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大初值下完整可压缩纳维–斯托克斯系统整体解在临界 Besov 空间中的低马赫数极限

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摘要

我们研究完整可压缩纳维–斯托克斯方程正则解的整体存在性，以及马赫数充分小时这些解的渐近行为。我们证明，只要初始速度的无散部分以及初始温度与密度之差能够产生不可压缩 Boussinesq 系统的整体正则解，那么对于任意大的初始数据，系统在临界 Besov 空间中存在整体解。此外，我们严格论证了当马赫数趋于零时，解收敛到不可压缩模型。证明依赖于对密度、速度和温度的高、中、低频进行细致分析。我们的结果可视为对 Danchin 和 He [《数学年刊》, 366 (2016), no. 3-4, pp. 1365-1402], 的改进，包括将小初值的情形推广到大初值，并给出在临界正则性层面成立的新收敛结果。

关键词： 完全可压缩Navier-Stokes方程，低马赫数极限，临界空间，大解，整体存在性

数学主题分类： 35Q30, 35B25, 76N10,76N06

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1 引言与主要结果

可压缩黏性热传导流体的运动由以下完整的可压缩纳维–斯托克斯系统所支配:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \\ \partial_t(\varrho \boldsymbol{u}) + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla P = \operatorname{div}\mathbb{S}(\nabla \boldsymbol{u}), \\ \partial_t(\varrho E) + \operatorname{div}\left(\varrho E \boldsymbol{u} + P\boldsymbol{u}\right) = \operatorname{div}(\kappa \nabla \theta) + \operatorname{div}(\mathbb{S}(\nabla \boldsymbol{u})\boldsymbol{u}), \\ (\varrho, \boldsymbol{u}, \vartheta)|_{t=0} = (\varrho_0, \boldsymbol{u}_0, \vartheta_0). \end{cases} \tag{1.1}$$

这里，未知量 $\varrho(t,\ x)$ $\boldsymbol{u}(t,\ x)$ $\theta(t,\ x)$分别表示流体的密度、速度场和温度，其中 $t \geq 0$ 是时间坐标，x 是空间坐标。$\mathbb{S}(\nabla \boldsymbol{u})$是黏性应力张量，定义为 $\mathbb{S}(\nabla \boldsymbol{u}) = 2\mu D\boldsymbol{u} + \lambda$ 散度$\boldsymbol{u}\mathbb{I}$，其中 $Du = \nabla u + (\nabla u\ ^{\top})$ 是变形张量，且黏度系数满足 $\mu > 0$，$2\mu + \lambda > 0$。总能量 $\overline{E = e + \frac{1}{2}|\boldsymbol{u}|^2}$，而 $e = \overset{2}{c_\nu}\theta = \overset{R}{\gamma - 1}\theta$是内能，其中 $\gamma > 1$是绝热常数，$R > 0$是一个通用气体常数。$\kappa > 0$是热导率。我们将研究范围限定在理想气体情形，此时压力为 $P = R\varrho\theta$；并考虑流体占据区域为全空间 $\mathbb{R}^d$ $(d \geq 3)$的情形。经过重新缩放，不失一般性，可取 $R = c_\nu = 1$。

显然，(1.1) 在如下变换下保持不变

$$(\varrho(t,x),\ \boldsymbol{u}(t,x),\ \theta(t,x)) \to (\varrho(l^2 t, lx),\ l\boldsymbol{u}(l^2 t, lx),\ l^2\theta(l^2 t, lx)),\ l > 0, \tag{1.2}$$

前提是压力项 P 已变为 $l^2 P$。对于所有 l，系统 (1.1) 的临界空间相对于变换 (1.2) 在范数意义下保持不变（至多相差一个无关紧要的常数）。

Danchin 证明了完整可压缩 Navier–Stokes 系统在 [11]中的局部适定性，以及在 [10]中的整体适定性，其中初始数据在临界 Besov 空间中且接近一个稳定的常数平衡态。关于适定性理论的更多精彩结果，参见 [12, 17, 21]及其中的参考文献。

我们聚焦于低马赫数流动的数学理论。因此，我们引入如下尺度变换（见 [1, 20]）

$$\varrho(t,x) = \varrho^\varepsilon(\varepsilon t, x),\quad \boldsymbol{u}(t,x) = \varepsilon \boldsymbol{u}^\varepsilon(\varepsilon t, x),\quad \theta(t,x) = \theta^\varepsilon(\varepsilon t, x)$$

and

$$\mu = \varepsilon\mu^\varepsilon, \lambda = \varepsilon\lambda^\varepsilon, \kappa = \varepsilon\kappa^\varepsilon,$$

其中 $\varepsilon > 0$ 是马赫数。因此，目标系统可写为

$$\begin{cases} \partial_t \varrho^\varepsilon + \operatorname{div}(\varrho^\varepsilon \boldsymbol{u}^\varepsilon) = 0, \\ \varrho^\varepsilon(\partial_t \boldsymbol{u}^\varepsilon + \boldsymbol{u}^\varepsilon \cdot \nabla \boldsymbol{u}^\varepsilon) - \mu^\varepsilon \Delta \boldsymbol{u}^\varepsilon - (\lambda^\varepsilon + \mu^\varepsilon)\nabla \operatorname{div} \boldsymbol{u}^\varepsilon + \dfrac{\nabla(\varrho^\varepsilon \theta^\varepsilon)}{\varepsilon^2} = 0, \\ \varrho^\varepsilon(\partial_t \theta^\varepsilon + \boldsymbol{u}^\varepsilon \cdot \nabla \theta^\varepsilon) - \kappa^\varepsilon \Delta \theta^\varepsilon + \varrho^\varepsilon \theta^\varepsilon \operatorname{div} \boldsymbol{u}^\varepsilon = \varepsilon^2 \left(2\mu^\varepsilon |D\boldsymbol{u}^\varepsilon|^2 + \lambda^\varepsilon (\operatorname{div} \boldsymbol{u}^\varepsilon)^2\right) \\ (\varrho^\varepsilon, \boldsymbol{u}^\varepsilon, \theta^\varepsilon)|_{t=0} = (\varrho_0^\varepsilon, \boldsymbol{u}_0^\varepsilon, \theta_0^\varepsilon). \end{cases} \tag{1.3}$$

我们假设系数 $\mu^\varepsilon, \lambda^\varepsilon, \kappa^\varepsilon$ 均为常数，且满足 $\mu^\varepsilon > 0, \nu^\varepsilon := 2\mu^\varepsilon + \lambda^\varepsilon > 0, \kappa^\varepsilon > 0$ 。为简单起见，我们省略角标 ε。取参考密度和温度为 1，并考虑如下形式的非 well-prepared 初值：$\varrho_0^\varepsilon = 1 + \varepsilon a_0$ 和$\theta^\varepsilon = 1 + \varepsilon\vartheta_0$。相应地，我们引入满足以下条件的未知量 a^ε 和 ϑ^ε

$$\varrho^\varepsilon = 1 + \varepsilon a^\varepsilon, \quad \theta^\varepsilon = 1 + \varepsilon\vartheta^\varepsilon.$$

于是，$(a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon)$满足以下方程组：

$$\begin{cases} \partial_t a^\varepsilon + \dfrac{\mathrm{div}\boldsymbol{u}^\varepsilon}{\varepsilon} = -\mathrm{div}(a^\varepsilon \boldsymbol{u}^\varepsilon), \\ \partial_t \boldsymbol{u}^\varepsilon + \boldsymbol{u}^\varepsilon \cdot \nabla \boldsymbol{u}^\varepsilon - \mathcal{L}\boldsymbol{u}^\varepsilon + \dfrac{\nabla(a^\varepsilon + \vartheta^\varepsilon)}{\varepsilon} = I(\varepsilon a^\varepsilon)\left(\dfrac{\nabla(a^\varepsilon + \vartheta^\varepsilon)}{\varepsilon} + \nabla(a^\varepsilon\vartheta^\varepsilon) - \mathcal{L}\boldsymbol{u}^\varepsilon\right) - \nabla(a^\varepsilon\vartheta^\varepsilon), \\ \partial_t \vartheta^\varepsilon - \kappa\Delta\vartheta^\varepsilon + \dfrac{\mathrm{div}\boldsymbol{u}^\varepsilon}{\varepsilon} = \varepsilon(1 - I(\varepsilon a^\varepsilon))\big(2\mu|D\boldsymbol{u}^\varepsilon|^2 + \lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big) - I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^\varepsilon - \mathrm{div}(\vartheta^\varepsilon\boldsymbol{u}^\varepsilon), \\ (a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon)\,|_{t=0} = (a_0, \boldsymbol{u}_0, \vartheta_0), \end{cases} \tag{1.4}$$

其中

$$\mathcal{L}(\boldsymbol{u}) := \mu\Delta\boldsymbol{u} + (\mu + \lambda)\nabla\mathrm{div}\boldsymbol{u}, \quad I(a) := \frac{a}{1+a}.$$

为了处理方程组（散度$_\varepsilon$ $\boldsymbol{u}^\varepsilon$ 和 ∇ $(a^\varepsilon+\vartheta^\varepsilon)$ ε 中的奇异项（1.4），我们引入未知量

$$\tau^\varepsilon = \frac{a^\varepsilon + \vartheta^\varepsilon}{\sqrt{2}} \quad \text{and} \quad \sigma^\varepsilon = \frac{\vartheta^\varepsilon - a^\varepsilon}{\sqrt{2}}.$$

其中 σ^ε 和 τ^ε 分别由方程（1.4）$_3$ 与（1.4）$_1$ 之和，以及（1.4）$_3$ 与（1.4）$_1$ 之差得到。因此，我们可以将 τ^ε 和 $\mathbb{Q}\boldsymbol{u}^\varepsilon$ 以及 σ^ε和 $\mathbb{P}\boldsymbol{u}^\varepsilon$ 相匹配，其中 $\mathbb{P}$ 和 $\mathbb{Q}$分别是到无散向量场和无旋向量场的标准投影，定义为

$$\mathbb{Q} := -\nabla(-\Delta)^{-1}\mathrm{div}, \quad \mathbb{P} := I - \mathbb{Q}.$$

于是，所得系统可写为

$$\begin{cases} \partial_t\tau^\varepsilon + \dfrac{\sqrt{2}}{\varepsilon}\mathrm{div}\mathbb{Q}\boldsymbol{u}^\varepsilon = \dfrac{\sqrt{2}\varepsilon}{2}\Big(1 - I(\varepsilon a^\varepsilon)\Big)\Big(2\mu|D\boldsymbol{u}^\varepsilon|^2 + \lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\Big) - \mathrm{div}(\tau^\varepsilon\boldsymbol{u}^\varepsilon) \\ \partial_t\mathbb{Q}\boldsymbol{u}^\varepsilon + \dfrac{\sqrt{2}}{\varepsilon}\nabla\tau^\varepsilon = \mathbb{Q}\left(I(\varepsilon a^\varepsilon)\left(\dfrac{\nabla(a^\varepsilon + \vartheta^\varepsilon)}{\varepsilon} + \nabla(a^\varepsilon\vartheta^\varepsilon) - \mathcal{L}\boldsymbol{u}^\varepsilon\right) + \nu\Delta\boldsymbol{u}^\varepsilon - \boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon - \nabla(a^\varepsilon\vartheta^\varepsilon)\right) \end{cases} \tag{1.5}$$

and

$$\begin{cases} \partial_t\sigma^\varepsilon + \mathbb{P}\boldsymbol{u}^\varepsilon\cdot\nabla\sigma^\varepsilon - \dfrac{\kappa}{2}\Delta\sigma^\varepsilon = \dfrac{\kappa}{2}\Delta\tau^\varepsilon - \mathrm{div}(\sigma^\varepsilon\mathbb{Q}\boldsymbol{u}^\varepsilon) - \dfrac{\sqrt{2}\kappa}{2}I(\varepsilon a^\varepsilon)\Delta\vartheta^\varepsilon \\ \qquad\qquad + \dfrac{\sqrt{2}\varepsilon}{2}(1 - I(\varepsilon a^\varepsilon))\big(2\mu|D\boldsymbol{u}^\varepsilon|^2 + \lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big), \\ \partial_t\mathbb{P}\boldsymbol{u}^\varepsilon - \mu\Delta\mathbb{P}\boldsymbol{u}^\varepsilon + \mathbb{P}(\mathbb{P}\boldsymbol{u}^\varepsilon\cdot\nabla\mathbb{P}\boldsymbol{u}^\varepsilon) = \mathbb{P}\left(I(\varepsilon a^\varepsilon)\left(\dfrac{\nabla(a^\varepsilon + \vartheta^\varepsilon)}{\varepsilon} + \nabla(a^\varepsilon\vartheta^\varepsilon) - \mathcal{L}\boldsymbol{u}^\varepsilon\right)\right) \\ \qquad\qquad - \mathbb{P}\Big(\mathbb{Q}\boldsymbol{u}^\varepsilon\cdot\nabla\mathbb{P}\boldsymbol{u}^\varepsilon + \boldsymbol{u}^\varepsilon\cdot\nabla\mathbb{Q}\boldsymbol{u}^\varepsilon\Big). \end{cases} \tag{1.6}$$

由波动方程的色散估计，可以预期当马赫数趋于零时，$(\tau^\varepsilon, \mathbb{Q}\boldsymbol{u}^\varepsilon)$在某些空间中趋于零。我们还预期$(\sigma^\varepsilon, \mathbb{P}\boldsymbol{u}^\varepsilon)$收敛于下述布西内斯克系统的解$(\Theta,\mathbf{v})$:

$$\begin{cases} \partial_t \Theta + \mathbf{v}\cdot\nabla\Theta - \dfrac{\kappa}{2}\Delta\Theta = 0, \\ \partial_t \mathbf{v} - \mu\Delta\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla\pi = 0, \quad \mathrm{div}\mathbf{v} = 0. \end{cases} \tag{1.7}$$

详细的收敛结果将在定理 1.1中给出。接下来，我们回顾若干与本研究密切相关的、关于 Besov 空间中低马赫数极限的结果，并参见[1, 5, 13, 14, 19, 20] Sobolev空间框架下的相关结果。对于全空间和周期区域中的可压缩纳维–斯托克斯系统，Danchin [6, 8]研究了当马赫数趋于零时解是否收敛到不可压缩纳维–斯托克斯系统的问题，并证明了在临界空间中的小初值以及超临界空间中的大初值情形下的收敛性。随后，Danchin和He 将他们的工作推广 [7]到 L^p 型临界空间以及完全可压缩纳维–斯托克斯系统，但仅限于小初值情形。然而，正如 [6, 8], 所提到的，对于具有临界正则性的大初值，收敛性是否成立仍是一个开放问题。直到最近，Fujii [15] 才解决了全空间中可压缩纳维–斯托克斯系统的这一情形[1]。对于纳维–斯托克斯–科特韦系统，Fujii和Li [16] 也证明了解在某些 L^p 型临界空间中收敛到不可压缩模型。除此之外，关于这一问题的研究结果寥寥无几。因此，我们在此进一步研究完全可压缩纳维–斯托克斯方程，包括大初值下临界 Besov 空间中正则解的整体存在性，以及马赫数充分小时解的渐近行为。

我们的主要结果如下。注意，涉及 $\|\cdot\|^l_{\varepsilon}$、$\|\cdot\|^h_{\varepsilon}$和$\beta_0$的范数定义将在第2节中给出。

定理1.1 设

$$2 < p < \min\{d, \frac{2d}{d-2}\}, \quad 0 < \frac{1}{q} \le \min\left\{\frac{1}{2} - \frac{d}{2}\left(\frac{1}{2}-\frac{1}{p}\right), \frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)\right\}$$

and

$$(a_0, \boldsymbol{u}_0, \vartheta_0) \in \left(\dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d) \cap \dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)\right) \times \left(\dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)\right)^d \times \left(\dot{B}^{\frac{d}{2}-2}_{2,1}(\mathbb{R}^d) \cap \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)\right). \tag{1.8}$$

且 $\inf_{x\in\mathbb{R}^d}(1+\varepsilon a_0(x)) > 0$.

若系统 (1.7) 配以初始数据 $\Theta_0 = \frac{\vartheta_0 - a_0}{\sqrt{2}}$和 $\mathbf{v}_0 = \mathbb{P}\boldsymbol{u}_0$存在全局正则解 $(\Theta, \mathbf{v})$ 属于类

$$(\Theta,\mathbf{v}) \in C\left([0,\infty;\dot{B}^{\frac{d}{p}-1}_{p,1}\right) \cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\right) \times C\left([0,\infty;\dot{B}^{\frac{d}{p}-1}_{p,1}\right) \cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\right), \tag{1.9}$$

则存在常数 ε_0和 C，它们依赖于 p、q、d、μ、λ、κ、a_0、$\boldsymbol{u}_0$、ϑ_0、Θ、$\mathbf{v}$，使得对于

[1] Th周期区域中的情况完全不同，因为缺乏色散估计。

任意 $\varepsilon \le \varepsilon_0$，系统 (1.4) 有唯一的整体解 $(a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon)$ 属于类[2]

$$a^\varepsilon \in C\Big([0,\infty;\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}\Big),\quad \boldsymbol{u}^\varepsilon \in C\Big([0,\infty;\dot{B}^{\frac{d}{2}-1}_{2,1}\Big)\cap L^1\Big(0,\infty;\dot{B}^{\frac{d}{2}+1}_{2,1}\Big),$$
$$\vartheta^\varepsilon \in C\Big([0,\infty;\dot{B}^{\frac{d}{2}-2}_{2,1}\Big)\cap L^1_{loc}\Big(0,\infty;\dot{B}^{\frac{d}{2}}_{2,1}\Big), \tag{1.10}$$

其中 无穷$(t,x)\in[0,\infty)\times\mathbb{R}^d$（$1+\varepsilon a^\varepsilon$（$t$、 x））>0，且该解满足：对于某个依赖于 d 和 ν 的正常数β_0，

$$\begin{aligned}&\varepsilon\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{2}}_{2,1}\right)}+\frac{1}{\varepsilon}\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(0,\infty;\dot{B}^{\frac{d}{2}}_{2,1}\right)}+\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{2}-2}_{2,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{2}}_{2,1}\right)}\\&+\|(a^\varepsilon,\vartheta^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(I;\dot{B}^{\frac{d}{2}+1}_{2,1}\right)}+\|\boldsymbol{u}^\varepsilon\|_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)\cap L^1\left(I;\dot{B}^{\frac{d}{2}+1}_{2,1}\right)}\le C,\end{aligned} \tag{1.11}$$

$$\begin{aligned}&\|\mathbb{P}\boldsymbol{u}^\varepsilon\|_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1-(\frac12-\frac1p)}_{p,1}\right)}+\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|_{L^2\left(0,\infty;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\\&+\|\tau^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(0,\infty;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}+\|\sigma^\varepsilon-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1})\cap L^2\left(0,\infty;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\\&+\varepsilon^{-\frac12-\frac1p}\|(\sigma^\varepsilon-\Theta,\tau^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-2}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\le C\varepsilon^{\frac12-\frac1p}.\end{aligned} \tag{1.12}$$

$$\begin{aligned}\lim_{\varepsilon\to0}\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|_{L^q\left(0,\infty:\dot{B}^{\frac{d}{p}-1+\frac2q}_{p,1}\right)\cap L^{q'}\left(0,\infty:\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}&+\varepsilon\|a\|_{L^\infty(0,\infty;\dot{B}^{\frac{d}{p}}_{p,1})}\\&+\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}=0.\end{aligned} \tag{1.13}$$

注 1.1 对于我们的研究而言，初始数据的临界正则性条件之一是

$$(a_0,\boldsymbol{u}_0,\vartheta_0)\in\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)\times\dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)\times\dot{B}^{\frac{d}{2}-2}_{2,1}(\mathbb{R}^d). \tag{1.14}$$

与（*1.14*）相比，假设（*1.8*）还增加了两项要求 $a_0\in\dot{B}^{\frac{d}{2}-1}_{2,1}$，$(\mathbb{R}^d)$ 和 $\vartheta_0\in\dot{B}^{\frac{d}{2}-1}_{2,1}$，$(\mathbb{R}^d)$。要求 $a_0\in\dot{B}^{\frac{d}{2}-1}_{2,1}$，$(\mathbb{R}^d)$ 是针对 a^ε 的低频和中频而设置的，*[9, 10]* 中也出于相同目的引入了这一要求。提出要求 $\vartheta_0\in\dot{B}^{\frac{d}{2}-1}_{2,1}$，$(\mathbb{R}^d)$ 主要有两个原因。首先是为了提高 Θ 的正则性。出于同样的目的，*Danchin*和*He* *[7]*假设 $\Theta_0\in\dot{B}^{\frac{d}{2}-1}_{2,1}$，$(\mathbb{R}^d)$，这使得 Θ 属于

$$\Theta\in C\Big([0,\infty;\dot{B}^{\frac{d}{2}-1}_{2,1}\Big)\cap L^1\Big(0,\infty;\dot{B}^{\frac{d}{2}+1}_{2,1}\Big).$$

[2] $\vartheta^\varepsilon\in L^1_{loc}\Big(0,\infty;\dot{B}^{\frac{d}{2}}_{2,1}\Big)$ means that for any $0<T<\infty$, $\|\vartheta^\varepsilon\|_{L^1\left(0,T;\dot{B}^{\frac{d}{2}}_{2,1}\right)}<\infty$.

