0|_H^{2p}\,e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}t} + \frac{C}{\varepsilon}\int_0^t e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}(t-s)}\left(1 + \mathbb{E}\,|j^\varepsilon|_H^{2p}\right)ds \\
\leq & C\left(1 + |\theta_0|_H^{2p}\right) + \int_0^t e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}(t-s)}\mathbb{E}\,|j^\varepsilon|_H^{2p}\,ds. \qquad (3.4)
\end{aligned}$$

此外，根据伊藤公式，我们有

$$\begin{aligned}
|j^\varepsilon|_H^{2p} = & |j_0|_H^{2p} + 2p\int_0^t |j^\varepsilon|_H^{2(p-1)}\,\langle \triangle j^\varepsilon, j^\varepsilon\rangle\,ds + 2p\int_0^t |j^\varepsilon|_H^{2(p-1)}\,(f(j^\varepsilon,\theta^\varepsilon), j^\varepsilon)\,ds \\
& + 2p\int_0^t |j^\varepsilon|_H^{2(p-1)}\,(\partial_x\theta^\varepsilon, j^\varepsilon)\,ds + 2p\int_0^t\int_{|z|_H<1} |j^\varepsilon|_H^{2(p-1)}\,(j^\varepsilon, \sigma_1(j^\varepsilon,\theta^\varepsilon,z))\,\tilde{\eta}_1(ds,dz) \\
& + \int_0^t\int_{|z|_H<1}\Big[|j^\varepsilon + \sigma_1(j^\varepsilon,\theta^\varepsilon,z)|_H^{2p} - |j^\varepsilon|_H^{2p} - 2p\,|j^\varepsilon|_H^{2(p-1)}\,(j^\varepsilon, \sigma_1(j^\varepsilon,\theta^\varepsilon,z))\Big]\,\eta_1(ds,dz).
\end{aligned}$$

因此

$$\begin{aligned}
|j^\varepsilon|_H^{2p} + 2p\lambda\int_0^t |j^\varepsilon|_H^{2(p-1)}\,|\nabla j^\varepsilon|_H^2\,ds \leq & |j_0|_H^{2p} + 2p\int_0^t |j^\varepsilon|_H^{2(p-1)}\,(f(j^\varepsilon,\theta^\varepsilon), j^\varepsilon)\,ds \\
& - 2p\int_0^t |j^\varepsilon|_H^{2(p-1)}\,(\theta^\varepsilon, \partial_x j^\varepsilon)\,ds + 2p\int_0^t\int_{|z|_H<1} |j^\varepsilon|_H^{2(p-1)}\,(j^\varepsilon, \sigma_1(j^\varepsilon,\theta^\varepsilon,z))\,\tilde{\eta}_1(ds,dz) \\
& + \int_0^t\int_{|z|_H<1}\Big[|j^\varepsilon + \sigma_1(j^\varepsilon,\theta^\varepsilon,z)|_H^{2p} - |j^\varepsilon|_H^{2p} - 2p\,|j^\varepsilon|_H^{2(p-1)}\,(j^\varepsilon, \sigma_1(j^\varepsilon,\theta^\varepsilon,z))\Big]\,\eta_1(ds,dz).
\end{aligned}$$

根据　　根据假设 1、2 及 Burkholder-Davis-Gundy (B-D-G) 不等式，我们有

$$\mathbb{E}\sup_{t\in[0,T]}|j^\varepsilon|_H^{2p} + 2p\lambda\mathbb{E}\int_0^T |j^\varepsilon|_H^{2(p-1)}\,|\nabla j^\varepsilon|_H^2\,ds$$

$$\leq |j_0|_H^{2p} + \gamma\mathbb{E}\int_0^T |j^\varepsilon|_H^{2(p-1)} |\nabla j^\varepsilon|_H^2 \, ds + C_\gamma \int_0^T \left(1 + \mathbb{E}\,|j^\varepsilon|_H^{2p} + \mathbb{E}\,|\theta^\varepsilon|_H^{2p}\right) ds$$
$$+ C_p\mathbb{E}\left(\int_0^T \int_{|z|_H<1} |j^\varepsilon|_H^{4p-4} |(j, \sigma_1(j^\varepsilon, \theta^\varepsilon, z))|_H^2 \, \eta_1(ds, dz)\right)^{\frac{1}{2}}.$$

注意到

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T \int_{|z|_H<1} |j^\varepsilon|_H^{4p-4} |(j, \sigma_1(j^\varepsilon, \theta^\varepsilon, z))|_H^2 \, \eta_1(ds, dz)\right)^{\frac{1}{2}} \\
\leq &\mathbb{E}\left(\int_0^T \int_{|z|_H<1} |j^\varepsilon|_H^{4p-2} |\sigma_1(j^\varepsilon, \theta^\varepsilon, z)|_H^2 \, \eta_1(ds, dz)\right)^{\frac{1}{2}} \\
\leq &\mathbb{E}\left(\sup_{t\in[0,T]} |j^\varepsilon|_H^{2p} \int_0^T \int_{|z|_H<1} |j^\varepsilon|_H^{2p-2} |\sigma_1(j^\varepsilon, \theta^\varepsilon, z)|_H^2 \, \eta_1(ds, dz)\right)^{\frac{1}{2}} \\
\leq &\gamma\mathbb{E}\sup_{t\in[0,T]} |j^\varepsilon|_H^{2p} + C_\gamma \int_0^T \left(1 + \mathbb{E}\,|j^\varepsilon|_H^{2p} + \mathbb{E}\,|\theta^\varepsilon|_H^{2p}\right) ds.
\end{aligned}$$

取 γ 充分小，我们得到

$$\mathbb{E}\sup_{t\in[0,T]} |j^\varepsilon|_H^{2p} + C_p\mathbb{E}\int_0^T |j^\varepsilon|_H^{2(p-1)} |\nabla j^\varepsilon|_H^2 \, ds \leq |j_0|_H^{2p} + C_p \int_0^T \left(1 + \mathbb{E}\,|j^\varepsilon|_H^{2p} + \mathbb{E}\,|\theta^\varepsilon|_H^{2p}\right) ds.$$

结合(3.4)，我们有

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]} |j^\varepsilon|_H^{2p} + C_p\mathbb{E}\int_0^T |j^\varepsilon|_H^{2(p-1)} |\nabla j^\varepsilon|_H^2 \, ds \\
\leq &C_T\left(1 + |j_0|_H^{2p} + |\theta_0|_H^{2p}\right) + C_p\int_0^T \mathbb{E}\,|j^\varepsilon|_H^{2p} \, ds + \frac{C_p}{\varepsilon}\int_0^T \int_0^s e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}(s-\tau)} \mathbb{E}\,|j^\varepsilon|_H^{2p} \, d\tau ds \\
= &C_T\left(1 + |j_0|_H^{2p} + |\theta_0|_H^{2p}\right) + C_p\int_0^T \mathbb{E}\,|j^\varepsilon|_H^{2p} \, ds + \frac{C_p}{\varepsilon}\int_0^T \int_\tau^T e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}(s-\tau)} ds \mathbb{E}\,|j^\varepsilon|_H^{2p} \, d\tau \\
\leq &C_T\left(1 + |j_0|_H^{2p} + |\theta_0|_H^{2p}\right) + C_p\int_0^T \mathbb{E}\sup_{r\in[0,s]} |j^\varepsilon|_H^{2p} \, ds.
\end{aligned}$$

利用格朗沃尔不等式，我们得到

$$\mathbb{E}\sup_{t\in[0,T]} |j^\varepsilon|_H^{2p} + C_p\mathbb{E}\int_0^T |j^\varepsilon|_H^{2(p-1)} |\nabla j^\varepsilon|_H^2 \, ds \leq C_T\left(1 + |j_0|_H^{2p} + |\theta_0|_H^{2p}\right).$$

因此

$$\mathbb{E}\,|\theta^\varepsilon|_H^{2p} + \frac{2p\gamma}{\varepsilon}\mathbb{E}\int_0^t |\theta^\varepsilon|_H^{2(p-1)} |\nabla\theta^\varepsilon|_H^2 \, ds \leq C_T\left(1 + |j_0|_H^{2p} + |\theta_0|_H^{2p}\right).$$

□

注 3.1. 在前一个定理中，我们只证明了关于ε的 $\sup_{t\in[0,T]}\mathbb{E}\,|\theta^\varepsilon|^{2p}$ 的一致界，而没有证明关于 $\mathbb{E}\sup_{t\in[0,T]}|\theta^\varepsilon|^{2p}$的一致界。事实上，我们只能证明

$$\mathbb{E}\sup_{t\in[0,T]}|\theta^\varepsilon|^{2p}\leqslant C_T\left(1+|j_0|_H^{2p}+|\theta_0|_H^{2p}+\varepsilon^{-1}\right). \tag{3.5}$$

接下来，我们考察涡量变量 j^ε 的正则性估计。下面给出定理:

定理 3.3. 对于任意 $T>0$, $M>0$ 和 $(j_0,\theta_0)\in H^1\times H$，有

$$\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}|j^\varepsilon|_{H^1}^2+\mathbb{E}\int_0^{\tau_M\wedge T}|j^\varepsilon|_{H^2}^2\,ds\leqslant C_{T,M}\left(1+|j_0|_{H^1}^2+|\theta_0|_H^2\right), \tag{3.6}$$

其中停时 $\tau_M:=\inf\{t\geq 0;\ |j^\varepsilon|_H>M\}$。

Proof. 我们将 (1.2) 中的第一个方程与 $-\triangle j^\varepsilon$ 取内积，得到 g

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla j^\varepsilon|_H^2+|\triangle j^\varepsilon|_H^2=&((u^\varepsilon\cdot\nabla)j^\varepsilon,\triangle j^\varepsilon)+(f(j^\varepsilon,\theta^\varepsilon),-\triangle j^\varepsilon)+(\partial_x\theta^\varepsilon,-\triangle j^\varepsilon)\\&+\int_{|z|_H<1}\left(-\triangle j^\varepsilon,\sigma_1(j^\varepsilon,\theta^\varepsilon,z)\dot{\tilde{\eta}}_1(t,dz)\right).\end{aligned}$$

注意到

$$((u^\varepsilon\cdot\nabla)j^\varepsilon,\triangle j^\varepsilon)=-\left(\nabla(u^\varepsilon\cdot\nabla)j^\varepsilon,\nabla j^\varepsilon\right)=-\left((\nabla u^\varepsilon\cdot\nabla)j^\varepsilon,\nabla j^\varepsilon\right),$$

由拉迪任斯卡娅不等式可得

$$|((u^\varepsilon\cdot\nabla)j^\varepsilon,\triangle j^\varepsilon)|\leq|\nabla u^\varepsilon|_H\,|\nabla j^\varepsilon|_{L^4}^2\leq|j^\varepsilon|_H\,|\nabla j^\varepsilon|_H\,|\triangle j^\varepsilon|_H\leq C_\gamma\,|j^\varepsilon|_H^2\,|\nabla j^\varepsilon|_H^2+\gamma\,|\triangle j^\varepsilon|_H^2.$$

因此

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\nabla j^\varepsilon|_H^2+|\triangle j^\varepsilon|_H^2\leq&C_\gamma\,|j^\varepsilon|_H^2\,|\nabla j^\varepsilon|_H^2+C_\gamma\left(1+|j^\varepsilon|_H^2+|\theta^\varepsilon|_{H^1}^2\right)+4\gamma\,|\triangle j^\varepsilon|_H^2\\&+C_\gamma\int_{|z|_H<1}|\sigma_1(j^\varepsilon,\theta^\varepsilon,z)|_H^2\,\dot{\tilde{\eta}}_1(t,dz).\end{aligned}$$

取足够小的 γ，由（3.1）、（3.2）可得

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}|\nabla j^\varepsilon|_H^2+\mathbb{E}\int_0^{\tau_M\wedge T}|\triangle j^\varepsilon|_H^2\,ds\\\leq&C\,|\nabla j_0|_H^2+C_M\mathbb{E}\int_0^{\tau_M\wedge T}|\nabla j^\varepsilon|_H^2\,ds+C\mathbb{E}\int_0^{\tau_M\wedge T}\left(1+|j^\varepsilon|_H^2+|\theta^\varepsilon|_{H^1}^2\right)ds\\\leq&C_T\left(1+|j_0|_{H^1}^2+|\theta_0|_H^2\right)+C_M\mathbb{E}\int_0^{\tau_M\wedge T}|\nabla j^\varepsilon|_H^2\,ds.\end{aligned}$$

由格朗沃尔不等式可得

$$\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}|\nabla j^\varepsilon|_H^2+\mathbb{E}\int_0^{\tau_M\wedge T}|\triangle j^\varepsilon|_H^2\,ds\leq C_{T,M}\left(1+|j_0|_{H^1}^2+|\theta_0|_H^2\right).$$

□

接下来，我们将证明涡量变量 j^ε 的紧性。为此，我们需要以下引理:

引理 3.1. 对于任意 $0 \le t-\delta < t \le \tau_M \wedge T$ 和 $(j_0, \theta_0) \in H^1 \times H$，有

$$\mathbb{E}\left|j_t^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^2 \leqslant C_{T,M}\delta^{\frac{1}{2}}. \tag{3.7}$$

此外，对于任意 $(j_0, \theta_0) \in H \times H$，以及某个 $p \ge 1$，有

$$\mathbb{E}\int_0^{\tau_M\wedge T}\left|j_t^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p} dt \leqslant C_{T,M}\delta^{\frac{1}{2}}. \tag{3.8}$$

证明. 对于任意 $0 \le t-\delta < t \le \tau_M \wedge T$，有

$$\begin{aligned} j_t^\varepsilon - j_{t-\delta}^\varepsilon =& -\int_{t-\delta}^t \left(u_s^\varepsilon \cdot \nabla\right) j_s^\varepsilon ds + \int_{t-\delta}^t \left(\triangle j_s^\varepsilon + f(j_s^\varepsilon, \theta_s^\varepsilon) + \partial_x \theta_s^\varepsilon\right) ds \\ &+ \int_{t-\delta}^t \int_{|z|_H<1} \sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z)\tilde{\eta}_1(ds, dz). \end{aligned}$$

由伊藤公式可得

$$\begin{aligned} \left|j_t^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^2 =& -2\int_{t-\delta}^t \left(\left(u_s^\varepsilon \cdot \nabla\right) j_s^\varepsilon, j_s^\varepsilon - j_{t-\delta}^\varepsilon\right) ds + 2\int_{t-\delta}^t \left\langle \triangle j_s^\varepsilon, j_s^\varepsilon - j_{t-\delta}^\varepsilon \right\rangle ds \\ &+ 2\int_{t-\delta}^t \left(f(j_s^\varepsilon, \theta_s^\varepsilon), j_s^\varepsilon - j_{t-\delta}^\varepsilon\right) ds + 2\int_{t-\delta}^t \left(\partial_x \theta_s^\varepsilon, j_s^\varepsilon - j_{t-\delta}^\varepsilon\right) ds \\ &+ 2\int_{t-\delta}^t \int_{|z|_H<1} \left(\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z), j_s^\varepsilon - j_{t-\delta}^\varepsilon\right) \tilde{\eta}_1(ds, dz) \\ &+ \int_{t-\delta}^t \int_{|z|_H<1} \left|\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z)\right|_H^2 \eta_1(ds, dz). \end{aligned} \tag{3.9}$$

因此

$$\begin{aligned} &\mathbb{E}\left|j_t^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^2 + 2\mathbb{E}\int_{t-\delta}^t \left|\nabla j_s^\varepsilon\right|_H^2 ds \\ =& \left\{2\mathbb{E}\int_{t-\delta}^t \left(\left(u_s^\varepsilon \cdot \nabla\right) j_s^\varepsilon, j_{t-\delta}^\varepsilon\right) ds\right\}_{I_1} + \left\{2\mathbb{E}\int_{t-\delta}^t \left(\nabla j_s^\varepsilon, \nabla j_{t-\delta}^\varepsilon\right) ds\right\}_{I_2} \\ &+ \left\{2\mathbb{E}\int_{t-\delta}^t \left(f(j_s^\varepsilon, \theta_s^\varepsilon), j_s^\varepsilon - j_{t-\delta}^\varepsilon\right) ds\right\}_{I_3} - \left\{2\mathbb{E}\int_{t-\delta}^t \left(\theta_s^\varepsilon, \partial_x j_s^\varepsilon - \partial_x j_{t-\delta}^\varepsilon\right) ds\right\}_{I_4} \\ &+ \left\{\mathbb{E}\int_{t-\delta}^t \int_{|z|_H<1} \left|\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z)\right|_H^2 \eta_1(ds, dz)\right\}_{I_5}. \end{aligned}$$

对于 I_1，根据以下有用的不等式（参见 [43]）

$$\left|\left((u\cdot\nabla)v, w\right)\right| \le C\left|u\right|_{H^{s_1}} \left|v\right|_{H^{1+s_2}} \left|w\right|_{H^{s_3}}, \tag{3.10}$$

其中 $s = (s_1, s_2, s_3) \in \mathbb{R}_+^3$、$\sum s_i \ge 1$ 和 $s \ne (1, 0, 0), (0, 1, 0), (0, 0, 1)$。我们有

$$\left|I_1\right| \le C\mathbb{E}\int_{t-\delta}^t \left|u_s^\varepsilon\right|_{H^1} \left|j_s^\varepsilon\right|_{H^1} \left|j_{t-\delta}^\varepsilon\right|_{H^1} ds \le C\mathbb{E}\int_{t-\delta}^t \left|j_s^\varepsilon\right|_H \left|j_s^\varepsilon\right|_{H^1} \left|j_{t-\delta}^\varepsilon\right|_{H^1} ds$$

$$\begin{aligned}
\leq & C_M \left(\mathbb{E} \left(\int_{t-\delta}^{t} |j_s^\varepsilon|_{H^1} \, ds \right)^2 \right)^{\frac{1}{2}} \cdot \left(\mathbb{E} \left| j_{t-\delta}^\varepsilon \right|_{H^1}^2 \right)^{\frac{1}{2}} \\
\leq & C_{T,M} \delta^{\frac{1}{2}}.
\end{aligned}$$

对于 I_2，我们有

$$|I_2| \leq \left(\mathbb{E} \left(\int_{t-\delta}^{t} |\nabla j_s^\varepsilon|_H \, ds \right)^2 \right)^{\frac{1}{2}} \cdot \left(\mathbb{E} \left| \nabla j_{t-\delta}^\varepsilon \right|_H^2 \right)^{\frac{1}{2}} \leq C_{T,M} \delta^{\frac{1}{2}}.$$

对于 I_3，根据假设**1**，容易得到

$$|I_3| \leq \int_{t-\delta}^{t} \left(1 + \mathbb{E} \left| j_s^\varepsilon \right|_H^2 + \mathbb{E} \left| \theta_s^\varepsilon \right|_H^2 + \mathbb{E} \left| j_{t-\delta}^\varepsilon \right|_H^2 \right) ds \leq C_T \delta.$$

对于 I_4，有

$$|I_4| \leq \int_{t-\delta}^{t} \left(\mathbb{E} \left| \theta_s^\varepsilon \right|_H^2 + \mathbb{E} \left| \nabla j_s^\varepsilon \right|_H^2 + \mathbb{E} \left| \nabla j_{t-\delta}^\varepsilon \right|_H^2 \right) ds \leq C_{T,M} \delta.$$

