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次着色与排除次图封闭图类中的弱着色数

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Abstract. Let $\mathcal{C}$ be a proper minor-closed class of graphs. Given the minors excluded in $\mathcal{C}$, we determine the maximum q-centered chromatic number and the maximum qth weak coloring number of graphs in $\mathcal{C}$ within an $\mathcal{O}(q)$-factor. Moreover, when $\mathcal{C}$ excludes a planar graph, we determine it within a constant factor. Our results imply that the q-centered chromatic number of K_t-minor-free graphs is in $\mathcal{O}(q^{t-1})$, improving on the previously known $\mathcal{O}(q^{h(t)})$ bound with a large and non-explicit function h. We include similar bounds for another family of parameters, the fractional treedepth fragility rates. All our bounds are proved via the same general framework.

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The results of this paper for weak coloring numbers were announced at SODA 2025 [21], see also the full version [20]; while the result for the centered chromatic numbers of K_t-minor-free graphs was announced at SODA 2026 [22]. A substantial portion of the material presented in this article is included in the PhD thesis of the fourth author [32].
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1. 引言

One of the driving forces in graph theory continues to be the development of efficient algorithms for computationally hard problems for sparse graph classes. Nešetřil and Ossona de Mendez [27] introduced the concepts of bounded expansion and nowhere denseness of classes of graphs. These notions cover many well-studied classes of graphs such as planar graphs, graphs of bounded treewidth, graphs excluding a fixed minor, graphs of bounded book-thickness, and graphs that admit drawings with a bounded number of crossings per edge; see the paper of Nešetřil, Ossona de Mendez, and Wood [28] and the lecture notes of Pilipczuk, Pilipczuk, and Siebertz [30]. Weak coloring numbers and centered chromatic numbers are key families of parameters capturing these concepts. In this work, we study bounds on these parameters in proper minor-closed classes of graphs.

我们首先介绍主要概念，以及一些作为我们在本文中构建的更大框架之推论而得出的定理。

设 G 是一个图，q 是一个正整数，C 是一个颜色集合。G 的一种着色 $\varphi\colon V(G) \to C$ 称为 q-中心着色，如果对于 G 的每个连通[1] 子图 H，要么 φ 在 H 上使用的颜色多于 q 种，要么存在一种颜色在 H 上恰好出现一次。G 的 q-中心色数记为 $\mathsf{cen}_q(G)$，它是使得 G 存在一种使用 k 种颜色的 q-中心着色的最小非负整数 k。以下结论是我们工作中最简单、也最重要的成果之一。

定理 1.1. 设 t 是满足 $t \geqslant 2$的整数。存在整数 c，使得对每个K_t-无子式图 G 以及每个正整数 q,

$$\mathrm{cen}_q(G) \leqslant c \cdot q^{t-1}.$$

This improves on the work of Pilipczuk and Siebertz [31] who proved that for every integer t with $t \geqslant 2$, K_t-minor-free graphs have q-centered chromatic number upper bounded by a polynomial in q. Contrary to Theorem 1.1, the degree of their polynomial is not explicitly given and arises from an application of the graph minor structure theorem by Robertson and Seymour [36]. On the other hand, Dębski, Micek, Schröder, and Felsner [10] showed that there exist K_t-minor-free graphs with q-centered chromatic number in $\Omega(q^{t-2})$. Hence, Theorem 1.1 is tight up to an $\mathcal{O}(q)$-factor.

Let G be a graph, let $\Pi(G)$ be the set of all vertex orderings of G, let $\sigma \in \Pi(G)$, and let q be a nonnegative integer. For all vertices u and v of G, we say that v is *weakly q-reachable from u in (G, σ)* if there exists a path between u and v in G containing at most q edges such that for every vertex w on the path, $v \leqslant_\sigma w$. Let $\mathrm{WReach}_q[G, \sigma, u]$ be the set of vertices that are weakly q-reachable from u in (G, σ). The *qth weak coloring number* of G is defined as

$$\mathrm{wcol}_q(G) = \min_{\sigma \in \Pi(G)} \max_{u \in V(G)} |\mathrm{WReach}_q[G, \sigma, u]|.$$

In order to show the robustness of our framework, we include another family of parameters, introduced by Dvořák and Sereni [9], also connected to concepts of sparsity, see the work of Dvořák [8]. The *treedepth* of a graph G, denoted by $\mathrm{td}(G)$, is defined recursively as 0 if G is the null graph, $\min\{\mathrm{td}(G - v) + 1 \mid v \in V(G)\}$ if G is connected, and $\max\{\mathrm{td}(C) \mid C \text{ component of } G\}$ otherwise. For a nonempty set $\mathcal{Y}$, we say that $\lambda\colon \mathcal{Y} \to [0, 1]$ is a *probability distribution* on $\mathcal{Y}$ if $\sum_{Y \in \mathcal{Y}} \lambda(Y) = 1$. Let G be a graph and let q be a positive integer. The *qth fractional treedepth fragility rate* of G is the minimum positive integer k such that there exists a family $\mathcal{Y}$ of subsets of $V(G)$ such that

[1]本文中的连通图均为非空图，也就是说，它们至少有一个顶点。

(i) $\mathrm{tw}(G[Y]) \leqslant k$ 对于每个 $Y \in \mathcal{Y}$，并且 (ii) 存在一个定义在 $\mathcal{Y}$ 上的概率分布 λ，使得对于每个 $u \in V(G)$，都有 $\sum_{u \in Y \in \mathcal{Y}} \lambda(Y) \leqslant \frac{1}{q}$。

我们将 q 的第 G 次分数树深度脆弱率记为 $\mathrm{ftdfr}_q(G)$。

弱着色数和分数树深脆弱性率的最新研究成果已经包括与定理1.1类似的结果。具体而言，Van den Heuvel、Ossona de Mendez、Quiroz、Rabinovich 和 Siebertz [19] 证明，对于每个整数 t 满足 $t \geqslant 2$，都存在一个整数 c，使得对于每个 K_t 无 G 的图 q，以及每个正整数

$$\mathrm{wcol}_q(G) \leqslant c \cdot q^{t-1}.$$

Dvořák and Sereni [9, Theorem 28] gave bounds on the fractional treedepth fragility rates for graphs of bounded treewidth. This, together with an idea of Esperet and Norin [13, Theorem 8] of applying a result of DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan [2]; and a product structure theorem by Illingworth, Scott, and Wood [25, Theorem 4] implies that for every integer t with $t \geqslant 2$, there exists an integer c such that for every K_t-minor-free graph G and for every positive integer q, we have

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1}.$$

由于这一结论在文献中从未得到明确证明，我们决定将其证明纳入本文。它也展示了我们的方法。参见小节 9.1 中的定理 9.1。

Centered colorings and weak coloring numbers are crucial tools in designing parameterized algorithms in classes of graphs of bounded expansion. For example, Pilipczuk and Siebertz [31] showed that if $\mathcal{C}$ is a class of graphs excluding a fixed minor, then it can be decided whether a given q-vertex graph H is a subgraph of an n-vertex graph G in $\mathcal{C}$ in time $2^{\mathcal{O}(q \log q)} \cdot n^{\mathcal{O}(1)}$ and space $n^{\mathcal{O}(1)}$. This algorithm relies on the fact that the union of any q color classes in a q-centered coloring induces a subgraph of treedepth at most q. Therefore, finding a q-centered coloring that uses $q^{\mathcal{O}(1)}$ colors allows us to reduce the problem to graphs of bounded treedepth, on which the subgraph isomorphism problem can be solved efficiently. The running times of algorithms based on q-centered colorings heavily depend on the number of colors used. Even more algorithmic problems were solved using the weak coloring numbers characterization of sparse graphs. Dvořák [7] showed a constant-factor approximation for distance versions of domination number and independence number, with further applications in fixed-parameter algorithms and kernelization by Eickmeyer, Giannopoulou, Kreutzer, Kwon, Pilipczuk, Rabinovich, and Siebertz [11]. Grohe, Kreutzer, and Siebertz [17] proved that deciding first-order properties is fixed-parameter tractable in nowhere dense graph classes. Reidl and Sullivan [33] presented an algorithm counting the number of occurrences of a fixed induced subgraph in sparse graphs. The time complexities of all these algorithms depend heavily on the asymptotics of wcol_q in respective graph classes.

The growth rates of centered chromatic numbers and weak coloring numbers have been extensively studied. Grohe, Kreutzer, Rabinovich, Siebertz, and Stavropoulos [16] proved that if $\mathrm{tw}(G) \leqslant t$, then $\mathrm{wcol}_q(G) \leqslant \binom{q+t}{t}$.[2] This is tight as for all nonnegative integers q, t they constructed a graph $G_{q,t}$ with $\mathrm{tw}(G_{q,t}) = t$ and $\mathrm{wcol}_q(G_{q,t}) = \binom{q+t}{t}$. Similarly, Pilipczuk and Siebertz [31] proved that if $\mathrm{tw}(G) \leqslant t$, then $\mathrm{cen}_q(G) \leqslant \binom{q+t}{t}$ which is again tight as proved by Dębski, Micek, Schröder, and Felsner [10]. Also, Dvořák and Sereni [9] proved that if $\mathrm{tw}(G) \leqslant t$, then $\mathrm{ftdfr}_q(G) = \mathcal{O}(q^t)$, which is also tight. In the class of planar graphs, Dębski, Felsner, Micek, and Schröder [10] proved that $\mathrm{cen}_q(G) = \mathcal{O}(q^3 \log q)$; van den Heuvel, Ossona

[2]对于图 G，令 $\mathrm{tw}(G)$、$\mathrm{pw}(G)$和 $\mathrm{vc}(G)$ 分别表示 G 的树宽、路径宽和顶点覆盖数。

de Mendez, Quiroz, Rabinovich, and Siebertz [19] proved that $\mathrm{wcol}_q(G) = \mathcal{O}(q^3)$; and Dvořák and Sereni [9] proved that $\mathrm{ftdfr}_q(G) = \mathcal{O}(q^3 \log q)$, while the best known lower bounds are in $\Omega(q^2 \log q)$ [9, 10, 26]. Interestingly, all these lower bound constructions in the planar case have bounded treewidth.

More generally, fix a nonnull graph X and let $\mathrm{par} \in \{\mathrm{cen}, \mathrm{wcol}, \mathrm{ftdfr}\}$. What is the growth rate, with respect to q, of the maximum of $\mathrm{par}_q(G)$ over all X-minor-free graphs G? Most previous work has focused on the case par = wcol. Van den Heuvel et al. [19] showed that $\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{|V(X)|-1}\right)$. Subsequently, van den Heuvel and Wood [18] proved that $\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{\mathrm{vc}(X)+1}\right)$. Dujmović, Hickingbotham, Hodor, Joret, La, Micek, Morin, Rambaud, and Wood [4] proved that there exists an exponential function g such that $\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{g(\mathrm{td}(X))}\right)$. For par = cen, we only have $\mathrm{cen}_q(G) = \mathcal{O}(q^{h(|V(X)|)})$ for some large and non-explicit function h as proved by Pilipczuk and Siebertz [31], while for par = ftdfr, we have $\mathrm{ftdfr}_q(G) = \mathcal{O}(q^{|V(X)|-1})$ as follows from the aforementioned results [9, 13, 25].

所有这些研究都可以看作是试图理解以下图参数。对于给定的非空图 X 和参数 $\in \{$cen,wcol, ftdfr$\}$，令

$f_{\mathrm{par}}(X)$ =下确界$\{\alpha \in \mathbb{R} \mid$ 存在 $c > 0$ 使得对于每个 X-无小图 G以及每个正整数 q，都有 偶数$_q(G) \leqslant c \cdot q^{\alpha}\}$.

问题是 f_{par} 是否与[3] 另一个广为人知的图参数有关。换言之， X 的哪种性质决定了par$_q(G)$ 对于 X- 次闭图的增长率 G? 回顾一下，对于每个图 X，

$$\mathrm{tw}(X) \leqslant \mathrm{pw}(X) \leqslant \mathrm{td}(X) - 1 \leqslant \mathrm{vc}(X) \leqslant |V(X)| - 1.$$

The aforementioned results imply that $\mathrm{tw}(X) - 1 \leqslant f_{\mathrm{wcol}}(X) \leqslant g(\mathrm{td}(X))$. However, f_{wcol} is not tied to any of these parameters. Indeed, neither pathwidth nor treedepth can lower bound f_{wcol}. For every positive integer k, let T_k be a complete ternary tree of vertex-height k. By Robertson-Seymour Excluded Tree Minor Theorem [34], there is a constant depending on k bounding pathwidth of T_k-minor-free graphs. Also, we have $\mathrm{wcol}_q(G) = \mathcal{O}(\mathrm{pw}(G)^2 \cdot \log q)$ (we provide a proof in Appendix C for completeness, see Lemma C.1), thus, $f_{\mathrm{wcol}}(T_k) = 0$ while $\mathrm{pw}(T_k) = k$ and $\mathrm{td}(T_k) = k + 1$. Next, we argue that neither treewidth nor pathwidth can upper-bound f_{wcol}. For every positive integer k, let L_k be a ladder with k rungs. There is a graph $G_{q,t}$ (constructed by Grohe et al. [16]) such that $\mathrm{wcol}_q(G_{q,t}) = \Omega(q^t)$, and if $k = \Omega(2^t)$, then $G_{q,t}$ excludes L_k as a minor. Therefore, $f_{\mathrm{wcol}}(L_k) = 2^{\Omega(k)}$, and $\mathrm{tw}(L_k) \leqslant \mathrm{pw}(L_k) \leqslant 2$.

Surprisingly, a key parameter in this context turns out to be 2-*treedepth* as defined by Huynh, Joret, Micek, Seweryn, and Wollan [24]. They used this parameter to characterize classes of graphs excluding a fixed ladder as a minor. In this paper, we prove that f_{par} is tied to 2-treedepth for each $\mathrm{par} \in \{\mathrm{cen}, \mathrm{wcol}, \mathrm{ftdfr}\}$. More precisely, we determine f_{par} up to ± 1 by introducing two 'rooted' versions of 2-treedepth called $\mathrm{rtd}_2(\cdot)$ and $\mathrm{srtd}_2(\cdot)$. They both admit treedepth-like recursive definitions that differ only in the base cases. We now present these definitions.