这里的要求 $\vartheta_0 \in \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$与 $a_0 \in \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$共同蕴含 $\Theta_0 \in \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$这也使得 Θ 属于更大的函数类 (1.9)，即

$$\Theta \in C\Big([0,\infty;\dot{B}^{\frac{d}{p}-1}_{p,1}\Big) \cap L^1\Big(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\Big),$$

由于嵌入 $\dot{B}^{\frac{d}{2}+s}_{2,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{p}+s}_{p,1}(\mathbb{R}^d)$对任意 $p > 2$，$s \in \mathbb{R}$均成立。第二个目标是建立关于 ε 的一致估计，(1.11) 其左端是我们的能量泛函，可视为 *[10]*中能量泛函的重标度形式。这就要求初始数据的范数关于 ε一致有界，即

$$\varepsilon\|a_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \frac{1}{\varepsilon}\|\vartheta_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-2}_{2,1}} + \|(a_0,\vartheta_0)\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\boldsymbol{u}_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \le C,$$

其中常数 C不必很小。困难来自 $\frac{1}{\varepsilon}\|\vartheta_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-2}_{2,1}}$以及 $\|\vartheta_0\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}$，[3]。

得益于假设 $\vartheta_0 \in \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$，可得

$$\frac{1}{\varepsilon}\|\vartheta_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-2}_{2,1}} + \|\vartheta_0\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \le \frac{1}{\varepsilon}\Big(\frac{\beta_0}{\varepsilon}\Big)^{-1}\|\vartheta_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\vartheta_0\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} = \frac{1}{\beta_0}\|\vartheta_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\vartheta_0\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \le C.$$

注1.2 对于 *[7]*, 满足以下条件的小初值

$$\varepsilon\|a_0^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}}_{p,1}} + \frac{1}{\varepsilon}\|\vartheta_0^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}-2}_{p,1}} + \|\mathbb{Q}\boldsymbol{u}_0^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|(a_0^\varepsilon,\vartheta_0^\varepsilon,\mathbb{Q}\boldsymbol{u}_0^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|\mathbb{P}\boldsymbol{u}_0^\varepsilon\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \ll 1 \tag{1.15}$$

在 $2 \le p < d$ 和 $p \le \frac{2d}{d-2}$的条件下，作者证明了以下全局估计

$$\begin{aligned}
&\varepsilon\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}}_{p,1}\right)} + \frac{1}{\varepsilon}\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(0,\infty;\dot{B}^{\frac{d}{p}}_{p,1}\right)} + \frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-2}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}}_{p,1}\right)} \\
&\quad + \|\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)} + \|(a^\varepsilon,\vartheta^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{2}+1}_{2,1}\right)} \\
&\quad + \|\mathbb{P}\boldsymbol{u}^\varepsilon\|_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)} < \infty
\end{aligned} \tag{1.16}$$

并证明了 (1.12)。显然， $p = 2$ 在 (1.16) 的情形下，估计为 (1.11)。我们还证明了 (1.16) 适用于 $p > 2$的情形，见 (6.33) 和 (6.35)。初始数据的大小不受限制，因此定理 *1.1*可视为将 *[7]* 从小初值推广到大初值的改进。新的收敛结果 (1.13) 在临界正则性层次上成立。

证明定理1.1 的关键思路是三部分分析方法，即高频、中频和低频分析。这一方法受 Fujii [15] 的启发。这引出了

[3] If $\vartheta_0 \in \dot{B}^{\frac{d}{2}-2}_{2,1}(\mathbb{R}^d)$, $\|\vartheta_0\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \le \frac{\beta_0}{\varepsilon}\|\vartheta_0\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-2}_{2,1}}$

对 $M^{\varepsilon,p}{}^{\alpha}{}_{,q}[a,\boldsymbol{u},\vartheta;\tau,\sigma](I)$ 的设计（更多细节见第 4 节）。对于合适的参数和较短的时间区间 I，该量将会很小。在某些乘积估计中，量 $M^{\varepsilon,p}{}^{\alpha}{}_{,q}[a,\boldsymbol{u},\vartheta;\tau,\sigma](I)$ 可以适当地替代能量泛函 $E^{\varepsilon}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](I)$，后者是 (1.14) 左端的量。这样便可将 $E^{\varepsilon}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](I)$ 的平方项转化为 $E^{\varepsilon}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](I)\cdot M^{\varepsilon,p}{}^{\alpha}{}_{,q}[a,\boldsymbol{u},\vartheta;\tau,\sigma](I)$。由于在大初值情形下，能量泛函 $E^{\varepsilon}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](I)$ 未必很小，因此这种转化对于建立全局估计 (1.14) 非常有效。量 $M^{\varepsilon,p}{}^{\alpha}{}_{,q}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon};\tau^{\varepsilon},\sigma^{\varepsilon}](I)$ 可分为两部分，使得

$$M^{\varepsilon,\alpha}_{p,q}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon};\tau^{\varepsilon},\sigma^{\varepsilon}](I)\le M^{\varepsilon,\alpha}_{p,q}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon}-\mathbf{v},\vartheta^{\varepsilon};\tau^{\varepsilon},\sigma^{\varepsilon}-\Theta](I)+\|(\Theta,\mathbf{v})\|_{L^q\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)\cap L^1\left(I;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}$$

对某个 $q\in[2,\infty)$。对于足够短的时间区间 I，我们有

$$\|(\Theta,\mathbf{v})\|_{L^q\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)\cap L^1\left(I;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\ll 1 \tag{1.17}$$

这使得证明分为 N 步，其中

$$\|(\Theta,\mathbf{v})\|_{L^q\left(T_{n-1},T_n;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)}\ll 1$$

对某个序列 $\{T_n\}_{n=0}^{N}$ 使得 $0=T_0<T_1<\cdots<T_{N-1}<T_N=\infty$。处理 $M^{\varepsilon,p}{}^{\alpha}{}_{,q}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon}-\mathbf{v}\ \vartheta^{\varepsilon};\tau^{\varepsilon},\sigma^{\varepsilon}-\Theta](I)$ 时最大的困难在于高频、中频（与未知量有关） $a^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}$ 与低频（涉及未知量 $\tau^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon},\sigma^{\varepsilon}-\Theta$）之间的差异。利用（1.17）、第 2 节中截断 Besov 范数的乘积估计，并选取足够大的 α 和足够小的 ε，使得

$$\alpha\varepsilon\ll 1\text{ and }\|(\Theta,\mathbf{v}\|^{h;\alpha}_{L^{\infty}\left(0,\infty;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\ll 1.$$

本文其余部分安排如下。第 2 节介绍 Besov 空间中的一些概念和分析工具。第 3 节建立线性估计。第 4 节引入能量泛函 $E^{\varepsilon}[a,\boldsymbol{u},\vartheta](I)$、量 $M^{\varepsilon,p}{}^{\alpha}{}_{,q}[a,\boldsymbol{u},\vartheta;\tau,\sigma](I)$，并介绍它们的一些性质。第 5 节建立先验估计。第 6 节完成定理1.1的证明。本文中，C 始终表示一个一般的正常数，该常数可能逐行变化；$C(a_1,\cdots,a_n)$表示 C 仅依赖于 $a_1,\cdots,a_n$。记 $C^{-1}a\le b\le Ca$ 为 $a\approx b$，并将 q 的共轭指数记为 q'，使得$\frac{1}{q'}+\frac{1}{q}=1$。对于两个巴拿赫空间 A 和 B，令 $\|\cdot\|_{A\cap B}=\|\cdot\|_A+\|\cdot\|_B$。

2 预备知识

2.1 函数空间

设 φ 是一个径向、非负且支撑于 $\{\xi\in\mathbb{R}^d$ 上的光滑函数：$\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$并满足 $\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1$ 对任意 $\xi\in\mathbb{R}^d\backslash\{0\}$ 都成立。（例如参见 [2]，其中给出了此类函数存在性的证明。）

$\mathcal{S}(\mathbb{R}^d)$ 和 $\mathcal{S}'(\mathbb{R}^d)$ 分别表示 $\mathbb{R}^d$ 上的施瓦茨函数集和缓增分布集。对于 $f \in \mathcal{S}'(\mathbb{R}^d)$, Littlewood-Paley 分解定义如下

$$\dot{\Delta}_j f := \mathcal{F}^{-1}(\varphi(2^{-j}|\cdot|)\mathcal{F}f), \quad \dot{S}_j f := \sum_{j' \le j-1} \dot{\Delta}_{j'} f$$

其中 $\mathcal{F}$ 和 $\mathcal{F}^{-1}$ 分别为傅里叶变换和傅里叶逆变换。设 $1 \le p, \ r \le \infty$ 和 $s \in \mathbb{R}$ 以及 $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$\dot{B}^s_{p,r}(\mathbb{R}^d) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{\dot{B}^s_{p,r}} < \infty \ \text{ and } \lim_{j \to -\infty} \|S_j f\|_{L^\infty(\mathbb{R}^d)} = 0 \right\},$$

$$\|f\|_{\dot{B}^s_{p,r}} := \left(\sum_{j \in \mathbb{Z}} 2^{jsr} \|\dot{\Delta}_j f\|^r_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{r}}.$$

对于 $1 \le q, \ p, \ r \le \infty$、 $s \in \mathbb{R}$、区间 $I \subset \mathbb{R}$ 以及取值于Besov 空间的时间相关函数 F，我们引入如下范数:

$$\|F\|_{L^q(I;\dot{B}^s_{p,r})} := \left\| \|F(t,\cdot)\|_{\dot{B}^s_{p,r}} \right\|_{L^q(I)}, \quad \|F\|_{\widetilde{L}^q(I;\dot{B}^s_{p,r})} := \left\| \left(2^{js} \|\dot{\Delta}_j F\|_{L^q(I;L^p)} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r}.$$

第二个范数由Chemin 和 Lerner在 [3]中首次引入。借助闵可夫斯基不等式，有

$$\|F\|_{L^q(I;\dot{B}^s_{p,r})} \le \|F\|_{\widetilde{L}^q(I;\dot{B}^s_{p,r})}, \ \text{ if } \ r \le q; \quad \|F\|_{L^q(I;\dot{B}^s_{p,r})} \ge \|F\|_{\widetilde{L}^q(I;\dot{B}^s_{p,r})}, \ \text{ if } \ r \ge q. \tag{2.1}$$

对于 $0 \le \alpha \le \beta \le +\infty$，我们定义

$$f^{l;\alpha} := \sum_{2^j < \alpha} \dot{\Delta}_j f, \quad f^{m;\alpha,\beta} := \sum_{\alpha \le 2^j < \beta} \dot{\Delta}_j f, \quad f^{h;\beta} := \sum_{2^j \ge \beta} \dot{\Delta}_j f.$$

接下来，我们引入截断 Besov 范数[4]

$$\|f\|^{h;\beta}_{\dot{B}^s_{p,r}} := \left\| (2^{js} \|\dot{\Delta}_j f\|_{L^p})_{2^j \ge \beta} \right\|_{\ell^r},$$

$$\|f\|^{m;\alpha,\beta}_{\dot{B}^s_{p,r}} := \left\| (2^{js} \|\dot{\Delta}_j f\|_{L^p})_{\alpha \le 2^j < \beta} \right\|_{\ell^r},$$

$$\|f\|^{l;\alpha}_{\dot{B}^s_{p,r}} := \left\| (2^{js} \|\dot{\Delta}_j f\|_{L^p})_{2^j < \alpha} \right\|_{\ell^r},$$

$$\|F\|^{h;\beta}_{L^q(I;\dot{B}^s_{p,r})} := \left\| \|F\|^{h;\beta}_{\dot{B}^s_{p,r}} \right\|_{L^q(I)} = \left\| \|(2^{js} \|\dot{\Delta}_j F\|_{L^p})_{2^j \ge \beta}\|_{\ell^r} \right\|_{L^q(I)},$$

$$\|F\|^{m;\alpha,\beta}_{L^q(I;\dot{B}^s_{p,r})} := \left\| \|F\|^{m;\alpha,\beta}_{\dot{B}^s_{p,r}} \right\|_{L^q(I)} = \left\| \|(2^{js} \|\dot{\Delta}_j F\|_{L^p})_{\alpha \le 2^j < \beta}\|_{\ell^r} \right\|_{L^q(I)},$$

[4]显然，$\|f\|^{l;\alpha}_{\dot{B}^s_{p,r}} \approx \|f^{l;\alpha}\|_{\dot{B}^s_{p,r}}$, $\|F\|^{l;\alpha}_{L^q(I;\dot{B}^s_{p,r})} \approx \|F^{l;\alpha}\|_{L^q(I;\dot{B}^s_{p,r})}$ $\|F\|^{l;\alpha}_{\widetilde{L}^q(I;\dot{B}^s_{p,r})} \approx \|F^{l;\alpha}\|_{\widetilde{L}^q(I;\dot{B}^s_{p,r})}$。中频和高频的相应不等式与低频的完全类似。

$$\|F\|^{l;\alpha}_{L^q(I;\dot{B}^s_{p,r})} := \left\| \|F\|^{l;\alpha}_{\dot{B}^s_{p,r}} \right\|_{L^q(I)} = \left\| \|(2^{js}\|\dot{\Delta}_j F\|_{L^p})_{2^j<\alpha}\|_{\ell^r} \right\|_{L^q(I)},$$

$$\|F\|^{h;\beta}_{\widetilde{L}^q(I;\dot{B}^s_{p,r})} := \left\| (2^{js}\|\dot{\Delta}_j F\|_{L^q(I;L^p)})_{2^j\geq\beta} \right\|_{\ell^r}$$

$$\|F\|^{m;\alpha,\beta}_{\widetilde{L}^q(I;\dot{B}^s_{p,r})} := \left\| (2^{js}\|\dot{\Delta}_j F\|_{L^q(I;L^p)})_{\alpha\leq 2^j<\beta} \right\|_{\ell^r},$$

$$\|F\|^{l;\alpha}_{\widetilde{L}^q(I;\dot{B}^s_{p,r})} := \left\| (2^{js}\|\dot{\Delta}_j F\|_{L^q(I;L^p)})_{2^j<\alpha} \right\|_{\ell^r}.$$

最后，我们引入 Bony 的抛物乘积分解。对于 f， $g\in\mathcal{S}'(\mathbb{R}^d)$,

$$fg = T_f g + T_g f + R(g,f),$$

其中

$$T_f g = \sum_{j\in\mathbb{Z}} \dot{S}_{j-2} f \dot{\Delta}_j g, \quad T_g f = \sum_{j\in\mathbb{Z}} \dot{S}_{j-2} g \dot{\Delta}_j f, \quad R(g,f) = \sum_{|j-j'|\leq 2} \dot{\Delta}_j g \dot{\Delta}_{j'} f.$$

以下几乎正交性质是众所周知的:

$$\dot{\Delta}_{j'}\dot{\Delta}_j f \equiv 0, \text{ if } |j'-j|\geq 2, \quad \dot{\Delta}_{j'}(\dot{S}_{j-2} f \dot{\Delta}_j g) \equiv 0, \text{ if } |j'-j|\geq 3. \tag{2.2}$$

and

$$\dot{\Delta}_k(\dot{\Delta}_j g \dot{\Delta}_{j'} f) \equiv 0, \text{ if } k-j\geq 5, |j'-j|\leq 2. \tag{2.3}$$

2.2 Besov 空间中的分析工具

本小节汇总一些有用的引理和辅助乘积估计。

引理 2.1 ([2]). 设 $1\leq p,p_1,p_2,r,r_1,r_2\leq\infty$ 和 $s,s_1,s_2\in\mathbb{R}$。若 $f\in\dot{B}^s_{p,r}(\mathbb{R}^d)$，则存在正常数 $C=C(d,s)$ 使得

$$C^{-1}\|f\|_{\dot{B}^s_{p,r}} \leq \|\nabla f\|_{\dot{B}^{s-1}_{p,r}} \leq C\|f\|_{\dot{B}^s_{p,r}};$$

若 $f\in\dot{B}^s_{p,\ r}(\mathbb{R}^d)$，则存在正常数 $C=C(d,s,s_1)$，使得

$$C^{-1}\|f\|_{\dot{B}^s_{p,r}} \leq \|\Lambda^{s_1} f\|_{\dot{B}^{s-s_1}_{p,r}} \leq C\|f\|_{\dot{B}^s_{p,r}};$$

其中 $\Lambda^{s_1} := (\sqrt{-\Delta})^{\ s_1}$;

若 $p_1\leq p_2$ 且 $f\in\dot{B}^s_{p_1,\ r}(\mathbb{R}^d)$，则存在正常数 $C=C(d,p_1,p_2)$，使得

$$\|f\|_{\dot{B}^{s-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r}} \leq C\|f\|_{\dot{B}^s_{p_1,r}};$$

若 $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $s=s_1+s_2$, $s_1\leq 0$ 和 $f\in\dot{B}^{s_1}_{p_1,\ 1}(\mathbb{R}^d)$ $g\in\dot{B}^{s_2}_{p_2,\ 1}(\mathbb{R}^d)$，则存在正常数 $C=C(d,s)$，使得

$$\|T_f g\|_{\dot{B}^s_{p,r}} \leq C\|f\|_{\dot{B}^{s_1}_{p_1,1}}\|g\|_{\dot{B}^{s_2}_{p_2,r}};$$

若 $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $s=s_1+s_2$, $s>0$, $\frac{1}{r}\le\frac{1}{r_1}+\frac{1}{r_2}$ 和 $f\in\dot{B}^{s_1}_{p_1,1}(\mathbb{R}^d)$ $g\in\dot{B}^{s_2}_{p_2,1}(\mathbb{R}^d)$，则存在正常数 $C=C(d,s_1,s_2)$，使得

$$\|R(g,f)\|_{\dot{B}^s_{p,r}}\le C\|f\|_{\dot{B}^{s_1}_{p_1,r_1}}\|g\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