对于 I_5，由假设2可得

$$|I_5| = \mathbb{E} \int_{t-\delta}^{t} \int_{|z|_H<1} \left| \sigma_1((j_s^\varepsilon, \theta_s^\varepsilon, z)) \right|_H^2 d\nu_1(dz) ds \leq C_T \delta.$$

因此，(3.7)成立。

此外，由 (3.1) 可得

$$\begin{aligned}
\mathbb{E} \int_0^{\tau_M \wedge T} \left| j_t^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p} dt = & \mathbb{E} \int_0^{\delta} \left| j_t^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p} dt + \mathbb{E} \int_\delta^{\tau_M \wedge T} \left| j_t^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p} dt \\
\leq & C_T \delta + \mathbb{E} \int_\delta^{\tau_M \wedge T} \left| j_t^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p} dt.
\end{aligned}$$

利用伊藤公式，我们得到

$$\begin{aligned}
& \left| j_t^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p} + 2p \int_{t-\delta}^{t} \left| j_s^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p-2} \left((u_s^\varepsilon \cdot \nabla) j_s^\varepsilon, j_s^\varepsilon - j_{t-\delta}^\varepsilon \right) ds \\
= & 2p \int_{t-\delta}^{t} \left| j_s^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p-2} \left\langle \triangle j_s^\varepsilon, j_s^\varepsilon - j_{t-\delta}^\varepsilon \right\rangle ds + 2p \int_{t-\delta}^{t} \left| j_s^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p-2} \left(f(j_s^\varepsilon, \theta_s^\varepsilon), j_s^\varepsilon - j_{t-\delta}^\varepsilon \right) ds \\
& + 2p \int_{t-\delta}^{t} \left| j_s^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p-2} \left(\partial_x \theta_s^\varepsilon, j_s^\varepsilon - j_{t-\delta}^\varepsilon \right) ds \\
& + 2p \int_{t-\delta}^{t} \int_{|z|_H<1} \left| j_s^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p-2} \left(\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z), j_s^\varepsilon - j_{t-\delta}^\varepsilon \right) \tilde{\eta}_1(ds, dz) \\
& + \int_{t-\delta}^{t} \int_{|z|_H<1} \Big[\left| (j_s^\varepsilon - j_{t-\delta}^\varepsilon) + \sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z) \right|_H^{2p} - \left| j_s^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p} \\
& - 2p \left| j_s^\varepsilon - j_{t-\delta}^\varepsilon \right|_H^{2p-2} \left(j_s^\varepsilon - j_{t-\delta}^\varepsilon, \sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z) \right) \Big] \eta_1(ds, dz).
\end{aligned}$$

根据 (2.1) 和 (3.10)，我们有

$$
\begin{aligned}
&\left|j_t^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p} + 2p\int_{t-\delta}^t \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |\nabla j_s^\varepsilon|_H^2 \, ds \\
\leq & 2p\int_{t-\delta}^t \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |j_s^\varepsilon|_{H^1}^2 \left|j_{t-\delta}^\varepsilon\right|_H ds + 2p\int_{t-\delta}^t \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |\nabla j_s^\varepsilon|_H \left|\nabla j_{t-\delta}^\varepsilon\right|_H ds \\
&+ 2p\int_{t-\delta}^t \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |\theta_s^\varepsilon|_H |\nabla j_s^\varepsilon|_H \, ds + 2p\int_{t-\delta}^t \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |\theta_s^\varepsilon|_H \left|\nabla j_{t-\delta}^\varepsilon\right|_H ds \\
&+ M_p\int_{t-\delta}^t \int_{|z|_H<1} \left(\left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z)|_H^2 + |\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z)|_H^{2p}\right) \eta_1(ds, dz) \\
&+ 2p\int_{t-\delta}^t \int_{|z|_H<1} \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} \left(\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z), j_s^\varepsilon - j_{t-\delta}^\varepsilon\right) \tilde{\eta}_1(ds, dz).
\end{aligned}
$$

由赫尔德不等式可得

$$
\begin{aligned}
&\left|j_t^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p} \\
\leq & 2p\left\{\int_{t-\delta}^t \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |j_s^\varepsilon|_{H^1}^2 \left|j_{t-\delta}^\varepsilon\right|_H ds\right\}_{J_1} + 2p\left\{\int_{t-\delta}^t \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} \left|\nabla j_{t-\delta}^\varepsilon\right|_H^2 ds\right\}_{J_2} \\
&+ 2p\left\{\int_{t-\delta}^t \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |\theta_s^\varepsilon|_H^2 \, ds\right\}_{J_3} \\
&+ M_p\left\{\int_{t-\delta}^t \int_{|z|_H<1} \left(\left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z)|_H^2 + |\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z)|_H^{2p}\right) \eta_1(ds, dz)\right\}_{J_4} \\
&+ 2p\left\{\int_{t-\delta}^t \int_{|z|_H<1} \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} \left(\sigma_1(j_s^\varepsilon, \theta_s^\varepsilon, z), j_s^\varepsilon - j_{t-\delta}^\varepsilon\right) \tilde{\eta}_1(ds, dz)\right\}_{J_5}.
\end{aligned}
$$

对于 J_1，由 (3.1) 和富比尼定理可得

$$
\begin{aligned}
\mathbb{E}\int_\delta^{\tau_M\wedge T} J_1 dt \leq & \mathbb{E}\int_0^{\tau_M\wedge T}\int_s^{s+\delta} \left|j_s^\varepsilon - j_{t-\delta}^\varepsilon\right|_H^{2p-2} |j_s^\varepsilon|_{H^1}^2 \left|j_{t-\delta}^\varepsilon\right|_H dtds \\
\leq & C_M\mathbb{E}\int_0^{\tau_M\wedge T}\int_s^{s+\delta} |j_s^\varepsilon|_H^{2p-2} |j_s^\varepsilon|_{H^1}^2 \, dtds \\
\leq & C_M\delta\mathbb{E}\int_0^{\tau_M\wedge T} |j_s^\varepsilon|_H^{2p-2} |j_s^\varepsilon|_{H^1}^2 \, ds \\
\leq & C_{T,M}\delta.
\end{aligned}
$$

对于 J_2，我们有

$$
\begin{aligned}
\mathbb{E}\int_\delta^{\tau_M\wedge T} J_2 dt \leq & C_M\mathbb{E}\int_\delta^{\tau_M\wedge T}\int_{t-\delta}^t \left|j_{t-\delta}^\varepsilon\right|_H^{2p-2} \left|\nabla j_{t-\delta}^\varepsilon\right|_H^2 dsdt \\
\leq & C_M\delta\mathbb{E}\int_\delta^{\tau_M\wedge T} \left|j_{t-\delta}^\varepsilon\right|_H^{2p-2} \left|\nabla j_{t-\delta}^\varepsilon\right|_H^2 dt \\
\leq & C_{T,M}\delta.
\end{aligned}
$$

对于 J_3，我们有

$$\begin{aligned}\mathbb{E}\int_{\delta}^{\tau_M\wedge T} J_3 dt \leq & C_p \int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\left(1+\mathbb{E}\left|j_s^{\varepsilon}\right|_H^{2p}+\mathbb{E}\left|j_{t-\delta}^{\varepsilon}\right|_H^{2p}+\mathbb{E}\left|\theta_s^{\varepsilon}\right|_H^{2p}\right) dsdt\\ \leq & C_T\delta.\end{aligned}$$

对于 J_4，我们有

$$\begin{aligned}\mathbb{E}\int_{\delta}^{\tau_M\wedge T} J_4 dt \leq & C_p\mathbb{E}\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\left|j_s^{\varepsilon}-j_{t-\delta}^{\varepsilon}\right|_H^{2p} dsdt\\ & + C_p\mathbb{E}\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\int_{|z|_H<1}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^{2p}\nu_1(dz)dsdt\\ \leq & C_p\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\left(1+\mathbb{E}\left|j_s^{\varepsilon}\right|_H^{2p}+\mathbb{E}\left|j_{t-\delta}^{\varepsilon}\right|_H^{2p}+\mathbb{E}\left|\theta_s^{\varepsilon}\right|_H^{2p}\right) dsdt\\ \leq & C_T\delta.\end{aligned}$$

对于 J_5，利用 B-D-G 不等式，我们有

$$\begin{aligned}&\mathbb{E}\int_{\delta}^{\tau_M\wedge T} J_5 dt\\ \leq & C\mathbb{E}\int_{\delta}^{\tau_M\wedge T}\left(\int_{t-\delta}^{t}\int_{|z|_H<1}\left|j_s^{\varepsilon}-j_{t-\delta}^{\varepsilon}\right|_H^{4p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(ds,dz)\right)^{\frac{1}{2}} dt\\ \leq & C_T\mathbb{E}\left(\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\int_{|z|_H<1}\left|j_s^{\varepsilon}-j_{t-\delta}^{\varepsilon}\right|_H^{4p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(ds,dz)dt\right)^{\frac{1}{2}}\\ \leq & C_{p,T}\mathbb{E}\left(\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\int_{|z|_H<1}\left|j_s^{\varepsilon}\right|_H^{4p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(ds,dz)dt\right)^{\frac{1}{2}}\\ & + C_{p,T}\mathbb{E}\left(\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\int_{|z|_H<1}\left|j_{t-\delta}^{\varepsilon}\right|_H^{4p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(ds,dz)dt\right)^{\frac{1}{2}}\\ \leq & C_{p,T}\mathbb{E}\left(\int_{0}^{\tau_M\wedge T}\int_{s}^{s+\delta}\int_{|z|_H<1}\left|j_s^{\varepsilon}\right|_H^{4p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(dt,dz)ds\right)^{\frac{1}{2}}\\ & + C_{p,T}\mathbb{E}\left(\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\int_{|z|_H<1}\left|j_{t-\delta}^{\varepsilon}\right|_H^{4p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(ds,dz)dt\right)^{\frac{1}{2}}\\ \leq & C_{p,T}\mathbb{E}\left(\sup_{s\in[0,\tau_M\wedge T]}\left|j_s^{\varepsilon}\right|_H^{2p}\int_{0}^{\tau_M\wedge T}\int_{s}^{s+\delta}\int_{|z|_H<1}\left|j_s^{\varepsilon}\right|_H^{2p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(dt,dz)ds\right)^{\frac{1}{2}}\\ & + C_{p,T}\mathbb{E}\left(\sup_{t\in[\delta,\tau_M\wedge T]}\left|j_{t-\delta}^{\varepsilon}\right|_H^{2p}\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\int_{|z|_H<1}\left|j_{t-\delta}^{\varepsilon}\right|_H^{2p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(ds,dz)dt\right)^{\frac{1}{2}}\\ \leq & C_{p,T}\left(\mathbb{E}\sup_{s\in[0,\tau_M\wedge T]}\left|j_s^{\varepsilon}\right|_H^{2p}\right)^{\frac{1}{2}}\left(\mathbb{E}\int_{0}^{\tau_M\wedge T}\int_{s}^{s+\delta}\int_{|z|_H<1}\left|j_s^{\varepsilon}\right|_H^{2p-2}\left|\sigma_1(j_s^{\varepsilon},\theta_s^{\varepsilon},z)\right|_H^2\eta_1(dt,dz)ds\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&+C_{p,T}\left(\mathbb{E}\sup_{t\in[\delta,\tau_M\wedge T]}\left|j^{\varepsilon}_{t-\delta}\right|_H^{2p}\right)^{\frac{1}{2}}\left(\mathbb{E}\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\int_{|z|_H<1}\left|j^{\varepsilon}_{t-\delta}\right|_H^{2p-2}|\sigma_1(j^{\varepsilon}_s,\theta^{\varepsilon}_s,z)|_H^2\,\eta_1(ds,dz)dt\right)^{\frac{1}{2}}\\
\leq& C_{p,T}\left(\mathbb{E}\int_0^{\tau_M\wedge T}\int_s^{s+\delta}\left(1+|j^{\varepsilon}_s|_H^{2p}+|\theta^{\varepsilon}_s|_H^{2p}\right)dtds\right)^{\frac{1}{2}}\\
&+C_{p,T}\left(\mathbb{E}\int_{\delta}^{\tau_M\wedge T}\int_{t-\delta}^{t}\left(1+\left|j^{\varepsilon}_{t-\delta}\right|_H^{2p}+|j^{\varepsilon}_s|_H^{2p}+|\theta^{\varepsilon}_s|_H^{2p}\right)dsdt\right)^{\frac{1}{2}}\\
\leq& C_{p,T}\delta^{\frac{1}{2}}.
\end{aligned}
$$

因此，(3.8) 成立。 □

根据 [42, 命题 10], (3.1) 和 (3.7)，可推出如下结果 :

定理 3.4. 对任意 $t\in[0,\tau_M\wedge T]$ 以及 $(j_0,\theta_0)\in H^1\times H$，族 $\{\mathscr{L}(j^{\varepsilon})\}_{\varepsilon\in(0,1]}$ 在 $\mathcal{D}([0,\tau_M\wedge T],H)$ 中是胎紧的，其中 $\mathscr{L}(j^{\varepsilon})$ 表示 j^{ε} 的分布，而 $\mathcal{D}([0,\tau_M\wedge T],H)$表示从 $[0,\tau_M\wedge T]\to H$出发的右连续且有左极限函数空间。

4 冻结方程的遍历性

本节我们关注 (1.2) 中第二个方程的冻结方程。对任意初值 $\theta\in H$ 以及固定的 $u\in H^1$，我们考虑如下方程:

$$
\begin{cases}
\dfrac{\partial\tilde{\theta}_t}{\partial t}+(u\cdot\nabla)\tilde{\theta}_t=\triangle\tilde{\theta}_t+\displaystyle\int_{|z|_H<1}\sigma_2(j,\tilde{\theta}_t,z)\dot{\tilde{\eta}}_2(t,dz),\\
\tilde{\theta}_0=\theta,
\end{cases}
\tag{4.1}
$$

其中 $j:=\nabla^{\perp}\cdot u$，补偿泊松随机测度 $\tilde{\eta}_2(t,x)=\eta_2(t,x)-\nu_2(x)\cdot t$，其强度测度为 ν_2。

容易知道，方程 (4.1) 存在唯一解 $\tilde{\theta}^{u,\theta}_{\cdot,t}$，并且容易验证，若取时间尺度变换 $_{\varepsilon}\,t\to\varepsilon t$ 和 $u=u^{\varepsilon},_{_}$ 则 $\theta_t=\tilde{\theta}^{u^{\varepsilon},\theta_0}_{\cdot,t}$ 是 (1.2) 的第二个方程的解。令 $\{P^u_t\}_{t\geqslant 0}$ 为 $\tilde{\theta}^{u,\theta}_{\cdot,t}$ 的马尔可夫转移半群，即对任意 $\varphi\in\mathcal{B}_b(H)$,

$$
P^u_t\varphi(\theta):=\mathbb{E}\varphi(\tilde{\theta}^{u,\theta}_t),\quad \theta\in H,\quad t\geqslant 0.
$$

对于方程 (4.1)，我们给出以下引理:

引理 4.1. 对任意 $(\theta,u)\in H\times H^1$，以及某个 $p\geq 1$，有

$$
\mathbb{E}\left|\tilde{\theta}^{u,\theta}_t\right|_H^{2p}+\mathbb{E}\int_0^t\left|\tilde{\theta}^{u,\theta}_s\right|_H^{2p-2}\left|\nabla\tilde{\theta}^{u,\theta}_s\right|_H^2ds\leqslant|\theta|_H^{2p}\,e^{-\lambda_{p,\gamma}t}+C_p\left(1+|j|_H^{2p}\right),
\tag{4.2}
$$

其中 $\lambda_{p,\,\gamma}>0$ 已在 *(3.3)* 中出现。此外，对于任意 $\theta_1,\theta_2\in H$,

$$
\mathbb{E}\left|\tilde{\theta}^{u,\theta_1}_t-\tilde{\theta}^{u,\theta_2}_t\right|_H^{2p}\leqslant|\theta_1-\theta_2|_H^{2p}\,e^{-\lambda_p t}.
\tag{4.3}
$$

此外，对于任意 $u_1, u_2 \in H^1$,

$$\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p} \leq C\int_0^t e^{-\lambda_{p,\gamma}(t-s)}\mathbb{E}\left|\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_s^{u_2,\theta}\right|_H^2 ds\cdot|j_1-j_2|_H^2 + C\kappa\left(|j_1-j_2|_H^{2p}\right). \tag{4.4}$$

证明。 根据伊藤公式，我们有

$$\begin{aligned}
&\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}\\
=&|\theta|_H^{2p}+2p\int_0^t\left|\tilde{\theta}_s^{u,\theta}\right|_H^{2(p-1)}\left\langle\triangle\tilde{\theta}_s^{u,\theta},\tilde{\theta}_s^{u,\theta}\right\rangle ds\\
&+2p\int_0^t\int_{|z|_H<1}\left|\tilde{\theta}_s^{u,\theta}\right|_H^{2(p-1)}\left(\tilde{\theta}_s^{u,\theta},\sigma_2(j,\tilde{\theta}_s^{u,\theta},z)\right)\tilde{\eta}_2(ds,dz)\\
&+\int_0^t\int_{|z|_H<1}\left[\left|\tilde{\theta}_s^{u,\theta}+\sigma_2(j,\tilde{\theta}_s^{u,\theta},z)\right|_H^{2p}-\left|\tilde{\theta}_s^{u,\theta}\right|_H^{2p}-2p\left|\tilde{\theta}_s^{u,\theta}\right|_H^{2(p-1)}\left(\tilde{\theta}_s^{u,\theta},\sigma_2(j,\tilde{\theta}_s^{u,\theta},z)\right)\right]\eta_2(ds,dz).
\end{aligned}$$

因此，我们有

$$\begin{aligned}
\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}=&|\theta|_H^{2p}+2p\mathbb{E}\int_0^t\left|\tilde{\theta}_s^{u,\theta}\right|_H^{2(p-1)}\left\langle\triangle\tilde{\theta}_s^{u,\theta},\tilde{\theta}_s^{u,\theta}\right\rangle ds+\mathbb{E}\int_0^t\int_{|z|_H<1}\left[\left|\tilde{\theta}_s^{u,\theta}+\sigma_2(j,\tilde{\theta}_s^{u,\theta},z)\right|_H^{2p}\right.\\
&\left.-\left|\tilde{\theta}_s^{u,\theta}\right|_H^{2p}-2p\left|\tilde{\theta}_s^{u,\theta}\right|_H^{2(p-1)}\left(\tilde{\theta}_s^{u,\theta},\sigma_2(j,\tilde{\theta}_s^{u,\theta},z)\right)\right]\nu_2(dz)ds.
\end{aligned}$$

于是

$$\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}+2p\gamma\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_t^{u,\theta}\right|_H^2\\
\leq&-2p(1-\gamma)\lambda\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}+M_p\mathbb{E}\int_{|z|_H<1}\left(\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2(p-1)}\left|\sigma_2(j,\tilde{\theta}_t^{u,\theta},z)\right|_H^2+\left|\sigma_2(j,\tilde{\theta}_t^{u,\theta},z)\right|_H^{2p}\right)\nu_2(dz)\\
\leq&-2p(1-\gamma)\lambda\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}+M_p\mathbb{E}\int_{|z|_H<1}\left(\frac{p-1}{p}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}+\frac{1}{p}\left|\sigma_2(j,\tilde{\theta}_t^{u,\theta},z)\right|_H^{2p}\right)\nu_2(dz)\\
&+M_pL_{\sigma_2}\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}+M_pC_1\left(1+|j|_H^{2p}\right)\\
\leq&-\lambda_{p,\gamma}\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}+M_pC_1\left(1+|j|_H^{2p}\right).
\end{aligned}$$