设 G 为一个图。 森林分解of G 是一个二元组 $\mathcal{W} = \big(F,\ (W_x \mid x \in V(F))\big)$，其中 F是一个森林，且 $W_x \subseteq V(G)$ 对每个 $x \in V(F)$均成立，并满足以下条件：

(i) 对于每个 $u \in V(G)$，图 $F[\{x \in V(F) \mid u \in W_x\}]$ 是 F 的连通子图；(ii) 对于每条边 $uv \in E(G)$，存在 $x \in V(F)$，使得 $u, v \in W_x$。

We call $\mathcal{W}$ a *tree decomposition* when F is a tree. The sets W_x are called the *bags* of $\mathcal{W}$. The sets $W_x \cap W_y$ for $xy \in E(F)$ are called the *adhesions* of $\mathcal{W}$ and the *adhesion* of $\mathcal{W}$ is the

[3]若存在两个函数 p, r，使得 两个图参数 α, β 满足 $p(G) \leqslant \alpha(r(G))$ 且 $r(G) \leqslant \beta(p(G))$ 对每个图 G都成立，则称它们是相互约束的。

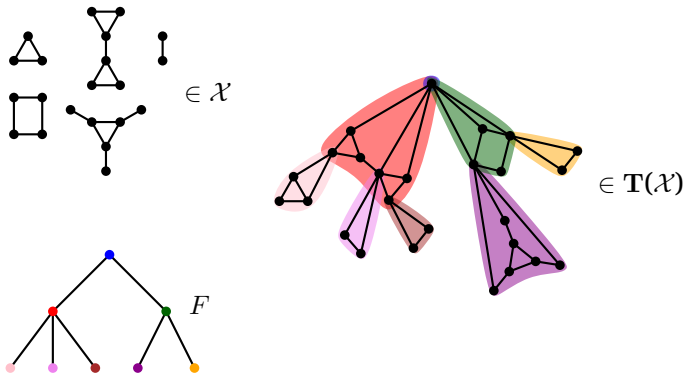


图 1. 一个属于 $\mathbf{T}(\mathcal{X})$ 的图，以及一个以 F 为索引、证明这一事实的有根树分解。彩色集合表示树分解的袋。

maximum size of an adhesion of $\mathcal{W}$. The *width* of $\mathcal{W}$ is $\max_{x\in V(F)} |W_x| - 1$, and the *treewidth* of G is the minimum width of a tree decomposition of G.

一个 线性森林 是若干条路径的不相交并。一个有根森林 是每个连通分量均为有根树的森林。当 F是一个有根森林时，对于每个 $x \in V(F)$ 且不是根的顶点，令 $\mathrm{p}(F,x)$为 x在 F中的父节点。森林分解 $\big(F,(W_x \mid x \in V(F))\big)$ 是 有根的 若 F是一个有根森林。

Let $\mathcal{X}$ be a class of graphs. We define $\mathbf{T}(\mathcal{X})$ as the class of all the graphs G such that there is a rooted forest decomposition $\big(F,(W_x \mid x \in V(F))\big)$ of G of adhesion at most 1 such that for every $x \in V(F)$, the graph $G[W_x - W_{\mathrm{p}(F,x)}] \in \mathcal{X}$ if x is not a root, and $|W_x| \leqslant 1$ if x is a root. See Figure 1.

注意到 $\mathcal{X} \subseteq \mathbf{T}(\mathcal{X})$。算子 $\mathbf{T}$ 具有单调性：对于所有满足 $\mathcal{X} \subseteq \mathcal{Y}$ 的图类$\mathcal{X}$ 和 $\mathcal{Y}$，都有 $\mathbf{T}(\mathcal{X}) \subseteq \mathbf{T}(\mathcal{Y})$。我们递归地定义每个非负整数 t 对应的图类

$$\mathcal{R}_t = \begin{cases} \text{only the null graph} & \text{if } t = 0,\\ \text{all edgeless graphs} & \text{if } t = 1,\\ \text{all forests} & \text{if } t = 2,\\ \mathbf{T}(\mathcal{R}_{t-1}) & \text{if } t \geqslant 3, \end{cases} \quad \text{and} \quad \mathcal{S}_t = \begin{cases} \text{only the null graph} & \text{if } t = 0,\\ \text{all edgeless graphs} & \text{if } t = 1,\\ \text{all linear forests} & \text{if } t = 2,\\ \mathbf{T}(\mathcal{S}_{t-1}) & \text{if } t \geqslant 3. \end{cases}$$

见图2。在 $\mathcal{R}_t$的定义中， $t \in \{1,2\}$ 的基础情形是多余的。将$\mathbf{T}$ 应用于仅由空图构成的图类，得到所有无边图构成的图类，并将$\mathbf{T}$ 应用于所有无边图构成的图类，得到所有森林构成的图类。然而，我们明确陈述这些基础情形，以便于与 $\mathcal{S}_t$进行比较。

我们现在可以定义两个关键参数，即根式2-树深度和简单根式2-树深度。对于每个图 G，令[4]

$$\mathrm{rtd}_2(G) = \min\{t \in \mathbb{N} \mid G \in \mathcal{R}_t\},$$
$$\mathrm{srtd}_2(G) = \min\{t \in \mathbb{N} \mid G \in \mathcal{S}_t\}.$$

容易验证， $\mathcal{S}_3$ 包含所有森林，因此， $\mathcal{R}_2 \subseteq \mathcal{S}_3$。所以，由于$\mathbf{T}$ 是单调算子，对于每个非负整数 t，

$$\mathcal{R}_{t+1} \supseteq \mathcal{S}_{t+1} \supseteq \mathcal{R}_t.$$

[4]We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}_1$ the set of positive integers.

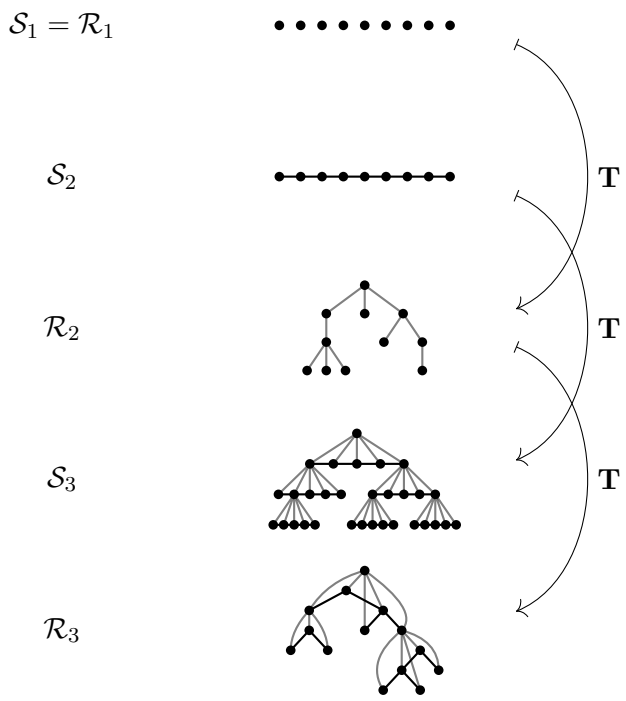


图 2. $\mathcal{S}_1 = \mathcal{R}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{R}_2 \subseteq \mathcal{S}_3 \subseteq \mathcal{R}_3$ 中图的示例。

Therefore, for every graph G, we have

$$\mathrm{rtd}_2(G) \leqslant \mathrm{srtd}_2(G) \leqslant \mathrm{rtd}_2(G) + 1.$$

这两个参数都与 Huynh 等人定义的2-树深度呈线性关联 [24]；参见附录B.4中的引理B。

现在，我们可以介绍本文的主要贡献。我们针对这三类参数分别给出一个结论。不过，所有证明都基于同一个抽象框架。

定理 1.2. 对于每个满足 $t \geqslant 3$ 的整数 t，对于每个图 X，都存在一个整数 c，使得对于每个 X-无 *minor* 图 G，对于所有满足 $q \geqslant 2$ 的整数 q，若 $\mathrm{srtd}_2(X) \leqslant t$，则

$$\mathrm{cen}_q(G) \leqslant c \cdot q^{t-1},$$
$$\mathrm{cen}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2};$$

并且若 $\mathrm{rtd}_2(X) \leqslant t$，则

$$\mathrm{cen}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\mathrm{cen}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q.$$

Theorem 1.3. *For every integer t with $t \geqslant 2$, for every graph X, there exists an integer c such that for every X-minor-free graph G, for all integers q with $q \geqslant 2$, if $\mathrm{srtd}_2(X) \leqslant t$, then*

$$\mathrm{wcol}_q(G) \leqslant c \cdot q^{t-1},$$
$$\mathrm{wcol}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2};$$

and if $\mathrm{rtd}_2(X) \leqslant t$, *then*

$$\mathrm{wcol}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\mathrm{wcol}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q.$$

定理 1.4。对于每个满足 $t \geqslant 2$ 的整数 t，对于每个图 X，都存在一个整数 c，使得对于每个无 X 次图的图 G，对于所有满足 $q \geqslant 2$ 的整数 q，若 $\mathrm{srtd}_2(X) \leqslant t$，则

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1},$$
$$\mathrm{ftdfr}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2};$$

且如果$\mathrm{rtd}_2(X) \leqslant t$，那么

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\mathrm{ftdfr}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q.$$

Let us briefly discuss the tightness of the bounds in Theorems 1.2 to 1.4. Fix an integer t and a graph X. All bounds involving $\mathrm{tw}(G)$ are tight up to a constant factor, while the remaining bounds are tight up to an $\mathcal{O}(q)$-factor. There are two setups for the lower bound constructions: the first for X-minor-free graphs when $\mathrm{rtd}_2(X) = t$, and the second for X-minor-free graphs when $\mathrm{srtd}_2(X) = t$. Both of them were already present in the literature, though not stated in terms of our new parameters. When $\mathrm{rtd}_2(X) = t$, this relates to a construction of graphs with bounded treewidth and large weak coloring numbers by Grohe et al. [16]. When $\mathrm{srtd}_2(X) = t$, the lower bound relies on a construction of graphs with bounded simple treewidth and large centered chromatic numbers by Dębski et al. [10]. We discuss the lower bounds in detail in Theorem 2.1 and Appendix A.

下一个命题概述了在适当的排除次图类中，每个 par_q par $\in$ {wcol,cen,ftdfr} 的增长率。

推论 1.5. 设 par $\in$ {cen、wcol、ftdfr}，设 $\mathcal{X}$ 是一个非空的非空图族，并令 $\mathcal{C}$ 为所有满足以下条件的图 G 所构成的类：对所有 $X \in \mathcal{X}$，X 都不是 G 的次图。令$t = \min\{\mathrm{rtd}_2(X) \mid X \in \mathcal{X}\}$，以及 $s = \min\{\mathrm{srtd}_2(X) \mid X \in \mathcal{X}\}$。

If $s \leqslant 2$, *then*

$$\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \Theta(1).$$

若 $(t, s) = (2,3)$，则

$$\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \begin{cases} \Theta(q) & \text{if par} = \text{cen}, \\ \Theta(\log q) & \text{if par} \in \{\text{wcol}, \text{ftdfr}\}. \end{cases}$$

若 $3 \leqslant t = s$，则

$$\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \Omega(q^{t-2}) \quad \textit{and} \quad \max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-1}).$$

若 $3 \leqslant t < s$，则

$$\max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \Omega(q^{t-2} \log q) \quad \textit{and} \quad \max_{G \in \mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-1} \log q).$$

Moreover, if $\mathcal{X}$ *contains a planar graph, then the lower bounds are tight.*

In particular, for every nonnull graph X and each par $\in \{\mathrm{cen}, \mathrm{wcol}, \mathrm{ftdfr}\}$, the value of f_{par} is tied to $\mathrm{rtd}_2(X)$ as follows:

$$f_{\mathrm{par}}(X) \in \{\mathrm{rtd}_2(X) - 2, \mathrm{rtd}_2(X) - 1\} \qquad \text{and}$$
$$f_{\mathrm{par}}(X) = \mathrm{rtd}_2(X) - 2 \qquad \text{当 } X \text{ 为平面图时。}$$

由推论1.5的一般性结论可以推出若干有趣的推论。下面列出其中一些。另请参见表1 和2。

首先，对于每个图 X，我们都有 $\mathrm{rtd}_2(X) \leqslant \mathrm{td}(X)$，见引理 2.3。因此，定理 1.2至 1.4 推出以下结论。

推论 1.6. 对于每个 参数 $\in \{\mathrm{wcol}、\mathrm{cen}、\mathrm{ftdfr}\}$，对于每个满足 $t \geqslant 3$ 的整数 t，对于每个满足 $\mathrm{td}(X) \leqslant t$ 的图 X，都存在一个整数 c，使得对于每个图 G，若 G 不含 X 子式，则对于每个满足 $q \geqslant 2$ 的整数 q，

$$\mathrm{par}_q(G) \leqslant c \cdot q^{t-1} \log q,$$
$$\mathrm{par}_q(G) \leqslant c \cdot (\mathrm{tw}(G) + 1) \cdot q^{t-2} \log q.$$

Since $\mathrm{td}(X) \leqslant \mathrm{vc}(X) + 1$, this improves the result by van den Heuvel and Wood [18] that $\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{\mathrm{vc}(X)+1}\right)$ for all X-minor-free graphs G. It also directly improves the result by Dujmović et al. [4] that there exists an exponential function g such that $\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{g(\mathrm{td}(X))}\right)$ for all X-minor-free graphs G. For par $\in \{\mathrm{cen}, \mathrm{ftdfr}\}$, no bounds in terms of $\mathrm{vc}(X)$ or $\mathrm{td}(X)$ were known.