引理 *2.2*（[15]）。设 G 是开区间$I\subset\mathbb{R}$上的光滑函数，在 *0* 处取值为 *0*，并满足（$-R,R$）$\subset I$，其中某个 $R>0$成立。若 $1\le p\le\infty,1\le p_1<\infty,s\in\mathbb{R}$且$s+\frac{d}{p_1}>0,\frac{1}{p}+\frac{1}{p_1}\le 1$，那么对于任意满足 $\|f\|_{L^\infty}<R$的 $f\in\dot{B}^s_{p,1}(\mathbb{R}^d)\cap\dot{B}^{p_1}_{p_1,1}(\mathbb{R}^d)$，存在一个正常数 $C=C(d,s,R,p,p_1,G)$，使得

$$\|G(f)\|_{\dot{B}^s_{p,1}}\le C\left(1+\|f\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1}}\right)\|f\|_{\dot{B}^s_{p,1}}.$$

引理 2.3 若 $1\le p_2,p_3\le p_1\le\infty$, $1\le q_2,q_3\le q_1\le\infty$, $s\in\mathbb{R}$, $\mu>0,0<\alpha\le\infty$ 且 $\boldsymbol{u}$ 是热方程的一个解：

$$\partial_t\boldsymbol{u}-\mu\Delta\boldsymbol{u}=f_1+f_2\quad and\quad \boldsymbol{u}|_{t=0}=\boldsymbol{u}_0,$$

那么存在一个正常数 $C=C(d,q_1,q_2,\mu)$ 使得

$$\|\boldsymbol{u}\|^{l;\alpha}_{\widetilde{L}^{q_1}(I;\dot{B}^{s+\frac{2}{q_1}}_{p_1,1})}\le C\Big(\|\boldsymbol{u}_0\|^{l;\alpha}_{\dot{B}^{s+\frac{d}{p_2}-\frac{d}{p_1}}_{p_2,1}}+\|f_1\|^{l;\alpha}_{\widetilde{L}^{q_2}\left(I;\dot{B}^{s+\frac{d}{p_2}-\frac{d}{p_1}-\frac{2}{q_2'}}_{p_2,1}\right)}+\|f_2\|^{l;\alpha}_{\widetilde{L}^{q_3}\left(I;\dot{B}^{s+\frac{d}{p_3}-\frac{d}{p_1}-\frac{2}{q_3'}}_{p_3,1}\right)}\Big).$$

证明： 引理 2.3 可直接由 [2]中的推论 2.5 得出。 □

接下来，我们给出截断 Besov 范数的几个乘积估计。对于 $0\le\alpha\le\beta\le\infty,j\in\mathbb{Z}$, $\mathbf{1}_{l;\alpha}(j)$, $\mathbf{1}_{m;\alpha,\beta}(j)$ 和 $\mathbf{1}_{h;\beta}(j)$ 定义如下

$$\mathbf{1}_{l;\alpha}(j)=1,\ \text{if } 2^j<\alpha,\quad \mathbf{1}_{l;\alpha}(j)=0,\ \text{if } 2^j\ge\alpha,$$

$$\mathbf{1}_{m;\alpha,\beta}(j)=1,\ \text{if } \alpha\le 2^j<\beta,\quad \mathbf{1}_{m;\alpha,\beta}(j)=0,\ \text{if } 2^j<\alpha \text{ or } 2^j\ge\beta.$$

$$\mathbf{1}_{h;\beta}(j)=1,\ \text{if } 2^j\ge\beta,\quad \mathbf{1}_{l;\alpha}(j)=0,\ \text{if } 2^j<\beta.$$

引理 2.4 设 $1\le p,p_1,p_2,r\le\infty$, $s,s_1,s_2\in\mathbb{R}$, $s=s_1+s_2$, $\sigma=d(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})$ 和 $0\le\alpha\le\beta\le\infty$。如果

$$\frac{1}{p}\le\frac{1}{p_1}+\frac{1}{p_2}\le 1,\quad \frac{1}{p_1}\le\frac{1}{p},\quad s_1\le\frac{d}{p_1},\quad s_1\le\sigma,\tag{2.4}$$

存在一个正常数 $C=C(d,p,p_1,p_2,s,s_1,s_2)$ 使得

$$\|T_fg\|^{m;\alpha,\beta}_{\dot{B}^s_{p,r}}\le C\|f\|^{l;\frac{\beta}{2}}_{\dot{B}^{s_1}_{p_1,1}}\|g\|^{m;\frac{\alpha}{4},4\beta}_{\dot{B}^{s_2+\sigma}_{p_2,r}}$$

对于 $f\in\dot{B}^{s_1}_{p_1,1}(\mathbb{R}^d)$ 以及 $g\in\dot{B}^{s_2+\sigma}_{p_2,r}(\mathbb{R}^d)$。

证明： 我们将证明分为以下两种情况： $s_1<0$ 和 $s_1\geq 0$。若 $s_1<0$，由条件$\frac{1}{p_1}\leq\frac{1}{p}$和(2.2)，对于 $\alpha\leq 2^j<\beta$，可得

$$\begin{aligned}2^{js}\|\dot{\Delta}_j(T_fg)\|_{L^p}&\leq C2^{js}\sum_{|j-j'|\leq 2}\|\dot{S}_{j'-2}f\|_{L^{p_1}}\|\dot{\Delta}_{j'}g\|_{L^{p_1^*}}\\&\leq C\sum_{|j-j'|\leq 2}2^{s_2(j-j')}\mathbf{1}_{m;\frac{\alpha}{4},4\beta}(j')2^{j's_2}\|\dot{\Delta}_{j'}g\|_{L^{p_1^*}}\sum_{k\leq j-1}2^{s_1(j-k)}2^{ks_1}\mathbf{1}_{l;\frac{\beta}{2}}(k)\|\dot{\Delta}_kf\|_{L^{p_1}},\end{aligned}$$

其中 $\frac{1}{p_1^*}=\frac{1}{p}-\frac{1}{p_1}$。条件（2.4）意味着 $\frac{1}{p_1^*}\leq\frac{1}{p_2}$，由杨不等式和伯恩斯坦不等式可得

$$\|T_fg\|^{m;\alpha,\beta}_{\dot{B}^s_{p,r}}\leq C\|f\|^{l;\frac{\beta}{2}}_{\dot{B}^{s_1}_{p_1,1}}\|g\|^{m;\frac{\alpha}{4},4\beta}_{\dot{B}^{s_2}_{p_1^*,r}}\leq C\|f\|^{l;\frac{\alpha}{2}}_{\dot{B}^{s_1}_{p_1,1}}\|g\|^{m;\frac{\alpha}{4},4\beta}_{\dot{B}^{s_2+\sigma}_{p_2,r}}$$

若 $s_1\geq 0$，我们采用如下分解

$$\frac{1}{p}=\frac{1}{p_1}-\frac{s_1}{d}+\frac{1}{p_2}-\frac{(\sigma-s_1)}{d}=\frac{1}{p_3}+\frac{1}{p_4}\text{ with }\frac{1}{p_3}=\frac{1}{p_1}-\frac{s_1}{d}\text{ and }\frac{1}{p_4}=\frac{1}{p_2}-\frac{(\sigma-s_1)}{d}.$$

于是 $s_1\geq 0$和条件（2.4）意味着 $0\leq\frac{1}{p_3}\leq\frac{1}{p_1}$，$0\leq\frac{1}{p_4}\leq\frac{1}{p_2}$。类似地，若 $\alpha\leq 2^j<\beta$，则有

$$\begin{aligned}2^{js}\|\dot{\Delta}_j(T_fg)\|_{L^p}&\leq C2^{js}\sum_{|j-j'|\leq 2}\|\dot{S}_{j'-2}f\|_{L^{p_3}}\|\dot{\Delta}_{j'}g\|_{L^{p_4}}\\&\leq C\sum_{|j-j'|\leq 2}2^{s(j-j')}\mathbf{1}_{m;\frac{\alpha}{4},4\beta}(j')2^{j's}\|\dot{\Delta}_{j'}g\|_{L^{p_4}}\sum_{k\leq j-1}\mathbf{1}_{l;\frac{\beta}{2}}(k)\|\dot{\Delta}_kf\|_{L^{p_3}}.\end{aligned}$$

于是，由杨不等式和伯恩斯坦不等式可得

$$\|T_fg\|^{m;\alpha,\beta}_{\dot{B}^s_{p,r}}\leq C\|f\|^{l;\frac{\beta}{2}}_{\dot{B}^0_{p_3,1}}\|g\|^{m;\frac{\alpha}{4},4\beta}_{\dot{B}^s_{p_4,r}}\leq C\|f\|^{l;\frac{\beta}{2}}_{\dot{B}^{s_1}_{p_1,1}}\|g\|^{m;\frac{\alpha}{4},4\beta}_{\dot{B}^{s_2+\sigma}_{p_2,r}}.\ \square$$

引理 2.5 设 $1\leq p,p_1,p_2,p_3,p_4,r,r_1,r_2,r_3,r_4\leq\infty$，$s_1,s_2,s_3,s_4\in\mathbb{R}$，$0\leq\alpha\leq\infty$和$s>0$。若

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4},\quad s=s_1+s_2=s_3+s_4,\quad \frac{1}{r}\leq\min\left\{\frac{1}{r_1}+\frac{1}{r_2},\frac{1}{r_3}+\frac{1}{r_4}\right\},$$

存在一个正常数 $C=C(d,s,s_1,s_2,s_3,s_4)$，使得

$$\|R(f,g)\|_{\dot{B}^s_{p,r}}\leq C\|f\|^{l;\alpha}_{\dot{B}^{s_1}_{p_1,r_1}}\|g\|^{l;4\alpha}_{\dot{B}^{s_2}_{p_2,r_2}}+C\|f\|^{h;\alpha}_{\dot{B}^{s_3}_{p_3,r_3}}\|g\|^{h;\frac{\alpha}{4}}_{\dot{B}^{s_4}_{p_4,r_4}}\tag{2.5}$$

and

$$\|R(f,g)\|^{h;\alpha}_{\dot{B}^s_{p,r}}\leq C\|f\|^{h;\frac{\alpha}{16}}_{\dot{B}^{s_1}_{p_1,r_1}}\|g\|^{h;\frac{\alpha}{64}}_{\dot{B}^{s_2}_{p_2,r_2}}\tag{2.6}$$

前提是右端有限。

证明： 等式（2.5）是 [15] 中引理 2.5 的直接结果。接下来我们证明（2.6）。对于$\alpha \le 2^j$，有

$$\begin{aligned}2^{js}\|\dot{\Delta}_j R(f,g)\|_{L^p} &\le C2^{js}\sum_{j-j'\le 4}\sum_{|k-j'|\le 2}\|\dot{\Delta}_{j'}f\|_{L^{p_1}}\|\dot{\Delta}_k g\|_{L^{p_2}}\\ &\le C\sum_{j-j'\le 4}2^{s(j-j')}\mathbf{1}_{h;\frac{\alpha}{16}}(j')2^{j's_1}\|\dot{\Delta}_{j'}f\|_{L^{p_1}}\sum_{k\ge j-6}\mathbf{1}_{h;\frac{\alpha}{64}}(k)\|\dot{\Delta}_k g\|_{L^{p_2}}.\end{aligned}$$

由杨氏不等式可得（2.6）。 □

3 线性估计

本节旨在研究系统（1.4）的线性化，该系统为

$$\begin{cases}\partial_t a^\varepsilon + \dfrac{\mathrm{div}\boldsymbol{u}^\varepsilon}{\varepsilon} = F,\\ \partial_t \boldsymbol{u}^\varepsilon - \mathcal{L}\boldsymbol{u}^\varepsilon + \dfrac{\nabla(a^\varepsilon+\vartheta^\varepsilon)}{\varepsilon} = G,\\ \partial_t \vartheta^\varepsilon - \kappa\Delta\vartheta^\varepsilon + \dfrac{\mathrm{div}\boldsymbol{u}^\varepsilon}{\varepsilon} = J,\\ (a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)\,|_{t=0} = (a_0,\boldsymbol{u}_0,\vartheta_0).\end{cases} \tag{3.1}$$

引理 3.1 令 $1<p<\infty$、$s\in\mathbb{R}$ 和 $I\subset\mathbb{R}$ 为一个初始时刻为 t_0 的时间区间。存在正常数 $\beta_0=\beta_0$（d、ν）、$C=C$（d、ν），使得对任意 $\varepsilon\le 1$，

$$\begin{aligned}&\varepsilon\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{s+1}_{p,1}\right)} + \frac{1}{\varepsilon}\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(0,\infty;\dot{B}^{s+1}_{p,1}\right)}\\ &\quad+\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{s-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{s+1}_{p,1}\right)} + \|\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{s}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{s+2}_{p,1}\right)}\\ &\le C\left(\varepsilon\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{s+1}_{p,1}}(t_0)+\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{s-1}_{p,1}}(t_0)+\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{s}_{p,1}}(t_0)\right)+C\|(F,\mathbb{Q}G)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{s}_{p,1}\right)}\\ &\quad+\frac{C}{\varepsilon}\|J\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{s-1}_{p,1}\right)}+C\varepsilon\sum_{2^j\ge\frac{\beta_0}{\varepsilon}}2^{j(s+1)}\left(\|\dot{S}_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}+\|\dot{\Delta}_j F+\dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}\right).\end{aligned} \tag{3.2}$$

证明： 该证明基于对有效速度 $\boldsymbol{\varpi}^\varepsilon := \mathbb{Q}\boldsymbol{u}^\varepsilon+\frac{1}{\varepsilon\nu}\nabla(-\Delta)^{-1}a^\varepsilon$（参见 [7, 15, 18] 以了解更多细节）的分析。目标系统如下

$$\begin{cases}\varepsilon\partial_t a^\varepsilon + \dfrac{a^\varepsilon}{\varepsilon\nu} = \varepsilon F - \mathrm{div}\boldsymbol{\varpi}^\varepsilon,\\ \partial_t\boldsymbol{\varpi}^\varepsilon - \nu\Delta\boldsymbol{\varpi}^\varepsilon + \dfrac{\nabla\vartheta^\varepsilon}{\varepsilon} = \dfrac{1}{\varepsilon\nu}\nabla(-\Delta)^{-1}F + \mathbb{Q}G + \dfrac{\boldsymbol{\varpi}^\varepsilon}{\nu\varepsilon^2} - \dfrac{1}{\nu^2\varepsilon^3}\nabla(-\Delta)^{-1}a^\varepsilon,\\ \partial_t\vartheta^\varepsilon - \kappa\Delta\vartheta^\varepsilon + \dfrac{a^\varepsilon}{\varepsilon\nu} = J - \mathrm{div}\boldsymbol{\varpi}^\varepsilon.\end{cases} \tag{3.3}$$

将 $\dot{\Delta}_j$ 应用于系统（3.3），则有

$$\begin{cases} \varepsilon\partial_t\dot{\Delta}_j a^\varepsilon + \dfrac{\dot{\Delta}_j a^\varepsilon}{\varepsilon\nu} + \varepsilon\dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_j a^\varepsilon = \varepsilon\left(\dot{\Delta}_j F + \dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_j a^\varepsilon\right) - \mathrm{div}\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon, \\ \partial_t\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon - \nu\Delta\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon + \dfrac{\nabla\dot{\Delta}_j\vartheta^\varepsilon}{\varepsilon} = \dfrac{1}{\varepsilon\nu}\nabla(-\Delta)^{-1}\dot{\Delta}_j F + \mathbb{Q}\dot{\Delta}_j G + \dfrac{\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon}{\varepsilon^2\nu} - \dfrac{\nabla(-\Delta)^{-1}\dot{\Delta}_j a^\varepsilon}{\nu^2\varepsilon^3}, \\ \partial_t\dot{\Delta}_j\vartheta^\varepsilon - \kappa\Delta\dot{\Delta}_j\vartheta^\varepsilon + \dfrac{\dot{\Delta}_j a^\varepsilon}{\varepsilon\nu} = \dot{\Delta}_j J - \mathrm{div}\dot{\Delta}_j\varpi^\varepsilon. \end{cases} \tag{3.4}$$

将方程（3.4）$_{1,2,3}$ 乘以 $\dot{\Delta}_j a^\varepsilon|\dot{\Delta}_j a^\varepsilon|^{p-2}$，$\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon|^{p-2}$ 和 $\dot{\Delta}_j\vartheta^\varepsilon|\dot{\Delta}_j\vartheta^\varepsilon|^{p-}$ 分别在 $\mathbb{R}^d$上积分，再分别除以 $|\dot{\Delta}_j a^\varepsilon|^{p-1}$、$|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon|^{p-1}$、$|\dot{\Delta}_j\vartheta^\varepsilon|^{p-1}$ 并在 I上积分，得到

$$\begin{aligned} \varepsilon\|\dot{\Delta}_j a^\varepsilon\|_{L^\infty(I;L^p)} + \frac{1}{\varepsilon}\|\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)} \le{}& C\varepsilon\|\dot{\Delta}_j a^\varepsilon(t_0)\|_{L^p} + C\varepsilon\|\dot{S}_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)} \\ &+ C\varepsilon\|\dot{\Delta}_j F + \dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)} + C2^j\|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon\|_{L^1(I;L^p)}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon\|_{L^\infty(I;L^p)} + 2^{2j}\|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon\|_{L^1(I;L^p)} \le{}& C\|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon(t_0)\|_{L^p} + C\frac{2^j}{\varepsilon}\|\dot{\Delta}_j\vartheta^\varepsilon\|_{L^1(I;L^p)} + \frac{C}{\varepsilon 2^j}\|\dot{\Delta}_j F\|_{L^1(I;L^p)} \\ &+ C\|\dot{\Delta}_j\mathbb{Q}G\|_{L^1(I;L^p)} + \frac{C}{(\varepsilon 2^j)^2}2^{2j}\|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon\|_{L^1(I;L^p)} + \frac{C}{(\varepsilon 2^j)^2}\frac{2^j}{\varepsilon}\|\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \|\dot{\Delta}_j\vartheta^\varepsilon\|_{L^\infty(I;L^p)} + 2^{2j}\|\dot{\Delta}_j\vartheta^\varepsilon\|_{L^1(I;L^p)} \le{}& C\|\dot{\Delta}_j\vartheta^\varepsilon(t_0)\|_{L^p} + \frac{C}{\varepsilon}\|\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)} \\ &+ C2^j\|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon\|_{L^1(I;L^p)} + C\|\dot{\Delta}_j J\|_{L^1(I;L^p)} \end{aligned} \tag{3.7}$$

设 δ 为一个较小的正常数，其值将在后文确定。将（3.5）两边乘以 $2^j\delta$，将（3.6）乘以 $C\delta$，并将（3.7）乘以 $\frac{1}{2^j\varepsilon}$，然后将所得公式相加，得到

$$\begin{aligned} &\varepsilon\delta 2^j\|\dot{\Delta}_j a^\varepsilon\|_{L^\infty(I;L^p)} + \Big(\delta - \frac{C^2\delta}{(\varepsilon 2^j)^2} - \frac{C\varepsilon}{(\varepsilon 2^j)^2}\Big)\frac{2^j}{\varepsilon}\|\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)} + C\delta\|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon\|_{L^\infty(I;L^p)} \\ &+ \Big(C\delta - \frac{C^2\delta}{(\varepsilon 2^j)^2} - \frac{C\varepsilon}{(2^j\varepsilon)^2}\Big)2^{2j}\|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon\|_{L^1(I;L^p)} + \frac{1}{2^j\varepsilon}\|\dot{\Delta}_j\vartheta^\varepsilon\|_{L^\infty(I;L^p)} + \Big(1 - C^2\delta\Big)\frac{2^{2j}}{2^j\varepsilon}\|\dot{\Delta}_j\vartheta^\varepsilon\|_{L^1(I;L^p)} \\ &\le C^*\Big(\varepsilon 2^j\|\dot{\Delta}_j a^\varepsilon(t_0)\|_{L^p} + \|\dot{\Delta}_j\boldsymbol{\varpi}^\varepsilon(t_0)\|_{L^p} + \frac{1}{2^j\varepsilon}\|\dot{\Delta}_j\vartheta^\varepsilon(t_0)\|_{L^p}\Big) + C^*\varepsilon 2^j\|\dot{S}_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)} \\ &+ C^*\varepsilon 2^j\|\dot{\Delta}_j F + \dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)} + C^*\Big(\frac{1}{\varepsilon 2^j}\|\dot{\Delta}_j F\|_{L^1(I;L^p)} + \|\dot{\Delta}_j\mathbb{Q}G\|_{L^1(I;L^p)} + \frac{1}{\varepsilon 2^j}\|\dot{\Delta}_j J\|_{L^1(I;L^p)}\Big). \end{aligned} \tag{3.8}$$