利用格朗沃尔不等式，我们有

$$\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}+\mathbb{E}\int_0^t\left|\tilde{\theta}_s^{u,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_s^{u,\theta}\right|_H^2 ds\leqslant|\theta|_H^{2p}e^{-\lambda_{p,\gamma}t}+M_pC_1\left(1+|j|_H^{2p}\right).$$

此外，对于任意 θ_1， $\theta_2\in H$，我们有

$$\begin{aligned}
&\left|\tilde{\theta}_t^{u,\theta_1}-\tilde{\theta}_t^{u,\theta_2}\right|_H^{2p}\\
=&|\theta_1-\theta_2|_H^{2p}+2p\int_0^t\left|\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2}\right|_H^{2p-2}\left\langle\triangle\left(\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2}\right),\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2}\right\rangle ds
\end{aligned}$$

$$
\begin{aligned}
&+2p\int_0^t\int_{|z|_H<1}\left|\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2}\right|_H^{2p-2}\left(\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2},\sigma_2(j,\tilde{\theta}_s^{u,\theta_1},z)-\sigma_2(j,\tilde{\theta}_s^{u,\theta_2},z)\right)\tilde{\eta}_2(ds,dz)\\
&+\int_0^t\int_{|z|_H<1}\left[\left|\left(\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2}\right)+\left(\sigma_2(j,\tilde{\theta}_s^{u,\theta_1},z)-\sigma_2(j,\tilde{\theta}_s^{u,\theta_2},z)\right)\right|_H^{2p}-\left|\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2}\right|_H^{2p}\right.\\
&\left.-2p\left|\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2}\right|_H^{2p-2}\left(\tilde{\theta}_s^{u,\theta_1}-\tilde{\theta}_s^{u,\theta_2},\sigma_2(j,\tilde{\theta}_s^{u,\theta_1},z)-\sigma_2(j,\tilde{\theta}_s^{u,\theta_2},z)\right)\right]\eta_2(ds,dz).
\end{aligned}
$$

根据假设3、4和（2.1），我们有

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left|\tilde{\theta}_t^{u,\theta_1}-\tilde{\theta}_t^{u,\theta_2}\right|_H^{2p}\\
\leq&-2p\lambda\mathbb{E}\left|\tilde{\theta}_t^{u,\theta_1}-\tilde{\theta}_t^{u,\theta_2}\right|_H^{2p}+M_p\mathbb{E}\int_{|z|_H<1}\left(\left|\tilde{\theta}_t^{u,\theta_1}-\tilde{\theta}_t^{u,\theta_2}\right|_H^{2p-2}\left|\sigma_2(j,\tilde{\theta}_t^{u,\theta_1},z)-\sigma_2(j,\tilde{\theta}_t^{u,\theta_2},z)\right|_H^2\right.\\
&\left.+\left|\sigma_2(j,\tilde{\theta}_t^{u,\theta_1},z)-\sigma_2(j,\tilde{\theta}_t^{u,\theta_2},z)\right|_H^{2p}\right)\nu_2(dz)\\
\leq&-\left(2p\lambda-\frac{M_p(p-1)}{p}\right)\mathbb{E}\left|\tilde{\theta}_t^{u,\theta_1}-\tilde{\theta}_t^{u,\theta_2}\right|_H^{2p}+\frac{M_p(p+1)}{p}\mathbb{E}\int_{|z|_H<1}\left|\sigma_2(j,\tilde{\theta}_t^{u,\theta_1},z)-\sigma_2(j,\tilde{\theta}_t^{u,\theta_2},z)\right|_H^{2p}\nu_2(dz)\\
\leq&-\lambda_p\mathbb{E}\left|\tilde{\theta}_t^{u,\theta_1}-\tilde{\theta}_t^{u,\theta_2}\right|_H^{2p}.
\end{aligned}
$$

由格朗沃尔不等式可得

$$
\mathbb{E}\left|\tilde{\theta}_t^{u,\theta_1}-\tilde{\theta}_t^{u,\theta_2}\right|_H^{2p}\leqslant|\theta_1-\theta_2|_H^{2p}\,e^{-\lambda_p t}.
$$

类似地，对于任意 $u_1,u_2\in H^2$，我们有

$$
\begin{aligned}
&\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}\\
=&-2p\int_0^t\left|\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right|_H^{2p-2}\left((u_1\cdot\nabla)\tilde{\theta}_s^{u_1,\theta}-(u_2\cdot\nabla)\tilde{\theta}_s^{u_2,\theta},\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right)ds\\
&+2p\int_0^t\left|\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right|_H^{2p-2}\left\langle\triangle\left(\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right),\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right\rangle ds\\
&+2p\int_0^t\int_{|z|_H<1}\left|\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right|_H^{2p-2}\left(\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta},\sigma_2(j_1,\tilde{\theta}_s^{u_1,\theta},z)-\sigma_2(j_2,\tilde{\theta}_s^{u_2,\theta},z)\right)\tilde{\eta}_2(ds,dz)\\
&+\int_0^t\int_{|z|_H<1}\left[\left|\left(\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right)+\left(\sigma_2(j_1,\tilde{\theta}_s^{u_1,\theta},z)-\sigma_2(j_2,\tilde{\theta}_s^{u_2,\theta},z)\right)\right|_H^{2p}-\left|\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right|_H^{2p}\right.\\
&\left.-2p\left|\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right|_H^{2p-2}\left(\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta},\sigma_2(j_1,\tilde{\theta}_s^{u_1,\theta},z)-\sigma_2(j_2,\tilde{\theta}_s^{u_2,\theta},z)\right)\right]\eta_2(ds,dz).
\end{aligned}
$$

由 (3.10) 可得

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}+2p\gamma\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p-2}\left|\nabla\left(\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right)\right|_H^2\\
\leq&-2p(1-\gamma)\lambda\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}+2p\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p-2}\left(((u_1-u_2)\cdot\nabla)\left(\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right),\tilde{\theta}_t^{u_2,\theta}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+M_p\mathbb{E}\int_{|z|_H<1}\left(\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p-2}\left|\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)-\sigma_2(j_2,\tilde{\theta}_t^{u_2,\theta},z)\right|_H^2\right.\\
&\left.+\left|\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)-\sigma_2(j_2,\tilde{\theta}_t^{u_2,\theta},z)\right|_H^{2p}\right)\nu_2(dz)\\
\leq&-\left(2p(1-\gamma)-\frac{M(p-1)}{p}\right)\lambda\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}+2p\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p-2}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_{H^1}\\
&\cdot\left|\nabla\tilde{\theta}_t^{u_2,\theta}\right|_H|u_1-u_2|_{H^1}+\frac{M_p(p+1)}{p}\mathbb{E}\int_{|z|_H<1}\left|\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)-\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta_2},z)\right|_H^{2p}\nu_2(dz).
\end{aligned}
$$

利用杨氏不等式，并取足够小的 γ，可得

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}\leq&-\left(2p(1-\gamma)-\frac{M(p-1)}{p}-\frac{M_p(p+1)L_{\sigma_2}}{p}\right)\lambda\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}\\
&+C_{p,\gamma}2p\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_t^{u_2,\theta}\right|_H^2|u_1-u_2|_{H^1}^2+\kappa\left(|j_1-j_2|_H^{2p}\right)\\
\leq&-\lambda_{p,\gamma}\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}+C_{p,\gamma}\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p-2}\left|\tilde{\theta}_t^{u_2,\theta}\right|_{H^1}^2|j_1-j_2|_H^2\\
&+C\kappa\left(|j_1-j_2|_H^{2p}\right).
\end{aligned}
$$

由格朗沃尔不等式和 (4.2)，我们有

$$
\begin{aligned}
\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}\leq&C\int_0^t e^{-\lambda_{p,\gamma}(t-s)}\mathbb{E}\left|\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_s^{u_2,\theta}\right|_H^2ds\cdot|j_1-j_2|_H^2\\
&+C\kappa\left(|j_1-j_2|_H^{2p}\right).
\end{aligned}
$$

□

接下来，我们将介绍 $\{P_t^u\}_{t\geqslant 0}$ 的遍历性。我们得到了以下结论：

定理4.1。 对于任意 $u\in H^1$，$\{P_t^u\}_{t\geqslant 0}$ 都有唯一的遍历不变测度 $\pi^u(\cdot)$。此外，

$$
\int_H|\theta|_H^{2p}\,\pi^u(d\theta)\leqslant C\left(1+|j|_H^{2p}\right). \tag{4.5}
$$

证明。 根据经典的 Bogoliubov-Krylov论证，(4.2) 意味着不变测度的存在。此外，设 $\pi^u(\cdot)$ 是 $\{P_t^u\}_{t\geqslant 0}$ 的一个不变测度，我们有

$$
\begin{aligned}
\int_H|\theta|_H^{2p}\,\pi^u(d\theta)=&\int_H\mathbb{E}\left|\tilde{\theta}_t^{u,\theta}\right|_H^{2p}\pi^u(d\theta)\\
\leq&\int_H|\theta|_H^{2p}\,e^{-\lambda_{p,\gamma}t}\pi^u(d\theta)+C_p\left(1+|j|_H^{2p}\right).
\end{aligned}
$$

选取 $t=-\frac{\ln\frac{1}{2}}{\lambda_{p,\gamma}}$，并且 (4.5) 成立。

对于不变测度的唯一性，如 [44], 所示，我们只需证明：对于任意 Lipschitz连续函数 $\varphi:\ H \to \mathbb{R}$，以及任意不变测度 $\pi^u(\cdot)$，下列公式成立：

$$\left| P_t^u \varphi(\theta_1) - \int_H \varphi(\theta_2)\pi^u(d\theta_2) \right| \leqslant C L_\varphi \left(1 + |\theta|_H + |j|_H\right) e^{-\frac{\lambda_p}{p}t}, \tag{4.6}$$

其中 $L_\varphi = \sup_{\theta_1,\theta_2 \in H} \frac{(\theta_1\ -\varphi\ \theta_2\ |\ )()_\circ}{|\theta_1 - \theta_2|_H}$

为此，由 (4.3) 和 (4.5)，我们有

$$\begin{aligned}
\left| P_t^u \varphi(\theta_1) - \int_H \varphi(\theta_2)\pi^u(d\theta_2) \right| &= \left| \int_H P_t^u \varphi(\theta_1) - P_t^u \varphi(\theta_2)\pi^u(d\theta_2) \right| \\
&\leq \int_H \mathbb{E} \left| \varphi(\tilde{\theta}_t^{u,\theta_1}) - \varphi(\tilde{\theta}_t^{u,\theta_2}) \right| \pi^u(d\theta_2) \\
&\leq L_\varphi \int_H \mathbb{E} \left| \tilde{\theta}_t^{u,\theta_1} - \tilde{\theta}_t^{u,\theta_2} \right|_H \pi^u(d\theta_2) \\
&\leq L_\varphi e^{-\frac{\lambda_p}{p}t} \int_H |\theta_1 - \theta_2|_H \, \pi^u(d\theta_2) \\
&\leq C L_\varphi \left(1 + |\theta|_H + |j|_H\right) e^{-\frac{\lambda_p}{p}t}.
\end{aligned}$$

□

5 方程 (1.2) 的平均原理

本节中，我们介绍方程 (1.2) 的平均方程，并证明本文的第一个主要结果。也就是说，对于任意 $T > 0$ and $(j_0, \theta_0) \in H \times H$，慢变量 j^{ε}收敛到以下平均方程的解 $\bar{j}$：

$$\begin{cases} \dfrac{\partial \bar{j}}{\partial t} + (\bar{u} \cdot \nabla)\,\bar{j} = \triangle \bar{j} + \bar{f}(\bar{j}, \bar{u}) + \partial_x g(\bar{u}) + \displaystyle\int_{|z|_H < 1} \bar{\sigma}_1(\bar{j}, \bar{u}, z)\dot{\bar{\eta}}_1(t, dz), \\ \bar{j}(0) = j_0, \end{cases} \tag{5.1}$$

其中 $\bar{j} := \nabla^\perp \cdot \bar{u}$, $g:\ H^1 \to H$ 由下式定义

$$g(u) := \int_H \hat{\theta}\pi^u d(\hat{\theta}),$$

and

$$\bar{f}(j, u) := \int_H f(j, \hat{\theta})\pi^u(d\hat{\theta}), \quad \bar{\sigma}_1(j, u, z) := \int_H \sigma(j, \hat{\theta}, z)\pi^u(d\hat{\theta}).$$

¯为了证明（5.1）的适定性，我们需要 g、 f 和 $\bar{\sigma}_2$ 满足非 Lipschitz 连续性。为此，我们引入以下引理:

引理 5.1. 设 $a > 0$和 $f(t) \geq 0$，若 $F(t) := \int_0^t f(s)ds$关于 t有界，则有

$$\lim_{t \to +\infty} \int_0^t e^{-a(t-s)} f(s)ds = 0.$$

该引理的证明见附录。

接下来，我们推导式 (5.1) 中系数的非 Lipschitz 连续性。我们断言以下性质：

命题 5.1. 在假设 *1-4* 下，对于任意 u_1、$u_2 \in H^1$ 以及某个 $p \geq 1$，有

$$|g(u_1) - g(u_2)|_H^{2p} \leq C_p \kappa \left(|j_1 - j_2|_H^{2p} \right),$$

and

$$\left| \bar{f}(j_1, u_1) - \bar{f}(j_2, u_2) \right|_H^{2p} \leq C_p \tilde{\kappa} \left(|j_1 - j_2|_H^{2p} \right),$$

其中 $\tilde{\kappa}(u) = \kappa(u + C\kappa(u))$ 也是一个凹连续非递减函数。此外，我们有

$$\int_{|z|_H<1} |\bar{\sigma}_1(j_1, u_1, z) - \bar{\sigma}_1(j_2, u_2, z)|_H^{2p} \nu_1(dz) \leq C_p \tilde{\kappa} \left(|j_1 - j_2|_H^{2p} \right).$$

证明。首先，回顾 (4.4) 中不等式右端的第一项，我们有 ave

$$\begin{aligned}
&\int_0^t e^{-\lambda_{p,\gamma}(t-s)} \mathbb{E} \left| \tilde{\theta}_s^{u_1,\theta} - \tilde{\theta}_s^{u_2,\theta} \right|_H^{2p-2} \left| \nabla \tilde{\theta}_s^{u_2,\theta} \right|_H^2 ds \\
\leq & C_p \left\{ \int_0^t e^{-\lambda_{p,\gamma}(t-s)} \mathbb{E} \left| \tilde{\theta}_s^{u_2,\theta} \right|_H^{2p-2} \left| \nabla \tilde{\theta}_s^{u_2,\theta} \right|_H^2 ds \right\}_{K_1} + C_p \left\{ \int_0^t e^{-\lambda_{p,\gamma}(t-s)} \mathbb{E} \left| \tilde{\theta}_s^{u_1,\theta} \right|_H^{2p-2} \left| \nabla \tilde{\theta}_s^{u_2,\theta} \right|_H^2 ds \right\}_{K_2}.
\end{aligned}$$

对于 K_1，根据 (4.2) 和引理 5.1，我们有

$$\lim_{t \to +\infty} \int_0^t e^{-\lambda_{p,\gamma}(t-s)} \mathbb{E} \left| \tilde{\theta}_s^{u_2,\theta} \right|_H^{2p-2} \left| \nabla \tilde{\theta}_s^{u_2,\theta} \right|_H^2 ds = 0. \tag{5.2}$$

对于 K_2，根据伊藤公式，我们有

$$\begin{aligned}
\frac{d}{dt} \left| \tilde{\theta}_t^{u_2,\theta} \right|_H^2 =& 2 \left\langle \triangle \tilde{\theta}_t^{u_2,\theta}, \tilde{\theta}_t^{u_2,\theta} \right\rangle + 2 \int_{|z|_H<1} \left(\tilde{\theta}_t^{u_2,\theta}, \sigma_2(j_2, \tilde{\theta}_t^{u_2,\theta}, z) \right) \dot{\tilde{\eta}}_2(t, dz) \\
&+ \int_{|z|_H<1} \left| \sigma_2(j_2, \tilde{\theta}_t^{u_2,\theta}, z) \right|_H^2 \dot{\eta}_2(t, dz),
\end{aligned}$$

因此

$$\begin{aligned}
\frac{d}{dt} \left| \tilde{\theta}_t^{u_2,\theta} \right|_H^2 =& -2 \left| \nabla \tilde{\theta}_t^{u_2,\theta} \right|_H^2 + 2 \int_{|z|_H<1} \left(\tilde{\theta}_t^{u_2,\theta}, \sigma_2(j_2, \tilde{\theta}_t^{u_2,\theta}, z) \right) \dot{\tilde{\eta}}_2(t, dz) \\
&+ \int_{|z|_H<1} \left| \sigma_2(j_2, \tilde{\theta}_t^{u_2,\theta}, z) \right|_H^2 \dot{\eta}_2(t, dz).
\end{aligned}$$

将上述方程两边同乘以 $\left| \tilde{\theta}^{u_1,}{}_t{}^{\theta} \right|_H^{2p-2}$，可得

$$\left| \tilde{\theta}_t^{u_1,\theta} \right|_H^{2p-2} \frac{d}{dt} \left| \tilde{\theta}_t^{u_2,\theta} \right|_H^2$$

$$
\begin{aligned}
&= -2\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_t^{u_2,\theta}\right|_H^2+2\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left(\tilde{\theta}_t^{u_2,\theta},\sigma_2(j_2,\tilde{\theta}_t^{u_2,\theta},z)\right)\dot{\tilde{\eta}}_2(t,dz)\\
&+\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\sigma_2(j_2,\tilde{\theta}_t^{u_2,\theta},z)\right|_H^2\dot{\eta}_2(t,dz),
\end{aligned}
$$

因此

$$
\begin{aligned}
&\frac{d}{dt}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2\\
&=-2\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_t^{u_2,\theta}\right|_H^2+2\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left(\tilde{\theta}_t^{u_2,\theta},\sigma_2(j_2,\tilde{\theta}_t^{u_2,\theta},z)\right)\dot{\tilde{\eta}}_2(t,dz)\\
&\quad+\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\sigma_2(j_2,\tilde{\theta}_t^{u_2,\theta},z)\right|_H^2\dot{\eta}_2(t,dz)+\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2\frac{d}{dt}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}.
\end{aligned}
$$

注意

$$
\begin{aligned}
&\frac{d}{dt}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\\
=&2(p-1)\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-4}\left\langle\triangle\tilde{\theta}_t^{u_1,\theta},\tilde{\theta}_t^{u_1,\theta}\right\rangle+2(p-1)\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-4}\left(\tilde{\theta}_t^{u_1,\theta},\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)\right)\dot{\tilde{\eta}}_2(t,dz)\\
&+\int_{|z|_H<1}\left[\left|\tilde{\theta}_t^{u_1,\theta}+\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)\right|_H^{2p-2}-\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\right.\\
&\left.-2(p-1)\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2(p-2)}\left(\tilde{\theta}_t^{u_1,\theta},\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)\right)\right]\dot{\eta}_2(t,dz).
\end{aligned}
$$