以下推论可由推论 1.6 以及对所有正整数 s，t均有 $\mathrm{td}(K_{s,\,t}) \leqslant s + 1$ 这一事实推得。

推论 1.7. 对于每一个par $\in \{\mathrm{wcol},\mathrm{cen},\mathrm{ftdfr}\}$，对于所有正整数 s, t 满足 $s \geqslant 2$，存在一个整数 c，使得对于每一个图 G，如果 G 是$K_{s,\,t}$-无子式的，则对于每一个整数 q 满足 $q \geqslant 2$，

$$\mathrm{par}_q(G) \leqslant c \cdot q^{s} \log q,$$
$$\mathrm{par}_q(G) \leqslant c \cdot (\mathrm{tw}(G) + 1) \cdot q^{s-1} \log q.$$

再次，这改进了 van den Heuvel 和 Wood 的结果 [18]，即对于所有$K_{s,\,t}$-小图免除图，$\mathrm{wcol}_q(G) = \mathcal{O}\left(q^{s+1}\right)$ G。对于 par $\in \{\mathrm{cen},\mathrm{ftdfr}\}$，此前尚无类似的界。
由于对于每个非负整数 g，属为 g 的图都是 $K_{3,\,2g+3}$-无小图，因此我们推得以下结论。

推论 1.8。对于每个参数 $\in \{\mathrm{wcol},\mathrm{cen},\mathrm{ftdfr}\}$，对于每个非负整数 g，存在一个整数 c，使得对于每个欧拉亏格至多为 g 的图 G，对于每个满足$q \geqslant 2$的整数 q，

$$\mathrm{par}_q(G) \leqslant c \cdot (\mathrm{tw}(G) + 1) \cdot q^2 \log q.$$

确定 $\mathrm{par}_q(G)$ 的精确渐近行为仍然是一项挑战。目前我们只知道它属于 $\Omega(q^2 \log q)$ 和 $\mathcal{O}(q^3 \log q)$，而在情形 $(\mathrm{par}_q \mid q \in \mathbb{N}_1) = (\mathrm{wcol}_q \mid q \in \mathbb{N}_1)$ 下，它属于 $\mathcal{O}(q^3)$。请再次参见表 1 和表 2。推论 1.8 在额外假设 G 具有有界树宽时弥合了这一差距。

论文结构。在第2节中，我们证明推论1.5 至1.7。在第3节中，我们引入本文中将会用到的一些记号和工具。在第4节中，我们介绍证明背后的关键概念，即富模型、聚焦

类 $\mathcal{C}$	**下界** [10]	**上界**	
平面	$\Omega(q^2 \log q)$	$\mathcal{O}(q^3 \log q)$	[10]
平面图和树宽 $\leqslant k$	$\Omega(q^2 \log q)$	$\mathcal{O}(q^2 \log q)$	推论 1.8
欧拉亏格 $\leqslant g$	$\Omega(q^2 \log q)$	$\mathcal{O}(q^3$ 对数 $q)$	[10]
欧拉亏格 $\leqslant g$ 和树宽 $\leqslant k$	$\Omega(q^2 \log q)$	$\mathcal{O}(q^2 \log q)$	推论 1.8
外平面	$\Omega(q \log q)$	$\mathcal{O}(q \log q)$	[10]
$K_{2,}$无$_t$-minor	$\Omega(q \log q)$	$\mathcal{O}(q \log q)$	推论 1.7
树宽 $\leqslant k$	$\binom{q+k}{k}$	$\binom{q+k}{k}$	[31]
无K_t子式	$\Omega(q^{t-2})$	$\mathcal{O}(q^{t-1})$	定理 1.1
$K_{s,}$无$_t$子式	$\Omega(q^{s-1} \log q)$	$\mathcal{O}(q^s \log q)$	推论 1.7
$K_{s,}$无$_t$子式且树宽 tw $\leqslant k$	$\Omega(q^{s-1} \log q)$	$\mathcal{O}(q^{s-1}$ 对数 $q)$	推论1.7

表 1. 若干小图封闭图类 $\mathcal{C}$ 的 $\max_{G\in\mathcal{C}}\mathrm{cen}_q(G)$ 的下界和上界。变量 g, k, s, t 为固定正整数，且 $s+3 \leqslant t \leqslant k$。

图类 $\mathcal{C}$	**下界** [16, 26]	**上界**	
平面	$\Omega(q^2 \log q)$	$\mathcal{O}(q^3)$	[19]
平面且 tw $\leqslant k$	$\Omega(q^2 \log q)$	$\mathcal{O}(q^2 \log q)$	推论 1.8
欧拉亏格 $\leqslant g$	$\Omega(q^2 \log q)$	$\mathcal{O}(q^3)$	[19]
欧拉亏格 $\leqslant g$ 和树宽 $\leqslant k$	$\Omega(q^2 \log q)$	$\mathcal{O}(q^2 \log q)$	推论 1.8
外平面	$\Omega(q \log q)$	$\mathcal{O}(q \log q)$	[26]
$K_{2,}$不含$_t$子式	$\Omega(q \log q)$	$\mathcal{O}(q \log q)$	推论 1.7
树宽 $\leqslant k$	$\binom{q+k}{k}$	$\binom{q+k}{k}$	[16]
无 K_t 次要图	$\Omega(q^{t-2})$	$\mathcal{O}(q^{t-1})$	[19]
$K_{s,}$不含$_t$ minor	$\Omega(q^{s-1} \log q)$	$\mathcal{O}(q^s \log q)$	推论 1.7
$K_{s,}$不含$_t$ minor 且树宽为 $\leqslant k$	$\Omega(q^{s-1} \log q)$	$\mathcal{O}(q^{s-1} \log q)$	推论 1.7

表 2. 某些排除次图的图类 $\mathcal{C}$ 上 $\max_{G\in\mathcal{C}}\mathrm{wcol}_q(G)$ 的下界和上界。变量 g， k， s， t 均为固定的正整数，且$s+3 \leqslant t \leqslant k$。

parameters, the coloring elimination property, and (par $\mathcal{X}$)-bounding functions. In Section 5, we show that the families of parameters $(\mathrm{cen}_q \mid q \in \mathbb{N}_1)$, $(\mathrm{wcol}_q \mid q \in \mathbb{N}_1)$, and $(\mathrm{ftdfr}_q \mid q \in \mathbb{N}_1)$ can be described as families of focused graph parameters which satisfy a few key properties and fit into our framework as a result. In Section 6, we show that the classes $\mathcal{R}_t$ and $\mathcal{S}_t$ have the coloring elimination property for every nonnegative integer t. In Section 7, we prove our main abstract theorems, which imply that passing from excluding a graph in $\mathcal{R}_t$ (resp. $\mathcal{S}_t$) to excluding a graph in $\mathcal{R}_{t+1}$ (resp. $\mathcal{S}_{t+1}$) multiplies the growth rates of the considered families of graph parameters by $\mathcal{O}(q)$. Next, we prove base cases, that is, we show bounds when a graph from $\mathcal{S}_2$, $\mathcal{R}_2$, or $\mathcal{R}_3$ is excluded (depending on the considered family of parameters). In Section 8, we provide general techniques to prove these base cases. In Sections 9 to 11, we apply this to each of the three considered families of parameters, and we conclude the proofs of Theorems 1.2 to 1.4. In Appendix A, we discuss the lower bounds in detail. In Appendix B, we give some basic facts on 2-treedepth and its variants. In Appendix C, we recall a proof of an upper bound on weak coloring numbers of graphs with bounded pathwidth.

2. 推论 1.5和 1.6 的证明

In this section, given our main theorems, i.e. Theorems 1.2 to 1.4, and the lower bounds, see Theorem 2.1, we wrap up the main corollaries from the introduction of the paper. As mentioned before, the lower bound constructions already appeared in the literature, though without the context of the new parameters: $\mathrm{srtd}_2(\cdot)$ and $\mathrm{rtd}_2(\cdot)$. For completeness, a uniform proof of Theorem 2.1 is given in Appendix A.

定理 2.1 ([9, 10, 16, 26])。 设 par $\in$ {wcol,cen, ftdfr}。对于所有正整数 t, s 满足$s \geqslant 2$,

$$\max_{G \in \mathcal{R}_t} \mathrm{par}_q(G) = \Omega(q^{t-1}),$$
$$\max_{G \in \mathcal{S}_s} \mathrm{par}_q(G) = \Omega(q^{s-2} \log q).$$

首先，我们证明以下众所周知的观察结果。

观察 2.2. 令par $\in$ {wcol,cen,ftdfr}，并令 q 为正整数。对于每个图G,

$$\mathrm{par}_q(G) \leqslant \mathrm{td}(G).$$

设 G 为一个图。若在 $\mathsf{ftdfr}_q(G)$ 的定义中取 $\mathcal{Y} = \{\emptyset\}$，我们立即得到对于所有正整数 q, $\mathsf{ftdfr}_q(G) \leqslant \mathsf{td}(G)$。为了证明 par $\in$ {wcol,cen} 的不等式，采用下面的树深度定义会更方便。

Let F be a rooted forest. The *vertex-height* of F is the maximum number of vertices on a path from a root to a leaf in F, and the *height* of a vertex u in F is the number of vertices in the path between u and the root of its component in F. For two vertices u and v in F, we say that u is a *descendant* of v and v is an *ancestor* of u in F if v lies on the path from a root to u in F. The *closure* of F is the graph with the vertex set $V(F)$ and the edge set $\{uv \mid u \text{ is ancestor of } v \text{ in } F\}$. The treedepth of G can be equivalently defined as the minimum vertex-height of a rooted forest whose closure contains G as a subgraph. An *elimination ordering* of a rooted forest F is an ordering $(x_1, \ldots, x_{|V(F)|})$ of $V(F)$ such that for all $i, j \in [|V(F)|]$, if x_i is an ancestor of x_j in F, then $i \leqslant j$.

设 F 是见证 $\mathsf{td}(G)$ 的有根森林，并设 σ 是 F 的消去序。设 q是一个正整数。设 u是 G 的一个顶点。$\mathsf{WReach}_q(G, \sigma, u)$ 中的每个顶点都必须是一个

u 在 F 中的祖先。由于 G 中的每个顶点在 F 中至多有 $\mathsf{td}(G)$ 个祖先，我们得出 $\mathsf{wcol}_q(G) \leqslant \mathsf{td}(G)$。
现在考虑 φ of $V(G)$ 的一种着色，使得 $\varphi(v)$ 是 v in F 中的高度。显然，φ 只使用 $[\mathsf{td}(G)]$ 中的颜色。对于 G 的每个连通子图 H，都存在一个顶点 v，使得v 在 F 中的高度小于 H 中所有其他顶点。因此，v 在 φ 下于 H 中具有唯一颜色。于是，$\mathsf{cen}_q(G) \leqslant \mathsf{td}(G)$。

另一个广为人知的观察如下：对于每个正整数 ℓ，如果 G 是一个不含有 ℓ 个顶点的路径的连通图，那么任何 DFS 生成树的顶点高度都至多为 $\ell-1$，因此可作为 $\mathsf{td}(G) \leqslant \ell-1$ 的证据。因此，以固定路径（或更一般地，以固定线性森林）作为次要图的排除条件的图，其树深度有界。

现在，我们可以证明推论 1.5。

推论的证明 *1.5*。若 $s \leqslant 2$，则 $\mathcal{X}$ 包含一个线性森林。特别地，$\mathcal{C}$ 排除了以线性森林为子式，因此 $\mathcal{C}$ 中的图的树深度有一个普适常数上界。由此可得 $\max_{G\in\mathcal{C}}\mathrm{par}_q(G)$ 有一个与 q无关的常数上界。

If $(t,s) = (2,3)$, then $\mathcal{X}$ contains a forest, and so by the Grid-Minor Theorem, $\mathcal{C}$ has bounded treewidth. First, suppose par $\in \{\mathrm{wcol}, \mathrm{ftdfr}\}$. Then, by Theorems 1.3 and 1.4, $\max_{G\in\mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(\log q)$. Moreover, since $s \geqslant 3$, every path is in $\mathcal{C}$. Paths are in $\mathcal{S}_2$ so by Theorem 2.1, $\max_{G\in\mathcal{C}} \mathrm{par}_q(G) = \Omega(\log q)$. Now, suppose $(\mathrm{par}_q \mid q \in \mathbb{N}_1) = (\mathrm{cen}_q \mid q \in \mathbb{N}_1)$. By Theorem 1.2 for $t = 3$, $\max_{G\in\mathcal{C}} \mathrm{cen}_q(G) = \mathcal{O}(q)$. Moreover, $\mathcal{C}$ contains every path. Since paths have unbounded treedepth, and because $\mathrm{cen}_q(G) \geqslant \min\{\mathrm{td}(G)\ q+1\}$ for every graph G and every positive integer q, we deduce that $\max_{G\in\mathcal{C}} \mathrm{cen}_q(G) \geqslant q+1$.

If $3 \leqslant t = s$, then, by Theorems 1.2 to 1.4, $\max_{G\in\mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-1})$. If $\mathcal{X}$ contains a planar graph, then $\mathcal{C}$ has bounded treewidth by the Grid-Minor Theorem, and so $\max_{G\in\mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-2})$. For the lower bound, by the definition of t and s, we have $\mathcal{R}_{t-1} \subseteq \mathcal{C}$. By Theorem 2.1, $\max_{G\in\mathcal{C}} \mathrm{par}_q(G) = \Omega(q^{t-2})$.

If $3 \leqslant t = s-1$, then, by Theorems 1.2 to 1.4, $\max_{G\in\mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-1}\log q)$. If moreover $\mathcal{X}$ contains a planar graph, then $\mathcal{C}$ has bounded treewidth by the Grid-Minor Theorem, and so $\max_{G\in\mathcal{C}} \mathrm{par}_q(G) = \mathcal{O}(q^{t-2}\log q)$. For the lower bound, by the definition of t and s, we have $\mathcal{S}_t \subseteq \mathcal{C}$. By Theorem 2.1, $\max_{G\in\mathcal{C}} \mathrm{par}_q(G) = \Omega(q^{t-2}\log q)$. □

推论 1.6 由定理 1.2 至 1.4 以及以下引理推出。对于一个图类$\mathcal{X}$，令 $\mathbf{A}(\mathcal{X})$ 为所有图 G的类，使得 $G \in \mathcal{X}$ 或 $G-u \in \mathcal{X}$ 对于某个 $u \in V(G)$。

引理 2.3. 对于每个图 X,

$$\mathrm{rtd}_2(X) \leqslant \mathrm{td}(X).$$

Proof. We prove by induction on t that graphs of treedepth at most t are in $\mathcal{R}_t$. This is true for $t = 0$. Now suppose that $t \geqslant 1$, and let X be a graph of treedepth at most t. Let C be a component of X. By the definition of treedepth, and by the induction hypothesis, $C \in \mathbf{A}(\{H \mid \mathrm{td}(H) \leqslant t-1\}) \subseteq \mathbf{A}(\mathcal{R}_{t-1})$. Since $\mathbf{A}(\mathcal{R}_{t-1}) \subseteq \mathbf{T}(\mathcal{R}_{t-1})$, we deduce that every component of X belongs to $\mathbf{T}(\mathcal{R}_{t-1})$, and since $\mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$ is closed under disjoint union, it follows that $X \in \mathcal{R}_t$. □

3. 预备知识

对于每个正整数 k，我们记 $[k] = \{1,\dots,k\}$ 和 $[0] = \emptyset$。除非另有明确说明，本文所考虑的所有图都是有限的、简单的和无向的。 空图 是

the graph with no vertices. Connected graphs are nonnull. A tree is defined as a connected forest, thus, trees and subtrees are also assumed to be nonnull. A graph G is *edgeless* if it has no edges. We denote by $\mathcal{E}$ the class of all the edgeless graphs. For a positive integer n, we denote by K_n the *complete graph* on the vertex set $[n]$; and we denote by $\overline{K_n}$ the edgeless graph on the vertex set $[n]$.

Let S be a finite set and let $\sigma = (x_1, \dots, x_{|S|})$ be an ordering of S. We write $\min_\sigma S = x_1$; $\max_\sigma S = x_{|S|}$; and $x_i \leqslant_\sigma x_j$ if and only if $i \leqslant j$, for all $i, j \in [|S|]$. A collection $\mathcal{P}$ of subsets of a nonempty set S is a *partition* of S if elements of $\mathcal{P}$ are nonempty, pairwise disjoint, and $\bigcup \mathcal{P} = S$.