现在令 $\delta = \frac{1}{4C^2}$和 $\varepsilon \le 1$。存在一个正常数 $\beta_0 = \beta_0(d,\nu)$，并且对于任意满足 $2^j\varepsilon \ge \beta_0$ 的 j，有

$$\delta - \frac{C^2\delta}{(\varepsilon 2^j)^2} - \frac{C}{(\varepsilon 2^j)^2} \ge \frac{\delta}{2} \quad \text{and} \quad C\delta - \frac{C^2\delta}{(\varepsilon 2^j)^2} - \frac{C}{(2^j\varepsilon)^2} \ge \frac{C\delta}{2}. \tag{3.9}$$

对于这样的 j，我们还可以得到，对于任意 $t\in I$，

$$\varepsilon 2^j\|\dot{\Delta}_j a^\varepsilon(t)\|_{L^p}+\|\dot{\Delta}_j\varpi^\varepsilon(t)\|_{L^p}\approx\varepsilon 2^j\|\dot{\Delta}_j a^\varepsilon(t)\|_{L^p}+\|\dot{\Delta}_j\mathbb{Q}\boldsymbol{u}^\varepsilon(t)\|_{L^p} \tag{3.10}$$

$$\frac{2^j}{\varepsilon}\|\dot{\Delta}_j a^\varepsilon(t)\|_{L^p}+2^{2j}\|\dot{\Delta}_j\varpi^\varepsilon(t)\|_{L^p}\approx\frac{2^j}{\varepsilon}\|\dot{\Delta}_j a^\varepsilon(t)\|_{L^p}+2^{2j}\|\dot{\Delta}_j\mathbb{Q}\boldsymbol{u}^\varepsilon(t)\|_{L^p}. \tag{3.11}$$

将（3.9）代入（3.8），并利用（3.10）、（3.11），得到

$$\begin{aligned}
&\varepsilon 2^j\|\dot{\Delta}_j a^\varepsilon\|_{L^\infty(I;L^p)}+\frac{2^j}{\varepsilon}\|\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}+\|\dot{\Delta}_j\varpi^\varepsilon\|_{L^\infty(I;L^p)}+2^{2j}\|\dot{\Delta}_j\mathbb{Q}\boldsymbol{u}^\varepsilon\|_{L^1(I;L^p)}\\
&\quad+\frac{1}{2^j\varepsilon}\|\dot{\Delta}_j\vartheta^\varepsilon\|_{L^\infty(I;L^p)}+\frac{2^{2j}}{2^j\varepsilon}\|\dot{\Delta}_j\vartheta^\varepsilon\|_{L^1(I;L^p)}\\
&\le C\Big(\varepsilon 2^j\|\dot{\Delta}_j a^\varepsilon(t_0)\|_{L^p}+\|\dot{\Delta}_j\mathbb{Q}\boldsymbol{u}^\varepsilon(t_0)\|_{L^p}+\frac{1}{2^j\varepsilon}\|\dot{\Delta}_j\vartheta^\varepsilon(t_0)\|_{L^p}\Big)+C\varepsilon 2^j\|\dot{S}_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}\\
&\quad+C\varepsilon 2^j\|\dot{\Delta}_j F+\dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}+C\Big(\|(\dot{\Delta}_j F,\dot{\Delta}_j\mathbb{Q}X)\|_{L^1(I;L^p)}+\frac{1}{\varepsilon 2^j}\|\dot{\Delta}_j J\|_{L^1(I;L^p)}\Big).
\end{aligned} \tag{3.12}$$

将（3.12）两边乘以 2^{js} 并对满足 $2^j\varepsilon\ge\beta_0$的 j 求和，得到（3.2）。 □

引理 3.2 设 $s\in\mathbb{R}$、$0\le\alpha\le\frac{\beta}{\varepsilon}<\infty$ 和 $I\subset\mathbb{R}$ 为一个以 t_0 为初始时刻的时间区间。存在常数 $C=C(d,\nu,\kappa)$，使得

$$\begin{aligned}
&\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)\|^{m;\alpha,\frac{\beta}{\varepsilon}}_{\widetilde{L}^\infty(I;\dot{B}^{s-1}_{2,1})}+\frac{1}{(1+\beta)^2}\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)\|^{m;\alpha,\frac{\beta}{\varepsilon}}_{L^1(I;\dot{B}^{s+1}_{2,1})}\\
&\qquad\le C\left(\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|^{m;\alpha,\frac{\beta}{\varepsilon}}_{\dot{B}^{s-1}_{2,1}}+\|(F,\mathbb{Q}G,J)\|^{m;\alpha,\frac{\beta}{\varepsilon}}_{L^1(I;\dot{B}^{s-1}_{2,1})}\right).
\end{aligned}$$

引理3.2 的证明与 [10], 第 4 节中的论证类似，因此省略证明。将方程（3.1）$_1$ 和（3.1）$_3$，相加，将所得方程除以$\sqrt{2}$，并对方程（3.1）应用算子 $\mathbb{Q}$）$_2$，，得到

$$\begin{cases}
\partial_t\tau^\varepsilon+\dfrac{\sqrt{2}}{\varepsilon}\mathrm{div}\mathbb{Q}\boldsymbol{u}^\varepsilon=\dfrac{1}{\sqrt{2}}(F+J),\\
\partial_t\mathbb{Q}\boldsymbol{u}^\varepsilon-\nu\Delta\mathbb{Q}\boldsymbol{u}^\varepsilon+\dfrac{\sqrt{2}}{\varepsilon}\nabla\tau^\varepsilon=\mathbb{Q}G.
\end{cases}$$

引理 3.3 设 $2\le p$、$q\le\infty$、$0<\alpha<\infty$、$\varsigma\in\mathbb{R}$ 和 $I\subset\mathbb{R}$ 为一个以 t_0 为初始时刻的时间区间。若

$$\frac{1}{q}\le\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right),\quad(p,q,d)\ne(\infty,2,3),$$

则存在正常数 $C=$（d、 p、 q、 ν、 κ），使得

$$\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\alpha}_{\widetilde{L}^q\left(I;\dot{B}^{\frac{d}{p}+\frac{1}{q}+\varsigma}_{p,1}\right)}\leq C\varepsilon^{\frac{1}{q}}(1+\varepsilon\alpha)^2\left(\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|^{l;\alpha}_{\dot{B}^{\frac{d}{2}+\varsigma}_{2,1}}+\|(F,\mathbb{Q}G,J)\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}+\varsigma}_{2,1})}\right),\tag{3.13}$$

$$\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{l;\alpha}_{\widetilde{L}^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\leq C\varepsilon^{\frac{1}{2}-\frac{1}{p}}\left(\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|^{l;\alpha}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|(F,\mathbb{Q}G,J)\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}\right).\tag{3.14}$$

证明： 由 [8], 中的命题 7.1 可得

$$\begin{aligned}\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\alpha}_{\widetilde{L}^q\left(I;\dot{B}^{\frac{d}{p}+\frac{1}{q}+\varsigma}_{p,1}\right)}\leq C\varepsilon^{\frac{1}{q}}&\left(\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)(t_0)\|^{l;\alpha}_{\dot{B}^{\frac{d}{2}+\varsigma}_{2,1}}+\|(F+J,\mathbb{Q}G)\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}+\varsigma}_{2,1})}\right.\\&\left.+\|\Delta\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}+\varsigma}_{2,1})}.\right)\end{aligned}\tag{3.15}$$

利用引理（3.2）可得

$$\begin{aligned}\|\Delta\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}+\varsigma}_{2,1})}&\leq C\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}+\varsigma+2}_{2,1})}\\&\leq C(1+\varepsilon\alpha)^2\left(\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|^{l;\alpha}_{\dot{B}^{\frac{d}{2}+\varsigma}_{2,1}}+\|(F,\mathbb{Q}G,J)\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}+\varsigma}_{2,1})}\right)\end{aligned}\tag{3.16}$$

结合（3.15）和（3.16）可得（3.13）。3.14）的证明与 [7],第 5 节中的论证类似，因此略去证明。
□

4 能量泛函 E^ε 和量 $M^{\varepsilon,p\,\alpha}_{\quad q,}$

考虑上一节的线性估计，我们引入能量泛函 $E^\varepsilon[a,\boldsymbol{u},\vartheta]$（$I$）以及量 $M^{\varepsilon,\ p\ ^{\alpha}_{,\ q}}[a,\boldsymbol{u},\vartheta;\ \tau,\ \sigma]$（$I$）。令 β_0 为引理3.1 中出现的正常数，且$I\subset\mathbb{R}$ 为一个区间。

定义 4.1 对于 $2\leq p<\infty$， $0<\alpha<\infty$ 和$\alpha\leq\frac{\beta_0}{\varepsilon}$， —

$$\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_p}:=\|(a_0,\mathbb{Q}\boldsymbol{u}_0,\vartheta_0)\|^{h;\alpha}_{X^\varepsilon_p}+\|(a_0,\mathbb{Q}\boldsymbol{u}_0,\vartheta_0)\|^{l;\alpha}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\mathbb{P}\boldsymbol{u}_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}$$

其中

$$\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|^{h;\alpha}_{X^\varepsilon_p}:=\varepsilon\|a_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}}_{p,1}}+\frac{1}{\varepsilon}\|\vartheta_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}-2}_{p,1}}+\|\mathbb{Q}\boldsymbol{u}_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|(a_0,\mathbb{Q}\boldsymbol{u}_0,\vartheta_0)\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}$$

对于 $(a_0,\ \boldsymbol{u}_0,\ \vartheta_0)\in\dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)\cap\dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)\times\left(\dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)\right)^d\times\dot{B}^{\frac{d}{2}-2}_{2,1}(\mathbb{R}^d)$。

显然,

$$\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^\varepsilon}\approx\varepsilon\|a_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}}_{2,1}}+\frac{1}{\varepsilon}\|\vartheta_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-2}_{2,1}}+\|(a_0,\vartheta_0)\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|\boldsymbol{u}_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}.$$

定义 4.2 令 $2\le p,\ q<\infty,\ 0<\alpha\le\frac{\beta_0}{\varepsilon}$。 —

- 令 $E^\varepsilon(I)$ 表示在 $I\times\mathbb{R}^d$ 上满足以下范数有限的所有时空分布 $(a、\ \boldsymbol{u}、\ \vartheta)$ 的集合:

$$\begin{aligned}E^\varepsilon[a,\boldsymbol{u},\vartheta](I):=&\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon}\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}}_{2,1})}+\frac{1}{\varepsilon}\|\vartheta\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{2}-2}_{2,1})\cap L^1(I;\dot{B}^{\frac{d}{2}}_{2,1})}\\&+\|(a,\vartheta)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(I;\dot{B}^{\frac{d}{2}+1}_{2,1})}+\|\boldsymbol{u}\|_{L^\infty(I;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(I;\dot{B}^{\frac{d}{2}+1}_{2,1})}\end{aligned}$$

- 令 $N^{\varepsilon,\ p\ \alpha}_{\ q,}(I)$ 表示在 $I\times\mathbb{R}^d$ 上满足以下范数有限的所有时空分布 $(a、\ \boldsymbol{u}、\ \vartheta、\ \tau、\ \sigma)$ 的集合:

$$\begin{aligned}\|(a,\boldsymbol{u},\vartheta;\tau,\sigma)\|_{N^{\varepsilon,\alpha}_{p,q}(I)}:=&\|(a,\mathbb{Q}\boldsymbol{u},\vartheta)\|^{h;\alpha}_{N^{\varepsilon,\alpha}_{p,q}(I)}+\|(\tau,\mathbb{Q}\boldsymbol{u},\sigma)\|^{l;\alpha}_{N^{\varepsilon,\alpha}_{p,q}(I)}\\&+\|\mathbb{P}\boldsymbol{u}\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})\cap L^1(I;\dot{B}^{\frac{d}{p}+1}_{p,1})}\end{aligned}$$

其中

$$\begin{aligned}\|(a,\boldsymbol{u},\vartheta)\|^{h;\alpha}_{N^{\varepsilon,\alpha}_{p,q}(I)}:=&\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|\vartheta\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{p}-2}_{p,1})\cap L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\&+\|\mathbb{Q}\boldsymbol{u}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{p}-1}_{p,1})\cap L^1(I;\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|(a,\mathbb{Q}\boldsymbol{u},\vartheta)\|^{m;\alpha;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(I;\dot{B}^{\frac{d}{2}+1}_{2,1})},\\\|(\tau,\boldsymbol{u},\sigma)\|^{l;\alpha}_{N^{\varepsilon,\alpha}_{p,q}(I)}:=&\|(\tau,\mathbb{Q}\boldsymbol{u})\|^{l;\alpha}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}+\|\sigma\|^{l;\alpha}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})\cap L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\end{aligned}$$

- 我们设

$$\begin{aligned}M^{\varepsilon,\alpha}_{p,q}[a,\boldsymbol{u},\vartheta;\tau,\sigma](I):=&\alpha\varepsilon E^\varepsilon[a,\mathbb{Q}\boldsymbol{u},\vartheta](I)+\|(a,\boldsymbol{u},\vartheta;\tau,\sigma)\|_{N^{\varepsilon,\alpha}_{p,q}(I)}\\&+\left(\|(\tau,\mathbb{Q}\boldsymbol{u})\|^{l;\alpha}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\right)^{\frac{1}{q-1}}\left(E^\varepsilon[a,\mathbb{Q}\boldsymbol{u},\vartheta](I)\right)^{\frac{q-2}{q-1}}.\end{aligned}$$

命题 **定理 4.1** 设 $2\le p、\ q<\infty、\ 0<\alpha\le\frac{\beta_0}{\varepsilon}$ 和 $(a、\ \boldsymbol{u}、\ \vartheta、\ \tau、\ \sigma)$ 是定义在

$I\times\mathbb{R}^d$ 上的时空分布。存在常数 $C_1=C_1(d、\ p、\ q)$ 使得

$$\varepsilon\|a\|_{L^\infty(I;\dot B^{\frac{d}{p}}_{2,1})}+\|a\|_{L^q(I;\dot B^{\frac{d}{2}-1+\frac{2}{q}}_{2,1})}+\|a\|^{l;\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot B^{\frac{d}{2}-1+\frac{2}{q'}}_{2,1})}+\|a\|_{L^2(I;\dot B^{\frac{d}{2}}_{2,1})}\le C_1E^\varepsilon[a,\boldsymbol{u},\vartheta](I),$$

$$\begin{aligned}&\varepsilon\|a\|_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}+\|a\|^{h;\alpha}_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+\|a\|^{h;\alpha}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}+\|a\|^{m;\alpha,\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot B^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\vartheta\|^{m;\alpha,\frac{4\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{p}-1}_{p,1})\cap L^1(I;\dot B^{\frac{d}{p}+1}_{p,1})}\\&+\|\boldsymbol{u}\|_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}+\|\boldsymbol{u}\|_{L^{q'}(I;\dot B^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\boldsymbol{u}\|_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}\le C_1M^{\varepsilon,\alpha}_{p,q}[a,\boldsymbol{u},\vartheta;\tau,\sigma](I),\end{aligned}$$

$$\begin{aligned}&\|a\|_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}+\|a\|_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+\|a\|^{l;\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot B^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\vartheta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})\cap L^{q'}(I;\dot B^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\&\quad\le C_1M^{\varepsilon,\alpha}_{p,q}[a,\boldsymbol{u},\vartheta;\frac{a+\vartheta}{\sqrt2},\frac{\vartheta-a}{\sqrt2}](I).\end{aligned}$$

证明： 命题的证明4.1 只需针对不同频率使用插值不等式和伯恩斯坦不等式；我们仅计算其中少数几项，其余项也可用类似方法得到。

$$\begin{aligned}\varepsilon\|a\|_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}&\le\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}+\varepsilon\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}+\varepsilon\|a\|^{l;\alpha}_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}\\&\le\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}+\varepsilon\cdot\frac{\beta_0}{\varepsilon}\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{p}-1}_{p,1})}+\varepsilon\alpha\|a\|^{l;\alpha}_{L^\infty(I;\dot B^{\frac{d}{p}-1}_{p,1})}\\&\le\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}+C_1\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{2}-1}_{2,1})}+C_1\varepsilon\alpha\|a\|^{l;\alpha}_{L^\infty(I;\dot B^{\frac{d}{2}-1}_{2,1})}\\&\le C_1M^{\varepsilon,\alpha}_{p,q}[a,\boldsymbol{u},\vartheta;\tau,\sigma](I)\end{aligned}$$

$$\begin{aligned}&\|a\|_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\\&\le\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}+\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}+\|a\|^{l;\alpha}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\\&\le C_1\left(\frac{\beta_0}{\varepsilon}\right)^{-1+\frac{2}{q}}\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^q(I;\dot B^{\frac{d}{p}}_{p,1})}+C_1\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^q(I;\dot B^{\frac{d}{2}-1+\frac{2}{q}}_{2,1})}+\|a\|^{l;\alpha}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\\&\le C_1\left(\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}\right)^{1-\frac1q}\left(\frac1\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac{d}{p}}_{p,1})}\right)^{\frac1q}+C_1\left(\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{2}-1}_{2,1})}\right)^{1-\frac1q}\left(\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac{d}{2}+1}_{2,1})}\right)^{\frac1q}\\&\quad+\|a\|^{l;\alpha}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\\&\le C_1\left(\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}+\frac1\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac{d}{p}}_{p,1})}\right)+C_1\|a\|^{m;\alpha;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot B^{\frac{d}{2}-1}_{2,1})\cap L^1(I;\dot B^{\frac{d}{2}+1}_{2,1})}+C_1\|\frac{a+\vartheta}{\sqrt2}-\frac{\vartheta-a}{\sqrt2}\|^{l;\alpha}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\\&\le C_1M^{\varepsilon,\alpha}_{p,q}[a,\boldsymbol{u},\vartheta;\frac{a+\vartheta}{\sqrt2},\frac{\vartheta-a}{\sqrt2}](I),\end{aligned}$$

$$
\begin{aligned}
&\|a\|^{l;\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})} \le \|a\|^{m;\frac{\beta_0}{\varepsilon},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})} + \|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})} + \|a\|^{l;\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\
&\le \left(\frac{4\beta_0}{\varepsilon}\right)^{-1+\frac{2}{q'}} \|a\|^{m;\frac{\beta_0}{\varepsilon},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}}_{p,1})} + C_1\|a\|^{m;\alpha;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(I;\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|\frac{a+\vartheta}{\sqrt{2}} - \frac{\vartheta-a}{\sqrt{2}}\|^{l;\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\
&\le C_1\left(\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})} + \frac{1}{\varepsilon}\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right) + C_1\|a\|^{m;\alpha;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(I;\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
&\quad + C_1\|\frac{\vartheta-a}{\sqrt{2}}\|^{l;\alpha}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})\cap L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})} + \left(\|\frac{a+\vartheta}{\sqrt{2}}\|^{l;\alpha}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\right)^{\frac{1}{q-1}}\left(\|\frac{a+\vartheta}{\sqrt{2}}\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)^{\frac{q-2}{q-1}}\\
&\le C_1 M^{\varepsilon,\alpha}_{p,q}[a,\boldsymbol{u},\vartheta;\frac{a+\vartheta}{\sqrt{2}},\frac{\vartheta-a}{\sqrt{2}}](I)
\end{aligned}
$$

$$
\begin{aligned}
&\|a\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\le \|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})} + \|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})} + \|a\|^{l;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\le \left(\varepsilon\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right)^{\frac12}\left(\frac{1}{\varepsilon}\|a\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right)^{\frac12}\\
&\quad + \left(\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}\right)^{\frac12}\left(\|a\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}+1}_{2,1})}\right)^{\frac12} + \left(\|a\|^{l;\alpha}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\right)^{\frac12}\left(\|a\|^{l;\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right)^{\frac12}\\
&\le C_1 M^{\varepsilon,\alpha}_{p,q}[a,\boldsymbol{u},\vartheta;\frac{a+\vartheta}{\sqrt{2}},\frac{\vartheta-a}{\sqrt{2}}](I).\ \square
\end{aligned}
$$