因此

$$
\begin{aligned}
&\frac{d}{dt}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2\\
&=-2\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_t^{u_2,\theta}\right|_H^2+2\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left(\tilde{\theta}_t^{u_2,\theta},\sigma_2(j_2,\tilde{\theta}_t^{u_2,\theta},z)\right)\dot{\tilde{\eta}}_2(t,dz)\\
&\quad+\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\sigma_2(j_2,\tilde{\theta}_t^{u_2,\theta},z)\right|_H^2\dot{\eta}_2(t,dz)\\
&\quad-2(p-1)\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-4}\left|\nabla\tilde{\theta}_t^{u_1,\theta}\right|_H^2\\
&\quad+2(p-1)\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-4}\left(\tilde{\theta}_t^{u_1,\theta},\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)\right)\dot{\tilde{\eta}}_2(t,dz)\\
&\quad+\int_{|z|_H<1}\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2\left[\left|\tilde{\theta}_t^{u_1,\theta}+\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)\right|_H^{2p-2}-\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\right.\\
&\quad\left.-2(p-1)\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2(p-2)}\left(\tilde{\theta}_t^{u_1,\theta},\sigma_2(j_1,\tilde{\theta}_t^{u_1,\theta},z)\right)\right]\dot{\eta}_2(t,dz).
\end{aligned}
$$

因此

$$
\begin{aligned}
&\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2\\
&=|\theta|_H^{2p}-2\mathbb{E}\int_0^t\left|\tilde{\theta}_s^{u_1,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_s^{u_2,\theta}\right|_H^2ds-2(p-1)\mathbb{E}\int_0^t\left|\tilde{\theta}_s^{u_2,\theta}\right|_H^2\left|\tilde{\theta}_s^{u_1,\theta}\right|_H^{2p-4}\left|\nabla\tilde{\theta}_s^{u_1,\theta}\right|_H^2ds
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^t\int_{|z|_H<1}\left|\tilde{\theta}_s^{u_1,\theta}\right|_H^{2p-2}\left|\sigma_2(j_2,\tilde{\theta}_s^{u_2,\theta},z)\right|_H^2\nu_2(dz)ds\\
&+\mathbb{E}\int_0^t\int_{|z|_H<1}\left|\tilde{\theta}_s^{u_2,\theta}\right|_H^2\left[\left|\tilde{\theta}_s^{u_1,\theta}+\sigma_2(j_1,\tilde{\theta}_s^{u_1,\theta},z)\right|_H^{2p-2}-\left|\tilde{\theta}_s^{u_1,\theta}\right|_H^{2p-2}\right.\\
&\left.-2(p-1)\left|\tilde{\theta}_s^{u_1,\theta}\right|_H^{2(p-2)}\left(\tilde{\theta}_s^{u_1,\theta},\sigma_2(j_1,\tilde{\theta}_s^{u_1,\theta},z)\right)\right]\nu_2(dz)ds.
\end{aligned}
$$

根据假设 3、(2.1) 和 (4.2)，容易验证

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2+\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_t^{u_2,\theta}\right|_H^2\\
\leq&-(2p-1)\lambda\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2+C_p\left(1+|\theta|_H^{2p}+|j_1|_H^{2p}+|j_2|_H^{2p}\right).
\end{aligned}
$$

由格朗沃尔不等式可得

$$
\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p-2}\left|\tilde{\theta}_t^{u_2,\theta}\right|_H^2+\mathbb{E}\int_0^t\left|\tilde{\theta}_s^{u_1,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_s^{u_2,\theta}\right|_H^2ds\leq C_p\left(1+|\theta|_H^{2p}+|j_1|_H^{2p}+|j_2|_H^{2p}\right).
$$

由引理 5.1 可得

$$
\lim_{t\to+\infty}\int_0^t e^{-\lambda_{p,\gamma}(t-s)}\mathbb{E}\left|\tilde{\theta}_s^{u_1,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_s^{u_2,\theta}\right|_H^2ds=0. \tag{5.3}
$$

结合（5.2）与（5.3），可得

$$
\lim_{t\to+\infty}\int_0^t e^{-\lambda_{p,\gamma}(t-s)}\mathbb{E}\left|\tilde{\theta}_s^{u_1,\theta}-\tilde{\theta}_s^{u_2,\theta}\right|_H^{2p-2}\left|\nabla\tilde{\theta}_s^{u_2,\theta}\right|_H^2ds=0.
$$

因此，我们证明了

$$
\lim_{t\to+\infty}\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}\leq C\kappa\left(|j_1-j_2|_H^{2p}\right). \tag{5.4}
$$

接下来，由 (4.6) 可知，对于任意 $u_1,u_2\in H^1$，有

$$
\begin{aligned}
&|g(u_1)-g(u_2)|_H^{2p}\\
=&\left|\int_H\hat{\theta}\pi^{u_1}(d\hat{\theta})-\int_H\hat{\theta}\pi^{u_2}(d\hat{\theta})\right|_H^{2p}\\
\leq&C_p\left|\int_H\hat{\theta}\pi^{u_1}(d\hat{\theta})-P_t^{u_1}\theta\right|_H^{2p}+C_p\left|P_t^{u_1}\theta-P_t^{u_2}\theta\right|_H^{2p}+C_p\left|P_t^{u_2}\theta-\int_H\hat{\theta}\pi^{u_2}(d\hat{\theta})\right|_H^{2p}\\
\leq&C_p\left|\int_H\left(P_t^{u_1}\hat{\theta}-P_t^{u_1}\theta\right)\pi^{u_1}(d\hat{\theta})\right|_H^{2p}+C_p\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}\\
&+C_p\left|\int_H\left(P_t^{u_2}\hat{\theta}-P_t^{u_2}\theta\right)\pi^{u_1}(d\hat{\theta})\right|_H^{2p}\\
\leq&C_p\left(1+|\theta|_H^{2p}+|j_1|_H^{2p}+|j_2|_H^{2p}\right)e^{-\lambda_p t}+C_p\mathbb{E}\left|\tilde{\theta}_t^{u_1,\theta}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}.
\end{aligned}
$$

利用 (5.4)，并令 $t \to +\infty$，可得

$$|g(u_1) - g(u_2)|_H^{2p} \leq C_p \kappa \left(|j_1 - j_2|_H^{2p} \right).$$

随后，由假设1可得

$$\begin{aligned}
&\left|\bar{f}(j_1, u_1) - \bar{f}(j_2, u_2)\right|_H^{2p} \\
=&\left| \int_H f(j_1, \hat{\theta}) \pi^{u_1} d(\hat{\theta}) - \int_H f(j_2, \hat{\theta}) \pi^{u_2}(d\hat{\theta}) \right|_H^{2p} \\
\leq& C_p \left| \int_H f(j_1, \hat{\theta}) \pi^{u_1} d(\hat{\theta}) - P_t^{u_1} f(j_1, \theta) \right|_H^{2p} + C_p \left| P_t^{u_1} f(j_1, \theta) - P_t^{u_2} f(j_2, \theta) \right|_H^{2p} \\
&+ C_p \left| \int_H f(j_2, \hat{\theta}) \pi^{u_2} d(\hat{\theta}) - P_t^{u_2} f(j_2, \theta) \right|_H^{2p} \\
\leq& C_p \left| \int_H \left(P_t^{u_1} f(j_1, \hat{\theta}) - P_t^{u_1} f(j_1, \theta) \right) \pi^{u_1} d(\hat{\theta}) \right|_H^{2p} + C_p \mathbb{E} \left| f(j_1, \tilde{\theta}^{u_1, \theta}) - f(j_2, \tilde{\theta}^{u_2, \theta}) \right|_H^{2p} \\
&+ C_p \left| \int_H \left(P_t^{u_2} f(j_2, \hat{\theta}) - P_t^{u_2} f(j_2, \theta) \right) \pi^{u_2} d(\hat{\theta}) \right|_H^{2p} \\
\leq& C_p \int_H \mathbb{E} \left| f(j_1, \tilde{\theta}_t^{u_1, \hat{\theta}}) - f(j_1, \tilde{\theta}_t^{u_1, \theta}) \right|_H^{2p} \pi^{u_1}(d\hat{\theta}) + C_p \int_H \mathbb{E} \left| f(j_2, \tilde{\theta}_t^{u_2, \hat{\theta}}) - f(j_2, \tilde{\theta}_t^{u_2, \theta}) \right|_H^{2p} \pi^{u_2}(d\hat{\theta}) \\
&+ C_p \kappa \left(|j_1 - j_2|_H^{2p} + \mathbb{E} \left| \tilde{\theta}^{u_1, \theta} - \tilde{\theta}^{u_2, \theta} \right|_H^{2p} \right) \\
\leq& C_p \kappa \left(\int_H \mathbb{E} \left| \tilde{\theta}_t^{u_1, \hat{\theta}} - \tilde{\theta}_t^{u_1, \theta} \right|_H^{2p} \pi^{u_1}(d\hat{\theta}) \right) + C_p \kappa \left(\int_H \mathbb{E} \left| \tilde{\theta}_t^{u_2, \hat{\theta}} - \tilde{\theta}_t^{u_2, \theta} \right|_H^{2p} \pi^{u_2}(d\hat{\theta}) \right) \\
&+ C_p \kappa \left(|j_1 - j_2|_H^{2p} + \mathbb{E} \left| \tilde{\theta}^{u_1, \theta} - \tilde{\theta}^{u_2, \theta} \right|_H^{2p} \right) \\
\leq& C_p \kappa \left(C \left(1 + |\theta|_H^{2p} + |j_1|_H^{2p} + |j_2|_H^{2p} \right) e^{-\lambda_p t} \right) + C_p \kappa \left(|j_1 - j_2|_H^{2p} + \mathbb{E} \left| \tilde{\theta}^{u_1, \theta} - \tilde{\theta}^{u_2, \theta} \right|_H^{2p} \right).
\end{aligned}$$

令 $t \to +\infty$，可得

$$\left|\bar{f}(j_1, u_1) - \bar{f}(j_2, u_2)\right|_H^{2p} \leq C_p \tilde{\kappa} \left(|j_1 - j_2|_H^{2p} \right),$$

其中 $\tilde{\kappa}(u) = \kappa(u + C\kappa(u))$。

最后，根据假设2，我们得到

$$\begin{aligned}
&\int_{|z|_H < 1} |\bar{\sigma}_1(j_1, u_1, z) - \bar{\sigma}_1(j_2, u_2, z)|_H^{2p} \nu_1(dz) \\
=& \int_{|z|_H < 1} \left| \int_H \sigma_1(j_1, \hat{\theta}, z) \pi^{u_1}(d\hat{\theta}) - \int_H \sigma_1(j_2, \hat{\theta}, z) \pi^{u_2}(d\hat{\theta}) \right|_H^{2p} \nu_1(dz) \\
\leq& C_p \int_{|z|_H < 1} \left| \int_H \sigma_1(j_1, \hat{\theta}, z) \pi^{u_1}(d\hat{\theta}) - P_t^{u_1} \sigma(j_1, \theta, z) \right|_H^{2p} \nu_1(dz) \\
&+ C_p \int_{|z|_H < 1} |P_t^{u_1} \sigma_1(j_1, \theta, z) - P_t^{u_2} \sigma_1(j_2, \theta, z)|_H^{2p} \nu_1(dz)
\end{aligned}$$

$$
\begin{aligned}
&+C_p\int_{|z|_H<1}\left|\int_H \sigma_1(j_2,\hat{\theta},z)\pi^{u_2}(d\hat{\theta})-P_t^{u_2}\sigma(j_2,\theta,z)\right|_H^{2p}\nu_1(dz)\\
\leq& C_p\kappa\left(\int_H \mathbb{E}\left|\tilde{\theta}_t^{u_1,\hat{\theta}}-\tilde{\theta}_t^{u_1,\theta}\right|_H^{2p}\pi^{u_1}(d\hat{\theta})\right)+C\kappa\left(\int_H \mathbb{E}\left|\tilde{\theta}_t^{u_2,\hat{\theta}}-\tilde{\theta}_t^{u_2,\theta}\right|_H^{2p}\pi^{u_2}(d\hat{\theta})\right)\\
&+C\kappa\left(|j_1-j_2|_H^{2p}+\mathbb{E}\left|\tilde{\theta}^{u_1,\theta}-\tilde{\theta}^{u_2,\theta}\right|_H^{2p}\right)\\
\leq& C_p\kappa\left(C\left(1+|\theta|_H^{2p}+|j_1|_H^{2p}+|j_2|_H^{2p}\right)e^{-\lambda_p t}\right)+C_p\kappa\left(|j_1-j_2|_H^{2p}+\mathbb{E}\left|\tilde{\theta}^{u_1,\theta}-\tilde{\theta}^{u_2,\theta}\right|_H^{2p}\right).
\end{aligned}
$$

令 $t\to+\infty$，我们得到

$$
\int_{|z|_H<1}|\bar{\sigma}_1(j_1,u_1,z)-\bar{\sigma}_1(j_2,u_2,z)|_H^{2p}\,\nu_1(dz)\leq C_p\tilde{\kappa}\left(|j_1-j_2|_H^{2p}\right).
$$

□

与定理3.2和定理3.3类似，我们直接陈述以下定理。

定理5.1。 对于任意 $j_0\in H$，方程*(5.1)*是适定的。此外，对于任意 $T>0$ 以及某个 $p\geq 1$，我们有

$$
\mathbb{E}\sup_{t\in[0,T]}|\bar{j}|_H^{2p}+\int_0^T\mathbb{E}\,|\bar{j}|_H^{2p-2}\,|\nabla\bar{j}|_H^2\,ds\leq C_T\left(1+|j_0|_H^{2p}\right). \tag{5.5}
$$

5.1 平均原理的弱收敛

在本小节中，我们证明以下结论，这是本文的第一个主要结果。

定理5.2。 假设假设 *1-4* 成立。对于任意 $T>0$、 $\eta>0$ 以及 $(j_0$、 $\theta_0)$ $\in H^1\times H$，我们有

$$
\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,T]}|j^\varepsilon-\bar{j}|_H>\eta\right)=0. \tag{5.6}
$$

我们考虑方程 (1.2) 在弱意义下的解，即对任意测试函数$\phi\in C_0^2(\mathbb{T}^2)$,

$$
\begin{aligned}
(j^\varepsilon,\phi)+\int_0^t\left((u^\varepsilon\cdot\nabla)\,j^\varepsilon,\phi\right)ds=&(j_0,\phi)+\int_0^t\langle j^\varepsilon,\triangle\phi\rangle\,ds+\int_0^t\left(f(j^\varepsilon,\theta^\varepsilon),\phi\right)ds\\
&+\int_0^t(\partial_x\theta^\varepsilon,\phi)\,ds+\int_0^t\int_{|z|_H<1}(\sigma_1(j^\varepsilon,\theta^\varepsilon,z),\phi)\,\tilde{\eta}_1(ds,dz).
\end{aligned}
$$

因此，我们有

$$
\begin{aligned}
(j^\varepsilon,\phi)+\int_0^t\left((u^\varepsilon\cdot\nabla)\,j^\varepsilon,\phi\right)ds=&(j_0,\phi)+\int_0^t\langle j^\varepsilon,\triangle\phi\rangle\,ds+\int_0^t\left(\bar{f}(j^\varepsilon,u^\varepsilon),\phi\right)ds\\
&+\int_0^t(\partial_x g(u^\varepsilon),\phi)\,ds+\int_0^t\int_{|z|_H<1}(\bar{\sigma}_1(j^\varepsilon,u^\varepsilon,z),\phi)\,\tilde{\eta}_1(ds,dz)
\end{aligned}
$$

$$+ R_\phi^\varepsilon(t).$$

其中余项 $R_\phi^\varepsilon(t)$ 为

$$R_\phi^\varepsilon(t) = \left\{\int_0^t (f(j^\varepsilon,\theta^\varepsilon),\phi)\,ds - \int_0^t \left(\bar{f}(j^\varepsilon,u^\varepsilon),\phi\right)ds\right\}_{P_1} + \left\{\int_0^t (\partial_x\theta^\varepsilon,\phi)\,ds - \int_0^t (\partial_x g(u^\varepsilon),\phi)\,ds\right\}_{P_2}$$
$$+ \left\{\int_0^t\int_{|z|_H<1} (\sigma_1(j^\varepsilon,\theta^\varepsilon,z),\phi)\,\tilde{\eta}_1(ds,dz) - \int_0^t\int_{|z|_H<1} (\bar{\sigma}_1(j^\varepsilon,u^\varepsilon,z),\phi)\,\tilde{\eta}_1(ds,dz)\right\}_{P_3}.$$

此外，我们还需要以下引理。

引理 5.2. 在假设 *1-4* 下，对于任意 $T>0$，有

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\left|R_\phi^\varepsilon(t)\right| = 0, \tag{5.7}$$

其中，停时 $\tau_M := \inf\{t\geq 0;\,|j^\varepsilon|_H > M\}$，已在定理 *3.3* 中出现。

证明。 对于 P_1，根据假设1、(4.3) 和 (4.5)，我们有

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}|P_1| \leq & C\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\int_0^t \left|f(j^\varepsilon,\theta^\varepsilon) - \bar{f}(j^\varepsilon,u^\varepsilon)\right|_H ds\\
= & C\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\int_0^t \left|f(j^\varepsilon,\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}}) - \int_H f(j^\varepsilon,\hat{\theta})\pi^{u^\varepsilon}d(\hat{\theta})\right|_H ds\\
= & C\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\int_0^t \left|\int_H f(j^\varepsilon,\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}}) - f(j^\varepsilon,\hat{\theta})\pi^{u^\varepsilon}d(\hat{\theta})\right|_H ds\\
\leq & C\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\int_0^t\int_H \left|f(j^\varepsilon,\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}}) - f(j^\varepsilon,\hat{\theta})\right|_H \pi^{u^\varepsilon}d(\hat{\theta})ds\\
\leq & C\mathbb{E}\int_0^{\tau_M\wedge T}\int_H \kappa\left(\left|\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}} - \hat{\theta}\right|_H\right)\pi^{u^\varepsilon}d(\hat{\theta})ds\\
\leq & C\int_0^{\tau_M\wedge T}\int_H \kappa\left(\mathbb{E}\left|\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}} - \tilde{\theta}^{u^\varepsilon,\hat{\theta}}_{\frac{s}{\varepsilon}}\right|_H\right)\pi^{u^\varepsilon}d(\hat{\theta})ds\\
\leq & C\int_0^{\tau_M\wedge T}\kappa\left(e^{-\frac{\lambda_p s}{2p\varepsilon}}\int_H\left|\theta_0-\hat{\theta}\right|_H\pi^{u^\varepsilon}d(\hat{\theta})\right)ds\\
\leq & C\int_0^{\tau_M\wedge T}\kappa\left(C_M e^{-\frac{\lambda_p s}{2p\varepsilon}}\right)ds\\
\leq & C_T\kappa\left(\frac{1}{\tau_M\wedge T}\int_0^{\tau_M\wedge T}e^{-\frac{\lambda_p s}{2p\varepsilon}}ds\right)\\
\leq & C_T\kappa\left(C_{T,M,p}\varepsilon\right).
\end{aligned}$$