设 F是一个有根森林。对于 F的子树 T，我们将 $\mathrm{root}(T)$ 定义为 T中距离 F中任意一个根最近的唯一顶点。对于顶点 x在 F中，我们用 $\mathrm{Sub}(F, x)$ 表示由 x 在 F中的所有后代所诱导的子树。

Let G be a graph. For all $S, T \subseteq V(G)$, an *S–T path* is a path in G that is either a one-vertex path with a vertex in $S \cap T$, or a path with one endpoint in S and the other endpoint in T such that no internal vertices are in $S \cup T$. Sometimes we say that such a path is a path *between* S and T. When S and T are singletons, say $S = \{u\}$ and $T = \{v\}$, we simplify the notation to *u–v path*.

The *distance* between two vertices u and v of G, denoted by $\mathrm{dist}_G(u, v)$, is equal to ∞ if there is no u–v path in G and otherwise, it is equal to the minimum number of edges in a u–v path. The *neighborhood* of a vertex u in a graph G, denoted by $N_G(u)$, is the set $\{v \in V(G) \mid uv \in E(G)\}$. For every set S of vertices of a graph G, let $N_G(S) = \bigcup_{u \in S} N_G(u) - S$. A *leaf addition* of a graph G is a graph that is obtained from G by adding one new vertex adjacent to at most one vertex.

设 G_1, G_2 为两个图。我们用 $G_1 \sqcup G_2$ 表示 G_1 与 G_2的不交并，用$G_1 \oplus G_2$ 表示通过添加所有可能的边得到的图：每条边的一个端点在$V(G_1)$中，另一个端点在 $V(G_2)$中。对于每个正整数 k，以及每个图 G，我们用 $k \cdot G$表示 G的 k 个互不相交的副本的并。

观察 3.1. 对于每个正整数 t，类 $\mathcal{R}_t$ 和 $\mathcal{S}_t$ 都在不交并运算下封闭。对于每个整数 t，若 $t \geqslant 2$，则 $\mathcal{R}_t$ 在添加叶节点运算下封闭；若 $t \geqslant 3$，则$\mathcal{S}_t$ 在添加叶节点运算下封闭。

For a graph G, a *layering* of G is a family $\mathcal{L} = (L_i \mid i \in \mathbb{N})$ of pairwise disjoint subsets of $V(G)$ such that $\bigcup_{i \in \mathbb{N}} L_i = V(G)$, and for every edge uv of G, there exists $i \in \mathbb{N}$ such that $u, v \in L_i \cup L_{i+1}$. A *tree partition* of a graph G is a pair $(T, \mathcal{P})$, where T is a rooted tree and $\mathcal{P} = (P_x \mid x \in V(T))$ is a partition of $V(G)$ such that for every edge uv in G either there is $x \in V(T)$ with $u, v \in P_x$ or there is an edge xy in T with $u \in P_x$ and $v \in P_y$. A *tree partition* of (G, S), where G is a graph and $S \subseteq V(G)$ is a tree partition $\big(T, (P_x \mid x \in V(T))\big)$ of $G[S]$ such that for every component C of $G - S$, there exists an edge xy in T such that $N_G(V(C)) \subseteq P_x \cup P_y$. A *path partition* of G (resp. (G, S)) is a tree partition $\big(T, (P_x \mid x \in V(T))\big)$ of G (resp. (G, S)) where T is a path.

给定一个图 G 及其划分 $\mathcal{P}$，其中 $V(G)$，商图 $G/\mathcal{P}$ 是顶点集为 $\mathcal{P}$的图，且两个不同的 P, $P' \in \mathcal{P}$在 $G/\mathcal{P}$中相邻，当且仅当存在 $u \in P$和 $u' \in P'$使得 uu' 是 G中的一条边。

对于所有图 A，B，令 $A \cup B = (V(A) \cup V(B)\ E(A) \cup E(B))$，并令 $A \cap B = (V(A) \cap V(B)\ E(A) \cap E(B))$。

对于图 H，图 H 在图 G 中的一个模型是由两两不相交的$(B_x \mid x \in V(H))$ 子集组成的族 $V(G)$，满足

(i) 对于每个 $x \in V(H)$， $G[B_x]$ 都是连通的，并且
(ii) 对于每个 $xy \in E$（H），G 中 B_x 和 B_y 之间都有一条边。
如果 G 有一个 H 的模型，则称 H 是 G 的一个小图。否则，称 G 是 H-小图自由的。令 $S \subseteq V(G)$。G 中 H 的一个模型（$B_x \mid x \in V(H)$）称为 S-根植的，如果对于每个 $x \in V(H)$， $B_x \cap S \neq \emptyset$。

Let $\mathcal{W} = \big(T, (W_x \mid x \in V(T))\big)$ be a tree decomposition of a graph G. For every subgraph H of G, we denote by $\mathcal{W}|_H$ the tree decomposition $\big(T, (W_x \cap V(H) \mid x \in V(T))\big)$ of H. An *elimination ordering* of $\mathcal{W}$ is an ordering $(u_1, \dots, u_{|V(G)|})$ of $V(G)$ such that for every $i \in [|V(G)|]$, there exists $x \in V(T)$ such that

$$\bigcup\{W_z \mid z \in V(T) \text{ and } u_i \in W_z\} \cap \{u_j \mid j \in [i-1]\} \subseteq W_x.$$[5]

A tree decomposition $\big(T, (W_x \mid x \in V(T))\big)$ of a graph G is *natural* if for every edge e in T, for each component T_0 of $T - e$, the graph $G\big[\bigcup\{W_z \mid z \in V(T_0)\}\big]$ is connected. The following lemma was first stated by Fraigniaud and Nisse [14], see also [15]. Later in the paper, we state and prove a variant of it: Lemma 11.3.

Lemma 3.2 ([14, Theorem 1])**.** *Let G be a connected graph and let $\big(T, (W_x \mid x \in V(T))\big)$ be a tree decomposition of G. There exists a natural tree decomposition $\big(T', (W'_{x'} \mid x' \in V(T'))\big)$ of G such that for every $x' \in V(T')$ there is $x \in V(T)$ with $W'_{x'} \subseteq W_x$.*

树宽有界的图满足以下 Helly 型性质，这在某种意义上构成了我们所用技术的基础。

引理 3.3 ([35, 陈述 (8.7)])。 对于每个图 G、 G 的每个树分解 $\mathcal{W}$、 G 的每个连通子图族 $\mathcal{F}$ 以及每个非负整数 d，以下两者必有其一：
(1) $\mathcal{F}$中有 $d+1$ 个两两顶点不相交的子图，或(2) 存在一个集合 Z，它是 $\mathcal{W}$ 中至多 d 个袋的并集，使得 $Z \cap V(F) \neq \emptyset$ 对每个 $F \in \mathcal{F}$ 都成立。

Let T be a tree. For every $x, y \in V(T)$ with $xy \in E(T)$, we denote by $T_{x|y}$ the component of x in $T - \{xy\}$. Observe that $T - \{xy\}$ is the disjoint union of $T_{x|y}$ and $T_{y|x}$. Suppose now that T is rooted, and let u and v be two (not necessarily distinct) vertices of T. The *lowest common ancestor* of u and v in T, denoted by $\mathrm{lca}(T, u, v)$, is the furthest vertex from the root that has u and v as descendants. Let $Y \subseteq V(T)$. The *lowest common ancestor closure* of Y in T is the set $\mathrm{LCA}(T, Y) = \{\mathrm{lca}(T, u, v) \mid u, v \in Y\}$. Observe that $\mathrm{LCA}(T, \mathrm{LCA}(T, Y)) = \mathrm{LCA}(T, Y)$. The following lemma is folklore.

引理 3.4 ([4, 引理 8])。 设 d 为正整数， T 为一棵有根树，且 Y 为 d 中 T个顶点组成的集合。若 $X = \mathrm{LCA}(T, Y)$，则 $|X| \leqslant 2d-1$，并且对于 C的每个连通分量$T - X$， $|N_T(V(C))| \leqslant 2$。

树分解也有类似的引理。

Lemma 3.5 ([4, Lemma 8])**.** *Let d be a positive integer. Let G be a graph and let $\mathcal{W} = \big(T, (W_x \mid x \in V(T))\big)$ be a rooted tree decomposition of G. Let X be a set of d vertices of T. Then $Z = \bigcup_{x \in \mathrm{LCA}(T,X)} W_x$ is the union of at most $2d-1$ bags of $\mathcal{W}$ such that for every component C of $G - Z$, $N_G(V(C))$ is a subset of the union of at most two bags of $\mathcal{W}$. Moreover, if $\mathcal{W}$ is natural, then $N_G(V(C))$ intersects at most two components of $G - V(C)$.*

[5]等价地，$(u_1,\dots,u_{|V(G)|})$ 是 $\mathcal{W}$ 的消去序，当且仅当它是由 G 添加每个包 W_x 中顶点之间所有可能的边后得到的弦图的完美消去序，其中对每个 $x \in V$（T）。

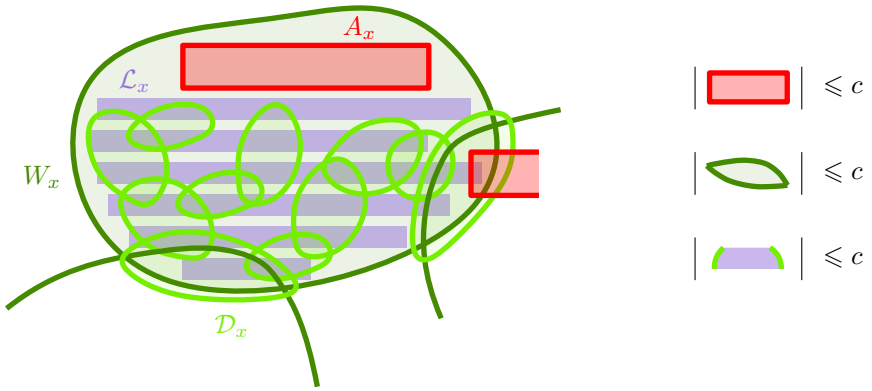


FIGURE 3. A bag W_x of $\mathcal{W}$ in a layered RS-decomposition $(\mathcal{T}, \mathcal{W}, \mathcal{A}, \mathcal{D}, \mathcal{L})$ of width at most c. The set A_x (in red) is included in W_x. The graph $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$ has a layering $\mathcal{L}_x$ (in purple) and a tree decomposition $\mathcal{D}_x$ (in green). Note that for every $y \in V(T) - \{x\}$, $(W_x \cap W_y) - A_x$ is a clique in $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$, and so, it is contained in a single bag of $\mathcal{D}_x$ and it at most two layers of $\mathcal{L}_x$.

We conclude this section by introducing the notion of layered tree decompositions, and layered Robertson-Seymour decompositions. A *layered tree decomposition* of a graph G is a pair $(\mathcal{W}, \mathcal{L})$ where $\mathcal{W} = \big(T, (W_x \mid x \in V(T))\big)$ is a tree decomposition of G, and $\mathcal{L} = (L_i \mid i \in \mathbb{N})$ is a *layering* of G. The *width* of $(\mathcal{W}, \mathcal{L})$ is $\max |W_x \cap L_i|$ over all $x \in V(T)$ and $i \in \mathbb{N}$. The *layered treewidth* of G, denoted by $\mathrm{ltw}(G)$, is the minimum width of a layered tree decomposition of G.

根据定义，对于每个图 G，都有 $\mathrm{ltw}(G) \leqslant \mathrm{tw}(G)$。平面图族是一个典型例子：其分层树宽有界，但树宽无界。事实上，基于 Eppstein 的思想， [12], Dujmović、Morin 和 Wood [6]证明了平面图的分层树宽至多为三。

For a graph G, a tree decomposition $\mathcal{W} = \big(T, (W_x \mid x \in V(T))\big)$, and $x \in V(T)$, the *torso* of W_x in G, and $\mathcal{W}$, denoted by $\mathrm{torso}_{G,\mathcal{W}}(W_x)$, is the graph with the vertex set W_x and where two distinct vertices u, v are adjacent if $uv \in E(G)$, or there exists $y \in N_T(x)$ such that $u, v \in W_x \cap W_y$. For every graph G and every positive integer c, a *layered Robertson-Seymour decomposition* of G (*layered RS-decomposition* for short) of width at most c is a tuple

$$(T, \mathcal{W}, \mathcal{A}, \mathcal{D}, \mathcal{L})$$

其中 T 是一棵树，并且

(lrs1) $\mathcal{W} = \big(T, (W_x \mid x \in V(T))\big)$ is a tree decomposition of G of adhesion at most c;
(lrs2) $\mathcal{A} = \big(A_x \mid x \in V(T)\big)$ where $A_x \subseteq W_x$ and $|A_x| \leqslant c$ for every $x \in V(T)$;
(lrs3) $\mathcal{D} = \big(\mathcal{D}_x \mid x \in V(T)\big)$ where $\mathcal{D}_x = \big(T_x, (D_{x,z} \mid z \in V(T_x))\big)$ is a tree decomposition of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$ for every $x \in V(T)$;
(lrs4) $\mathcal{L} = \big(\mathcal{L}_x \mid x \in V(T)\big)$ where $\mathcal{L}_x = (L_{x,i} \mid i \in \mathbb{N})$ is a layering of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$ for every $x \in V(T)$; and
(lrs5) $|D_{x,z} \cap L_{x,i}| \leqslant c$ for all $x \in V(T)$, $z \in V(T_x)$, and $i \in \mathbb{N}$.