5 先验估计

本节给出先验估计。令 β_0 为引理 3.1 中出现的正常数，$I\subset\mathbb{R}$ 为一个初始时刻为 t_0的时间区间。假设 $(a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)$ 是系统 (1.4) 在 I 上的一个正则解，且满足 $\varepsilon\|a^\varepsilon\|_{L^\infty(I;L^\infty)}\le\frac12$；同时，$(\Theta,\mathrm{v})$ 是系统 (1.7) 的一个正则解，其初始数据为 $\Theta_0=\frac{\vartheta_0-a_0}{\sqrt{2}}$，且 $\mathrm{v}_0=\mathbb{P}\boldsymbol{u}_0$。显然，$(a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)$ 满足系统 (3.1)，其中

$$
F=-\mathrm{div}(a^\varepsilon\boldsymbol{u}^\varepsilon),\quad G=I(\varepsilon a^\varepsilon)\left(\frac{\nabla(a^\varepsilon+\vartheta^\varepsilon)}{\varepsilon}+\nabla(a^\varepsilon\vartheta^\varepsilon)-\mathcal{L}\boldsymbol{u}^\varepsilon\right)-\nabla(a^\varepsilon\vartheta^\varepsilon)-\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon,
$$

$$
J=\varepsilon(1-I(\varepsilon a^\varepsilon))\big(2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big)-I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^\varepsilon-\mathrm{div}(\vartheta^\varepsilon\boldsymbol{u}^\varepsilon).
$$

引理 5.1 设 $0<\alpha<\infty, \varepsilon\le\min\{\frac{\beta_0}{\alpha},1\}, 2\le p<d, 2\le q<\infty$。则存在常数 $C_2=C_2(d,p,\mu,\kappa,\nu)$ 使得

$$E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\le C_2\|(a^\varepsilon,\ \boldsymbol{u}^\varepsilon,\ \vartheta^\varepsilon)(t_0)\|_{X_2^\varepsilon}+C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\cdot\left(M_{p,q}^{\varepsilon,\alpha}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)+\left(M_{p,q}^{\varepsilon,\alpha}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\right)^2\right).$$

证明： 我们发现

$$\partial_t\mathbb{P}\boldsymbol{u}^\varepsilon-\mu\Delta\mathbb{P}\boldsymbol{u}^\varepsilon=\mathbb{P}G.$$

由引理2.3、引理 3.1 和引理 3.2，可得

$$E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\le C_2\|(a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|_{X_2^\varepsilon}+C_2\|(F,G)\|_{L^1(I;\dot B^{\frac d2-1}_{2,1})}+\frac{C_2}{\varepsilon}\|J\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac d2-2}_{2,1})}\tag{5.1}$$
$$+C_2\|J\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac d2-1}_{2,1})}+C_2\varepsilon\sum_{2^j\ge\frac{\beta_0}{\varepsilon}}2^{j\frac d2}\left(\|\dot S_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot\Delta_j a^\varepsilon\|_{L^1(I;L^2)}+\|\dot\Delta_jF+\dot S_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot\Delta_ja^\varepsilon\|_{L^1(I;L^2)}\right).$$

在后续计算中，我们将频繁使用命题 4.1 。我们有

$$\begin{aligned}&\|\big(\mathrm{div}(a^\varepsilon\boldsymbol{u}^\varepsilon),I(\varepsilon a^\varepsilon)\mathcal{L}\boldsymbol{u}^\varepsilon,\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon\big)\|_{L^1(I;\dot B^{\frac d2-1}_{2,1})}\le C_2\|a^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}\|\boldsymbol{u}^\varepsilon\|_{L^2(I;\dot B^{\frac d2}_{2,1})}\\&\quad+C_2\|a^\varepsilon\|_{L^2(I;\dot B^{\frac d2}_{2,1})}\|\boldsymbol{u}^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}+C_2\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot B^{\frac d2}_{2,1})}\|\boldsymbol{u}^\varepsilon\|_{L^1(I;\dot B^{\frac dp+1}_{p,1})}\\&\quad+C_2\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot B^{\frac dp}_{p,1})}\|\boldsymbol{u}^\varepsilon\|_{L^1(I;\dot B^{\frac d2+1}_{2,1})}+\|\boldsymbol{u}^\varepsilon\|_{L^2(I;\dot B^{\frac d2}_{2,1})}\|\boldsymbol{u}^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}\\&\quad\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I),\end{aligned}\tag{5.2}$$

$$\begin{aligned}&\|I(\varepsilon a^\varepsilon)\frac{\nabla(a^\varepsilon+\vartheta^\varepsilon)}{\varepsilon}\|_{L^1(I;\dot B^{\frac d2-1}_{2,1})}\le C_2\|a^\varepsilon\|_{L^2(I;\dot B^{\frac d2}_{2,1})}\|a^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}+C_2\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot B^{\frac dp}_{p,1})}\frac1\varepsilon\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac d2}_{2,1})}\\&\quad+C_2\|a^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}\|\vartheta^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac d2}_{2,1})}+C_2\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot B^{\frac d2}_{2,1})}\frac1\varepsilon\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac dp}_{p,1})}+\|a^\varepsilon\|_{L^2(I;\dot B^{\frac d2}_{2,1})}\|\vartheta^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}\\&\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I).\end{aligned}\tag{5.3}$$

通过与上述类似的计算可得

$$\|\big(I(\varepsilon a^\varepsilon)\nabla(a^\varepsilon\vartheta^\varepsilon),\nabla(a^\varepsilon\vartheta^\varepsilon)\big)\|_{L^1(I;\dot B^{\frac d2-1}_{2,1})}\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\big)^2\right).\tag{5.4}$$

计算可得

$$\begin{aligned}&\|(1-I(\varepsilon a^\varepsilon))\big(2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac d2-2}_{2,1})}+\varepsilon\|(1-I(\varepsilon a^\varepsilon))\big(2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac d2-1}_{2,1})}\\&\le\|(1-I(\varepsilon a^\varepsilon))\big(2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big)\|_{L^1(I;\dot B^{\frac d2-2}_{2,1})}\\&\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\big)^2\Big),\end{aligned}\tag{5.5}$$

$$\frac{1}{\varepsilon}\|I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-2}_{2,1})}+\|I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}$$
$$\le\frac{C_2}{\varepsilon}\|I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^\varepsilon\|_{L^1(I;\dot{B}^{\frac{d}{2}-2}_{2,1})}\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I),\tag{5.6}$$

$$\begin{aligned}&\frac{1}{\varepsilon}\|\mathrm{div}(\vartheta^\varepsilon\boldsymbol{u}^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-2}_{2,1})}+\|\mathrm{div}(\vartheta^\varepsilon\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}\\&\le\frac{1}{\varepsilon}\|\mathrm{div}(\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\boldsymbol{u}^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-2}_{2,1})}+C_2\|\mathrm{div}(\vartheta^{\varepsilon,l;\frac{\beta_0}{\varepsilon}}\boldsymbol{u}^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}+\frac{C_2}{\varepsilon}\|\mathrm{div}(\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-2}_{2,1})}\\&\quad+\|\mathrm{div}(\vartheta^{\varepsilon,l;\frac{\beta_0}{\varepsilon}}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I).\end{aligned}\tag{5.7}$$

我们还可得

$$\begin{aligned}&\varepsilon\sum_{2^j\ge\frac{\beta_0}{\varepsilon}}2^{j\frac{d}{2}}\left(\|\dot{S}_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot{\Delta}_ja^\varepsilon\|_{L^1(I;L^2)}+\|\dot{\Delta}_jF+\dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_ja^\varepsilon\|_{L^1(I;L^2)}\right)\\&\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+C_2\varepsilon\sum_{j\in\mathbb{Z}}2^{j\frac{d}{2}}\|\dot{\Delta}_jT_{\boldsymbol{u}^\varepsilon}\nabla a^\varepsilon-\dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_ja^\varepsilon\|_{L^1(I;L^2)}\\&\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\\&\quad+C_2\varepsilon\sum_{j\in\mathbb{Z}}\sum_{|j-j'|\le2}2^{j\frac{d}{2}}\left(\|(\dot{S}_{j'-2}\boldsymbol{u}^\varepsilon-\dot{S}_{j-2}\boldsymbol{u}^\varepsilon)\nabla\dot{\Delta}_j\dot{\Delta}_{j'}a^\varepsilon\|_{L^1(I;L^2)}+\|[\dot{\Delta}_j,\dot{S}_{j'-2}\boldsymbol{u}^\varepsilon]\dot{\Delta}_{j'}\nabla a^\varepsilon\|_{L^1(I;L^2)}\right)\\&\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I),\end{aligned}\tag{5.8}$$

其中我们使用了引理 2.97 的交换子估计，该引理来自 [2]，并用于最后一个不等式。综合不等式（5.2）-（5.8），我们得到

$$\begin{aligned}&\|(F,G)\|_{L^1(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}+\frac{1}{\varepsilon}\|J\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-2}_{2,1})}+\|J\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}\\&\quad+\varepsilon\sum_{2^j\ge\frac{\beta_0}{\varepsilon}}2^{j\frac{d}{2}}\left(\|\dot{S}_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot{\Delta}_ja^\varepsilon\|_{L^1(I;L^2)}+\|\dot{\Delta}_jF+\dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_ja^\varepsilon\|_{L^1(I;L^2)}\right)\\&\le C_2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\right)^2\right).\end{aligned}\tag{5.9}$$

将（5.9）代入（5.1），我们便完成了证明。□

引理 5.2 设 $0<\alpha<\infty$，$\varepsilon\le\min\{\frac{\beta_0}{16\alpha},\cancel{1}\}$ 和

$$2<p<\min\{d,\frac{2d}{d-2}\},\quad 0<\frac{1}{q}\le\min\left\{\frac{1}{2}-\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p}\right),\frac{d-1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)\right\}.$$

则存在一个常数 $C_3 = C_3(d, p, q, \mu, \kappa, \nu)$, 使得

$$
\begin{aligned}
&\|(a^\varepsilon, \mathbb{Q}\boldsymbol{u}^\varepsilon, \vartheta^\varepsilon)\|^{h;\alpha}_{N^{\varepsilon,\alpha}_{p,q}(I)} \qquad (5.10)\\
&\le C_3\|(a^\varepsilon, \mathbb{Q}\boldsymbol{u}^\varepsilon, \vartheta^\varepsilon)(t_0)\|^{h;\alpha}_{X^\varepsilon_p} + C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I)\Big)^2\\
&\quad + C_3 M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon - \mathbf{v}, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon - \Theta](I) + \|(\Theta, \mathbf{v})\|^{h;\frac{\alpha}{64}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})\cap L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right),
\end{aligned}
$$

$$
\begin{aligned}
\|(\tau^\varepsilon, \mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\alpha}_{L^q\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)} &\le C_3(\varepsilon\alpha)^{\frac{1}{q}}\|(a^\varepsilon, \mathbb{Q}\boldsymbol{u}^\varepsilon, \vartheta^\varepsilon)(t_0)\|_{X^\varepsilon_2}\\
&\quad + C_3(\varepsilon\alpha)^{\frac{1}{q}}E^\varepsilon[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon](I)\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon](I) + \left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I)\right)^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|\sigma^\varepsilon - \Theta\|^{l;\alpha}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})\cap L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\
&\le C_3\|(\sigma^\varepsilon - \Theta)(t_0)\|^{l;\alpha}_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + C_3(\varepsilon\alpha)^{\frac{1}{q(q-1)}}\|(a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon)(t_0)\|_{X^\varepsilon_2}\\
&\quad + C_3(\varepsilon\alpha)^{\frac{1}{q(q-1)}}E^\varepsilon[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon](I)\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I) + \left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I)\right)^2\right)\\
&\quad + C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I)\Big)^2 + C_3\|\mathbf{v}\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\Theta\|^{h;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\quad + C_3 M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon - \mathbf{v}, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon - \Theta](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I) + \|\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right.\\
&\quad \left. + \left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I)\right)^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathbb{P}\boldsymbol{u}^\varepsilon - \mathbf{v}\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(I;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\\
&\le C_3\|(\mathbb{P}\boldsymbol{u}^\varepsilon - \mathbf{v})(t_0)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I)\Big)^2\\
&\quad + C_3 M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon - \mathbf{v}, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon - \Theta](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I) + \|\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right.\\
&\quad \left. + \left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon](I)\right)^2\right).
\end{aligned}
$$

证明： 由引理 3.1，我们得到

$$
\begin{aligned}
&\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)\|^{h;\alpha}_{N^{\varepsilon,\alpha}_{p,q}(I)}\\
&\le C_3\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|^{h;\alpha}_{X^\varepsilon_p}+C_3\|(F,\mathbb{Q}G)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot B^{\frac dp-1}_{p,1}\right)}+\frac{C_3}{\varepsilon}\|J\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot B^{\frac dp-2}_{p,1}\right)}+C_3\|(F,\mathbb{Q}G,J)\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot B^{\frac d2-1}_{2,1}\right)}\\
&\quad+C_3\varepsilon\sum_{2^j\ge\frac{\beta_0}{\varepsilon}}2^{j(\frac dp)}\left(\|\dot S_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot\Delta_j a^\varepsilon\|_{L^1(I;L^p)}+\|\dot\Delta_j F+\dot S_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot\Delta_j a^\varepsilon\|_{L^1(I;L^p)}\right).\quad(5.11)
\end{aligned}
$$

接下来，我们计算（5.11）右端的各项，这基于引理 2.4、引理 2.5和命题 4.1，并且还利用了这一事实：$a^\varepsilon=\frac{1}{\sqrt{2}}(\tau^\varepsilon-\sigma^\varepsilon+\Theta-\Theta)$ $\vartheta^\varepsilon=\frac{1}{\sqrt{2}}(\tau^\varepsilon+\sigma^\varepsilon-\Theta+\Theta)$。

- F 的估计：

$$
\begin{aligned}
&\|\mathrm{div}(a^\varepsilon\boldsymbol{u}^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot B^{\frac dp-1}_{p,1}\right)}\\
&\le C_3\|a^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}+C_3\|\boldsymbol{u}^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}\|a^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}+C_3\|a^\varepsilon\|^{h;\frac{\beta_0}{16\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{64\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}\\
&\le C_3\|a^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}\left(\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}+\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}+\|\mathbf{v}\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}\right)\\
&\quad+C_3\|\boldsymbol{u}^\varepsilon\|_{L^2(I;\dot B^{\frac dp}_{p,1})}\|a^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}+C_3\|a^\varepsilon\|^{h;\frac{\beta_0}{16\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{64\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}\qquad(5.12)\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)+\|\mathbf{v}\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot B^{\frac dp}_{p,1})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathrm{div}(a^\varepsilon \boldsymbol{u}^\varepsilon)\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\|a^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+C_3\|\boldsymbol{u}^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\|a^\varepsilon\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\
&\quad+C_3\|a^\varepsilon\|^{h;\frac{\alpha}{16}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{h;\frac{\alpha}{64}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\le C_3\|a^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\left(\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\mathbf{v}\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right)\\
&\quad+C_3\|\boldsymbol{u}^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\left(\|\tau^\varepsilon\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\sigma^\varepsilon-\Theta\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\Theta\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right.\\
&\quad\left.+\|a^\varepsilon\|^{m;\alpha,\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right)+C_3\|\boldsymbol{u}^\varepsilon\|^{h;\frac{\alpha}{64}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\left(\|\tau^\varepsilon\|^{m;\frac{\alpha}{16},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\sigma^\varepsilon-\Theta\|^{m;\frac{\alpha}{16},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right.\\
&\quad\left.+\|\Theta\|^{m;\frac{\alpha}{16},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|a^\varepsilon\|^{h;\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right)\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)\right.\\
&\quad\left.+\|\mathbf{v}\|^{m;\frac{\alpha}{4},\frac{\beta_0}{4\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\Theta\|^{h;\frac{\alpha}{16}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})\cap L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right).
\end{aligned}
$$