对于 P_2，我们有

$$\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}|P_2| = \mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\left|\int_0^t(\theta^\varepsilon - g(u^\varepsilon),\partial_x\phi)\,ds\right|$$

$$
\begin{aligned}
\leq & C\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\int_0^t |\theta^\varepsilon - g(u^\varepsilon)|_H\, ds \\
= & C\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\int_0^t \left|\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}} - \int_H \hat{\theta}\pi^{u^\varepsilon} d(\hat{\theta})\right|_H ds \\
\leq & C\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\int_0^t \int_H \left|\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}} - \hat{\theta}\right|_H \pi^{u^\varepsilon} d(\hat{\theta}) ds \\
\leq & C\int_0^{\tau_M\wedge T}\int_H \mathbb{E}\left|\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}} - \tilde{\theta}^{u^\varepsilon,\hat{\theta}}_{\frac{s}{\varepsilon}}\right|_H \pi^{u^\varepsilon} d(\hat{\theta}) ds \\
\leq & C\int_0^{\tau_M\wedge T} e^{-\frac{\lambda_p s}{2p\varepsilon}}\int_H \left|\theta_0 - \hat{\theta}\right|_H \pi^{u^\varepsilon} d(\hat{\theta}) ds \\
\leq & C_M\int_0^{\tau_M\wedge T} e^{-\frac{\lambda_p s}{2p\varepsilon}} ds \\
\leq & C_{M,p}\varepsilon.
\end{aligned}
$$

对于 P_3，根据假设3和 B-D-G 不等式，我们有

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}|P_3| \leq & C\mathbb{E}\left(\int_0^{\tau_M\wedge T}\int_{|z|_H<1}|\sigma_1(j^\varepsilon,\theta^\varepsilon,z) - \bar{\sigma}_1(j^\varepsilon,u^\varepsilon,z)|_H^2\,\eta_1(ds,dz)\right)^{\frac{1}{2}} \\
\leq & C\left(\mathbb{E}\int_0^{\tau_M\wedge T}\int_{|z|_H<1}\left|\sigma_1(j^\varepsilon,\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}},z) - \int_H \sigma_1(j^\varepsilon,\hat{\theta},z)\pi^{u^\varepsilon}d(\hat{\theta})\right|_H^2 \eta_1(ds,dz)\right)^{\frac{1}{2}} \\
\leq & C\left(\mathbb{E}\int_0^{\tau_M\wedge T}\int_{|z|_H<1}\int_H\left|\sigma_1(j^\varepsilon,\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}},z) - \sigma_1(j^\varepsilon,\hat{\theta},z)\right|_H^2 \pi^{u^\varepsilon}d(\hat{\theta})\eta_1(ds,dz)\right)^{\frac{1}{2}} \\
= & C\left(\int_0^{\tau_M\wedge T}\int_H\mathbb{E}\int_{|z|_H<1}\left|\sigma_1(j^\varepsilon,\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}},z) - \sigma_1(j^\varepsilon,\tilde{\theta}^{u^\varepsilon,\hat{\theta}}_{\frac{s}{\varepsilon}},z)\right|_H^2 \eta_1(ds,dz)\pi^{u^\varepsilon}d(\hat{\theta})\right)^{\frac{1}{2}} \\
\leq & C\left(\int_0^{\tau_M\wedge T}\int_H \kappa\left(\mathbb{E}\left|\tilde{\theta}^{u^\varepsilon,\theta_0}_{\frac{s}{\varepsilon}} - \tilde{\theta}^{u^\varepsilon,\hat{\theta}}_{\frac{s}{\varepsilon}}\right|_H^2\right)\pi^{u^\varepsilon}d(\hat{\theta})ds\right)^{\frac{1}{2}} \\
\leq & C\left(\int_0^{\tau_M\wedge T}\int_H \kappa\left(e^{-\frac{\lambda_p s}{p\varepsilon}}\left|\theta_0 - \hat{\theta}\right|_H^2\right)\pi^{u^\varepsilon}d(\hat{\theta})ds\right)^{\frac{1}{2}} \\
\leq & C\left(\int_0^{\tau_M\wedge T}\kappa\left(e^{-\frac{\lambda_p s}{p\varepsilon}}\int_H\left|\theta_0 - \hat{\theta}\right|_H^2\pi^{u^\varepsilon}d(\hat{\theta})\right)ds\right)^{\frac{1}{2}} \\
\leq & C\left(\int_0^{\tau_M\wedge T}\kappa\left(C_M e^{-\frac{\lambda_p s}{p\varepsilon}}\right)ds\right)^{\frac{1}{2}} \\
\leq & C_T\left(\kappa\left(\frac{1}{\tau_M\wedge T}\int_0^{\tau_M\wedge T}e^{-\frac{\lambda_p s}{p\varepsilon}}ds\right)\right)^{\frac{1}{2}} \\
\leq & C_T\left[\kappa\left(C_{T,M,p}\varepsilon\right)\right]^{\frac{1}{2}}.
\end{aligned}
$$

令 $\varepsilon \to 0$，我们得到

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{t\in[0,\tau_M\wedge T]}\left|R_{\phi}^{\varepsilon}(t)\right|=0.$$

□

定理5.2的证明。根据定理 3.4，分布族 $\{\mathscr{L}(j^{\varepsilon})\}_{\epsilon\in(0,1]}$ 在 $\mathcal{D}([0,T\wedge\tau_M];\,H)$ 中是紧的。根据斯科罗霍德表示定理，对于任意两个子列 $\left\{\varepsilon_n^1\right\}_{n\in\mathbb{N}}$ 和 $\left\{\varepsilon_n^2\right\}_{n\in\mathbb{N}}$，存在两个序列 $\left\{\hat{}j^{\varepsilon_n^1}\right\}_{n\in\mathbb{N}}$, $\left\{\hat{}j^{\varepsilon_n^2}\right\}_{n\in\mathbb{N}}$ 以及随机过程 $\hat{}j^1,\hat{}j^2\in\mathcal{D}([0,T\wedge\tau_M];\,H)$，定义在某个概率空间 $\left(\hat{}\Omega,\ \hat{}\mathcal{F},\ \hat{}\mathbb{P}\right)$ 上，使得

$$\left\{j^{\varepsilon_n^k}\right\}_{\varepsilon_n^k\in(0,1]}\overset{in\ law}{=\!=\!=}\quad\left\{\hat{j}^{\varepsilon_n^k}\right\}_{n\in\mathbb{N}},\ k=1,2,$$

and

$$\hat{j}^{\varepsilon_n^k}\to\hat{j}^k,\ \hat{\mathbb{P}}-a.s.\ on\ \mathcal{D}\left([0,T\wedge\tau_M];H\right),\ as\ n\to\infty,\ k=1,2.$$

正如 Cerrai 和 Freidlin 所示，[45], (5.7) 表明 $\hat{}j^1$ 和 $\hat{}j^2$ 均满足平均方程 (5.1)。根据方程 (5.1) 解的唯一性，我们有 $\hat{}j^1=\hat{}j^2$。根据 Gyöngy 和 Krylov 的著名论证 [46],，存在某个 $\bar{j}\in\mathcal{D}([0,T\wedge\tau_M],\ H)$，使得

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,\tau_M\wedge T]}|j^{\varepsilon}-\bar{j}_t|_H>\eta\right)=0. \tag{5.8}$$

由切比雪夫不等式、(3.1) 和 (5.5)，可得

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T]}|j^{\varepsilon}-\bar{j}|_H\,\chi\{t>\tau_M\}>\eta\right)\leq&\frac{\mathbb{E}\left(\sup_{t\in[0,T]}|j^{\varepsilon}-\bar{j}|_H\,\chi\{t>\tau_M\}\right)}{\eta}\\ \leq&\frac{1}{\eta}\left(\mathbb{E}\sup_{t\in[0,T]}|j^{\varepsilon}-\bar{j}|_H^2\right)^{\frac{1}{2}}\left(\mathbb{P}\{t>\tau_M\}\right)^{\frac{1}{2}}\\ \leq&\frac{C_T}{\eta}\frac{\left(\mathbb{E}\sup_{t\in[0,T]}|j^{\varepsilon}|_H^{2p}\right)^{\frac{1}{2}}}{M^p}\\ \leq&\frac{C_{T,p}}{\eta}\cdot M^{-p},\end{aligned}$$

其中 $\chi\{\cdot\}$ 是示性函数。

结合（5.8），令 $\varepsilon\to 0$ 首先、 $M\to\infty$ 其次，我们得到

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,T]}|j^{\varepsilon}-\bar{j}|_H>\eta\right)=0.$$

□

5.2 平均原理的强收敛

本节将证明第二个主要结果，即慢变量可以在 p 阶均值意义下强收敛到平均方程（5.1）的解。首先需要强调的是，我们将作出假设 $\sigma_1(j,\ \theta,\ z) \equiv \sigma_1(j,\ z)$，因此 $\bar{\sigma}_1(j,\ u,\ z) \equiv \sigma_1(j,\ z)$。一些反例表明，当慢方程的噪声项依赖于快变量时，强收敛并不成立，参见 [28, 47]。

强收敛论证方法源自 Khasminskii 的 [24] 。特别地，对于任意 $T > 0$ 和 $t \in [k\delta, min\{(k+1)\delta,\ T\}],\quad k = 0,\ 1,\ 2,\ \cdots$，我们构造辅助过程

$$\hat{\theta}_t^\varepsilon + \frac{1}{\varepsilon}\int_{k\delta}^t (u_{k\delta}^\varepsilon \cdot \nabla)\,\hat{\theta}_s^\varepsilon ds = \hat{\theta}_{k\delta}^\varepsilon + \frac{1}{\varepsilon}\int_{k\delta}^t \triangle\hat{\theta}_s^\varepsilon ds + \int_{k\delta}^t\int_{|z|_H<1} \sigma_2(j_{k\delta}^\varepsilon, \hat{\theta}_s^\varepsilon, z)\tilde{\eta}_2^\varepsilon(ds,dz)$$

$\hat{}$，其中 $\theta_0^\varepsilon = \theta_0$。这意味着

$$\hat{\theta}_t^\varepsilon + \frac{1}{\varepsilon}\int_0^t \left(u_{s(\delta)}^\varepsilon \cdot \nabla\right)\hat{\theta}_s^\varepsilon ds = \theta_0 + \frac{1}{\varepsilon}\int_0^t \triangle\hat{\theta}_s^\varepsilon ds + \int_0^t\int_{|z|_H<1} \sigma_2(j_{s(\delta)}^\varepsilon, \hat{\theta}_s^\varepsilon, z)\tilde{\eta}_2^\varepsilon(ds,dz),$$

其中 $s - \delta \le s\ (\delta) := \left[\frac{s}{\delta}\right]\delta < s$，且 $[\cdot]$ 表示下取整函数。此外，我们定义

$$\begin{aligned}\hat{j}_t^\varepsilon + \int_0^t (u_s^\varepsilon \cdot \nabla)\,\hat{j}_s^\varepsilon ds =& j_0 + \int_0^t \triangle\hat{j}_s^\varepsilon ds + \int_0^t f(j_{s(\delta)}^\varepsilon, \hat{\theta}_s^\varepsilon)ds + \int_0^t \partial_x\hat{\theta}_s^\varepsilon ds \\ &+ \int_0^t\int_{|z|_H<1} \sigma_1(j_s^\varepsilon, z)\tilde{\eta}_1(ds,dz),\end{aligned}$$

其中 $\hat{}\ j^\varepsilon := \nabla^\perp \cdot \hat{}u^\varepsilon$。容易得到，对任意 $T>0$，$(j_0,\theta_0) \in H\times H$，且对某个$p \ge 1$,

$$\sup_{t\in[0,T]} \mathbb{E}\left|\hat{\theta}_t^\varepsilon\right|_H^{2p} + \mathbb{E}\int_0^T \left|\hat{\theta}_s^\varepsilon\right|_H^{2p-2}\left|\nabla\hat{\theta}_s^\varepsilon\right|_H^2 ds \le C_T\left(1 + |\theta_0|_H^{2p} + |j_0|_H^{2p}\right), \tag{5.9}$$

and

$$\mathbb{E}\sup_{t\in[0,T]} \left|\hat{j}_t^\varepsilon\right|_H^{2p} + \mathbb{E}\int_0^T \left|\hat{j}_s^\varepsilon\right|_H^{2p-2}\left|\nabla\hat{j}_s^\varepsilon\right|_H^2 ds \le C_T\left(1 + |\theta_0|_H^{2p} + |j_0|_H^{2p}\right). \tag{5.10}$$

我们给出如下引理:

引理 5.3. 在假设 *1-4* 下，对任意 $T>0$，$(j_0,\theta_0) \in H\times H$ 且对某个 $p \ge 1$，我们有

$$\mathbb{E}\int_0^{\tau_{M,L}\wedge T} \left|\hat{\theta}_t^\varepsilon - \theta_t^\varepsilon\right|_H^{2p} dt \le \mathcal{O}(\delta), \tag{5.11}$$

其中 $\tau_{M,L} := \tau_M \wedge \inf\left\{t\ge 0;\ |\theta_t^\varepsilon|_H + \left|\hat{\theta}_t^\varepsilon\right|_H > L\right\}$ 且 $\mathcal{O}(\delta) := C_{p,T}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}\right) + C_{p,T,M,L}\delta^{\frac{1}{2p}}$.

证明。 根据伊藤公式，我们有

$$
\begin{aligned}
&\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p}+\frac{2p}{\varepsilon}\int_0^t\left|\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right|_H^{2p-2}\left(\left(u_{s(\delta)}^\varepsilon\cdot\nabla\right)\hat{\theta}_s^\varepsilon-\left(u_s^\varepsilon\cdot\nabla\right)\theta_s^\varepsilon,\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right)ds\\
=&\frac{2p}{\varepsilon}\int_0^t\left|\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right|_H^{2p-2}\left\langle\triangle\left(\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right),\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right\rangle ds\\
&+2p\int_0^t\int_{|z|_H<1}\left|\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right|_H^{2p-2}\left(\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon,\sigma_2(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon,z)-\sigma_2(j_s^\varepsilon,\theta_s^\varepsilon,z)\right)\tilde{\eta}_2^\varepsilon(ds,dz)\\
&+\int_0^t\int_{|z|_H<1}\left[\left|\left(\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right)+\left(\sigma_2(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon,z)-\sigma_2(j_s^\varepsilon,\theta_s^\varepsilon,z)\right)\right|_H^{2p}-\left|\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right|_H^{2p}\right.\\
&\left.-2p\left|\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right|_H^{2p-2}\left(\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon,\sigma_2(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon,z)-\sigma_2(j_s^\varepsilon,\theta_s^\varepsilon,z)\right)\right]\eta_2^\varepsilon(ds,dz).
\end{aligned}
$$

由 (3.10) 和杨氏不等式可得

$$
\begin{aligned}
&\frac{d}{dt}\mathbb{E}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p}+\frac{p\gamma}{\varepsilon}\mathbb{E}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p-2}\left|\nabla\left(\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right)\right|_H^2\\
\leq&-\frac{1}{\varepsilon}\left(2p\lambda\left(1-\frac{\gamma}{2}\right)-\frac{M_p(p-1)}{p}\right)\mathbb{E}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p}+\frac{2p}{\varepsilon}\mathbb{E}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p-2}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_{H^1}|\theta_t^\varepsilon|_{H^1}\\
&+\frac{M_p(p+1)}{\varepsilon p}\mathbb{E}\int_{|z|_H<1}\left|\sigma_2(j_{t(\delta)}^\varepsilon,\hat{\theta}_t^\varepsilon,z)-\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z)\right|_H^{2p}\nu_2(dz),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p}\leq&-\frac{\lambda_{p,\gamma}}{\varepsilon}\mathbb{E}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p}+\frac{C_p}{\varepsilon}\kappa\left(\mathbb{E}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^{2p}\right)\\
&+\frac{C_p}{\varepsilon}\mathbb{E}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p-2}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\nabla\theta_t^\varepsilon|_H^2
\end{aligned}
$$

其中 γ 足够小。利用格朗沃尔不等式，可得

$$
\begin{aligned}
\mathbb{E}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p}\leq&\frac{C_p}{\varepsilon}\int_0^t e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}(t-s)}\kappa\left(\mathbb{E}\left|j_s^\varepsilon-j_{s(\delta)}^\varepsilon\right|_H^{2p}\right)ds\\
&+\frac{C_p}{\varepsilon}\int_0^t e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}(t-s)}\mathbb{E}\left|\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right|_H^{2p-2}\left|j_s^\varepsilon-j_{s(\delta)}^\varepsilon\right|_H^2|\nabla\theta_s^\varepsilon|_H^2\,ds.
\end{aligned}
$$

根据杨卷积不等式和（3.8），我们有

$$
\begin{aligned}
&\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p}dt\\
\leq&\frac{C_p}{\varepsilon}\int_0^{\tau_{M,L}\wedge T}\int_0^t e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}(t-s)}\kappa\left(\mathbb{E}\left|j_s^\varepsilon-j_{s(\delta)}^\varepsilon\right|_H^{2p}\right)dsdt\\
&+\frac{C_p}{\varepsilon}\int_0^{\tau_{M,L}\wedge T}\int_0^t e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}(t-s)}\mathbb{E}\left|\hat{\theta}_s^\varepsilon-\theta_s^\varepsilon\right|_H^{2p-2}\left|j_s^\varepsilon-j_{s(\delta)}^\varepsilon\right|_H^2|\nabla\theta_s^\varepsilon|_H^2\,dsdt\\
\leq&\frac{C_p}{\varepsilon}\int_0^{\tau_{M,L}\wedge T}e^{-\frac{\lambda_{p,\gamma}}{\varepsilon}t}dt\cdot\left(\int_0^{\tau_{M,L}\wedge T}\kappa\left(\mathbb{E}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^{2p}\right)dt\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p-2}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\nabla\theta_t^\varepsilon|_H^2\,dt\Big)\\
\leq& C_{p,T}\kappa\left(\frac{1}{\tau_{M,L}\wedge T}\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^{2p}dt\right)+C_{p,T,L}\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\nabla\theta_t^\varepsilon|_H^2\,dt\\
\leq& C_{p,T}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}\right)+C_{p,T,L}\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\nabla\theta_t^\varepsilon|_H^2\,dt.
\end{aligned}
$$

接下来，我们将处理上述不等式右侧的最后一项。具体而言，由于

$$
\frac{d}{dt}|\theta_t^\varepsilon|_H^2+\frac{2}{\varepsilon}|\nabla\theta_t^\varepsilon|_H^2=2\int_{|z|_H<1}(\theta_t^\varepsilon,\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z))\,\dot{\tilde{\eta}}_2(t,dz)+\int_{|z|_H<1}|\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z)|_H^2\,\dot{\eta}_2(\frac{1}{\varepsilon}t,dz).
$$

容易得到

$$
\left\langle|\theta_t^\varepsilon|_H^2\right\rangle_t=\int_0^t\int_{|z|_H<1}\left(2\left(\theta_s^\varepsilon,\sigma_2(j_s^\varepsilon,\theta_s^\varepsilon,z)\right)+|\sigma_2(j_s^\varepsilon,\theta_s^\varepsilon,z)|_H^2\right)^2\eta_2(\frac{1}{\varepsilon}ds,dz),
$$

其中 $\langle\cdot\rangle_t$ 表示二次变差，并且

$$
\begin{aligned}
&\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2\frac{d}{dt}|\theta_t^\varepsilon|_H^2+\frac{2}{\varepsilon}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\nabla\theta_t^\varepsilon|_H^2\\
=&2\int_{|z|_H<1}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2(\theta_t^\varepsilon,\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z))\,\dot{\tilde{\eta}}_2(t,dz)+\int_{|z|_H<1}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z)|_H^2\,\dot{\eta}_2(\frac{1}{\varepsilon}t,dz).
\end{aligned}
$$