见图 3。Dujmović、Morin 和 Wood [6] 证明了，对于每个固定的正整数 t，K_t-无小图的图允许具有有界宽度的此类分解。

定理 3.6 ([6, 定理22 和引理 26])。 对于每个正整数 t，存在一个正整数 $c_{\mathrm{LRS}}(t)$，使得每个不含 K_t *minor* 的图都存在一个宽度至多为 $c_{\mathrm{LRS}}(t)$的分层 *RS* 分解。

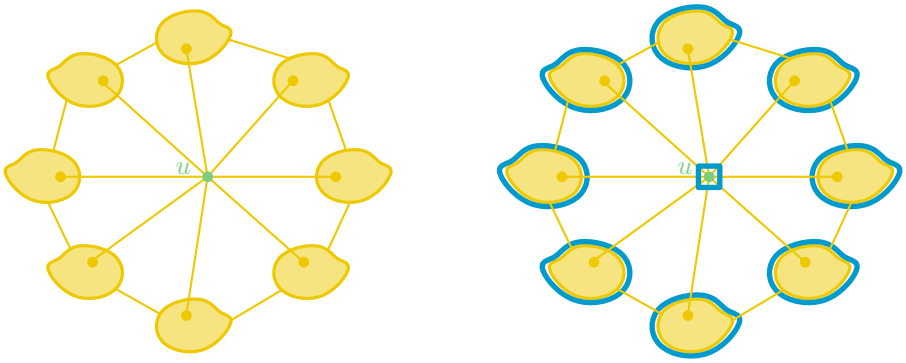


图 4. 在左侧，我们描绘了 X 的一个 $\mathcal{F}$-富模型，其中 X是一个有 8 个顶点的圈图，而 $\mathcal{F}$ 是 $G - \{u\}$ 的所有连通子图所构成的族，这些子图包含 G 中 u 的一个邻居。在右侧，我们展示了如何在给定 X 的一个 $\mathcal{F}$-富模型的情况下构造 $K_1 \oplus X$ 的一个模型。

我们以一个简单的观察结束预备知识部分。

观察 3.7。 设 G 是一个图。对于每个树分解 $\big(T, (W_x \mid x \in V(T))\big)$ *of* G，我们有

$$\mathrm{tw}(G) \leqslant \max\{\mathrm{tw}(\mathrm{torso}_{G,\mathcal{W}}(W_x)) \mid x \in V(T)\}.$$

4. 抽象框架

我们主要定理（定理 1.2 至 1.4）的证明采用归纳法，按根化2-树深度或简单根化 2-树深度进行归纳。我们提出一个抽象框架，用以概括针对以下三类参数族的归纳步骤：$(\mathrm{cen}_q \mid q \in \mathbb{N}_1)$、$(\mathrm{wcol}_q \mid q \in \mathbb{N}_1)$和 $(\mathrm{ftdfr}_q \mid q \in \mathbb{N}_1)$。

4.1. **富模型**。 设 G 和 H 为两个图。设 $\mathcal{F}$ 为 G的一个连通子图族。称模型 $(B_x \mid x \in V(H))$ 是 H 在 G 中的 $\mathcal{F}$-富的，如果对每个 $x \in V(H)$，都存在 $F \in \mathcal{F}$ 使得 F 是 $G[B_x]$的一个子图。

例如，如果 H 有 k 个顶点且没有边，那么 G 当且仅当 $\mathcal{F}$含有 H 个 G 两两不相交的 k 成员时，才具有一个 $\mathcal{F}$- 丰富模型。一个极端情形是 $\mathcal{F}$ 包含 G 的每个单顶点子图。此时， H 在 G 中的每个模型都是 $\mathcal{F}$-丰富的。另一个例子见图4。

定义一种将连通子图族限制到某个子图的记号会很方便：如果 $\mathcal{F}$ 是图 G的连通子图族，并且 H 是 G的一个子图，我们用 $\mathcal{F}|_H$ 表示族 $\{F \in \mathcal{F} \mid F \subseteq H\}$。

4.2. **聚焦参数族**。 令 $\mathcal{Z}$ 为所有二元组 (G, S) 构成的族，其中 G 是一个图，且 $S \subseteq V(G)$。我们称一个族为

$$\mathrm{par} = (\mathrm{par}_q \colon \mathcal{Z} \to [0, \infty) \mid q \in \mathbb{N}_1)$$

一个聚焦参数族。[6]

以下定义概括了一族聚焦参数必须满足的性质，使得抽象归纳步骤能够进行。

[6]稍微滥用一下记号，我们用 $\mathrm{par}_q(G, S)$ 代替 $\mathrm{par}_q((G, S))$。

我们称一族聚焦参数为良好的，如果存在正整数 b 和 b'，使得对于每个正整数 q，对于每个 $(G,\ S) \in \mathcal{Z}$，都有

(n1) 对于 G 的某个连通分量 C，$\mathsf{par}_q(G,S) \leqslant \mathsf{par}_q(C, S \cap V(C))$；(n2) $\mathsf{par}_q(G,S) \leqslant |S|$；(n3) 对所有互不相交的 $S_0, S_1 \subseteq V(G)$，$\mathsf{par}_q(G, S_0 \cup S_1) \leqslant \mathsf{par}_q(G, S_0) + \mathsf{par}_q(G - S_0, S_1)$；以及 (n4) 对于 (G,S) 的每个树划分 $\big(T,\ (P_x \mid x \in V(T))\big)$，

$$\mathrm{par}_q(G,S) \leqslant b'(q+1) \cdot \max_{x \in V(T)} \mathrm{par}_{bq}(G_x, P_x)$$

其中，对于每个 $x \in V(T)$，G_x 是由 G 的以下部分诱导的子图：$U_x = \bigcup\{P_z \mid z \in V(\mathsf{Sub}(T,x))\}$ 的并集，以及 $G - S$ 中所有与 U_x 中的某个顶点相邻的连通分支的顶点集。

在第5节中，我们定义了$\mathrm{cen}_q(\cdot)$、$\mathrm{wcol}_q(\cdot)$和$\mathrm{ftdfr}_q(\cdot)$的聚焦版本。我们证明这些聚焦版本满足(n1)–(n4)。因此，我们将在本节末尾陈述的抽象定理可用于证明主要定理的归纳步骤。

4.3. **Coloring elimination property.** To bound the growth rate of our families of focused parameters in minor-closed graph classes, we rely on the structure of the excluded minors, namely, rooted 2-treedepth and/or simple rooted 2-treedepth. The families with bounded rtd_2 and srtd_2 are built inductively in the same manner and they differ only in the base cases. Therefore, we extract the main property of these two parameters to characterize a family $\mathcal{X}$ of graphs for which the inductive step of our main theorems would apply.

Let $\mathcal{X}$ be a class of graphs. We say that $\mathcal{X}$ has the *coloring elimination property* if, for every positive integer k, for every $X \in \mathcal{X}$, there exists $Y \in \mathcal{X}$ such that for all sets $S_1, \ldots, S_k \subseteq V(Y)$ such that $\bigcup_{i \in [k]} S_i = V(Y)$, there exists $i \in [k]$ such that there is an S_i-rooted model of X in Y. In this case, we say that Y *witnesses* the coloring elimination property of $\mathcal{X}$ for X and k.

在引理6.5中，在第6节，我们证明 $\mathcal{R}_t$ 和 $\mathcal{S}_t$ 具有着色消除性质。这里，我们证明该性质在“次闭包”下保持不变。

引理 4.1。设 $\mathcal{X}$ 是一个图类，并且让

$$\mathcal{X}' = \{X' \mid X' \text{ is a minor of } X \text{ for some } X \in \mathcal{X}\}.$$

当且仅当 $\mathcal{X}'$ 具有着色消除性质时，类 $\mathcal{X}$ 才具有着色消除性质。

Proof. Suppose that $\mathcal{X}'$ has the coloring elimination property. Let k be a positive integer and let $X \in \mathcal{X}$. Since $\mathcal{X} \subseteq \mathcal{X}'$, we can fix $Y' \in \mathcal{X}'$ that witness the coloring elimination property of $\mathcal{X}'$ for X and k. By the definition of $\mathcal{X}'$, there is $Y \in \mathcal{X}$ such that Y contains a model $(B'_y \mid y \in V(Y'))$ of Y'. Fix $S_1, \ldots, S_k \subseteq V(Y)$ whose union is $V(Y)$. For every $i \in [k]$, let $S'_i = \{y \in V(Y') \mid B'_y \cap S_i \neq \emptyset\}$. Since $\bigcup_{i \in [k]} S_i = V(Y)$, we have $\bigcup_{i \in [k]} S'_i = V(Y')$. Fix $j \in [k]$ and an S'_j-rooted model $(B_x \mid x \in V(X))$ of X in Y'. For every $x \in V(X)$, let $D_x = \bigcup_{y \in B_x} B'_y$. By construction, $(D_x \mid x \in V(X))$ is an S_j-rooted model of X in Y. Therefore, $\mathcal{X}$ has the coloring elimination property.

Next, suppose that $\mathcal{X}$ has the coloring elimination property. Let k be a positive integer and let $X' \in \mathcal{X}'$. By the definition of $\mathcal{X}'$, there is $X \in \mathcal{X}$ such that X contains a model $(B'_x \mid x \in V(X'))$ of X'. Let $Y \in \mathcal{X}$ witness the coloring elimination property of $\mathcal{X}$ for X and k. Observe that $Y \in \mathcal{X}'$. Fix $S_1, \ldots, S_k \subseteq V(Y)$ whose union is $V(Y)$. Fix $j \in [k]$ and an S_j-rooted model $(B_x \mid x \in V(X))$ of X in Y. Finally, $(\bigcup_{y \in B'_x} B_y \mid x \in V(X'))$ is an S_j-rooted model of X' in Y, which shows that $\mathcal{X}'$ has the coloring elimination property. □

4.4. **The abstract induction step.** Let $g\colon \mathbb{N}_1 \to \mathbb{N}_1$ be a function, let $\mathrm{par} = (\mathrm{par}_q \mid q \in \mathbb{N}_1)$ be a family of focused parameters, let X be a graph, and let α and β be nonnegative integers. We say that a graph G is $(g, \mathrm{par}\ X, \alpha, \beta)$-*good* if for every positive integer q, for every family $\mathcal{F}$ of connected subgraphs of G, if G has no $\mathcal{F}$-rich model of X, then there exists $S \subseteq V(G)$ such that

(g1) $S \cap V(F) \neq \emptyset$ 对每个 $F \in \mathcal{F}$；(g2) 对 $G-S$的每个连通分量 C， $N_G(V(C))$至多与 $G-V(C)$的 α 个连通分量相交；并且(g3) $\mathrm{par}_q(G,S) \leqslant \beta \cdot g(q)$。

对于一类图 $\mathcal{X}$，我们称函数 $g\colon \mathbb{N}_1 \to \mathbb{N}_1$ 是 (par $\mathcal{X}$)-有界的 如果对每个正整数 k，存在一个非负整数 α 使得对每个图 $X \in \mathcal{X}$，存在一个非负整数 $\beta(X)$ 使得每个 K_k-无子式图 G 是 $(g, \mathrm{par}\ X, \alpha, \beta(X))$-好的。

In this case, when k is fixed, we say that α and $\beta(\cdot)$ *witness* g being (par $\mathcal{X}$)-bounding for k. Essentially, we require that excluding an $\mathcal{F}$-rich model of a graph X in G implies the existence of a set $S \subseteq V(G)$ hitting[7] $\mathcal{F}$ such that the parameter focused on S is bounded. If for two classes of graphs $\mathcal{X}$ and $\mathcal{X}'$ we have $\mathcal{X} \subseteq \mathcal{X}'$, then the fact that a function g is (par $\mathcal{X}'$)-bounding implies that g is (par $\mathcal{X}$)-bounding.

以下是本节的主要抽象陈述。

定理 4.2。 设$(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一个良好的聚焦参数族，设 $\mathcal{X}$ 是一个具有着色消除性质，且在不相交并和叶添加下封闭的图类。设 $g\colon \mathbb{N}_1 \to \mathbb{N}_1$ 是一个函数。若 g是(par $\mathcal{X}$)-有界的，则存在正整数 b，使得 $q \mapsto q \cdot g(bq)$ 是(par, $\mathbf{T}(\mathcal{X})$)-有界的。

在实践中，我们将这一归纳分为以下两个步骤。

定理 4.3。 设$(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一个良好的聚焦参数族，设 $\mathcal{X}$ 是一个在不交并运算下封闭且具有着色消除性质的图类，并且$g\colon \mathbb{N}_1 \to \mathbb{N}_1$。若 g是(par $\mathcal{X}$)-有界的，则存在一个正整数 b，使得 $q \mapsto q \cdot g(bq)$ 是(par, $\mathbf{A}(\mathcal{X})$)-有界的。

定理 4.4. 设$(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一个良好的聚焦参数族， $\mathcal{X}$ 是一个对不交并和叶添加封闭的非空图类，且 $g\colon \mathbb{N}_1 \to \mathbb{N}_1$。若 g 是(par, $\mathbf{A}(\mathcal{X})$)-有界的，则 g是(par, $\mathbf{T}(\mathcal{X})$)-有界的。

Clearly, Theorems 4.3 and 4.4 imply Theorem 4.2. We prove Theorems 4.3 and 4.4 in Section 7. We conclude the proofs of Theorems 1.2 to 1.4 by proving the base cases in Sections 8 to 11. We wrap up this section by a simple observation which will be reused several times to relate (par $\mathcal{X}$)-bounding of certain functions with bounds such as in Theorems 1.2 to 1.4.

引理 4.5. 设 $\mathrm{par} = (\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 为一族聚焦参数，设 $\mathcal{X}$ 为一类图，并设 $g\colon \mathbb{N}_1 \to \mathbb{N}_1$ 为一个 (par $\mathcal{X}$)-有界函数。则对每个 $X \in \mathcal{X}$，存在常数 $\beta(X)$，使得对每个正整数 q 以及每个 X-无子式图 G，我们有

$$\mathrm{par}(G, V(G)) \leqslant \beta(X) \cdot g(q).$$

证明。 令 $X \in \mathcal{X}$和 $k = |V(X)|$。由于 g是 (par $\mathcal{X}$)-有界的，因此存在非负整数 α和 $\beta(X)$，使得每个 K_k-无关图 G都是 $(g, \mathrm{par}\ X, \alpha, \beta(X))$-良好的。令 G 为一个 X-无关图。由此可得， G 也是 K_k-无关的。令 $\mathcal{F}$ 为 the

[7]一个命中集对于图 $\mathcal{F}$的一族连通子图 G 是一个集合 $S \subseteq V(G)$ 使得 $S \cap V(F)$ 对于每个 $F \in \mathcal{F}$。

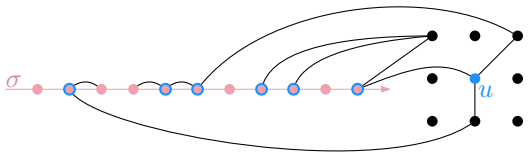


图 5。粉色顶点对应于集合 S。以蓝色突出显示的 S 顶点属于 $\mathsf{WReach}_3[G,S,\sigma,u]$。

所有单顶点子图所成的族，即 G。由于 G 是 X-无子式的，故不存在 $\mathcal{F}$-富模型（对象为 X，位于 G）。因此，存在 $S \subseteq V(G)$ 使得 (g1)–(g3) 成立，并且特别地，

(i) 对于每个 $F \in \mathcal{F}$，$S \cap V(F) \neq \emptyset$；以及 (ii) 对

于每个正整数 q，$\mathsf{par}_q(G,S) \leqslant \beta(X) \cdot g(q)$。

注意，(i) 意味着 $S = V$（G）。因此，对于每个正整数 q，

$$\mathrm{par}_q(G, V(G)) \leqslant \beta(X) \cdot g(q). \qquad \square$$

5. 专注参数的优良族系

5.1. **Weak coloring numbers.** Let G be a graph, let q be a nonnegative integer, let $S \subseteq V(G)$, let σ be an ordering of S, let $u \in V(G)$, and let $v \in S$. We say that *v is weakly q-reachable from u in (G,S,σ)* if there is a u–v path P in G of length at most q such that $\min_\sigma(V(P) \cap S) = v$. We denote by $\mathrm{WReach}_q[G,S,\sigma,u]$ the set of all the weakly q-reachable vertices from u in (G,S,σ) and we write $\mathrm{wcol}_q(G,S,\sigma) = \max_{u \in V(G)} |\mathrm{WReach}_q[G,S,\sigma,u]|$. Finally, let $\mathrm{wcol}_q(G,S)$ be the minimum value of $\mathrm{wcol}_q(G,S,\sigma)$ among all orderings σ of S. For each of the defined objects, we drop S when $S = V(G)$. Namely, v is weakly q-reachable from u in (G,σ) whenever v is weakly q-reachable from u in $(G, V(G)\ \sigma)$, $\mathrm{WReach}_q[G,\sigma,u] = \mathrm{WReach}_q[G, V(G)\ \sigma, u]$, $\mathrm{wcol}_q(G,\sigma) = \mathrm{wcol}_q(G, V(G)\ \sigma)$, and $\mathrm{wcol}_q(G) = \mathrm{wcol}_q(G, V(G))$. This matches the definition given in Section 1. See an illustration in Figure 5.