- G的估计：

$$
\begin{aligned}
&\|I(\varepsilon a^\varepsilon)\frac{\nabla(a^\varepsilon+\vartheta^\varepsilon)}{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\\
&\le C_3\|a^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\left(\|a^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\|\vartheta^\varepsilon\|^{m;\frac{\beta_0}{4\varepsilon},\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right)+C_3\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\quad+C_3\|a^\varepsilon+\vartheta^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|a^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+C_3\|a^\varepsilon+\vartheta^\varepsilon\|^{h;\frac{\beta_0}{16\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|a^\varepsilon\|^{h;\frac{\beta_0}{64\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I),
\end{aligned}
$$

$$
\begin{aligned}
&\|I(\varepsilon a^\varepsilon)\frac{\nabla(a^\varepsilon+\vartheta^\varepsilon)}{\varepsilon}\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\|a^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\left(\|a^\varepsilon+\vartheta^\varepsilon\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|a^\varepsilon\|^{m;\alpha,\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\vartheta^\varepsilon\|^{m;\alpha,\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right)\\
&\quad+C_3\left(\|a^\varepsilon+\vartheta^\varepsilon\|^{l;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\|a^\varepsilon\|^{h;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\|\vartheta^\varepsilon\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right)\|a^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\quad+C_3\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}+C_3\left(\|a^\varepsilon+\vartheta^\varepsilon\|^{m;\frac{\alpha}{16},\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\|a^\varepsilon\|^{h;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right.\\
&\quad\left.+\|\vartheta^\varepsilon\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right)\|a^\varepsilon\|^{h;\frac{\alpha}{64}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+C_3\varepsilon\|a^\varepsilon\|^{h;\frac{\alpha}{64}}_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I),
\end{aligned}
$$

$$
\begin{aligned}
&\|\nabla(a^\varepsilon\vartheta^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\\
&\le C_3\|a^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\vartheta^\varepsilon\|^{m;\frac{\beta_0}{4\varepsilon},\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+C_3\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}+C_3\|\vartheta^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|a^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\quad+C_3\|a^\varepsilon\|^{h;\frac{\beta_0}{16\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{64\varepsilon},\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+C_3\varepsilon\|a^\varepsilon\|^{h;\frac{\beta_0}{16\varepsilon}}_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I),
\end{aligned}
$$

$$
\begin{aligned}
&\|\nabla(a^\varepsilon\vartheta^\varepsilon)\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&=\|\nabla(T_{a^\varepsilon}\vartheta^\varepsilon+T_{\vartheta^{\varepsilon,l;\frac{\beta_0}{\varepsilon}}}a^\varepsilon+T_{\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}}a^\varepsilon+R(\vartheta^{\varepsilon,l;\frac{\beta_0}{\varepsilon}},a^\varepsilon)+R(\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}},a^\varepsilon))\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\|a^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\left(\|\tau^\varepsilon\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\sigma^\varepsilon-\Theta\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right.\\
&\quad\left.+\|\Theta\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\vartheta^\varepsilon\|^{m;\alpha,\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right)+C_3\left(\|\vartheta^{\varepsilon,l;\frac{\beta_0}{\varepsilon}}\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\right.\\
&\left.+\frac{1}{\varepsilon}\|\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\|_{L^q(I;\dot{B}^{\frac{d}{p}-2+\frac{2}{q}}_{p,1})}\right)\cdot\left(\|\tau^\varepsilon\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|\sigma^\varepsilon-\Theta\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right.\\
&\quad\left.+\|\Theta\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|a^\varepsilon\|^{m;\alpha,\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right)+C_3\left(\|\tau^\varepsilon\|^{m;\frac{\alpha}{16},\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\|\sigma^\varepsilon-\Theta\|^{m;\frac{\alpha}{16},\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right.\\
&\quad\left.+\|\Theta\|^{m;\frac{\alpha}{16},\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\|a^\varepsilon\|^{h;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right)\left(\|\vartheta^{\varepsilon,l;\frac{\beta_0}{\varepsilon}}\|^{h;\frac{\alpha}{64}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\frac{1}{\varepsilon}\|\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\|^{h;\frac{\alpha}{64}}_{L^2(I;\dot{B}^{\frac{d}{p}-1}_{p,1})}\right)\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)+\|\Theta\|^{m;\frac{\alpha}{16},\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})\cap L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|I(\varepsilon a^\varepsilon)\nabla(a^\varepsilon\vartheta^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\\
&\le\|\varepsilon a^\varepsilon\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\|\nabla(a^\varepsilon\vartheta^\varepsilon)\|_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\\
&\le\|\varepsilon a^\varepsilon\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\left(\|a^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\vartheta^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+C_3\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right)\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I),
\end{aligned}
$$

$$
\begin{aligned}
&\|I(\varepsilon a^\varepsilon)\nabla(a^\varepsilon\vartheta^\varepsilon)\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le\|\varepsilon a^\varepsilon\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\|\nabla(a^\varepsilon\vartheta^\varepsilon)\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}+\|\varepsilon a^\varepsilon\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\|\nabla(a^\varepsilon\vartheta^\varepsilon)\|_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\\
&\le C_3\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\right)^2\right)\\
&\quad\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)+\|\Theta\|^{m;\frac{\alpha}{64},\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})\cap L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|I(\varepsilon a^\varepsilon)\mathcal{L}\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\\
&\le \|I(\varepsilon a^\varepsilon)\mathcal{L}\boldsymbol{u}^\varepsilon\|_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\\
&\le C_3(\alpha\varepsilon)\|a^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{l;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\boldsymbol{u}^\varepsilon\|^{h;\alpha}_{L^1(I;\dot{B}^{\frac{d}{p}+1}_{p,1})}\\
&\le C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2+M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta]\,(I)\,,
\end{aligned}
$$

$$
\begin{aligned}
&\|I(\varepsilon a^\varepsilon)\mathcal{L}\boldsymbol{u}^\varepsilon\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\|a^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+C_3(\alpha\varepsilon)\|a^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{l;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\\
&\quad+\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\boldsymbol{u}^\varepsilon\|^{h;\alpha}_{L^1(I;\dot{B}^{\frac{d}{p}+1}_{p,1})}\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)+\|\mathbf{v}\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right)\\
&\quad+C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2,
\end{aligned}
$$

$$
\begin{aligned}
&\|\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}+\|\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\|\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon\|^{h;\alpha}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\|\boldsymbol{u}^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{h,\frac{\alpha}{4}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+C_3\|\boldsymbol{u}^\varepsilon\|^{h,\frac{\alpha}{64}}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{h,\frac{\alpha}{16}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)+\|\mathbf{v}\|^{h;\frac{\alpha}{16}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right).
\end{aligned}
$$

- Estimates for J :

$$
\begin{aligned}
&\|2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-2}_{p,1}\right)}+\varepsilon\|2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\|2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\|^{h;\alpha}_{L^1\left(I;\dot{B}^{\frac{d}{2}-2}_{2,1}\right)}\\
&\le C_3\|\boldsymbol{u}^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{h,\frac{\alpha}{4}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+C_3\|\boldsymbol{u}^\varepsilon\|^{h,\frac{\alpha}{64}}_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\|\boldsymbol{u}^\varepsilon\|^{h,\frac{\alpha}{16}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)+\|\mathbf{v}\|^{h;\frac{\alpha}{16}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|I(\varepsilon a^\varepsilon)\big(2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-2}_{p,1}\right)}+\varepsilon\|I(\varepsilon a^\varepsilon)\big(2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big)\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\|_{L^1\left(I;\dot{B}^{\frac{d}{2}-2}_{2,1}\right)}\\
&\le C_3\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\varepsilon}\|I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-2}_{p,1}\right)}\\
&\le C_3\|I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^{\varepsilon,l;\alpha}\|_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}+\frac{C_3}{\varepsilon}\|I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^{\varepsilon,h;\alpha}\|_{L^1\left(I;\dot{B}^{\frac{d}{p}-2}_{p,1}\right)}\\
&\le C_3\left((\alpha\varepsilon)\|a^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\vartheta^\varepsilon\|^{l;\alpha}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\|a^\varepsilon\|_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\|\vartheta^\varepsilon\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}+\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot{B}^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right)\\
&\le C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2\\
&\quad+M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I),
\end{aligned}
$$

$$
\begin{aligned}
&\|I(\varepsilon a^\varepsilon)\kappa\Delta\vartheta^\varepsilon\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3\|a^\varepsilon\|_{L^q(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\|\vartheta^\varepsilon\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+\|T_{\kappa\Delta\vartheta^{\varepsilon,l;\alpha}}I(\varepsilon a^\varepsilon)\|_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\quad+\frac{C_3}{\varepsilon}\|T_{\kappa\Delta\vartheta^{\varepsilon,h;\alpha}}I(\varepsilon a^\varepsilon)\|_{L^1\left(I;\dot{B}^{\frac{d}{2}-2}_{2,1}\right)}+\|R\big(\kappa\Delta\vartheta^{\varepsilon,l;\alpha},I(\varepsilon a^\varepsilon)\big)\|_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\quad+\|R\big(\kappa\Delta\vartheta^{\varepsilon,h;\alpha},I(\varepsilon a^\varepsilon)\big)\|_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2+C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(\|\Theta\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\right.\\
&\quad\left.+M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\varepsilon}\|\mathrm{div}(\vartheta^\varepsilon \boldsymbol{u}^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-2}_{p,1}\right)}\\
&\le \frac{C_3}{\varepsilon}\|\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\|\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}+C_3\|\vartheta^{\varepsilon,l;\frac{\beta_0}{\varepsilon}}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\|\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\\
&\quad+C_3\|\boldsymbol{u}^\varepsilon\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\left(\|\vartheta^\varepsilon\|^{m;\frac{\beta_0}{4\varepsilon},\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}+\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\right)\\
&\quad+C_3\|\boldsymbol{u}^\varepsilon\|^{\frac{\beta_0}{64\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\left(\|\vartheta^\varepsilon\|^{m;\frac{\beta_0}{16\varepsilon},\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}+\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\right)\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)+\|\mathbf{v}\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathrm{div}(\vartheta^\varepsilon \boldsymbol{u}^\varepsilon)\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{2}-1}_{2,1}\right)}\\
&\le \frac{C_3}{\varepsilon}\|\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\|_{L^q\left(I;\dot{B}^{\frac{d}{p}-2+\frac{2}{q}}_{p,1}\right)}\|\boldsymbol{u}^\varepsilon\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}+C_3\|\vartheta^{\varepsilon,l;\frac{\beta_0}{\varepsilon}}\|_{L^q\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)}\|\boldsymbol{u}^\varepsilon\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}\\
&\quad+C_3\|\boldsymbol{u}^\varepsilon\|_{L^q\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)}\left(\|\vartheta^\varepsilon\|^{m;\frac{\alpha}{4},\alpha}_{L^{q'}\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}+\|\vartheta^\varepsilon\|^{m;\alpha,\frac{4\beta_0}{\varepsilon}}_{L^{q'}\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}\right)\\
&\quad+C_3\|\boldsymbol{u}^\varepsilon\|^{\frac{\alpha}{64}}_{L^q\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)}\left(\|\vartheta^\varepsilon\|^{m;\frac{\alpha}{16},\alpha}_{L^{q'}\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}+\|\vartheta^\varepsilon\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}+\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^{q'}\left(I;\dot{B}^{\frac{d}{p}-2+\frac{2}{q'}}_{p,1}\right)}\right)\\
&\le C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)\right.\\
&\quad\left.+\|\mathbf{v}\|^{m;\frac{\alpha}{4},\frac{4\beta_0}{\varepsilon}}_{L^{q'}\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}+\|\Theta\|^{m;\frac{\alpha}{16},\alpha}_{L^{q'}\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1}\right)}\right).
\end{aligned}
$$

- 对 $\varepsilon\sum\limits_{2^j\ge\frac{\beta_0}{\varepsilon}}2^{j(\frac{d}{p})}\left(\|\dot{S}_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}+\|\dot{\Delta}_j F+\dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}\right)$ 的估计：仿照式（5.12）进行类似计算，并使用引理 2.97 中的交换子估计，可得

[2],我们有

$$\begin{aligned}
&\varepsilon\sum_{2^j\geq\frac{\beta_0}{\varepsilon}}2^{j(\frac{d}{p})}\Big(\|\dot{S}_{j-2}\mathrm{div}\boldsymbol{u}^\varepsilon\cdot\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}+\|\dot{\Delta}_j F+\dot{S}_{j-2}\boldsymbol{u}^\varepsilon\nabla\dot{\Delta}_j a^\varepsilon\|_{L^1(I;L^p)}\Big)\\
&\leq C_3\|\boldsymbol{u}^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}+C_3\|\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\varepsilon\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)} \qquad (5.13)\\
&\quad+C_3\varepsilon\|T_{\nabla a^\varepsilon}\boldsymbol{u}^\varepsilon+R(\nabla a^\varepsilon,\boldsymbol{u}^\varepsilon)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\\
&\quad+C_3\varepsilon\sum_{2^j\geq\frac{\beta_0}{\varepsilon}}\sum_{|j-j'|\leq 2}2^{j\frac{d}{2}}\Big(\|(\dot{S}_{j'-2}\boldsymbol{u}^\varepsilon-\dot{S}_{j-2}\boldsymbol{u}^\varepsilon)\nabla\dot{\Delta}_j\dot{\Delta}_{j'}a^\varepsilon\|_{L^1(I;L^2)}+\|[\dot{\Delta}_j,\dot{S}_{j'-2}\boldsymbol{u}^\varepsilon]\dot{\Delta}_{j'}\nabla a^\varepsilon\|_{L^1(I;L^2)}\Big)\\
&\leq C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)+\|\mathbf{v}\|^{h;\frac{\beta_0}{4\varepsilon}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})}\right).
\end{aligned}$$

代入 F、 G、 J和 (5.13) 的估计到 (5.11) 中，得到 (5.10)。

注意到 $\varepsilon\leq\frac{\beta_0}{16\alpha}$，再利用引理 3.3 和 (5.9) 可得

$$\begin{aligned}
\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\alpha}_{L^q\left(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)}&\leq C_3(\varepsilon\alpha)^{\frac{1}{q}}\left(\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|^{l;\alpha}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|(F,\mathbb{Q}G,J)\|^{l;\alpha}_{L^1(I;\dot{B}^{\frac{d}{2}-1}_{2,1})}\right)\\
&\leq C_3(\varepsilon\alpha)^{\frac{1}{q}}\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|_{X^\varepsilon_2}+C_3(\varepsilon\alpha)^{\frac{1}{q}}E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\\
&\quad\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)+\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\right)^2\right). \qquad (5.14)
\end{aligned}$$

我们看到

$$\partial_t(\sigma^\varepsilon-\Theta)-\frac{\kappa}{2}\Delta(\sigma^\varepsilon-\Theta)=K, \qquad (5.15)$$

其中

$$\begin{aligned}
K=&\frac{\kappa}{2}\Delta\tau^\varepsilon-\mathrm{div}(\sigma^\varepsilon\mathbb{Q}\boldsymbol{u}^\varepsilon)-\frac{\sqrt{2}\kappa}{2}I(\varepsilon a^\varepsilon)\Delta\vartheta^\varepsilon+\frac{\sqrt{2}\varepsilon}{2}(1-I(\varepsilon a^\varepsilon))\big(2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big)\\
&-(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})\nabla\sigma^\varepsilon-\mathbf{v}\nabla(\sigma^\varepsilon-\Theta) \qquad (5.16)
\end{aligned}$$

由引理2.3、引理5.1 和（5.14），

$$
\begin{aligned}
&\|\sigma^\varepsilon-\Theta\|^{l;\alpha}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})\cap L^{q'}(I;\dot B^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\
&\le C_3\|(\sigma^\varepsilon-\Theta)(t_0)\|^{l;\alpha}_{\dot B^{\frac{d}{p}-1}_{p,1}}+C_3\|\tau^\varepsilon\|^{l;\alpha}_{L^{q'}(I;\dot B^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}+C_3(\alpha\varepsilon)\|(1-I(\varepsilon a^\varepsilon))\big(2\mu|D\boldsymbol{u}^\varepsilon|^2+\lambda|\mathrm{div}\boldsymbol{u}^\varepsilon|^2\big)\|^{l;\alpha}_{L^1(I;\dot B^{\frac{d}{p}-2}_{p,1})}\\
&\quad+C_3\|\Big(\mathrm{div}(\sigma^{\varepsilon,l;\frac{\beta_0}{\varepsilon}}\mathbb{Q}\boldsymbol{u}^\varepsilon),(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})\nabla\sigma^{\varepsilon,l;\frac{\beta_0}{\varepsilon}},\mathbf{v}\nabla(\sigma^\varepsilon-\Theta)^{l;\frac{\beta_0}{\varepsilon}}\Big)\|^{l;\alpha}_{L^1(I;\dot B^{\frac{d}{p}-1}_{p,1})}\\
&\quad+C_3\|\Big(\mathrm{div}(a^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\mathbb{Q}\boldsymbol{u}^\varepsilon),(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})\nabla a^{\varepsilon,h;\frac{\beta_0}{\varepsilon}},\mathbf{v}\nabla a^{\varepsilon,h;\frac{\beta_0}{\varepsilon}},\mathbf{v}\nabla\Theta^{h;\frac{\beta_0}{\varepsilon}}\Big)\|^{l;\alpha}_{L^1(I;\dot B^{\frac{d}{p}-1}_{p,1})}\\
&\quad+C_3\alpha\|\Big(\mathrm{div}(\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\mathbb{Q}\boldsymbol{u}^\varepsilon),(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})\nabla\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}},\mathbf{v}\nabla\vartheta^{\varepsilon,h;\frac{\beta_0}{\varepsilon}}\Big)\|^{l;\alpha}_{L^1(I;\dot B^{\frac{d}{p}-2}_{p,1})}+C_3\alpha\|I(\varepsilon a^\varepsilon)\Delta\vartheta^\varepsilon\|^{l;\alpha}_{L^1(I;\dot B^{\frac{d}{p}-2}_{p,1})}\\
&\le C_3\|(\sigma^\varepsilon-\Theta)(t_0)\|^{l;\alpha}_{\dot B^{\frac{d}{p}-1}_{p,1}}+C_3\left(\|\tau^\varepsilon\|^{l;\alpha}_{L^q(I;\dot B^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\right)^{\frac{1}{q-1}}\left(\|\tau^\varepsilon\|^{l;\alpha}_{L^1(I;\dot B^{\frac{d}{p}+1}_{p,1})}\right)^{\frac{q-2}{q-1}}+C_3(\alpha\varepsilon)\big(1+\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}\big)\\
&\quad\cdot\|\boldsymbol{u}^\varepsilon\|^2_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+C_3\left(\|\sigma^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+(\alpha\varepsilon)\frac{1}{\varepsilon}\|\sigma^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac{d}{p}-1}_{p,1})}\right)\cdot\|(\mathbb{Q}\boldsymbol{u}^\varepsilon,\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})\|_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}\\
&\quad+C_3\|\mathbf{v}\|_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}\left(\|\sigma^\varepsilon-\Theta\|^{l;\alpha}_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+\|\sigma^\varepsilon-\Theta\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+\|(a^\varepsilon,\Theta)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+(\alpha\varepsilon)\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac{d}{p}-1}_{p,1})}\right)\\
&\quad+C_3(\alpha\varepsilon)\left(\|a^\varepsilon\|_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}\|\vartheta^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}+\varepsilon\|a^\varepsilon\|_{L^\infty(I;\dot B^{\frac{d}{p}}_{p,1})}\frac{1}{\varepsilon}\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1(I;\dot B^{\frac{d}{p}}_{p,1})}\right)\\
&\le C_3\|(\sigma^\varepsilon-\Theta)(t_0)\|^{l;\alpha}_{\dot B^{\frac{d}{p}-1}_{p,1}}+C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2+C_3\|\mathbf{v}\|_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}\|\Theta\|^{h;\alpha}_{L^2(I;\dot B^{\frac{d}{p}}_{p,1})}\\
&\quad+C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\|\mathbf{v}\|_{L^2\left(I;\dot B^{\frac{d}{p}}_{p,1}\right)}\right.\\
&\quad+\left.\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2\right)+C_3(\varepsilon\alpha)^{\frac{1}{q(q-1)}}\|(a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|_{X^\varepsilon_2}\\
&\quad+C_3(\varepsilon\alpha)^{\frac{1}{q(q-1)}}E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)+\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2\right),
\end{aligned}
$$

我们发现

$$
\partial_t(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})-\mu\Delta(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})=H, \tag{5.17}
$$

其中

$$
\begin{aligned}
H:=&\,\mathbb{P}\left(I(\varepsilon a^\varepsilon)\left(\frac{\nabla(a^\varepsilon+\vartheta^\varepsilon)}{\varepsilon}+\nabla(a^\varepsilon\vartheta^\varepsilon)-\mathcal{L}\boldsymbol{u}^\varepsilon\right)\right)\\
&-\mathbb{P}\Big(\mathbb{Q}\boldsymbol{u}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon+(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})\cdot\nabla\mathbb{P}\boldsymbol{u}^\varepsilon+\mathbf{v}\cdot\nabla(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})+\mathbb{P}\boldsymbol{u}^\varepsilon\cdot\nabla\mathbb{Q}\boldsymbol{u}^\varepsilon)\Big)
\end{aligned} \tag{5.18}
$$

由引理2.3 并按照与 G 的估计类似的方法计算，可得

$$
\begin{aligned}
&\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^\infty\left(I;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\\
&\le C_3\|(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})(t_0)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+C_3\|H\|_{L^1\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\\
&\le C_3\|(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})(t_0)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+C_3(\alpha\varepsilon)\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2\\
&+C_3M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](I)\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\|\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right.\\
&\quad\left.+\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2\right).\ \square
\end{aligned}
$$