因此，由 B-D-G 不等式可得

$$
\begin{aligned}
&\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\nabla\theta_t^\varepsilon|_H^2\,dt\\
=&-\frac{\varepsilon}{2}\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2d\,|\theta_t^\varepsilon|_H^2+\varepsilon\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\int_{|z|_H<1}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z)|_H^2\,\eta_2(\frac{1}{\varepsilon}dt,dz)\\
\leq& C\varepsilon\mathbb{E}\left(\int_0^{\tau_{M,L}\wedge T}\int_{|z|_H<1}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^4\left(2\left(\theta_t^\varepsilon,\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z)\right)+|\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z)|_H^2\right)^2\eta_2(\frac{1}{\varepsilon}dt,dz)\right)^{\frac{1}{2}}\\
&+\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\int_{|z|_H<1}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2|\sigma_2(j_t^\varepsilon,\theta_t^\varepsilon,z)|_H^2\,\nu_2(dz)dt\\
\leq& C_{M,L}\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_t^\varepsilon-j_{t(\delta)}^\varepsilon\right|_H^2dt\\
\leq& C_{T,M,L}\delta^{\frac{1}{2p}}.
\end{aligned}
$$

因此

$$
\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|\hat{\theta}_t^\varepsilon-\theta_t^\varepsilon\right|_H^{2p}dt\leq C_{p,T}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}\right)+C_{p,T,M,L}\delta^{\frac{1}{2p}}.
$$

□

此外，我们还可以得到如下引理：

引理 5.4. 设 $\sigma_1(j,\theta,z)\equiv\sigma_1(j,z)$，对于任意 $T>0$，$(j_0,\theta_0)\in H\times H$ 以及某个 $p\geq 1$，有

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_{M,L}\wedge T]}\left|j_t^\varepsilon-\hat{j}_t^\varepsilon\right|_H^{2p}+\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left|\nabla\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right)\right|_H^2ds\\&\leq\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right).\end{aligned}\tag{5.12}$$

证明。由于

$$\begin{aligned}\left(j_t^\varepsilon-\hat{j}_t^\varepsilon\right)+\int_0^t(u_s^\varepsilon\cdot\nabla)\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right)=&\int_0^t\triangle\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right)ds\\&+\int_0^t f(j_s^\varepsilon,\theta_s^\varepsilon)-f(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon)ds+\int_0^t\partial_x\theta_s^\varepsilon-\partial_x\hat{\theta}_s^\varepsilon ds,\end{aligned}$$

我们有

$$\begin{aligned}\left|j_t^\varepsilon-\hat{j}_t^\varepsilon\right|_H^{2p}=&2p\int_0^t\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left\langle\triangle\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right),j_s^\varepsilon-\hat{j}_s^\varepsilon\right\rangle ds\\&+2p\int_0^t\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left(f(j_s^\varepsilon,\theta_s^\varepsilon)-f(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon),j_s^\varepsilon-\hat{j}_s^\varepsilon\right)ds\\&+2p\int_0^t\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left(\partial_x\theta_s^\varepsilon-\partial_x\hat{\theta}_s^\varepsilon,j_s^\varepsilon-\hat{j}_s^\varepsilon\right)ds,\end{aligned}$$

and

$$\begin{aligned}&\left|j_t^\varepsilon-\hat{j}_t^\varepsilon\right|_H^{2p}+2p\int_0^t\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left|\nabla\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right)\right|_H^2ds\\=&2p\int_0^t\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left(f(j_s^\varepsilon,\theta_s^\varepsilon)-f(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon),j_s^\varepsilon-\hat{j}_s^\varepsilon\right)ds\\&-2p\int_0^t\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left(\theta_s^\varepsilon-\hat{\theta}_s^\varepsilon,\partial_x j_s^\varepsilon-\partial_x\hat{j}_s^\varepsilon\right)ds.\end{aligned}$$

因此，利用杨氏不等式，我们有

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_{M,L}\wedge T]}\left|j_t^\varepsilon-\hat{j}_t^\varepsilon\right|_H^{2p}+2p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left|\nabla\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right)\right|_H^2ds\\\leq&C_p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p}ds+C_p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|f(j_s^\varepsilon,\theta_s^\varepsilon)-f(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon)\right|_H^{2p}ds\\&+p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left|\nabla\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right)\right|_H^2ds+C_p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|\theta_s^\varepsilon-\hat{\theta}_s^\varepsilon\right|_H^{2p}ds.\end{aligned}$$

根据（3.8）和（5.11），可得

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_{M,L}\wedge T]}\left|j_t^\varepsilon-\hat{j}_t^\varepsilon\right|_H^{2p}+p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left|\nabla\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right)\right|_H^2ds\\\leq&C_p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p}ds+C_p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|\theta_s^\varepsilon-\hat{\theta}_s^\varepsilon\right|_H^{2p}ds\end{aligned}$$

$$
\begin{aligned}
&+C_p\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\kappa\left(\left|j_s^\varepsilon-j_{s(\delta)}^\varepsilon\right|_H^{2p}+\left|\theta_s^\varepsilon-\hat{\theta}_s^\varepsilon\right|_H^{2p}\right)ds\\
\leq& C_p\int_0^{\tau_{M,L}\wedge T}\mathbb{E}\sup_{r\in[0,\tau_{M,L}\wedge s]}\left|j_r^\varepsilon-\hat{j}_r^\varepsilon\right|_H^{2p}ds+\mathcal{O}_{p,T,M,L}(\delta)\\
&+C_{p,T}\kappa\left(\frac{1}{\tau_{M,L}\wedge T}\int_0^{\tau_{M,L}\wedge T}\left|j_s^\varepsilon-j_{s(\delta)}^\varepsilon\right|_H^{2p}+\left|\theta_s^\varepsilon-\hat{\theta}_s^\varepsilon\right|_H^{2p}ds\right)\\
\leq& C_p\int_0^{\tau_{M,L}\wedge T}\mathbb{E}\sup_{r\in[0,\tau_{M,L}\wedge s]}\left|j_r^\varepsilon-\hat{j}_r^\varepsilon\right|_H^{2p}ds+\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right).
\end{aligned}
$$

格朗沃尔不等式表明

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,\tau_{M,L}\wedge T]}\left|j_t^\varepsilon-\hat{j}_t^\varepsilon\right|_H^{2p}+\mathbb{E}\int_0^{\tau_{M,L}\wedge T}\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_H^{2p-2}\left|\nabla\left(j_s^\varepsilon-\hat{j}_s^\varepsilon\right)\right|_H^2ds\\
\leq&\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right).
\end{aligned}
$$

□

接下来，我们给出 $j_t^\varepsilon-j_t$ 的估计，即以下引理:

引理5.5。 令 $\sigma_1(j,\theta,z)\equiv\sigma_1(j,z)$，对于任意 $T>0$，$(j_0,\theta_0)\in H\times H$ 且对于某个 $p\geq 1$，有

$$
\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\left|\hat{j}_t^\varepsilon-\bar{j}_t\right|_H^{2p}\leq\hat{\Omega}^{-1}\left(\hat{\Omega}\left(\mathcal{R}(\varepsilon,\delta)\right)+C_{p,T,N}T\right),\tag{5.13}
$$

其中 $\tau_{M,\ L,\ N}:=\tau_{M,\ L}\wedge$ 下确界 $\left\{t\geq 0;\quad\left|\hat{j}_t^\varepsilon\right|_H+|\bar{j}_t|_H>N\right\}$。 $\mathcal{R}(\varepsilon,\delta)$ 和 $\Omega(\cdot)$的定义见式*(5.14)*和*(5.15)*。

证明。 由伊藤公式可得

$$
\begin{aligned}
&\left|\hat{j}_t^\varepsilon-\bar{j}_t\right|_H^{2p}+2p\int_0^t\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p-2}\left(\left(u_s^\varepsilon\cdot\nabla\right)\hat{j}_s^\varepsilon-\left(\bar{u}_s\cdot\nabla\right)\bar{j}_s,\hat{j}_s^\varepsilon-\bar{j}_s\right)ds\\
=&2p\int_0^t\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p-2}\left\langle\triangle\left(\hat{j}_s^\varepsilon-\bar{j}_s\right),\hat{j}_s^\varepsilon-\bar{j}_s\right\rangle ds\\
&+2p\int_0^t\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p-2}\left(f(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon)-\bar{f}(\bar{j}_s,\bar{u}_s),\hat{j}_s^\varepsilon-\bar{j}_s\right)ds\\
&+2p\int_0^t\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p-2}\left(\partial_x\hat{\theta}_s^\varepsilon-\partial_x g(\bar{u}_s),\hat{j}_s^\varepsilon-\bar{j}_s\right)ds\\
&+2p\int_0^t\int_{|z|_H<1}\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p-2}\left(\sigma_1(j_s^\varepsilon,z)-\sigma_1(\bar{j}_s,z),\hat{j}_s^\varepsilon-\bar{j}_s\right)\tilde{\eta}_1(ds,dz)\\
&+\int_0^t\int_{|z|_H<1}\Bigg[\left|(\hat{j}_s^\varepsilon-\bar{j}_s)+(\sigma_1(j_s^\varepsilon,z)-\sigma_1(\bar{j}_s,z))\right|_H^{2p}-\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p}\\
&-2p\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p-2}\left(\sigma_1(j_s^\varepsilon,z)-\sigma_1(\bar{j}_s,z),\hat{j}_s^\varepsilon-\bar{j}_s\right)\Bigg]\eta_1(ds,dz).
\end{aligned}
$$

因此

$$
\begin{aligned}
&\left|\hat{j}_t^{\varepsilon}-\bar{j}_t\right|_H^{2p}+2p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left|\nabla(\hat{j}_s^{\varepsilon}-\bar{j}_s)\right|_H^2 ds\\
\leq&2p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left(((u_s^{\varepsilon}-\hat{u}_s^{\varepsilon})\cdot\nabla)\left(\hat{j}_s^{\varepsilon}-\bar{j}_s\right),\bar{j}_s\right)ds\\
&+2p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left(((\hat{u}_s^{\varepsilon}-\bar{u}_s)\cdot\nabla)\left(\hat{j}_s^{\varepsilon}-\bar{j}_s\right),\bar{j}_s\right)ds\\
&+C_p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}ds+C_p\int_0^t\left|f(j_{s(\delta)}^{\varepsilon},\hat{\theta}_s^{\varepsilon})-\bar{f}(\bar{j}_s,\bar{u}_s)\right|_H^{2p}ds\\
&+p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left|\nabla(\hat{j}_s^{\varepsilon}-\bar{j}_s)\right|_H^2 ds+C_p\int_0^t\left|\hat{\theta}_s^{\varepsilon}-g(\bar{u}_s)\right|_H^{2p}ds\\
&+2p\int_0^t\int_{|z|_H<1}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left(\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z),\hat{j}_s^{\varepsilon}-\bar{j}_s\right)\tilde{\eta}_1(ds,dz)\\
&+M_p\int_0^t\int_{|z|_H<1}\left(\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}|\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z)|_H^2+\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\right)\eta_1(ds,dz).
\end{aligned}
$$

根据 (3.10) 和杨氏不等式，我们有

$$
\begin{aligned}
&\left|\hat{j}_t^{\varepsilon}-\bar{j}_t\right|_H^{2p}+p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left|\nabla(\hat{j}_s^{\varepsilon}-\bar{j}_s)\right|_H^2 ds\\
\leq&2p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left|j_s^{\varepsilon}-\hat{j}_s^{\varepsilon}\right|_{H^1}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_{H^1}|\bar{j}_s|_H\,ds\\
&+2p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_{H^1}|\bar{j}_s|_{H^1}\,ds\\
&+C_p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}ds+C_p\int_0^t\left|f(j_{s(\delta)}^{\varepsilon},\hat{\theta}_s^{\varepsilon})-\bar{f}(\bar{j}_s,\bar{u}_s)\right|_H^{2p}ds+C_p\int_0^t\left|\hat{\theta}_s^{\varepsilon}-g(\bar{u}_s)\right|_H^{2p}ds\\
&+2p\int_0^t\int_{|z|_H<1}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left(\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z),\hat{j}_s^{\varepsilon}-\bar{j}_s\right)\tilde{\eta}_1(ds,dz)\\
&+C_p\int_0^t\int_{|z|_H<1}|\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z)|_H^{2p}\,\eta_1(ds,dz)\\
\leq&2p\gamma\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_{H^1}^2 ds+C_{p,\gamma}\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left|j_s^{\varepsilon}-\hat{j}_s^{\varepsilon}\right|_{H^1}^2|\bar{j}_s|_H^2\,ds\\
&+C_{p,\gamma}\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}|\bar{j}_s|_{H^1}^2\,ds+C_p\int_0^t\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}ds\\
&+C_p\int_0^t\left|f(j_{s(\delta)}^{\varepsilon},\hat{\theta}_s^{\varepsilon})-\bar{f}(\bar{j}_s,\bar{u}_s)\right|_H^{2p}ds+C_p\int_0^t\left|\hat{\theta}_s^{\varepsilon}-g(\bar{u}_s)\right|_H^{2p}ds\\
&+2p\int_0^t\int_{|z|_H<1}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left(\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z),\hat{j}_s^{\varepsilon}-\bar{j}_s\right)\tilde{\eta}_1(ds,dz)\\
&+C_p\int_0^t\int_{|z|_H<1}|\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z)|_H^{2p}\,\eta_1(ds,dz).
\end{aligned}
$$

选取 γ 足够小，我们得到

$$\begin{aligned}\left|\hat{j}_t^\varepsilon-\bar{j}_t\right|_H^{2p}\leq & C_p\int_0^t\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p}ds+C_p\left\{\int_0^t\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p-2}\left|j_s^\varepsilon-\hat{j}_s^\varepsilon\right|_{H^1}^2|\bar{j}_s|_H^2\,ds\right\}_{Q_1}\\&+C_p\left\{\int_0^t\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p}|\bar{j}_s|_{H^1}^2\,ds\right\}_{Q_2}+C_p\left\{\int_0^t\left|f(j_{s(\delta)}^\varepsilon,\hat{\theta}_s^\varepsilon)-\bar{f}(\bar{j}_s,\bar{u}_s)\right|_H^{2p}ds\right\}_{Q_3}\\&+C_p\left\{\int_0^t\left|\hat{\theta}_s^\varepsilon-g(\bar{u}_s)\right|_H^{2p}ds\right\}_{Q_4}+C_p\left\{\int_0^t\int_{|z|_H<1}|\sigma_1(j_s^\varepsilon,z)-\sigma_1(\bar{j}_s,z)|_H^{2p}\,\eta_1(ds,dz)\right\}_{Q_5}\\&+2p\left\{\int_0^t\int_{|z|_H<1}\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p-2}\left(\sigma_1(j_s^\varepsilon,z)-\sigma_1(\bar{j}_s,z),\hat{j}_s^\varepsilon-\bar{j}_s\right)\tilde{\eta}_1(ds,dz)\right\}_{Q_6}.\end{aligned}$$

我们将逐一讨论 Q_1 至 Q_6。具体而言，对于 Q_1，根据 (5.12)，我们有

$$\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}Q_1\leq C_{p,T,N}\mathcal{O}(\delta)+C_{p,T,M,N}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right).$$

对于 Q_2，注意到

$$\begin{aligned}\frac{d}{dt}|\bar{j}_t|_H^2+2|\nabla\bar{j}_t|_H^2=&2\left(\bar{f}(\bar{j}_t,\bar{u}_t),\bar{j}_t\right)+2\left(\partial_x g(\bar{u}),\bar{j}_t\right)+2\int_{|z|_H<1}(\bar{j}_t,\sigma_1(\bar{j}_t,z))\,\dot{\tilde{\eta}}_1(t,dz)\\&+\int_{|z|_H<1}|\sigma_1(\bar{j}_t,z)|_H^2\,\dot{\eta}_1(t,dz),\end{aligned}$$

我们有

$$\left\langle|\bar{j}_t|_H^2\right\rangle_t=\int_0^t\int_{|z|_H<1}\left(2\left(\bar{j}_t,\sigma_1(\bar{j}_t,z)\right)+|\sigma_1(\bar{j}_t,z)|_H^2\right)^2\eta_1(ds,dz),$$

and

$$\begin{aligned}\frac{d}{dt}|\bar{j}_t|_H^2+|\nabla\bar{j}_t|_H^2\leq&C\left(1+|\bar{j}_t|_H^2\right)+2\int_{|z|_H<1}(\bar{j}_t,\sigma_1(\bar{j}_t,z))\,\dot{\tilde{\eta}}_1(t,dz)\\&+\int_{|z|_H<1}|\sigma_1(\bar{j}_t,z)|_H^2\,\dot{\eta}_1(t,dz),\end{aligned}$$

因此

$$\begin{aligned}&\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p}|\nabla\bar{j}_s|_H^2\,ds\\\leq&C\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\left(1+|\bar{j}_s|_H^2\right)\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p}ds\\&+2\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\int_{|z|_H<1}(\bar{j}_s,\sigma_1(\bar{j}_s,z))\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p}\tilde{\eta}_1(ds,dz)\\&+\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\int_{|z|_H<1}|\sigma_1(\bar{j}_s,z)|_H^2\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p}\eta_1(ds,dz)-\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\left|\hat{j}_s^\varepsilon-\bar{j}_s\right|_H^{2p}d\,|\bar{j}_s|_H^2\end{aligned}$$

$$
\begin{aligned}
\leq & C_N \mathbb{E} \int_0^{\tau_{M,L,N} \wedge T} \left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{2p} ds + C \mathbb{E} \left(\int_0^{\tau_{M,L,N} \wedge T} \int_{|z|_H < 1} \left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{4p} |\bar{j}_s|_H^2 \, |\sigma_1(\bar{j}_s, z)|_H^2 \, \eta_1(ds, dz) \right)^{\frac{1}{2}} \\
& + C \mathbb{E} \left(\int_0^{\tau_{M,L,N} \wedge T} \int_{|z|_H < 1} \left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{4p} d \left\langle |\bar{j}_s|_H^2 \right\rangle_s \right)^{\frac{1}{2}} \\
\leq & C_N \mathbb{E} \int_0^{\tau_{M,L,N} \wedge T} \left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{2p} ds \\
& + C \mathbb{E} \left(\int_0^{\tau_{M,L,N} \wedge T} \int_{|z|_H < 1} \left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{4p} \left(2 \left(\bar{j}_s, \sigma_1(\bar{j}_s, z) \right) + |\sigma_1(\bar{j}_s, z)|_H^2 \right)^2 \eta_1(ds, dz) \right)^{\frac{1}{2}} \\
\leq & C_N \mathbb{E} \int_0^{\tau_{M,L,N} \wedge T} \left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{2p} ds.
\end{aligned}
$$

因此

$$
\mathbb{E} \sup_{t \in [0, \tau_{M,L,N} \wedge T]} Q_2 \leq C_N \mathbb{E} \int_0^{\tau_{M,L,N} \wedge T} \left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{2p} ds.
$$