接下来，我们陈述一些关于弱着色数的简单观察。最终，我们将证明 $(\mathsf{wcol}_q \mid q \in \mathbb{N}_1)$ 是一族优美的聚焦参数。

观察 5.1. 设 G 是一个图，令 $S \subseteq V(G)$，并令 $\mathcal{C}$ 为 G 的连通分量族。对于每个正整数 q，有

$$\mathrm{wcol}_q(G,S) = \max_{C \in \mathcal{C}} \mathrm{wcol}_q(C, S \cap V(C)).$$

观察 5.2. 设 G 是一个图，且令 $S, U \subseteq V(G)$。对于每个非负整数 q，有

$$\mathrm{wcol}_q(G - U, S - U) \leqslant \mathrm{wcol}_q(G,S).$$

观察 5.3. 设 G 是一个图，且 S_1, S_2 是 $V(G)$ 的不相交子集。对于每个正整数 q，有

$$\mathrm{wcol}_q(G, S_1 \cup S_2) \leqslant \mathrm{wcol}_q(G,S_1) + \mathrm{wcol}_q(G - S_1, S_2).$$

观察 5.1 和5.2由定义显然可得。为证明观察 5.3，只需取一个 σ_1 的顶点排列 S_1，使其见证 $\mathrm{wcol}_q(G,S_1)$；再取一个 σ_2 的顶点排列 S_2，使其见证$\mathrm{wcol}_q(G - S_1, S_2)$，然后将二者串接，使 S_1的所有顶点都排在 S_2的所有顶点之前。

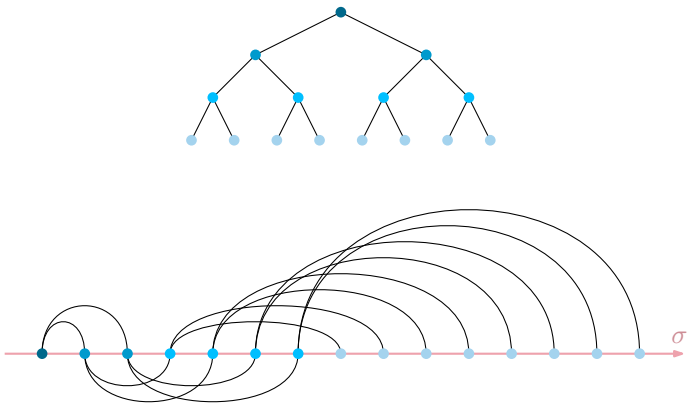


图 6. 高度为 3 的完全二叉树的一种消去序。

要为树的弱着色数推导一个上界，只需将给定的树选定根，并对其顶点取一个消去序。在这种序中，对于每个正整数 q，一个顶点只能弱 q可达其 $q+1$ 个最近的祖先（包括其自身）。见图 6。

观察 5.4. 设 T 是一棵有根树。设 σ 是 T 的消除序。则

$$\mathrm{wcol}_q(T,\sigma) \leqslant q+1.$$

我们将这一思想扩展到图的树分割。

Lemma 5.5. *Let G be a graph, let $S \subseteq V(G)$, let $\big(T,(P_x \mid x \in V(T))\big)$ be a tree partition of (G,S). For every $x \in V(T)$, let G_x be the subgraph of G induced by the union of $U_x = \bigcup\{P_z \mid z \in V(\mathrm{Sub}(T,x))\}$ and all the vertex sets of the components of $G-S$ having a neighbor in U_x. For every positive integer q, we have*

$$\mathrm{wcol}_q(G,S) \leqslant (q+1)\cdot \max_{x\in V(T)} \mathrm{wcol}_q(G_x,P_x).$$

Proof. Let q be a positive integer and let $k = \max_{x\in V(T)} \mathrm{wcol}_q(G_x,P_x)$. Let $(x_1,\dots,x_{|V(T)|})$ be an elimination ordering of T. For every $x \in V(T)$, let σ_x witness $\mathrm{wcol}_q(G_x,P_x) \leqslant k$. Let σ be an ordering of $S = \bigcup_{x\in V(T)} P_x$ obtained by concatenating $\sigma_{x_1},\dots,\sigma_{x_{|V(T)|}}$.

It suffices to show that $\mathrm{wcol}_q(G,S,\sigma) \leqslant (q+1)k$. Let $u \in V(G)$ and consider $W = \mathrm{WReach}_q[G,S,\sigma,u]$. Assume that $W \neq \emptyset$. If $u \in S$, let $x(u) \in V(T)$ such that $u \in P_{x(u)}$. Otherwise, let C be the component of u in $G-S$ and let $x(u)$ be the vertex of T furthest from $\mathrm{root}(T)$ such that $N_G(V(C))$ intersects $P_{x(u)}$. Observe that for every $x \in V(T)$, $W \cap P_x \subseteq \mathrm{WReach}_q[G_x,P_x,\sigma_x,u]$. Moreover, if x is not among the closest $q+1$ ancestors of $x(u)$ (including $x(u)$ itself), then $W \cap P_x = \emptyset$. Altogether, this implies that $|W| \leqslant (q+1)k$ as desired. □

引理 5.6. 族 $(\mathrm{wcol}_q \mid q \in \mathbb{N}_1)$ 是良好的。

证明。 观察 5.1 推出 (n1)；(n2) 由定义显然成立；观察 5.3 推出(n3)；并且 (n4) 结合 $b' = b = 1$ 可由引理 5.5得出。□

在定理 1.3中，我们通过用一个（ q-因子换取一个（$\mathrm{tw}(G)+1$）-因子，给出了每个上界的变体。为了获得依赖于树宽的界，我们将所采用的技术应用于以下聚焦参数族。

Lem引理 5.7。 对于每个图 G、每个 $S \subseteq V(G)$ 以及每个正整数 q，有 t

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\,\mathrm{wcol}_q(G,S)$$

聚焦参数族$(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是良好的。

证明。 由于 $\mathrm{tw}(G) \geqslant -1$ 对每个图 G都成立，这些参数是定义良好的。由子图上的树宽单调性，以及族 $(\mathrm{wcol}_q \mid q \in \mathbb{N}_1)$由引理5.6证明是优良的可知，该族也是优良的。 □

5.2. **Centered colorings.** Let S be a set. A *coloring* of S is a function $\varphi\colon S \to C$ for some set C. For each $u \in S$, we say that $\varphi(u)$ is the *color* of u and subsequently we say that $\varphi(S)$ is the *set of colors used by* φ. For two colorings φ_1 and φ_2 of S, we define the coloring $\varphi_1 \times \varphi_2$ of S, called the *product coloring* of φ_1 and φ_2, by $(\varphi_1 \times \varphi_2)(u) = (\varphi_1(u)\ \varphi_2(u))$ for every $u \in S$. Given $S' \subseteq S$, an element $u \in S'$ is a *φ-center* of S' if the color of u is unique in S' under φ, in other words, $\varphi(u) \notin \varphi(S' - \{u\})$. Let G be a graph. A *coloring* of G is a coloring of $V(G)$. When $X \subseteq V(G)$ and φ is a coloring of G, we denote by $\varphi|_X$, the restriction of φ to X. Recall that a coloring φ of G is a q-centered coloring of G for a positive integer q if for every connected subgraph H of G, either $|\varphi(V(H))| > q$ or $V(H)$ has a φ-center.

对于中心着色，我们考虑定义在预着色图上的一族聚焦参数。

设 q 是一个正整数， G 是一个图， φ 是 G 的一种着色。对于每个 $S \subseteq V(G)$，我们将 $\mathrm{cen}_q(G,\varphi,S)$ 定义为最小的非负整数 k，使得存在 S 的一种着色ψ ，使用至多 k 种颜色，并且对于 G 的每个连通子图 H ，只要满足$V(H) \cap S \neq \emptyset$，以下条件之一成立：

(i) $|\varphi(V(H))| > q$，或 (ii) $|(\varphi \times \psi)(V(H) \cap S)| > q$，或 (iii) 存在一个 $V(H) \cap S$ 的（$\varphi \times \psi$）中心。

以下结论几乎可由定义直接得出。

引理 5.8。 设 q 为正整数， G 为图，且 φ 为 G 的一种使用k种颜色的着色。那么，

$$\mathrm{cen}_q(G) \leqslant \mathrm{cen}_q(G,\varphi,V(G)) \cdot k.$$

证明。 只需取 G 的着色 $\varphi \times \psi$，其中 ψ 是 $\mathrm{cen}_q(G,\varphi,V(G$ 的见证)). □

下面我们对 $\mathrm{cen}_q(\cdot,\cdot,\cdot)$ 作一些简单的观察。

观察 5.9。 设 G 是一个图， φ 是 G 的一种着色， $S \subseteq V(G)$，且 $\mathcal{C}$ 是 G 的分支构成的族。对于每个正整数 q，有

$$\mathrm{cen}_q(G,\varphi,S) = \max_{C \in \mathcal{C}} \mathrm{cen}_q(C, \varphi|_{V(C)}, S \cap V(C)).$$

观察 5.10。 设 G 是一个图， φ 是 G 的一种着色，且 S_1、S_2 是 $V(G)$ 的不相交子集。对于每个正整数 q，有

$$\mathrm{cen}_q(G,\varphi,S_1 \cup S_2) \leqslant \mathrm{cen}_q(G,\varphi,S_1) + \mathrm{cen}_q(G - S_1, \varphi|_{V(G)-S_1}, S_2).$$

观察 5.11. 设 G 是一个图， φ 是 G 的一种着色，并且 $S \subseteq V(G)$。

$$\mathrm{cen}_q(G,\varphi,S) \leqslant |S|.$$

观察 5.9 直接由 $\mathrm{cen}_q(G,\varphi,S)$ 的定义可得，即 G 的每个连通子图H 都是 G 的某个连通分支的子图。观察 5.10 成立，因为我们可以使用两组互不相交的颜色：$\mathrm{cen}_q(G,\varphi,S_1)$ 和 $\mathrm{cen}_q(G-S_1,\varphi|_{V(G)-S_1},S_2)$，分别为 S_1 和 S_2 着色。最后，对于观察 5.11，只需令 ψ 为 S 的一种单射着色。

引理 5.12. 设 $\mathcal{U}$ 是一个无限图，并且对每一个正整数 q，设 φ_q 是 $\mathcal{U}$的一个着色。对每一个正整数 q，对每一个图 G，以及对每一个 $S\subseteq V(G)$，设

$$\mathrm{par}_q(G,S)=\begin{cases}\mathrm{cen}_q(G,\varphi_q|_{V(G)},S) & \text{if } V(G)\subseteq V(\mathcal{U}) \text{ and } E(G)\subseteq E(\mathcal{U}),\\ 0 & \text{otherwise.}\end{cases}$$

聚焦图参数族 $(\mathrm{par}_q \mid q\in\mathbb{N}_1)$ 是良好的。

证明。设 q 为正整数， G 为图。若并非 $V(G)\subseteq V(\mathcal{U})$ 且 $E(G)\subseteq E(\mathcal{U})$，则对每个 $S\subseteq V(G)$，$\mathrm{par}_q(G,S)=0$，因此 (n1)–(n4) 显然成立。于是，我们假设 $V(G)\subseteq V(\mathcal{U})$ 且 $E(G)\subseteq E(\mathcal{U})$

令 $S\subseteq V(G)$。注意，对于 G 的每个分量 C，都有 $V(C)\subseteq V(G)$，且$E(C)\subseteq E(G)$。此外， $V(G-S)\subseteq V(G)$，且 $E(G-S)\subseteq E(G)$。因此，(n1)由观察 5.9 得出，(n2)由观察 5.11 得出，而 (n3)由观察 5.10 得出。

We now show (n4). Let $\big(T,(P_x \mid x\in V(T))\big)$ be a tree partition of (G,S) with T rooted in $r\in V(T)$. For every $x\in V(T)$, let G_x be the subgraph of G induced by the union of $U_x=\bigcup\{P_z \mid z\in V(\mathrm{Sub}(T,x))\}$ and all the vertex sets of the components of $G-S$ having a neighbor in U_x. Let $k=\max\{\mathrm{cen}_q(G_x,\varphi_q|_{V(G_x)}\ P_x) \mid x\in V(T)\}$ and note that $V(G_x)\subseteq V(G)$ and $E(G_x)\subseteq E(G)$ for every $x\in V(T)$. For every $x\in V(T)$, let $\psi_x\colon P_x\to[k]$ witnesses $\mathrm{cen}_q(G_x,\varphi_q|_{V(G_x)}\ P_x)\leqslant k$. Let $\psi\colon S\to[k]\times\{0,\dots,q\}$ be defined by

$$\psi(u)=\big(\psi_x(u),\mathrm{dist}_T(r,x)\bmod(q+1)\big)$$

对于每个 $x\in V(T)$，以及每个 $u\in P_x$。

We claim that ψ witnesses $\mathrm{cen}_q(G,\varphi_q|_{V(G)}\ S)\leqslant(q+1)k$. Consider a connected subgraph H of G intersecting S. Let x be the vertex of T closest to the root such that $V(H)\cap P_x\neq\emptyset$. Observe that $H\subseteq G_x$. Since ψ_x witnesses $\mathrm{cen}_q(G_x,\varphi_q|_{V(G_x)}\ P_x)\leqslant k$, one of the following is true: $|\varphi_q(V(H))|>q$ or $|(\varphi_q\times\psi_x)(V(H)\cap P_x)|>q$ or there is a $(\varphi_q\times\psi_x)$-center u of $V(H)\cap P_x$. If one of the first two cases holds, the claim holds. Thus, assume that the latter holds. If u is a $(\varphi_q\times\psi)$-center of $V(H)\cap S$, then again the claim holds. Otherwise, there exists $y\in V(T)$ distinct from x such that

$$\mathrm{dist}_T(r,x)\equiv\mathrm{dist}_T(r,y)\pmod{(q+1)},$$

and $V(H)\cap P_y\neq\emptyset$. Since H is connected, x is an ancestor of y by the choice of x. Altogether, this implies that $\mathrm{dist}_T(x,y)>q$ and $V(H)$ intersects P_z for every z in the x–y path in T. This implies that $|(\varphi_q\times\psi)(V(H)\cap S)|\geqslant|\psi(V(H)\cap S)|>q$. Therefore, ψ witnesses $\mathrm{cen}_q(G,\varphi_q|_{V(G)}\ S)\leqslant(q+1)k$. This proves (n4) with $b'=b=1$, and concludes the proof of the lemma. □