引理 5.3 令 $2<p<\min\{d,\frac{2d}{d-2}\}$，那么存在常数 $C_4=C_4(d,p,\mu,\kappa,\nu)$ 使得对于 $\varepsilon\le 1$,

$$
\begin{aligned}
&\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^1\left(I;\dot{B}^{\frac{d}{p}+1-(\frac12-\frac1p)}_{p,1}\right)}\\
&+\|\sigma^\varepsilon-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(I;\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}+\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\\
&\le C_4\left(\|(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})(t_0)\|_{\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}}+\|(\sigma^\varepsilon-\Theta)(t_0)\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}}+\varepsilon^{\frac12-\frac1p}\|(a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|_{X^\varepsilon_2}\right)\\
&+C_4\left(\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}+\|\sigma^\varepsilon-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}+\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\right)\\
&\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2+\|\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right)\\
&+C_4\varepsilon^{\frac12-\frac1p}\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\|(\Theta,\mathbf{v})\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right)\\
&\cdot\left(\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2+M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)+\|(\Theta,\mathbf{v})\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right)\\
&+C_4\varepsilon^{\frac12-\frac1p}E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)+\Big(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\Big)^2\right).\qquad(5.19)
\end{aligned}
$$

证明： 由引理 2.3, （5.17）和（5.18），

$$
\begin{aligned}
&\|\mathbb{P}\boldsymbol{u}^{\varepsilon}-\mathbf{v}\|_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}-1-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)\cap L^{1}\left(I;\dot{B}^{\frac{d}{p}+1-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)} \qquad (5.20)\\
&\leq C_4\|(\mathbb{P}\boldsymbol{u}^{\varepsilon}-\mathbf{v})(t_0)\|_{\dot{B}^{\frac{d}{p}-1-(\frac{1}{2}-\frac{1}{p})}_{p,1}}\\
&+C_4\|(\boldsymbol{u}^{\varepsilon},\mathbf{v})\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\left(\|\mathbb{P}\boldsymbol{u}^{\varepsilon}-\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}+\|\mathbb{Q}\boldsymbol{u}^{\varepsilon}\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}+\|\mathbb{Q}\boldsymbol{u}^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\right)\\
&+C_4\|a^{\varepsilon}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\left(\|\tau^{\varepsilon}\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}+\|a^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}+\|\mathbb{Q}\boldsymbol{u}^{\varepsilon}\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\right)\\
&+C_4\varepsilon\|a^{\varepsilon}\|_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\cdot\left(\frac{1}{\varepsilon}\|\vartheta^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}+\frac{1}{\varepsilon}\|\vartheta^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\varepsilon\|a^{\varepsilon}\|_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right.\\
&\left.+\|\vartheta^{\varepsilon}\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\|a^{\varepsilon}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}+\|\mathbb{P}\boldsymbol{u}^{\varepsilon}\|_{L^1\left(I;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}+\|\mathbb{Q}\boldsymbol{u}^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\right).
\end{aligned}
$$

由引理2.3, （5.15）, （5.16）, 我们有

$$
\begin{aligned}
&\|\sigma^{\varepsilon}-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}-1-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)\cap L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\\
&\leq C_4\|(\sigma^{\varepsilon}-\Theta)(t_0)\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}-1-(\frac{1}{2}-\frac{1}{p})}_{p,1}}+C_4\|\tau^{\varepsilon}\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\\
&+C_4\left(\|\mathbb{P}\boldsymbol{u}^{\varepsilon}-\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}+\|\mathbb{Q}\boldsymbol{u}^{\varepsilon}\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}+\|\mathbb{Q}\boldsymbol{u}^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\right)\\
&\cdot\left(\|\sigma^{\varepsilon}\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}+\|a^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right)\\
&+C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}\|(\mathbb{P}\boldsymbol{u}^{\varepsilon}-\mathbf{v},\mathbb{Q}\boldsymbol{u}^{\varepsilon})\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\frac{1}{\varepsilon}\|\vartheta^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}+C_4\|\sigma^{\varepsilon}-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\|\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\\
&+C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}\left(\|(a^{\varepsilon},\Theta)\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}+\frac{1}{\varepsilon}\|\vartheta^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)}\right)\|\mathbf{v}\|_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\\
&+C_4\varepsilon\|a^{\varepsilon}\|_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\|\vartheta^{\varepsilon}\|^{l;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}+C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}\varepsilon\|a^{\varepsilon}\|_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\frac{1}{\varepsilon}\|\vartheta^{\varepsilon}\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\\
&+C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}\left(1+\varepsilon\|a^{\varepsilon}\|_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\right)\|\boldsymbol{u}^{\varepsilon}\|^2_{L^2\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)} \qquad (5.21)
\end{aligned}
$$

注意到

$$\begin{aligned}\varepsilon\|a^\varepsilon\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)} &\le C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}\|a^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(I;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)} + C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(I;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\\ &\le C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\end{aligned}\tag{5.22}$$

由引理3.3 和（5.9），可得

$$\begin{aligned}&\|\tau^\varepsilon\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)} \le C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}\|(a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(t_0)\|_{X^\varepsilon_2}\\ &+C_4\varepsilon^{\frac{1}{2}-\frac{1}{p}}E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)\cdot\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](I)+\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](I)\right)^2\right)\end{aligned}\tag{5.23}$$

将（5.20）与（5.21）相加，再利用（5.22）、（5.23），命题4.1 可推出（5.19）。 □

6 定理1.1 的证明

由 [4, 6], 可知，若初始数据（a_0， $\boldsymbol{u}_0$， ϑ_0）满足（1.8），且 $\inf_{x\in\mathbb{R}^d}(1+\varepsilon a_0(x)) > 0$，则存在一个正时间 $0<T<\infty$，使得系统（1.4）存在唯一的正则解（a^ε， $\boldsymbol{u}^\varepsilon$， ϑ^ε），其定义在 $[0,T]$上，并满足

$$\begin{aligned}&a^\varepsilon\in C\left([0,T];\dot{B}^{\frac{d}{2}}_{2,1}\cap\dot{B}^{\frac{d}{2}-1}_{2,1}\right),\quad \boldsymbol{u}^\varepsilon\in C\left([0,T];\dot{B}^{\frac{d}{2}-1}_{2,1}\right)\cap L^1\left(0,T;\dot{B}^{\frac{d}{2}+1}_{2,1}\right),\\ &\vartheta^\varepsilon\in C\left([0,T];\dot{B}^{\frac{d}{2}-2}_{2,1}\right)\cap L^1\left(0,T;\dot{B}^{\frac{d}{2}}_{2,1}\right),\quad \inf_{(t,x)\in[0,T]\times\mathbb{R}^d}(1+\varepsilon a(t,x))>0.\end{aligned}\tag{6.1}$$

设 $0<T_1<\infty$为一个正时间，且（a^ε， $\boldsymbol{u}^\varepsilon$， ϑ^ε）是系统（1.4）在 $[0,T_1)$ 上的解，它满足任意6.1），则该解可以延拓至 T_1，前提是

$$\begin{aligned}&\|a^\varepsilon\|_{L^\infty\left(0,T_1;\dot{B}^{\frac{d}{2}}_{2,1}\right)}+\|\boldsymbol{u}^\varepsilon\|_{L^1\left(0,T_1;\dot{B}^{\frac{d}{2}+1}_{2,1}\right)}+\|\vartheta^\varepsilon\|_{L^1\left(0,T_1;\dot{B}^{\frac{d}{2}}_{2,1}\right)}<\infty,\\ &\inf_{(t,x)\in[0,T_1)\times\mathbb{R}^d}(1+\varepsilon a(t,x))>0.\end{aligned}\tag{6.2}$$

我们定义

$$T^*:=\sup\{0<T<\infty:(a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)\text{ satisfies (6.1) for }T\}.$$

利用上述延拓准则[5]，，我们将证明 $T^*=\infty$。

设 $(\Theta,\mathbf{v})$ 是系统 (1.7) 的正则解，并辅以初始数据 $\Theta_0=\frac{\vartheta_0-a_0}{\sqrt{2}}$和 $\mathbf{v}_0=\mathbb{P}\boldsymbol{u}_0$，且属于类——(1.9)。注意到 $\|f\|_{L^\infty}\le C_5(d,p)\|f\|_{\dot{B}^{\frac{d}{p}}_{p,1}}$，则我们令

[5]延拓准则的证明与 [2]中的命题 10.10 类似。

$C_0 = \max\{1, C_1, C_2, C_3, C_4, C_5\}$ 其中 C_1, C_2, C_3, C_4 出现在前面的引理中，并且令

$$\delta_0 := \min\left\{\frac{1}{96C_0^2},\ \frac{1}{2}\|(\mathbf{v},\Theta)\|_{L^q\left(0,\infty;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\right\}.$$

根据 [15], 引理 2.10，存在一个正数 $N_0 = N_0(\delta_0,\ p,\ q,\ \mathbf{v}\ \Theta)$以及一个时间序列 $\{T_n = T_n(\delta_0,\ p,\ q,\ \mathbf{v}\ \Theta)\}_{n=0}^{N_0}$ 使得 $0 = T_0 < T_1 < \cdots < T_{N_0-1} < T_{N_0} = \infty$ 并且

$$\|(\mathbf{v},\Theta)\|_{L^q\left(T_{n-1},T_n;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)\cap L^1\left(T_{n-1},T_n;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)} \le \delta_0,\quad n = 1, 2, \cdots, N_0. \tag{6.3}$$

注意，如果 $\varepsilon \le 1$,

$$\|(a_0, \mathbb{Q}\boldsymbol{u}_0, \vartheta_0)\|^{h;\alpha}_{X^\varepsilon_p} \le C\left(\|a_0\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|(\vartheta_0, \mathbb{Q}\boldsymbol{u}_0)\|^{h;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|(a_0, \vartheta_0, \mathbb{Q}\boldsymbol{u}_0)\|^{m;\alpha,\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\right),$$

其中常数 C 取决于 $d,\ p_\circ$

存在一个正常数 $\alpha_0 = \alpha_0(d,\ p,\ q,\ C_0,\ a_0,\ \boldsymbol{u}_0,\ \vartheta_0,\ N_0,\ \delta_0,\ \mathbf{v}\ \Theta)$，使得

$$\|(\Theta, \mathbf{v})\|^{h;\frac{\alpha_0}{64}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})\cap L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})} \le \frac{\delta_0}{(4C_0)^{N_0}}. \tag{6.4}$$

并且对于 $\varepsilon \le 1$,

$$\|(a_0, \mathbb{Q}\boldsymbol{u}_0, \vartheta_0)\|^{h;\alpha_0}_{X^\varepsilon_p} \le \frac{\delta_0}{(4C_0)^{N_0}}. \tag{6.5}$$

那么，对于这样的 α_0，我们选取一个正常数 $\varepsilon_0 = \varepsilon_0(\alpha_0,\ \delta_0,\ N_0,\ q,\ C_0,\ a_0,\ \boldsymbol{u}_0,\ \vartheta_0) \le \min\{1, \frac{1}{\alpha_0}, \frac{\beta_0}{16\alpha_0}\}$，使得

$$(\alpha_0\varepsilon)^{\frac{1}{q(q-1)}}\left(\|(a_0, \boldsymbol{u}_0, \vartheta_0)\|_{X^\varepsilon_2} + \delta_0\right) \le \frac{1}{36}\frac{\delta_0}{(4C_0)^{N_0}} \tag{6.6}$$

对于任意 $0 < \varepsilon \le \varepsilon_{0\circ}$

现在我们固定一个穆赫数 $0 < \varepsilon \le \varepsilon_{0\circ}$ 我们将使用归谬法来证明$T^* \ge T_{1\circ}$ 我们定义一个时间

$$T_1^* := \sup\{0 < T < T^*, (6.7) - (6.10) \text{ hold at } T\}$$

其中

$$E^\varepsilon[a^\varepsilon, \boldsymbol{u}^\varepsilon, \vartheta^\varepsilon](0,T) \le 4C_0\|(a_0, \boldsymbol{u}_0, \vartheta_0)\|_{X^\varepsilon_2}, \tag{6.7}$$

$$M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon, \boldsymbol{u}^\varepsilon - \mathbf{v}, \vartheta^\varepsilon; \tau^\varepsilon, \sigma^\varepsilon - \Theta](0,T) \le \frac{\delta_0}{(4C_0)^{N_0-1}}, \tag{6.8}$$

$$\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(0,T;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(0,T;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\le\frac{1}{4}\frac{\delta_0}{(4C_0)^{N_0-1}}. \tag{6.9}$$

$$\begin{aligned}&\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(I;\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^1\left(I;\dot{B}^{\frac{d}{p}+1-(\frac12-\frac1p)}_{p,1}\right)}+\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\\&+\|\sigma^\varepsilon-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(I;\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\le 2\varepsilon^{\frac12-\frac1p}4C_0\Big(\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}+1\Big)\end{aligned} \tag{6.10}$$

显然， $T_1^*>0$。首先假设 $T^*<T_1$。令 $0<T<T_1^*$。由命题4.1、式（6.3）、式（6.3），可得

$$\begin{aligned}&M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](0,T)\\&\le M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](0,T)+2\|(\mathbf{v},\Theta)\|_{L^q\left(0,T;\dot{B}^{\frac{d}{p}-1+\frac2q}_{p,1}\right)\cap L^1\left(0,T;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\le 3\delta_0,\end{aligned}$$

$$\begin{aligned}\varepsilon\|a^\varepsilon\|_{L^\infty(0,T;L^\infty)}&\le C_0\varepsilon\|a^\varepsilon\|_{L^\infty(0,T;\dot{B}^{\frac{d}{p}}_{p,1})}\\&\le C_0^2M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](0,T)\le 3C_0^2\delta_0\le\frac12,\end{aligned}$$

这意味着

$$\frac12\le 1+\varepsilon a(t,x)\le\frac32,\quad\forall(t,x)\in[0,T]\times\mathbb{R}^d. \tag{6.11}$$

由引理5.1,

$$\begin{aligned}E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](0,T)&\le C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}+4C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}(3\delta_0+9\delta_0^2)\\&\le 2C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}\end{aligned} \tag{6.12}$$

由引理 3.3、引理 5.2、（6.4）、（6.5）以及（6.6）

$$\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)\|^{h;\alpha}_{N^{\varepsilon,\alpha}_{p,q}(0,T)}\le C_0\frac{\delta_0}{(4C_0)^{N_0}}+C_0(\alpha\varepsilon)9\delta_0^2+C_03\delta_0\Big(\frac{\delta_0}{(4C_0)^{N_0}}+\frac{\delta_0}{(4C_0)^{N_0-1}}\Big), \tag{6.13}$$

$$\begin{aligned}&\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\alpha}_{L^q\left(0,T;\dot{B}^{\frac{d}{p}-1+\frac2q}_{p,1}\right)}+\|\sigma^\varepsilon-\Theta\|^{l;\alpha}_{L^q(0,T;\dot{B}^{\frac{d}{p}-1+\frac2q}_{p,1})\cap L^{q'}(0,T;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\&\le C_0(\alpha\varepsilon)^{\frac1q}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}+C_0(\alpha\varepsilon)^{\frac1q}4C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}(3\delta_0+9\delta_0^2)+C_0(\alpha\varepsilon)^{\frac{1}{(q-1)}}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}\\&+C_0(\alpha\varepsilon)^{\frac{1}{(q-1)}}4C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}(3\delta_0+9\delta_0^2)+C_0(\alpha\varepsilon)9\delta_0^2+C_0\delta_0\frac{\delta_0}{(4C_0)^{N_0}}\\&+C_0\frac{\delta_0}{(4C_0)^{N_0-1}}(4\delta_0+9\delta_0^2)\end{aligned} \tag{6.14}$$

$$
\begin{aligned}
&(\alpha\varepsilon)E^{\varepsilon}[a^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](0,T)+\left(\|(\tau^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon})\|^{l;\alpha}_{L^q(0,T;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\right)^{\frac{1}{q-1}}\left(E^{\varepsilon}[a^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](0,T)\right)^{\frac{q-2}{q-1}}\\
&\leq(\alpha\varepsilon)4C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^{\varepsilon}_2}+C_0(\alpha\varepsilon)^{\frac{1}{q(q-1)}}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^{\varepsilon}_2}\\
&\quad+C_0(\alpha\varepsilon)^{\frac{1}{q(q-1)}}4C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^{\varepsilon}_2}(3\delta_0+9\delta_0^2),
\end{aligned}\tag{6.15}
$$

$$
\begin{aligned}
\|\mathbb{P}\boldsymbol{u}^{\varepsilon}-\mathbf{v}\|_{L^{\infty}\left(0,T;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(0,T;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}&\leq C_0(\alpha\varepsilon)9\delta_0^2+C_0\frac{\delta_0}{(4C_0)^{N_0-1}}(4\delta_0+9\delta_0^2)\\
&\leq\frac{5}{96}\frac{\delta_0}{(4C_0)^{N_0-1}}
\end{aligned}\tag{6.16}
$$

将（6.13）-（6.15）与（6.16）结合使用，我们得到

$$
\begin{aligned}
&M^{\varepsilon,\alpha}_{p,q}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon}-\mathbf{v},\vartheta^{\varepsilon};\tau^{\varepsilon},\sigma^{\varepsilon}-\Theta](0,T)\\
&\leq\frac{\delta_0}{(4C_0)^{N_0-1}}\left(\frac{1}{4}+\delta_0+9C_0\delta_0^2+7C_0\delta_0\right)+C_0(\alpha\varepsilon)^{\frac{1}{q(q-1)}}18\delta_0^2\\
&\quad+(\alpha\varepsilon)^{\frac{1}{q(q-1)}}4C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^{\varepsilon}_2}\left(\frac{7}{4}+9C_0\delta_0+27C_0\delta_0^2\right)+2\cdot\frac{5}{96}\frac{\delta_0}{(4C_0)^{N_0-1}}\\
&\leq\frac{\delta_0}{(4C_0)^{N_0-1}}\left(\frac{1}{4}+\frac{1}{96}+\frac{1}{96}+\frac{7}{96}\right)+\frac{\delta_0}{(4C_0)^{N_0-1}}\frac{1}{96}+\frac{\delta_0}{(4C_0)^{N_0-1}}\frac{1}{18}+\frac{10}{96}\frac{\delta_0}{(4C_0)^{N_0-1}}\\
&\leq\frac{3}{4}\frac{\delta_0}{(4C_0)^{N_0-1}}.
\end{aligned}\tag{6.17}
$$

由引理5.3,

$$
\begin{aligned}
&\|\mathbb{P}\boldsymbol{u}^{\varepsilon}-\mathbf{v}\|_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}-1-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)\cap L^1\left(I;\dot{B}^{\frac{d}{p}+1-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}+\|(\tau^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon})\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\\
&+\|\sigma^{\varepsilon}-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^{\infty}\left(I;\dot{B}^{\frac{d}{p}-1-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)\cap L^2\left(I;\dot{B}^{\frac{d}{p}-(\frac{1}{2}-\frac{1}{p})}_{p,1}\right)}\\
&\leq\varepsilon^{\frac{1}{2}-\frac{1}{p}}C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^{\varepsilon}_2}+2C_0\varepsilon^{\frac{1}{2}-\frac{1}{p}}4C_0(\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^{\varepsilon}_2}+1)(4\delta_0+9\delta_0^2)+C_0\varepsilon^{\frac{1}{2}-\frac{1}{p}}4\delta_0(4\delta_0+9\delta_0^2)\\
&+C_0\varepsilon^{\frac{1}{2}-\frac{1}{p}}4C_0\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^{\varepsilon}_2}(3\delta_0+9\delta_0^2)\\
&\leq\varepsilon^{\frac{1}{2}-\frac{1}{p}}4C_0\left(\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^{\varepsilon}_2}+1\right)
\end{aligned}\tag{6.18}
$$