对于 Q_3，根据命题 5.1、(3.8) 和 (5.12)，我们有

$$
\begin{aligned}
\mathbb{E} \sup_{t \in [0, \tau_{M,L,N} \wedge T]} Q_3 \leq & C_p \mathbb{E} \sup_{t \in [0, \tau_{M,L,N} \wedge T]} \int_0^t \left| f(j_{s(\delta)}^{\varepsilon}, \hat{\theta}_s^{\varepsilon}) - \bar{f}(j_{s(\delta)}^{\varepsilon}, u_{s(\delta)}^{\varepsilon}) \right|_H^{2p} ds \\
& + C_p \mathbb{E} \sup_{t \in [0, \tau_{M,L,N} \wedge T]} \int_0^t \left| \bar{f}(j_{s(\delta)}^{\varepsilon}, u_{s(\delta)}^{\varepsilon}) - \bar{f}(j_s^{\varepsilon}, u_s^{\varepsilon}) \right|_H^{2p} ds \\
& + C_p \mathbb{E} \sup_{t \in [0, \tau_{M,L,N} \wedge T]} \int_0^t \left| \bar{f}(j_s^{\varepsilon}, u_s^{\varepsilon}) - \bar{f}(\hat{j}_s^{\varepsilon}, \hat{u}_s^{\varepsilon}) \right|_H^{2p} ds \\
& + C_p \mathbb{E} \sup_{t \in [0, \tau_{M,L,N} \wedge T]} \int_0^t \left| \bar{f}(\hat{j}_s^{\varepsilon}, \hat{u}_s^{\varepsilon}) - \bar{f}(\bar{j}_s, \bar{u}_s) \right|_H^{2p} ds \\
\leq & C_p \mathbb{E} \sup_{t \in [0, \tau_{M,L,N} \wedge T]} \int_0^t \left| f(j_{s(\delta)}^{\varepsilon}, \hat{\theta}_s^{\varepsilon}) - \bar{f}(j_{s(\delta)}^{\varepsilon}, u_{s(\delta)}^{\varepsilon}) \right|_H^{2p} ds \\
& + C_p \mathbb{E} \int_0^{\tau_{M,L,N} \wedge T} \tilde{\kappa} \left(\left| j_s^{\varepsilon} - j_{s(\delta)}^{\varepsilon} \right|_H^{2p} \right) ds + C_p \mathbb{E} \int_0^{\tau_{M,L,N} \wedge T} \tilde{\kappa} \left(\left| j_s^{\varepsilon} - \hat{j}_s^{\varepsilon} \right|_H^{2p} \right) ds \\
& + C_p \mathbb{E} \int_0^{\tau_{M,L,N} \wedge T} \tilde{\kappa} \left(\left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{2p} \right) ds \\
\leq & C_{p,T} \int_0^{\tau_{M,L,N} \wedge T} \tilde{\kappa} \left(\mathbb{E} \sup_{r \in [0, \tau_{M,L,N} \wedge s]} \left| \hat{j}_s^{\varepsilon} - \bar{j}_s \right|_H^{2p} \right) ds \\
& + C_p \mathbb{E} \sup_{t \in [0, \tau_{M,L,N} \wedge T]} \int_0^t \left| f(j_{s(\delta)}^{\varepsilon}, \hat{\theta}_s^{\varepsilon}) - \bar{f}(j_{s(\delta)}^{\varepsilon}, u_{s(\delta)}^{\varepsilon}) \right|_H^{2p} ds \\
& + C_{p,T} \tilde{\kappa} \left(\frac{1}{\tau_{M,L,N} \wedge T} \int_0^{\tau_{M,L,N} \wedge T} \left| j_s^{\varepsilon} - j_{s(\delta)}^{\varepsilon} \right|_H^{2p} ds \right) \\
& + C_{p,T} \tilde{\kappa} \left(\frac{1}{\tau_{M,L,N} \wedge T} \int_0^{\tau_{M,L,N} \wedge T} \left| j_s^{\varepsilon} - \hat{j}_s^{\varepsilon} \right|_H^{2p} ds \right)
\end{aligned}
$$

$$
\begin{aligned}
\leq & C_{p,T}\int_0^{\tau_{M,L,N}\wedge T}\tilde{\kappa}\left(\mathbb{E}\sup_{r\in[0,\tau_{M,L,N}\wedge s]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\right)ds\\
&+C_p\left\{\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\int_0^t\left|f(j_{s(\delta)}^{\varepsilon},\hat{\theta}_s^{\varepsilon})-\bar{f}(j_{s(\delta)}^{\varepsilon},u_{s(\delta)}^{\varepsilon})\right|_H^{2p}ds\right\}_{Q_{3,1}}\\
&+C_{p,T}\tilde{\kappa}\left(\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right)\right)\\
&+C_{p,T}\tilde{\kappa}\left(C_{T,M}\delta^{\frac{1}{2}}\right).
\end{aligned}
$$

对于 $Q_{3,\ 1}$，我们有

$$
\begin{aligned}
Q_{3,1}=&\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\sum_{k=0}^{\left[\frac{t}{\delta}\right]-1}\int_{k\delta}^{(k+1)\delta}\left|f(j_{k\delta}^{\varepsilon},\hat{\theta}_s^{\varepsilon})-\bar{f}(j_{k\delta}^{\varepsilon},u_{k\delta}^{\varepsilon})\right|_H^{2p}ds\\
&+\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\int_{\left[\frac{t}{\delta}\right]\delta}^{t}\left|f(j_{\left[\frac{t}{\delta}\right]\delta}^{\varepsilon},\hat{\theta}_s^{\varepsilon})-\bar{f}(j_{\left[\frac{t}{\delta}\right]\delta}^{\varepsilon},u_{\left[\frac{t}{\delta}\right]\delta}^{\varepsilon})\right|_H^{2p}ds\\
\leq&\left[\frac{T}{\delta}\right]\max_{0\leq k\leq\left[\frac{t}{\delta}\right]-1}\mathbb{E}\int_{k\delta}^{(k+1)\delta}\left|f(j_{k\delta}^{\varepsilon},\hat{\theta}_s^{\varepsilon})-\bar{f}(j_{k\delta}^{\varepsilon},u_{k\delta}^{\varepsilon})\right|_H^{2p}ds+C_T\delta\\
\overset{\text{s=r+k}\delta}{\leq}&\left[\frac{T}{\delta}\right]\max_{0\leq k\leq\left[\frac{t}{\delta}\right]-1}\mathbb{E}\int_0^{\delta}\left|f(j_{k\delta}^{\varepsilon},\hat{\theta}_{r+k\delta}^{\varepsilon})-\bar{f}(j_{k\delta}^{\varepsilon},u_{k\delta}^{\varepsilon})\right|_H^{2p}dr+C_T\delta.
\end{aligned}
$$

注意，$\hat{\theta}_{r+k\delta}^{\varepsilon}=\tilde{\theta}^{u_{k\delta}^{\varepsilon},\ \underline{\theta_0}}_{\underline{r+k\delta}}$，由（4.3）可得$_{\varepsilon}$

$$
\begin{aligned}
&\mathbb{E}\int_0^{\delta}\left|f(j_{k\delta}^{\varepsilon},\hat{\theta}_{r+k\delta}^{\varepsilon})-\bar{f}(j_{k\delta}^{\varepsilon},u_{k\delta}^{\varepsilon})\right|_H^{2p}dr\\
=&\mathbb{E}\int_0^{\delta}\left|f(j_{k\delta}^{\varepsilon},\tilde{\theta}^{u_{k\delta}^{\varepsilon},\theta_0}_{\frac{r+k\delta}{\varepsilon}})-\int_H f(j_{k\delta}^{\varepsilon},\theta)\pi^{u_{k\delta}^{\varepsilon}}(d\theta)\right|_H^{2p}dr\\
\leq&C_p\mathbb{E}\int_0^{\delta}\int_H\left|f(j_{k\delta}^{\varepsilon},\tilde{\theta}^{u_{k\delta}^{\varepsilon},\theta_0}_{\frac{r+k\delta}{\varepsilon}})-f(j_{k\delta}^{\varepsilon},\theta)\right|_H^{2p}\pi^{u_{k\delta}^{\varepsilon}}(d\theta)dr\\
=&C_p\mathbb{E}\int_0^{\delta}\int_H\left|f(j_{k\delta}^{\varepsilon},\tilde{\theta}^{u_{k\delta}^{\varepsilon},\theta_0}_{\frac{r+k\delta}{\varepsilon}})-f(j_{k\delta}^{\varepsilon},\tilde{\theta}^{u_{k\delta}^{\varepsilon},\theta}_{\frac{r+k\delta}{\varepsilon}})\right|_H^{2p}\pi^{u_{k\delta}^{\varepsilon}}(d\theta)dr\\
\leq&C_p\int_0^{\delta}\int_H\kappa\left(\mathbb{E}\left|\tilde{\theta}^{u_{k\delta}^{\varepsilon},\theta_0}_{\frac{r+k\delta}{\varepsilon}}-\tilde{\theta}^{u_{k\delta}^{\varepsilon},\theta}_{\frac{r+k\delta}{\varepsilon}}\right|_H^{2p}\right)\pi^{u_{k\delta}^{\varepsilon}}(d\theta)dr\\
\leq&C_p\int_0^{\delta}\int_H\kappa\left(e^{-\frac{\lambda_p(r+k\delta)}{\varepsilon}}|\theta_0-\theta|_H^{2p}\right)\pi^{u_{k\delta}^{\varepsilon}}(d\theta)dr\\
\leq&C_p\int_0^{\delta}\kappa\left(e^{-\frac{\lambda_p(r+k\delta)}{\varepsilon}}\int_H|\theta_0-\theta|_H^{2p}\pi^{u_{k\delta}^{\varepsilon}}(d\theta)\right)dr\\
\leq&C_p\int_0^{\delta}\kappa\left(Ce^{-\frac{\lambda_p(r+k\delta)}{\varepsilon}}\right)dr\\
\leq&C_p\delta\kappa\left(\frac{C}{\delta}\int_0^{\delta}e^{-\frac{\lambda_p(r+k\delta)}{\varepsilon}}dr\right)
\end{aligned}
$$

$$\leq C_p\delta\kappa\left(\frac{C\varepsilon}{\delta}\right).$$

因此

$$Q_{3,1} \leq C_{p,T}\kappa\left(\frac{C\varepsilon}{\delta}\right) + C_T\delta.$$

因此

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]} Q_3 \leq & C_{p,T}\int_0^{\tau_{M,L,N}\wedge T}\tilde{\kappa}\left(\mathbb{E}\sup_{r\in[0,\tau_{M,L,N}\wedge s]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\right)ds\\ &+C_{p,T}\tilde{\kappa}\left(\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right)\right)\\ &+C_{p,T}\tilde{\kappa}\left(C_{T,M}\delta^{\frac{1}{2}}\right)+C_{p,T}\kappa\left(\frac{C\varepsilon}{\delta}\right)+C_T\delta.\end{aligned}$$

与 Q_3 类似，但比 Q_3 更容易，我们可以得到

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]} Q_4 \leq & C_{p,T}\int_0^{\tau_{M,L,N}\wedge T}\kappa\left(\mathbb{E}\sup_{r\in[0,\tau_{M,L,N}\wedge s]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\right)ds\\ &+C_{p,T}\kappa\left(\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right)\right)\\ &+C_{p,T}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}\right)+C_{p,T}\kappa\left(\frac{C\varepsilon}{\delta}\right)+C_T\delta.\end{aligned}$$

对于 Q_5，我们有

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]} Q_5 =&\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\int_{|z|_H<1}|\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z)|_H^{2p}\,\nu_1(dz)ds\\ \leq&C_p\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\int_{|z|_H<1}\left|\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\hat{j}_s^{\varepsilon},z)\right|_H^{2p}\nu_1(dz)ds\\ &+C_p\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\int_{|z|_H<1}\left|\sigma_1(\hat{j}_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z)\right|_H^{2p}\nu_1(dz)ds\\ \leq&C_p\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\kappa\left(\left|j_s^{\varepsilon}-\hat{j}_s^{\varepsilon}\right|_H^{2p}\right)ds+C_p\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\kappa\left(\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\right)ds\\ \leq&C_{p,T}\int_0^{\tau_{M,L,N}\wedge T}\kappa\left(\mathbb{E}\sup_{r\in[0,\tau_{M,L,N}\wedge s]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\right)ds\\ &+C_{p,T}\kappa\left(\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right)\right).\end{aligned}$$

对于 Q_6，利用 B-D-G 不等式，我们有

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]} Q_6\\ \leq&C\mathbb{E}\left(\int_0^{\tau_{M,L,N}\wedge T}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{4p-2}|\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z)|_H^2\,\eta_1(ds,dz)\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
\leq & C\mathbb{E}\left(\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\int_0^{\tau_{M,L,N}\wedge T}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p-2}\left|\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z)\right|_H^2\eta_1(ds,dz)\right)^{\frac{1}{2}}\\
\leq & C\gamma\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}+C_{p,\gamma}\int_0^{\tau_{M,L,N}\wedge T}\mathbb{E}\sup_{r\in[0,\tau_{M,L,N}\wedge s]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}ds\\
& +C_{p,\gamma}\mathbb{E}\int_0^{\tau_{M,L,N}\wedge T}\int_{|z|_H<1}\left|\sigma_1(j_s^{\varepsilon},z)-\sigma_1(\bar{j}_s,z)\right|_H^{2p}\nu_1(dz)ds\\
\leq & C\gamma\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\left|\hat{j}_t^{\varepsilon}-\bar{j}_t\right|_H^{2p}+C_{p,\gamma}\int_0^{\tau_{M,L,N}\wedge T}\mathbb{E}\sup_{r\in[0,\tau_{M,L,N}\wedge s]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}ds\\
& +C_{p,\gamma,T}\int_0^{\tau_{M,L,N}\wedge T}\kappa\left(\mathbb{E}\sup_{r\in[0,\tau_{M,L,N}\wedge s]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\right)ds\\
& +C_{p,\gamma,T}\kappa\left(\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right)\right).
\end{aligned}
$$

综上，取足够小的 γ，我们有

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\left|\hat{j}_t^{\varepsilon}-\bar{j}_t\right|_H^{2p}\leq & C_{p,T,N}\int_0^{\tau_{M,L,N}\wedge T}\hat{\kappa}\left(\mathbb{E}\sup_{r\in[0,\tau_{M,L,N}\wedge s]}\left|\hat{j}_s^{\varepsilon}-\bar{j}_s\right|_H^{2p}\right)ds\\
& +C_{p,T}\hat{\kappa}\left(\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right)\right)\\
& +C_{p,T}\hat{\kappa}\left(C_{T,M}\delta^{\frac{1}{2}}\right)+C_{p,T}\kappa\left(\frac{C\varepsilon}{\delta}\right)+C_T\delta,
\end{aligned}
$$

其中 $\hat{\kappa}(u):=u+\kappa(u)+\tilde{\kappa}(u)$ 也是一个凹连续非递减函数。Le t

$$
\begin{aligned}
\mathcal{R}(\varepsilon,\delta):=&C_{p,T}\hat{\kappa}\left(\mathcal{O}_{p,T,M,L}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}_{p,T,M,L}(\delta)\right)\right)\\
&+C_{p,T}\hat{\kappa}\left(C_{T,M}\delta^{\frac{1}{2}}\right)+C_{p,T}\kappa\left(\frac{C\varepsilon}{\delta}\right)+C_T\delta.
\end{aligned}
\tag{5.14}
$$

应用 Bihari不等式（见 [48, 第 3] 节），我们得到

$$
\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\left|\hat{j}_t^{\varepsilon}-\bar{j}_t\right|_H^{2p}\leq\hat{\Omega}^{-1}\left(\hat{\Omega}\left(\mathcal{R}(\varepsilon,\delta)\right)+C_{p,T,N}T\right),
$$

其中

$$
\hat{\Omega}(v):=\int_0^v\frac{du}{\hat{\kappa}(u)},\ \forall v>0.
\tag{5.15}
$$

□

在本小节末尾，我们给出以下主要结果：

定理5.3。 设 $\sigma_1(j,\theta,z)\equiv\sigma_1(j,z)$，在假设 ***1-4*** 下，对于任意 $T>0$，$(j_0,\theta_0)\in H\times H$ 且对于某个 $p\geq1$，有

$$
\lim_{\varepsilon\to0}\mathbb{E}\sup_{t\in[0,T]}|j_t^{\varepsilon}-\bar{j}_t|_H^{2p}=0.
\tag{5.16}
$$

(5.12) 和 (5.13)，我们有

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}|j_t^\varepsilon-\bar{j}_t|_H^{2p}\leq&C_p\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\left|j_t^\varepsilon-\hat{j}_t^\varepsilon\right|_H^{2p}+\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}\left|\hat{j}_t^\varepsilon-\bar{j}_t\right|_H^{2p}\\ \leq&\mathcal{O}(\delta)+C_{p,T,M}\kappa\left(C_{T,M}\delta^{\frac{1}{2}}+\mathcal{O}(\delta)\right)\\&+\hat{\Omega}^{-1}\left(\hat{\Omega}\left(\mathcal{R}(\varepsilon,\delta)\right)+C_{p,T,N}T\right).\end{aligned}$$

设 $\delta=\varepsilon^{\frac{1}{2}}$，注意到

$$\mathcal{O}\left(\varepsilon^{\frac{1}{2}}\right)\to 0,\quad and\quad \mathcal{R}\left(\varepsilon,\varepsilon^{\frac{1}{2}}\right)\to 0,$$

如 $\varepsilon\to 0$。根据 [49, 引理 2.3], ，我们有

$$\hat{\Omega}^{-1}\left(\hat{\Omega}\left(\mathcal{R}(\varepsilon,\delta)\right)+C_{p,T,N}T\right)\to 0,\quad as\quad \mathcal{R}\left(\varepsilon,\varepsilon^{\frac{1}{2}}\right)\to 0.$$

因此

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{t\in[0,\tau_{M,L,N}\wedge T]}|j_t^\varepsilon-\bar{j}_t|_H^{2p}=0.$$

由于 (3.1)、(3.2)、(5.5)、(5.9) 和 (5.11) 可推出 $\tau_{M,\ L,\ N}\to\infty$ 如 $M,\ L,\ N\to\infty$，首先令 $\varepsilon\to 0,\ M,\ L,\ N\to\infty$ 其次，我们得到

$$\lim_{\varepsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}|j_t^\varepsilon-\bar{j}_t|_H^{2p}=0.$$

□

6 示例与数值模拟

本节给出一个示例，以验证本文的理论结果。

6.1 示例

考虑以下双尺度系统:

$$\begin{cases}\dfrac{\partial j^\varepsilon}{\partial t}+(u^\varepsilon\cdot\nabla)j^\varepsilon=\triangle j^\varepsilon-\dfrac{1}{2}\mathrm{sgn}(j^\varepsilon)|j^\varepsilon|^{2/3}-\dfrac{3}{10}\mathrm{sgn}(\theta^\varepsilon)|\theta^\varepsilon|^{1/3}+\partial_x\theta^\varepsilon\\ \qquad\qquad+\displaystyle\int_{|z|_H<1}\frac{\left(1+(j^\varepsilon)^2\right)^{1/3}}{\sqrt{1+|z|_H^2}}\dot{\tilde{\eta}}_1(t,dz),\\ \dfrac{\partial\theta^\varepsilon}{\partial t}+\dfrac{1}{\varepsilon}(u^\varepsilon\cdot\nabla)\theta^\varepsilon=\dfrac{1}{\varepsilon}\triangle\theta^\varepsilon-\displaystyle\int_{|z|_H<1}\frac{1}{2}\theta^\varepsilon e^{-|z|_H^2}\dot{\tilde{\eta}}_2^\varepsilon(t,dz),\\ j^\varepsilon(0)=j_0,\quad\theta^\varepsilon(0)=\theta_0,\end{cases}\tag{6.1}$$