5.3. **分数树深脆弱率**。 首先，我们定义树深的一个聚焦版本。注意，这不同于 $\mathrm{td}(\cdot,\cdot)$，亦即我们在另一篇论文 [23] 以及 Claus、Hodor、Joret 和 Morin [1]中所考虑的参数。对于每个图 G 和 $S\subseteq V(G)$，我们递归地定义

$$\mathrm{ftd}(G,S)=\begin{cases}0 & \text{若 } S=\emptyset,\\ \min_{u\in S}\mathrm{ftd}(G,S-\{u\})+1 & \text{若 } G \text{ 是连通的，}\\ \max^{i\in[\ell]}\mathrm{ftd}(C_i,S\cap V(C_i)) & \text{若 } G \text{ 由以下分支组成：}\end{cases}$$

C_1，……，C_ℓ和 $\ell>1$。

根据定义，我们可以直接得到以下观察结论。

观察 5.13. 设 G 为一个图，令 $S \subseteq V(G)$，并设 $\mathcal{C}$ 为 G的所有连通分支组成的族。我们有

$$\mathrm{ftd}(G,S) = \max_{C\in\mathcal{C}} \mathrm{ftd}(C, S\cap V(C)).$$

引理 5.14。 设 G 为图，且 S_1 和 S_2 是 $V(G)$ 的不相交子集。我们有

$$\mathrm{ftd}(G, S_1\cup S_2) \leqslant \mathrm{ftd}(G,S_1) + \mathrm{ftd}(G-S_1, S_2).$$

Proof. We proceed by induction on $|V(G)|$. If $S_1=\emptyset$, then the result is clear. Thus, assume that $|S_1|\geqslant 1$. If G is not connected, then for every component C of G, by the induction hypothesis, $\mathrm{ftd}(C,(S_1\cup S_2)\cap V(C)) \leqslant \mathrm{ftd}(C, S_1\cap V(C)) + \mathrm{ftd}(C-(V(C)\cap S_1)\ S_2\cap V(C)) \leqslant \mathrm{ftd}(G,S_1)+\mathrm{ftd}(G-S_1,S_2)$. This implies the result by Observation 5.13. If G is connected, then by the definition of $\mathrm{ftd}(G,S_1)$, there exists $u\in S_1$ such that $\mathrm{ftd}(G,S_1) = 1+\mathrm{ftd}(G-u, S_1-\{u\})$. By the induction hypothesis, we have $\mathrm{ftd}(G-u,(S_1-\{u\})\cup S_2) \leqslant \mathrm{ftd}(G-u,S_1-\{u\})+\mathrm{ftd}(G-S_1,S_2) = \mathrm{ftd}(G,S_1)-1+\mathrm{ftd}(G-S_1,S_2)$. Since $\mathrm{ftd}(G,S_1\cup S_2)\leqslant 1+\mathrm{ftd}(G-u,(S_1-\{u\})\cup S_2)$, this implies the result. □

引理 5.15. 设 G 是一个图，设 $S\subseteq V(G)$，并且设 $\big(T,\ (P_x \mid x\in V(T))\big)$ 是 (G,S) 的一个树划分。我们有

$$\mathrm{ftd}(G,S) \leqslant h\cdot \max_{x\in V(T)} \mathrm{ftd}(G_x,P_x)$$

其中 h 是 T 的顶点高度，并且对于每个 $x\in V(T)$，G_x 是 G 的由 $U_x = \bigcup\{P_z \mid z\in V(\mathrm{Sub}(T,x))\}$ 以及 $G-S$ 中所有与 U_x 有邻点的连通分支的顶点集之并所诱导的子图。

Proof. The statement is clear when $S=\emptyset$, hence, assume otherwise. We proceed by induction on h. The statement is clear again when $h=1$, hence, assume $h\geqslant 2$. Let r be the root of T. By possibly removing the vertex sets of all the components of G disjoint from S, we suppose that every component of G intersects S, and so $G=G_r$. Let $k=\max_{x\in V(T)}\mathrm{ftd}(G_x,P_x)$. For every child x of r, $\big(\mathrm{Sub}(T,x),(P_z \mid z\in V(T_x))\big)$ is a tree partition of $(G_x, S\cap V(G_x))$. Since $\mathrm{Sub}(T,x)$ has vertex-height at most $h-1$, by induction, $\mathrm{ftd}(G_x,S\cap V(G_x)) \leqslant (h-1)\max_{z\in V(\mathrm{Sub}(T,x))}\mathrm{ftd}(G_z,P_z) \leqslant (h-1)k$. By Observation 5.13, $\mathrm{ftd}(G-P_r, S-P_r) \leqslant \max\{\mathrm{ftd}(G_x, S\cap V(G_x)) \mid x\in V(T-r), \mathrm{p}(T,x)=r\} \leqslant (h-1)k$. By Lemma 5.14, $\mathrm{ftd}(G,S)\leqslant \mathrm{ftd}(G_r,P_r)+\mathrm{ftd}(G-P_r,S-P_r)\leqslant k+(h-1)k = h\cdot k$ since $G_r=G$. This concludes the proof. □

设 q 为正整数，G 为图，且 $S\subseteq V(G)$。图 q阶分数树深度脆弱率 of (G,S)，记为 $\mathrm{ftdfr}_q(G,S)$，是满足以下条件的最小非负整数 k：存在一个 $\mathcal{Y}$ 的子集族 S，使得

(i) 对每个 $Y\in\mathcal{Y}$，$\mathrm{ftd}(G-Y,S-Y)\leqslant k$，且 (ii) 存在定义在 $\mathcal{Y}$ 上的概率分布 λ，使得对每个 $u\in S$，都有$\sum_{u\in Y\in\mathcal{Y}}\lambda(Y)\leqslant \frac{1}{q}$。

我们称满足 (ii) 的概率分布为 q-薄。换言之，若随机变量 $\mathbf{Y}$ 以 S 的子集为取值，则其概率分布为 q-薄，当且仅当对每个 $u\in S$，都有 $\Pr[u\in\mathbf{Y}]\leqslant\frac{1}{q}$。我们将使用以下简写。对于正整数$k$，写作“随机变量 $\mathbf{Y}$ 见证 $\mathrm{ftdfr}(G,S)\leqslant k$”，意指 $\mathbf{Y}$ 的取值属于集合 $\{Y\subseteq S \mid \mathrm{ftd}(G-Y,S-Y)\leqslant k\}$，且 $\mathbf{Y}$ 的概率分布为 q-薄。此外，稍微滥用一下记号，我们有时会说随机变量 $\mathbf{Y}$“满足某个陈述”，意指该陈述对每个可能作为 $\mathbf{Y}$ 取值的元素都成立。

Lemma 5.16. 对于每个图 G、每个 $S \subseteq V(G)$，以及每个正整数 q，l *et*

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\,\mathrm{ftdfr}_q(G,S).$$

聚焦参数族 $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是良好的。

证明。首先注意，对于每个图 G，$1/(\mathsf{tw}(G)+2)$ 都是定义良好的，因为 $\mathsf{tw}(G) \geqslant -1$。设 G 为一个图，设 $S \subseteq V(G)$，并设 q 为一个正整数。

Let $\mathcal{C}$ be the family of components of G. Let $k = \max_{C\in\mathcal{C}} \mathrm{ftdfr}_q(C, S\cap V(C))$. For every $C \in \mathcal{C}$, let Y_C witness $\mathrm{ftdfr}_q(C, S\cap V(C)) \leqslant k$. Let $\mathsf{Y} = \bigcup_{C\in\mathcal{C}} \mathsf{Y}_C$. Using Observation 5.13, it is easy to verify that $\mathrm{ftd}(G-\mathsf{Y}\ S-\mathsf{Y}) \leqslant k$. Let $u \in S$ and let $C \in \mathcal{C}$ be such that $u \in V(C)$. Since the probability distribution of Y_C is q-thin, we have $\Pr[u \in \mathsf{Y}] = \Pr[u \in \mathsf{Y}_C] \leqslant \frac{1}{q}$. Thus, Y witnesses $\mathrm{ftdfr}_q(G,S) \leqslant k$. By definition of k, there is $C \in \mathcal{C}$ such that $\mathrm{ftdfr}_q(C, S\cap V(C)) = k$. Therefore, by monotonicity of treewidth, $\mathrm{par}_q(G,S) \leqslant \mathrm{par}_q(C, S\cap V(C))$, and so, (n1) holds.

为证明 (n2)，考虑一个随机变量Y，其恒等于空集。于是，由于$\mathrm{ftd}(G,S) \leqslant |S|$，并且对每个 $u \in S$都有 $\Pr[u \in \mathsf{Y}] = 0 \leqslant \frac{1}{q}$，我们推得 Y见证了 $\mathrm{ftdfr}^q(G,S) \leqslant |S|$，由此得到 (n2)。

Let S_1 and S_2 be disjoint subsets of S whose union is S. By monotonicity of treewidth, in order to obtain (n3), it suffices to prove that $\mathrm{ftdfr}_q(G, S_1\cup S_2) \leqslant \mathrm{ftdfr}_q(G,S_1)+\mathrm{ftdfr}_q(G-S_1,S_2)$. Let Y_1 and Y_2 witness $\mathrm{ftdfr}_q(G,S_1) = k_1$ and $\mathrm{ftdfr}_q(G-S_1,S_2) = k_2$ respectively. Let $\mathsf{Y} = \mathsf{Y}_1\cup\mathsf{Y}_2$. By Lemma 5.14,

$$\mathrm{ftd}(G-\mathsf{Y}, S-\mathsf{Y}) \leqslant \mathrm{ftd}(G-\mathsf{Y}_1, S_1-\mathsf{Y}_1) + \mathrm{ftd}\left((G-S_1)-\mathsf{Y}_2, S_2-\mathsf{Y}_2\right) \leqslant k_1+k_2.$$

设 $u \in S$ 并假设 $u \in S_i$ 对某个 $i \in [2]$成立。由于 Y_i 的概率分布是q-稀疏的，我们有 $\Pr[u \in \mathsf{Y}] = \Pr[u \in \mathsf{Y}_i] \leqslant \frac{1}{q}$。由此可得 Y 见证了 $\mathrm{ftdfr}^q(G,S) \leqslant k_1+k_2$，这就完成了 (n3)的证明。

To show (n4), consider a tree partition $\left(T, (P_x \mid x \in V(T))\right)$ of (G,S) with T rooted in $r \in V(T)$. For every $x \in V(T)$, let G_x be the subgraph of G induced by the union of $U_x = \bigcup\{P_z \mid z \in V(\mathrm{Sub}(T,x))\}$ and all the vertex sets of the components of $G-S$ having a neighbor in U_x. Let $k = \max_{x\in V(T)} \mathrm{ftdfr}_{2q}(G_x,P_x)$. For every $x \in V(T)$, let Y_x witness $\mathrm{ftdfr}_{2q}(G_x,P_x) \leqslant k$.

For each $i \in [2q]$, let $L_i = \{x \in V(T) \mid \mathrm{dist}_T(r,x) \equiv i \bmod 2q\}$. Let Y' be a random variable with the uniform distribution on the set $\left\{\bigcup_{x\in L_i} P_x \mid i \in [2q]\right\}$, and let $\mathsf{Y} = \mathsf{Y}' \cup \bigcup_{x\in V(T)} \mathsf{Y}_x$. For every $x \in V(T)$, let $\mathsf{P}_x = P_x - \mathsf{Y}$. Let F be the forest obtained from T by removing every vertex $x \in V(T)$ such that $\mathsf{P}_x = \emptyset$. We root each component of F in its vertex which is the closest to r in T. Let C be a component of $G-\mathsf{Y}$, and let T_C be the subtree of F such that $V(\mathsf{C}) \cap \bigcup_{x\in V(\mathsf{T}_\mathsf{C})} \mathsf{P}_x \neq \emptyset$. Observe that $\left(\mathsf{T}_\mathsf{C}, (\mathsf{P}_x \mid x \in V(\mathsf{T}_\mathsf{C}))\right)$ is a tree partition of $(\mathsf{C}, (S-\mathsf{Y}) \cap V(\mathsf{C}))$. By the definition of Y', all trees in F have vertex-height at most $2q-1$. Therefore, by Lemma 5.15,

$$\begin{aligned}\mathrm{ftd}(\mathsf{C}, (S-\mathsf{Y})\cap V(\mathsf{C})) &\leqslant (2q-1) \max_{x\in V(\mathsf{T}_\mathsf{C})} \mathrm{ftd}(\mathsf{G}_x, \mathsf{P}_x)\\ &\leqslant (2q-1) \max_{x\in V(T)} \mathrm{ftd}(G_x-\mathsf{Y}_x, P_x-\mathsf{Y}_x) \leqslant (2q-1)k,\end{aligned}$$

其中，对于每个 $x \in V(T)$，G_x 是 C 的子图，由 $\mathsf{U}_x = \bigcup\{\mathsf{P}_z \mid z \in V((\mathsf{T}_\mathsf{C})_x)\}$ 与 $\mathsf{C} - (S-\mathsf{Y})$ 中所有与 U_x 相邻的连通分支的顶点集之并所诱导。

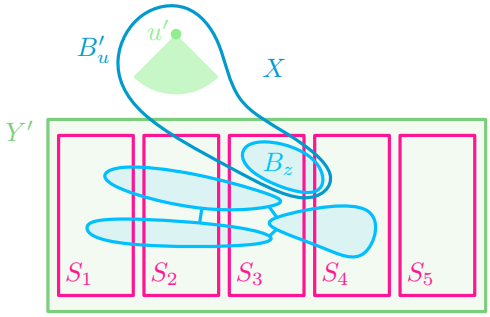


图 7。顶点 u'对应于 $K_1 \oplus Y'$中的 K_1。 S_1、……、S_5是 $V(Y')$的一个划分。图中有一个 Y' （浅蓝色）中 $K_1 \sqcup (X-u)$的 S_3-根模型。我们将 u' 添加到分支集 B_z 中，其中 z 对应于 $K_1 \sqcup (X-u)$中的 K_1，从而得到 Y中 X 的一个 S_3-根模型。