显然

$$
\|a^{\varepsilon}\|_{L^{\infty}\left(0,T;\dot{B}^{\frac{d}{2}}_{2,1}\right)}+\|\boldsymbol{u}^{\varepsilon}\|_{L^1\left(0,T;\dot{B}^{\frac{d}{2}+1}_{2,1}\right)}+\|\vartheta^{\varepsilon}\|_{L^1\left(0,T;\dot{B}^{\frac{d}{2}}_{2,1}\right)}\leq C(T,\varepsilon)E^{\varepsilon}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](0,T).\tag{6.19}
$$

于是由（6.12）、（6.16）、（6.17）、（6.18），我们有 $T_1^*=T^*$。此外，由（6.11）、（6.19）以及延拓准则，解可以延拓到 T^* 之外，这与 T^*的定义相矛盾。因此我们推断 $T^*\geq T_1^*\geq T_1$。

接下来我们用归谬法证明 $T^*\geq T_k,\ \ k=2,\ 3,\ \cdots\cdots,\ N_0$。我们定义时间

$$T_k^*:=\sup\{T_{k-1}<T<T^*,(6.20)-(6.23)\text{ hold at }T\}$$

其中

$$E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](T_{k-1},T)\leq(4C_0)^k\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^\varepsilon},\tag{6.20}$$

$$M_{p,q}^{\varepsilon,\alpha}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](T_{k-1},T)\leq\frac{\delta_0}{(4C_0)^{N_0-k}},\tag{6.21}$$

$$\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(T_{k-1},T;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(T_{k-1},T;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\leq\frac{1}{4}\frac{\delta_0}{(4C_0)^{N_0-k}}.\tag{6.22}$$

$$\begin{aligned}&\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(T_{k-1},T;\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^1\left(T_{k-1},T;\dot{B}^{\frac{d}{p}+1-(\frac12-\frac1p)}_{p,1}\right)}+\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(T_{k-1},T;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\\&+\|\sigma^\varepsilon-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(T_{k-1},T;\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^2\left(T_{k-1},T;\dot{B}^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\leq 2\varepsilon^{\frac12-\frac1p}(4C_0)^k\Big(\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^\varepsilon}+1\Big)\end{aligned}\tag{6.23}$$

接下来假设 $T_{k-1}<T^*<T_k$。令 $T_{k-1}<T<T^*$。我们有

$$\begin{aligned}&M_{p,q}^{\varepsilon,\alpha}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](T_{k-1},T)\\&\leq M_{p,q}^{\varepsilon,\alpha}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](T_{k-1},T)+2\|(\mathbf{v},\Theta)\|_{L^q\left(T_{k-1},T;\dot{B}^{\frac{d}{p}-1+\frac2q}_{p,1}\right)\cap L^1\left(T_{k-1},T;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\leq 3\delta_0,\end{aligned}$$

$$\begin{aligned}\varepsilon\|a^\varepsilon\|_{L^\infty(T_{k-1},T;L^\infty)}&\leq C_0\varepsilon\|a^\varepsilon\|_{L^\infty(T_{k-1},T;\dot{B}^{\frac{d}{p}}_{p,1})}\\&\leq C_0^2M_{p,q}^{\varepsilon,\alpha}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](T_{k-1},T)\leq 3C_0^2\delta_0\leq\frac12,\end{aligned}$$

这意味着

$$\frac12\leq 1+\varepsilon a(t,x)\leq\frac32,\quad\forall(t,x)\in[T_k-1,T]\times\mathbb{R}^d.\tag{6.24}$$

显然,

$$\begin{aligned}&\|(a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(T_{k-1})\|_{X_2^\varepsilon}\leq 2E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](T_{k-2},T_{k-1}),\\&\|(a^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon)(T_{k-1})\|^{h;\alpha}_{X_p^\varepsilon}\leq M_{p,q}^{\varepsilon,\alpha}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](T_{k-2},T_{k-1}),\\&\|(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})(T_{k-1})\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\leq\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(T_{k-2},T_{k-1};\dot{B}^{\frac{d}{p}-1}_{p,1}\right)},\\&\|(\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v})(T_{k-1})\|_{\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}}+\|(\sigma^\varepsilon-\Theta)(T_{k-1})\|^{l;\frac{\beta_0}{\varepsilon}}_{\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}}\\&\leq\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(T_{k-2},T_{k-1};\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)}+\|\sigma^\varepsilon-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(T_{k-2},T_{k-1};\dot{B}^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)}.\end{aligned}$$

由引理5.1,

$$\begin{aligned}E^{\varepsilon}[a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](T_{k-1},T)&\le C_0\|(a^{\varepsilon},\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon})(T_{k-1})\|_{X_2^{\varepsilon}}+(4C_0)^{k-1}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}(3\delta_0+9\delta_0^2)\\&\le 2C_0(4C_0)^{k-1}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}+(4C_0)^{k-1}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}\\&\le\frac{3}{4}(4C_0)^{k}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}},\end{aligned}\tag{6.25}$$

由引理5.2,

$$\|(a^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon})\|^{h;\alpha}_{N^{\varepsilon,\alpha}_{p,q}(T_{k-1},T)}\le C_0\frac{\delta_0}{(4C_0)^{N_0-(k-1)}}+C_0(\alpha\varepsilon)9\delta_0^2+C_03\delta_0\Big(\frac{\delta_0}{(4C_0)^{N_0-k}}+\frac{\delta_0}{(4C_0)^{N_0}}\Big),\tag{6.26}$$

$$\begin{aligned}&\|(\tau^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon})\|^{l;\alpha}_{L^q\left(T_{k-1},T;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)}+\|\sigma^{\varepsilon}-\Theta\|^{l;\alpha}_{L^q(0,T;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})\cap L^{q'}(t_{k-1},T;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\&\le C_0(\alpha\varepsilon)^{\frac{1}{q}}2(4C_0)^{k-1}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}+C_0(\alpha\varepsilon)^{\frac{1}{q}}(4C_0)^{k}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}(3\delta_0+9\delta_0^2)\\&\quad+C_0\frac{\delta_0}{(4C_0)^{N_0-(k-1)}}+C_0(\alpha\varepsilon)^{\frac{1}{q(q-1)}}2(4C_0)^{k-1}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}\\&\quad+C_0(\alpha\varepsilon)^{\frac{1}{q(q-1)}}(4C_0)^{k}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}(3\delta_0+9\delta_0^2)\\&\quad+C_0(\alpha\varepsilon)9\delta_0^2+C_0\delta_0\frac{\delta_0}{(4C_0)^{N_0}}+C_0\frac{\delta_0}{(4C_0)^{N_0-k}}(4\delta_0+9\delta_0^2)\end{aligned}\tag{6.27}$$

$$\begin{aligned}&(\alpha\varepsilon)E^{\varepsilon}[a^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](T_{k-1},T)+\left(\|(\tau^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon})\|^{l;\alpha}_{L^q(T_{k-1},T;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})}\right)^{\frac{1}{q-1}}\Big(E^{\varepsilon}[a^{\varepsilon},\mathbb{Q}\boldsymbol{u}^{\varepsilon},\vartheta^{\varepsilon}](T_{k-1},T)\Big)^{\frac{q-2}{q-1}}\\&\le(\alpha\varepsilon)(4C_0)^{k}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}+C_0(\alpha\varepsilon)^{\frac{1}{q(q-1)}}2(4C_0)^{k-1}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}\\&\quad+C_0(\alpha\varepsilon)^{\frac{1}{q(q-1)}}(4C_0)^{k}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X_2^{\varepsilon}}(3\delta_0+9\delta_0^2),\end{aligned}\tag{6.28}$$

$$\begin{aligned}&\|\mathbb{P}\boldsymbol{u}^{\varepsilon}-\mathbf{v}\|_{L^{\infty}\left(T_{k-1},T;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(0,T;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\\&\le C_0\frac{1}{4}\frac{\delta_0}{(4C_0)^{N_0-(k-1)}}+C_0(\alpha\varepsilon)9\delta_0^2+C_0\frac{\delta_0}{(4C_0)^{N_0-k}}(4\delta_0+9\delta_0^2)\\&\le\frac{11}{96}\frac{\delta_0}{(4C_0)^{N_0-k}}\end{aligned}\tag{6.29}$$

综合（6.26）-（6.28）并利用（6.29），我们得到

$$
\begin{aligned}
&M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](T_{k-1},T)\\
&\le\frac{\delta_0}{(4C_0)^{N_0-k}}\left(\frac12+\delta_0+7C_0\delta_0+9C_0\delta_0^2\right)+C_0(\alpha\varepsilon)^{\frac{1}{q(q-1)}}18\delta_0^2\\
&\quad+(\alpha\varepsilon)^{\frac{1}{q(q-1)}}(4C_0)^k\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}\left(\frac52+9C_0\delta_0+27C_0\delta_0^2\right)+2\cdot\frac{11}{96}\frac{\delta_0}{(4C_0)^{N_0-k}}\\
&\le\frac{\delta_0}{(4C_0)^{N_0-k}}\left(\frac12+\frac{9}{96}\right)+\frac{1}{96}\frac{\delta_0}{(4C_0)^{N_0-k}}+\frac{1}{12}\frac{\delta_0}{(4C_0)^{N_0-k}}+\frac{22}{96}\frac{\delta_0}{(4C_0)^{N_0-k}}\\
&\le\frac{88}{96}\frac{\delta_0}{(4C_0)^{N_0-k}}.
\end{aligned}\tag{6.30}
$$

$$
\begin{aligned}
&\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(T_{k-1},T;\dot B^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^1\left(T_{k-1},T;\dot B^{\frac{d}{p}+1-(\frac12-\frac1p)}_{p,1}\right)}+\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(T_{k-1},T;\dot B^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\\
&+\|\sigma^\varepsilon-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(T_{k-1},T;\dot B^{\frac{d}{p}-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^2\left(T_{k-1},T;\dot B^{\frac{d}{p}-(\frac12-\frac1p)}_{p,1}\right)}\\
&\le C_0 2\varepsilon^{\frac12-\frac1p}(4C_0)^{k-1}\Big(\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}+1\Big)+C_0\varepsilon^{\frac12-\frac1p}2(4C_0)^{k-1}\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}\\
&+C_0 2\varepsilon^{\frac12-\frac1p}(4C_0)^{k}\Big(\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}+1\Big)(4\delta_0+9\delta_0^2)+C_0\varepsilon^{\frac12-\frac1p}4\delta_0(4\delta_0+9\delta_0^2)\\
&+C_0\varepsilon^{\frac12-\frac1p}(4C_0)^k\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}(3\delta_0+9\delta_0^2)\\
&\le\frac34\cdot2\varepsilon^{\frac12-\frac1p}(4C_0)^k\Big(\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}+1\Big)
\end{aligned}\tag{6.31}
$$

显然，

$$
\begin{aligned}
&\|a^\varepsilon\|_{L^\infty\left(T_{k-1},T;\dot B^{\frac{d}{2}}_{2,1}\right)}+\|\boldsymbol{u}^\varepsilon\|_{L^1\left(T_{k-1},T;\dot B^{\frac{d}{2}+1}_{2,1}\right)}+\|\vartheta^\varepsilon\|_{L^1\left(T_{k-1},T;\dot B^{\frac{d}{2}}_{2,1}\right)}\\
&\le C(T,\varepsilon)E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](T_{k-1},T).
\end{aligned}\tag{6.32}
$$

由 (6.25)、(6.29)、(6.30)、(6.31) 可得 $T^*_k=T^*$。由 (6.24)、(6.32) 以及延拓准则，解可以延拓到 T^* 之后，这与 T^* 的定义矛盾。因此我们推得 $T^*\ge T^*_k\ge T_k$并得到

$$
\frac12\le1+\varepsilon a(t,x)\le\frac32,\quad\forall(t,x)\in[0,\infty)\times\mathbb{R}^d,
$$

$$
\begin{aligned}
E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](0,\infty)&\le\sum_{1\le k\le T_{N_0}}E^\varepsilon[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon](T_{k-1},T_k)\\
&\le\sum_{1\le k\le T_{N_0}}(4C_0)^k\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}\le C\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2},
\end{aligned}\tag{6.33}
$$

$$M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](0,\infty)\le\sum_{1\le k\le T_{N_0}}M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](T_{k-1},T)$$
$$\le\sum_{1\le k\le T_{N_0}}\frac{\delta_0}{(4C_0)^{N_0-k}}\le 2\delta_0, \tag{6.34}$$

$$M^{\varepsilon,\ \alpha}_{p\ ,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\ \tau^\varepsilon,\sigma^\varepsilon](0,\infty) \tag{6.35}$$
$$\le M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](0,\infty)+2\|(\mathbf{v},\Theta)\|_{L^q\left(0,\infty;\dot B^{\frac dp-1+\frac 2q}_{p,1}\right)\cap L^1\left(0,\infty;\dot B^{\frac dp+1}_{p,1}\right)}\le C,$$

$$\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(0,\infty;\dot B^{\frac dp-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot B^{\frac dp+1}_{p,1}\right)}\le\sum_{1\le k\le T_{N_0}}\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(T_{k-1},T_k;\dot B^{\frac dp-1}_{p,1}\right)\cap L^1\left(T_{k-1},T_k;\dot B^{\frac dp+1}_{p,1}\right)}$$
$$\le\frac14\sum_{1\le k\le T_{N_0}}\frac{\delta_0}{(4C_0)^{N_0-k}}\le\frac12\delta_0. \tag{6.36}$$

$$\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(0,\infty;\dot B^{\frac dp-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^1\left(0,\infty;\dot B^{\frac dp+1-(\frac12-\frac1p)}_{p,1}\right)}+\|(\tau^\varepsilon,\mathbb{Q}\boldsymbol{u}^\varepsilon)\|^{l;\frac{\beta_0}{\varepsilon}}_{L^2\left(0,\infty;\dot B^{\frac dp-(\frac12-\frac1p)}_{p,1}\right)}$$
$$+\|\sigma^\varepsilon-\Theta\|^{l;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot B^{\frac dp-1-(\frac12-\frac1p)}_{p,1}\right)\cap L^2\left(0,\infty;\dot B^{\frac dp-(\frac12-\frac1p)}_{p,1}\right)}\le C\varepsilon^{\frac12-\frac1p} \tag{6.37}$$

由（6.33），可推知（a^ε，$\boldsymbol{u}^\varepsilon$，$\vartheta^\varepsilon$）属于类（1.10），并满足（1.11）。利用（6.35），可得

$$\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(0,\infty;\dot B^{\frac dp-(\frac12-\frac1p)}_{p,1}\right)}\le C\varepsilon^{\frac12-\frac1p}\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^2\left(0,\infty;\dot B^{\frac dp}_{p,1}\right)}$$
$$\le C\varepsilon^{\frac12-\frac1p}M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](0,\infty)\le C\varepsilon^{\frac12-\frac1p}, \tag{6.38}$$

$$\|\sigma^\varepsilon-\Theta\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot B^{\frac dp-2}_{p,1}\right)\cap L^1\left(0,\infty;\dot B^{\frac dp}_{p,1}\right)}$$
$$\le C\varepsilon\left(\varepsilon\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot B^{\frac dp}_{p,1}\right)}+\frac1\varepsilon\|a^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^1\left(0,\infty;\dot B^{\frac dp}_{p,1}\right)}\right.$$
$$\left.+\frac1\varepsilon\|\vartheta^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot B^{\frac dp-2}_{p,1}\right)\cap L^1\left(0,\infty;\dot B^{\frac dp}_{p,1}\right)}+\|\Theta\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot B^{\frac dp-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot B^{\frac dp+1}_{p,1}\right)}\right)$$
$$\le C\varepsilon\left(M^{\varepsilon,\alpha}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon,\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon](0,\infty)+\|\Theta\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot B^{\frac dp-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot B^{\frac dp+1}_{p,1}\right)}\right)$$
$$\le C\varepsilon. \tag{6.39}$$

同理，我们有

$$\|\tau^\varepsilon\|^{h;\frac{\beta_0}{\varepsilon}}_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-2}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}}_{p,1}\right)}\leq C\varepsilon. \tag{6.40}$$

由（6.37）-（6.40），可得（1.12）。

最后指出，（6.34）和（6.36）对任意 $\delta\leq\delta_0$ 及充分小的 ε均成立。更具体地，对于任意 $\delta\leq\delta_0$，由 [15], 中的引理 2.10，可知存在一个正数 N_δ和一个时间序列 $\{T_{\delta;n}\}_{n=0}^{N_\delta}$，使得 $0=T_{\delta;0}<T_{\delta;1}<\cdots<T_{\delta;N_\delta-1}<T_{\delta;N_\delta}=\infty$并且

$$\|(\mathbf{v},\Theta)\|_{L^q\left(T_{\delta;n-1},T_{\delta;n};\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1}\right)\cap L^1\left(T_{\delta;n-1},T_{\delta;n};\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\leq\delta,\quad n=1,2,\cdots,N_\delta.$$

于是，我们可以选取 α_δ，使得对于 $\varepsilon\leq1$

$$\|(\Theta,\mathbf{v})\|^{h;\frac{\alpha_\delta}{64}}_{L^2(I;\dot{B}^{\frac{d}{p}}_{p,1})\cap L^{q'}(I;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\leq\frac{\delta}{(4C_0)^{N_\delta}},\qquad\|(a_0,\mathbb{Q}\boldsymbol{u}_0,\vartheta_0)\|^{h;\alpha_\delta}_{X^\varepsilon_p}\leq\frac{\delta}{(4C_0)^{N_\delta}}.$$

对于这样的 α_δ，我们还可以选取 $\varepsilon_\delta\leq\min\{\varepsilon_0,\frac{1}{\alpha_\delta},\frac{\beta_0}{16\alpha_\delta}\}$，使得

$$(\alpha_\delta\varepsilon)^{\frac{1}{q(q-1)}}\left(\|(a_0,\boldsymbol{u}_0,\vartheta_0)\|_{X^\varepsilon_2}+\delta\right)\leq\frac{1}{36}\frac{\delta}{(4C_0)^{N_\delta}}$$

对于任意 $0<\varepsilon\leq\varepsilon_\delta$。

完全类似于前面的论证，我们可以得到与(6.34) 和 (6.36) 类似的结果，即

$$\begin{aligned}&M^{\varepsilon,\alpha_\delta}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](0,\infty)\leq2\delta,\\&\|\mathbb{P}\boldsymbol{u}^\varepsilon-\mathbf{v}\|_{L^\infty\left(0,\infty;\dot{B}^{\frac{d}{p}-1}_{p,1}\right)\cap L^1\left(0,\infty;\dot{B}^{\frac{d}{p}+1}_{p,1}\right)}\leq\frac{1}{2}\delta,\end{aligned} \tag{6.41}$$

对任意 $0<\varepsilon\leq\varepsilon_\delta$。注意

$$\begin{aligned}&\varepsilon\|a\|_{L^\infty(0,\infty;\dot{B}^{\frac{d}{p}}_{p,1})}+\|\mathbb{Q}\boldsymbol{u}^\varepsilon\|_{L^q(0,\infty;\dot{B}^{\frac{d}{p}-1+\frac{2}{q}}_{p,1})\cap L^{q'}(0,\infty;\dot{B}^{\frac{d}{p}-1+\frac{2}{q'}}_{p,1})}\\&\leq CM^{\varepsilon,\alpha_\delta}_{p,q}[a^\varepsilon,\boldsymbol{u}^\varepsilon-\mathbf{v},\vartheta^\varepsilon;\tau^\varepsilon,\sigma^\varepsilon-\Theta](0,\infty)\\&\leq C\delta\end{aligned} \tag{6.42}$$

其中常数 C取决于 d,p,q。由 (6.41) 和 (6.42)，可得 (1.13)。 □

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Conflict of interest

作者声明不存在利益冲突。

数据可用性声明

由于本研究期间未生成或分析任何数据集，因此本文不涉及数据共享。

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