其中，sgn(·)表示符号函数，$z=\sum_{k=1}^{\infty}z_ke_k$ 是 z 在 H的基$\{e_k\}$ 上的展开。补偿泊松随机测度 $\tilde{\eta}_1(t,\ z)=\eta_1(t,\ z)-\nu_1(z)\cdot t$ 和$\tilde{\eta}_2^{\varepsilon}(t,\ z)=\eta_2(\frac{1}{\varepsilon}t,\ z)-\frac{1}{\varepsilon}\nu_2(z)\cdot t$，以及强度测度 $\nu_i(i=1,\ 2)$定义为

$$\nu_i(dz)=\frac{C_{\nu_i}}{|z|_H^{1+\beta_i}}dz,\quad i=1,2,\quad for\quad |z|_H<1,$$

其中 $\beta_i\in(0,1)$，且 C_{ν_i} 为归一化常数。

与原方程(1.2)相比，容易看出

$$\begin{aligned}
&f(j,\theta)==-\frac{1}{2}\mathrm{sgn}(j)\,|j|^{2/3}-\frac{3}{10}\mathrm{sgn}(\theta)\,|\theta|^{1/3},\\
&\sigma_1(j,z)=\left(1+j^2\right)^{1/3}\frac{1}{\sqrt{1+|z|_H^2}},\\
&\sigma_2(j,\theta,z)\equiv\sigma_2(\theta,z)=-\frac{1}{2}\theta e^{-|z|_H^2},
\end{aligned}$$

并且在 $p=1$和 $\kappa(u)=Cu^{\frac{2}{3}}$下满足假设 1-4。

对任意固定的 $u\in H^1$，冻结方程为:

$$\begin{cases}\dfrac{\partial\tilde{\theta}_t}{\partial t}+(u\cdot\nabla)\tilde{\theta}_t=\triangle\tilde{\theta}_t-\displaystyle\int_{|z|_H<1}\frac{1}{2}\tilde{\theta}_te^{-|z|_H^2}\dot{\tilde{\eta}}_2(t,dz),\\ \tilde{\theta}_0=\theta.\end{cases}\tag{6.2}$$

容易验证，(6.2)在 H上存在唯一的遍历不变测度 π^u。此外，由于 $\nu_2(\cdot)$是对称的，故 π^u 也是对称的。因此，奇函数关于 π^u 的积分为零。从而

$$\bar{f}(j,u)=-\frac{1}{2}\mathrm{sgn}(j)|j|^{2/3}-\frac{3}{10}\int_H\mathrm{sgn}(\hat{\theta})|\hat{\theta}|^{1/3}\pi^u(d\hat{\theta})=-\frac{1}{2}\mathrm{sgn}(j)|j|^{2/3},$$

and

$$g(u)=\int_H\hat{\theta}\,\pi^u(d\hat{\theta})=0.$$

因此，给出的平均方程为

$$\begin{cases}\dfrac{\partial\bar{j}}{\partial t}+(\bar{u}\cdot\nabla)\bar{j}=\triangle\bar{j}-\frac{1}{2}\mathrm{sgn}(\bar{j})|\bar{j}|^{2/3}+\displaystyle\int_{|z|_H<1}\frac{(1+\bar{j}^2)^{1/3}}{\sqrt{1+|z|_H^2}}\dot{\tilde{\eta}}_1(t,dz),\\ \bar{j}(0)=j_0.\end{cases}\tag{6.3}$$

6.2 数值模拟

令 $\beta_1=0.8$、$\beta_2=0.6$，对于 $\varepsilon=1$，0。5，0。25，0。1，定义

$$error(\varepsilon):=\sup_{t\in[0,1]}|j^{\varepsilon}-\bar{j}|_H,$$

¯并针对 $(j^{\varepsilon}, \theta^{\varepsilon})$ 和 j 选取 100 个样本，我们得到不同 ε 下 100 个样本的误差分布，见图 1。

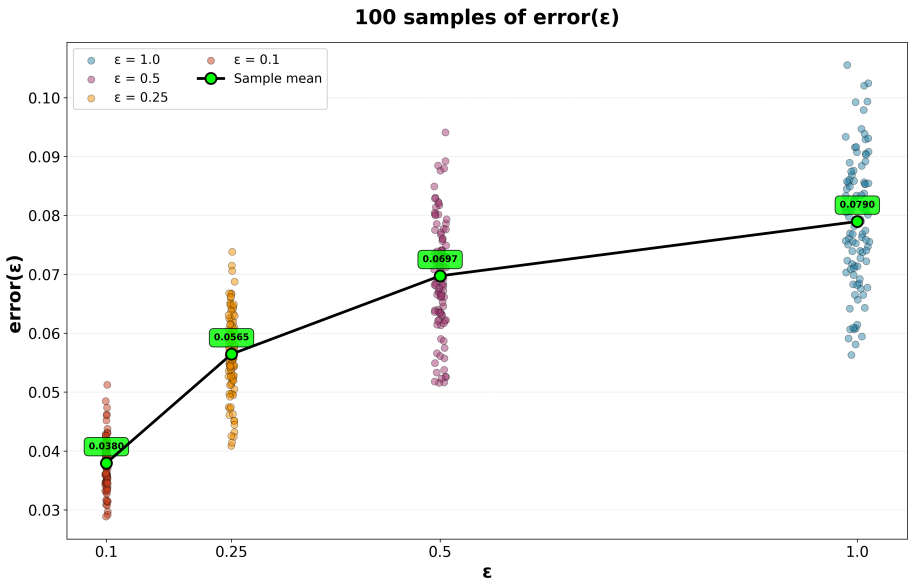

图1：ε的100个样本误差

从图1中可以很容易看出，100个样本的数值模拟结果与定理5.2一致。

此外， ε 的均方误差如图2所示。

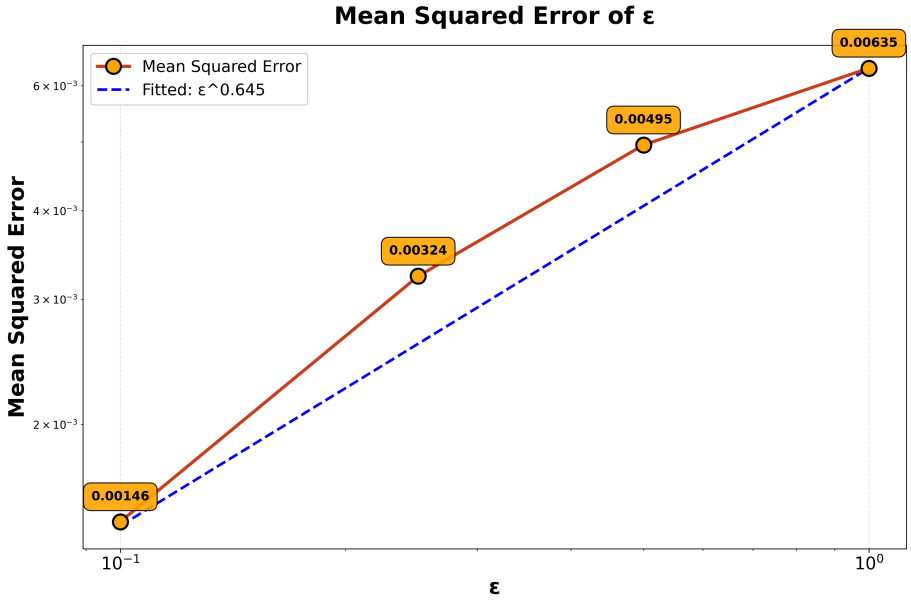

图2： ε的均方误差

观察图2可知，模拟结果与 $p=1$的定理5.3结论一致。此外，在本例中，通过幂律拟合得到的均方误差为

$$\text{mean squared error} \approx C\varepsilon^{0.645}.$$

附录

(1.2)的适定性

为方便起见，令 $\varepsilon=1$。根据 (3.1) 和 (3.5)，通过标准的伽辽金逼近，可以直接证明解的存在性。下面考虑唯一性问题。

设 (j_1,θ_1) 和 (j_2,θ_2) 是 (1.2) 的两个解，且具有相同的初值 (j_0,θ_0)。对于任意 $T>0$ 和 $R>0$, 我们定义停时

$$\tau_R := \{t\le 0; |j_2|_H + |\theta_2|_H > R\}.$$

根据伊藤公式，我们有

$$\begin{aligned}
|j_1-j_2|_H^2 + 2\int_0^t \left((u_1\cdot\nabla)j_1-(u_2\cdot\nabla)j_2, j_1-j_2\right)ds &= 2\int_0^t \left(\triangle(j_1-j_2), j_1-j_2\right)ds \\
&+ 2\int_0^t \left(f(j_1,\theta_1)-f(j_2,\theta_2), j_1-j_2\right)ds + 2\int_0^t \left(\partial_x\theta_1-\partial_x\theta_2, j_1-j_2\right)ds \\
&+ \int_0^t\int_{|z|_H<1} |\sigma_1(j_1,\theta_1,z)-\sigma_1(j_2,\theta_2,z)|_H^2\,\eta_1(ds,dz) \\
&+ 2\int_0^t\int_{|z|_H<1} \left(\sigma_1(j_1,\theta_1,z)-\sigma_1(j_2,\theta_2,z), j_1-j_2\right)\tilde{\eta}_1(ds,dz).
\end{aligned}$$

因此

$$\begin{aligned}
|j_1-j_2|_H^2 + 2\int_0^t |\nabla(j_1-j_2)|_H^2\,ds &= 2\left\{\int_0^t \left(((u_1-u_2)\cdot\nabla)(j_1-j_2), j_2\right)ds\right\}_{H_1} \\
&+ 2\left\{\int_0^t \left(f(j_1,\theta_1)-f(j_2,\theta_2), j_1-j_2\right)ds\right\}_{H_2} - 2\left\{\int_0^t \left(\theta_1-\theta_2, \partial_x j_1-\partial_x j_2\right)ds\right\}_{H_3} \\
&+ \left\{\int_0^t\int_{|z|_H<1} |\sigma_1(j_1,\theta_1,z)-\sigma_1(j_2,\theta_2,z)|_H^2\,\eta_1(ds,dz)\right\}_{H_4} \\
&+ 2\left\{\int_0^t\int_{|z|_H<1} \left(\sigma_1(j_1,\theta_1,z)-\sigma_1(j_2,\theta_2,z), j_1-j_2\right)\tilde{\eta}_1(ds,dz)\right\}_{H_5}.
\end{aligned}$$

由赫尔德不等式、拉迪任斯卡娅不等式和杨氏不等式可得

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]} H_1 \le& \mathbb{E}\int_0^{\tau_R\wedge T} |u_1-u_2|_{L^4}\,|\nabla(j_1-j_2)|_H\,|j_2|_{L^4}\,ds \\
\le& \mathbb{E}\int_0^{\tau_R\wedge T} |\nabla(u_1-u_2)|_H\,|\nabla(j_1-j_2)|_H\,|\nabla j_2|_H\,ds
\end{aligned}$$

$$\begin{aligned}
\leq & \mathbb{E}\int_0^{\tau_R\wedge T} |j_1-j_2|_H \,|\nabla(j_1-j_2)|_H \,|\nabla j_2|_H \,ds \\
\leq & \gamma\mathbb{E}\int_0^{\tau_R\wedge T} |\nabla(j_1-j_2)|_H^2 \,ds + C_\gamma\mathbb{E}\int_0^{\tau_R\wedge T} |j_1-j_2|_H^2 \,|\nabla j_2|_H^2 \,ds.
\end{aligned}$$

注意到

$$\begin{aligned}
\frac{d}{dt}|j_2|_H^2 + 2|\nabla j_2|_H^2 =& 2(f(j_2,\theta_2),j_2) - 2(\theta_2,\partial_x j_2) + \int_{|z|_H<1} |\sigma_1(j_2,\theta_2,z)|_H^2 \,\dot\eta_1(t,dz) \\
&+ 2\int_{|z|_H<1} (\sigma_1(j_2,\theta_2,z),j_2)\,\dot{\tilde\eta}_1(t,dz),
\end{aligned}$$

我们有

$$\left\langle |j_2|_H^2 \right\rangle_t = \int_0^t \int_{|z|_H<1} \left(|\sigma_1(j_2,\theta_2,z)|_H^2 + 2(\sigma_1(j_2,\theta_2,z),j_2)\right)^2 \eta_1(ds,dz),$$

and

$$\begin{aligned}
\frac{d}{dt}|j_2|_H^2 + |\nabla j_2|_H^2 \leq & C\left(1+|j_2|_H^2+|\theta_2|_H^2\right) + \int_{|z|_H<1} |\sigma_1(j_2,\theta_2,z)|_H^2 \,\dot\eta_1(t,dz) \\
&+ 2\int_{|z|_H<1} (\sigma_1(j_2,\theta_2,z),j_2)\,\dot{\tilde\eta}_1(t,dz).
\end{aligned}$$

利用 B-D-G 不等式，我们有

$$\begin{aligned}
& \mathbb{E}\int_0^{\tau_R\wedge T} |j_1-j_2|_H^2 \,|\nabla j_2|_H^2 \,ds \\
\leq & -\mathbb{E}\int_0^{\tau_R\wedge T} |j_1-j_2|_H^2 \,d\,|j_2|_H^2 + C\mathbb{E}\int_0^{\tau_R\wedge T} \left(1+|j_2|_H^2+|\theta_2|_H^2\right)|j_1-j_2|_H^2 \,ds \\
& + \mathbb{E}\int_0^{\tau_R\wedge T}\int_{|z|_H<1} |\sigma_1(j_2,\theta_2,z)|_H^2 \,|j_1-j_2|_H^2 \,\eta_1(ds,dz) \\
& + 2\mathbb{E}\int_0^{\tau_R\wedge T}\int_{|z|_H<1} (\sigma_1(j_2,\theta_2,z),j_2)\,|j_1-j_2|_H^2 \,\tilde\eta_1(ds,dz) \\
\leq & C\mathbb{E}\left(\int_0^{\tau_R\wedge T}\int_{|z|_H<1} |j_1-j_2|_H^4 \left(|\sigma_1(j_2,\theta_2,z)|_H^2 + 2(\sigma_1(j_2,\theta_2,z),j_2)\right)^2 \eta_1(ds,dz)\right)^{\frac12} \\
& + C_R\mathbb{E}\int_0^{\tau_R\wedge T} |j_1-j_2|_H^2 \,ds + \mathbb{E}\int_0^{\tau_R\wedge T}\int_{|z|_H<1} |\sigma_1(j_2,\theta_2,z)|_H^2 \,|j_1-j_2|_H^2 \,\nu_1(dz)ds \\
& + C\mathbb{E}\left(\int_0^{\tau_R\wedge T}\int_{|z|_H<1} |(\sigma_1(j_2,\theta_2,z),j_2)|^2 \,|j_1-j_2|_H^4 \,\eta_1(ds,dz)\right)^{\frac12} \\
\leq & C\gamma\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]} |j_1-j_2|_H^2 + C_{R,\gamma}\mathbb{E}\int_0^{\tau_R\wedge T} |j_1-j_2|_H^2 \,ds
\end{aligned}$$

因此

$$\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]} H_1 \leq C\gamma\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]} |j_1-j_2|_H^2 + \gamma\mathbb{E}\int_0^{\tau_R\wedge T} |\nabla(j_1-j_2)|_H^2 \,ds$$

$$+ C_{R,\gamma}\mathbb{E}\int_0^{\tau_R\wedge T}|j_1-j_2|_H^2\,ds$$

根据假设1，我们有

$$\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}H_2\le C\mathbb{E}\int_0^{\tau_R\wedge T}|j_1-j_2|_H^2\,ds+C\mathbb{E}\int_0^{\tau_R\wedge T}\kappa\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)ds.$$

容易得到

$$\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}H_3\le\gamma\mathbb{E}\int_0^{\tau_R\wedge T}|\nabla\left(j_1-j_2\right)|_H^2\,ds+C_\gamma\mathbb{E}\int_0^{\tau_R\wedge T}|\theta_1-\theta_2|_H^2\,ds$$

根据假设2，我们可以得到

$$\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}H_4\le C\mathbb{E}\int_0^{\tau_R\wedge T}\kappa\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)ds,$$

and

$$\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}H_5\le C\gamma\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}|j_1-j_2|_H^2+C_\gamma\mathbb{E}\int_0^{\tau_R\wedge T}\kappa\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)ds.$$

选取足够小的 γ，我们有

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}|j_1-j_2|_H^2\le&C_R\mathbb{E}\int_0^{\tau_R\wedge T}|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\,ds\\&+C\mathbb{E}\int_0^{\tau_R\wedge T}\kappa\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)ds.\end{aligned}$$

通过类似的推导过程，我们可以得到

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}|\theta_1-\theta_2|_H^2\le&C_R\mathbb{E}\int_0^{\tau_R\wedge T}|\theta_1-\theta_2|_H^2\,ds\\&+C\mathbb{E}\int_0^{\tau_R\wedge T}\kappa\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)ds.\end{aligned}$$

因此

$$\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)\le C_R\int_0^{\tau_R\wedge T}\check{\kappa}\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)ds,$$

其中 $\check{\kappa}(u):=u+\kappa(u)$。根据Bihari不等式，我们有

$$\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)\le\check{\Omega}^{-1}\left(\check{\Omega}(0)+C_RT\right),$$

其中

$$\check{\Omega}(v):=\int_0^v\frac{du}{\check{\kappa}(u)},\forall v\ge0.$$

根据 [49, 引理 2.3], 我们有

$$\mathbb{E}\sup_{t\in[0,\tau_R\wedge T]}\left(|j_1-j_2|_H^2+|\theta_1-\theta_2|_H^2\right)\equiv0.$$

由于 (3.1) 和 (3.5) 推出 $\tau_R\to\infty$ 当 $R\to\infty$ 时，因此由控制收敛定理并令 $R\to\infty$，可得唯一性。

引理 5.1 的证明

注意到 F 关于 t 单调递增且有界，因此我们有

$$\lim_{t\to+\infty} F(t) = \sup_{t\geq 0} F(t) := M.$$

因此，对于任意 $\varepsilon > 0$，存在某个 $\delta > 0$，使得

$$e^{-a\delta} M < \frac{\varepsilon}{2}.$$

Let

$$I(t) := \int_0^t e^{-a(t-s)} f(s) ds.$$

当 t 足够大时，我们有

$$I(t) = \left\{ \int_0^{t-\delta} e^{-a(t-s)} f(s) ds \right\}_{I_1(t)} + \left\{ \int_{t-\delta}^{t} e^{-a(t-s)} f(s) ds \right\}_{I_2(t)}.$$

对于 $I_1(t)$，由于 $\delta \leq t - s \leq t$，我们有

$$I_1(t) \leq e^{-a\delta} \int_0^{t-\delta} f(s) ds \leq e^{-a\delta} M < \frac{\varepsilon}{2}.$$

对于 $I_2(t)$，我们有

$$I_2(t) \leq \int_{t-\delta}^{t} f(s) ds = F(t) - F(t-\delta) < \frac{\varepsilon}{2},$$

当 t 充分大时。因此，引理 5.1 成立。

□

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