最后，令 $u \in S$，并令 $x \in V(T)$ 满足 $u \in P_x$。根据 $\mathbf{Y}'$ 的定义，且由于 $\mathbf{Y}_x$ 的概率分布是 q-薄的，由并集界可得

$$\Pr[u \in \mathbf{Y}] = \Pr[u \in \mathbf{Y}'] + \Pr[u \in \mathbf{Y}_x] \leqslant \frac{1}{2q} + \frac{1}{2q} = \frac{1}{q}.$$

因此，(n4) 成立，其中 $b' = b = 2$。引理得证。 □

6. Coloring elimination property of $\mathcal{R}_t$ and $\mathcal{S}_t$

在本节中，我们证明 $\mathcal{R}_t$ 和 $\mathcal{S}_t$ 对于每个非负整数 t都具有着色消除性质。首先，我们证明算子 $\mathbf{A}$ 和 $\mathbf{T}$ 在一些温和的假设下保持该性质，见引理 6.1。因此，为得出结论，只需检查基本情形。

引理 6.1. 设 $\mathcal{X}$ 是一个在取子图和不交并运算下封闭的图类。若 $\mathcal{X}$ 具有着色消去性质，则 $\mathbf{A}(\mathcal{X})$ 和 $\mathbf{T}(\mathcal{X})$ 具有着色消去性质。

证明。当 $\mathcal{X}$ 仅由空图构成时，该命题显然成立，因此假设 $\mathcal{X}$包含一个非空图。设 k 为一个正整数。

For every $X \in \mathbf{T}(\mathcal{X})$, we say that a vertex $u \in V(X)$ is a *root* of X if there exists a rooted forest decomposition $\big(F, (W_a \mid a \in V(F))\big)$ of X of adhesion at most 1 such that for every $a \in V(F)$ that is not a root, $G[W_a - W_{\mathrm{p}(F,a)}] \in \mathcal{X}$; for every a that is a root, $|W_a| \leqslant 1$; and there exists a root r of F such that $W_r = \{u\}$. For every $X \in \mathbf{T}(\mathcal{X})$, X has a root u unless X is the null graph because $\mathcal{X}$ is closed under taking minors (in particular, for every $X \in \mathcal{X}$ and $u \in V(X)$, we have $X - u \in \mathcal{X}$). This definition also applies to the members of $\mathbf{A}(\mathcal{X})$ since $\mathbf{A}(\mathcal{X}) \subseteq \mathbf{T}(\mathcal{X})$. For every $X \in \mathbf{A}(\mathcal{X})$, for every root u of X, $X - u \in \mathcal{X}$.

命题 6.2。设 X是 $\mathbf{A}(\mathcal{X})$ 中的一个非空图，且 u是 X 的一个根。存在一个带有根 u' 的 $Y \in \mathbf{A}(\mathcal{X})$，使得对于任意一族集合 $S_1, \ldots, S_k$，若其并集为 $V(Y)$，则 Y包含一个 S_i-根模型 $(B_x \mid x \in V(X))$，该模型是某个 $i \in [k]$ 的 X，并且满足 $u' \in B_u$。

该断言的证明。证明过程如图 7 所示。令 $X' = K_1 \sqcup (X-u)$。由于 $\mathcal{X}$包含一个非空图且对取子式封闭，因此它包含 K_1。由于 u 是X的根，且 $X \in \mathbf{A}(\mathcal{X})$，$X - u \in \mathcal{X}$。特别地，$X' \in \mathcal{X}$，因为 $\mathcal{X}$对不交并封闭。我们

将 z 记为在 X'中对应于 K_1 的顶点。令 $Y' \in \mathcal{X}$ 见证 $\mathcal{X}$ 中关于 X' 和 k的着色消去性质。此外，令 $Y = K_1 \oplus Y'$，并令 u' 为 Y中对应于 K_1的顶点。特别地，$Y \in \mathbf{A}(\mathcal{X})$，且 u' 是 Y的根。

考虑一族集合 S_1、.....、$S_k \subseteq V(Y)$，其并集为 $V(Y) = V(Y') \cup \{u'\}$。存在 $i \in [k]$，使得 Y' 包含一个以（$S_i - \{u'\}$）为根的 X' 的模型（$B'_x \mid x \in V(X')$）。令$B_u = B'_z \cup \{u'\}$，并对每个 $x \in V(X') - \{z\}$ 取 $B_x = B'_x$。由此可得（$B_x \mid x \in V(X)$）是 Y 中以 S_i 为根的 X 的模型。由于 $u' \in B_u$，这就完成了该断言的证明。$\Diamond$

断言 6.2 表明 $\mathbf{A}(\mathcal{X})$ 具有着色消除性质。要证明下一个断言，需要对断言 6.2 中的根作出更强的断言。

断言6.3。设 $X \in \mathbf{T}(\mathcal{X})$是连通的，且 u 是 X的一个根。存在 $Y \in \mathbf{T}(\mathcal{X})$和 Y的一个根 u'，使得对于任意一族集合 $S_1, \ldots, S_k$，其并集为 $V(Y)$，存在某个 $i \in [k]$，使得 Y中有一个 X 的S_i-根模型($B_x \mid x \in V(X)$)，且 $u' \in B_u$。

Proof of the claim. The objects defined in the proof of this claim are depicted in Figure 8. There exists a rooted forest decomposition $\big(T, (W_a \mid a \in V(T))\big)$ of X witnessing the fact that $X \in \mathbf{T}(\mathcal{X})$ and with $\{u\}$ as a root bag. Without loss of generality, $W_a \neq \emptyset$ for every $a \in V(T)$. Since X is connected, this implies that T is a tree. Fix such a rooted forest decomposition. Let r be the unique root of T. We proceed by induction on the vertex-height of T. When $T = K_1$, $X = K_1$ and the statement is clear. Now, suppose that $T \neq K_1$ and that the result holds for trees with smaller vertex-heights.

Let T_0 be the subtree of T induced by $\{a \in V(T) \mid u \in W_a\}$. Observe that for every $a \in V(T_0)$, $X[W_a - \{u\}] \in \mathcal{X}$. Since $\mathcal{X}$ is closed under disjoint union, this implies that $X\left[\bigcup_{a \in V(T_0)} (W_a - \{u\})\right] \in \mathcal{X}$. So, $X_0 = X\left[\bigcup_{a \in V(T_0)} W_a\right]$ is in $\mathbf{A}(\mathcal{X})$ and u is a root of X_0. By Claim 6.2, there exists $Y_0 \in \mathbf{A}(\mathcal{X})$ and a root u' of Y_0 such that for every family of sets $S_1, \ldots, S_k \subseteq V(Y_0)$ whose union is $V(Y_0)$, Y_0 contains an S_i-rooted model of X_0 for some $i \in [k]$ such that the branch set corresponding to u contains u'.

令 X_1 为通过将 X 中的 $V(X_0)$所有顶点合并为一个顶点 u_1所得到的图。注意，由于 $\mathcal{X}$ 对取子图操作封闭，因此 $X_1 \in \mathcal{X}$。令 T_1 为通过将 T 中的 $V(T_0)$顶点合并为一个顶点 r_1所得到的树。对于每个$a \in V(T_1)$，令

$$W_{1,a} = \begin{cases} \{u_1\} & \text{if } a = r_1, \\ W_a & \text{otherwise.} \end{cases}$$

The rooted tree decomposition $\big(T_1, (W_{1,a} \mid a \in V(T_1))\big)$ of X_1 witnesses the fact that $X_1 \in \mathbf{T}(\mathcal{X})$. We claim that the vertex-height of T_1 is smaller than the vertex-height of T. Indeed, for every neighbor s or r in T, since X is connected and $W_s \neq \emptyset$, we have $u \in W_s$, and so $s \in V(T_0)$. This proves that $N_T(r) \subseteq V(T_0)$, and so T_1 has smaller vertex-height than T. Therefore, by the induction hypothesis, fix a graph $Y_1 \in \mathbf{T}(\mathcal{X})$ and a root u'_1 of Y_1 such that for every family of sets $S_1, \ldots, S_k \subseteq V(Y_1)$ whose union is $V(Y_1)$, there is an S_i-rooted model of X_1 in Y_1 such that the branch set corresponding to u_1 contains u'_1. Since X_1 is connected, we assume without loss of generality that Y_1 is also connected.

For every $v \in V(Y_0)$, let $Y_{1,v}$ be a copy of Y_1. We label vertices in $Y_{1,v}$ in the following way. The vertex corresponding to u'_1 is labeled as v and a vertex corresponding to each $y \in V(Y_1) - \{u'_1\}$ is labeled as (y, v). Finally, let Y be obtained from the disjoint union of Y_0 and $Y_{1,v}$ for each $v \in V(Y_0)$ by identifying vertices with the same labels. We claim that Y witnesses the assertion of the claim.

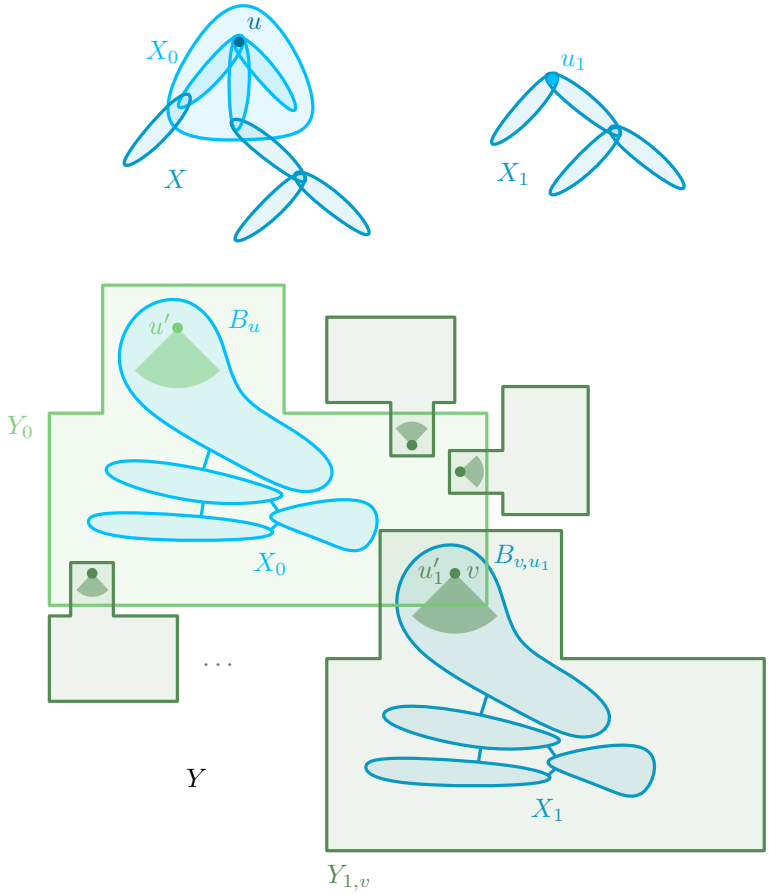


FIGURE 8. The graph X is split into two graphs according to its rooted tree decomposition witnessing $X \in \mathbf{T}(\mathcal{X})$: $X_0 \in \mathbf{A}(\mathcal{X})$ induced by all vertices of the bags containing the root u of X and $X_1 \in \mathbf{T}(\mathcal{X})$ obtained by contracting X_0. Since $X_0 \in \mathbf{A}(\mathcal{X})$, Claim 6.2 gives $Y_0 \in \mathbf{A}(\mathcal{X})$ which witnesses the coloring elimination property in $\mathbf{A}(\mathcal{X})$ for X_0. Since X_1 has a rooted tree decomposition witnessing $X_1 \in \mathbf{T}(\mathcal{X})$ with smaller vertex-height, induction gives $Y_1 \in \mathbf{T}(\mathcal{X})$ which witness the coloring elimination property in $\mathbf{T}(\mathcal{X})$ for X_1. We make a graph $Y \in \mathbf{T}(\mathcal{X})$ that witness the coloring elimination property in $\mathbf{T}(\mathcal{X})$ for X by taking, for every $v \in V(Y_0)$, a copy $Y_{1,v}$ of Y_1, and identifying v and the vertex corresponding to a root u'_1 of Y_1. This operation is allowed by the operator $\mathbf{T}$. Moreover, for any family of k sets covering $V(Y)$, a rooted model of X can be found using a rooted model of X_0 in Y_0 and rooted models of X_1 in the copies $Y_{1,v}$ for each $v \in V(Y_0)$.

First, we argue that $Y \in \mathbf{T}(\mathcal{X})$ and u' is a root of Y. For each $v \in V(Y_0)$, let $\big(T'_{1,v}, (W'_{1,v,a} \mid a \in V(T'_{1,v}))\big)$ be a rooted tree decomposition of $Y_{1,v}$ witnessing $Y_1 \in \mathbf{T}(\mathcal{X})$, and with $T'_{1,v}$ rooted in $r'_{1,v}$ such that $W'_{1,r'_{1,v}} = \{(u'_1, v)\}$. Let T' be a tree obtained from a disjoint union of $T'_{1,v}$ for each $v \in V(Y_0)$ by identifying the roots of all copies into a vertex r'', and then adding the root r' with a single neighbor r''. We define $U_{r'} = \{u'\}$, $U_{r''} = V(Y_0)$, and $U_a = W'_{1,v,a}$ for every $a \in V(T') - \{r', r''\}$ where $a \in V(T'_{1,v})$. Since $Y_0 \in \mathbf{A}(\mathcal{X})$, $Y_0 - u' \in \mathcal{X}$. Therefore, $\big(T', (U_a \mid a \in V(T'))\big)$ is a rooted tree decomposition of Y witnessing that $Y \in \mathbf{T}(\mathcal{X})$. Additionally, $U_{r'} = \{u'\}$, hence, u' is a root of Y.

设 $S_1, \ldots, S_k$ 是一族集合，其并集为 $V(Y)$。设 $v \in V(Y_0)$。由于 $Y_{1,v}$ 是 Y_1 的一个副本，且 v 是 $Y_{1,v}$ 的一个根，因此存在 $i(v) \in [k]$，以及 X_1 在 $Y_{1,v}$ 中的一个 $(S_{i(v)} \cap V(Y_{1,v}))$-根模型 $\mathcal{M}_v = (B_{v,x} \mid x \in V(X_1))$，满足 $v \in B_{v,u_1}$。

For each $j \in [k]$, let $S'_j = \{v \in V(Y_0) \mid i(v) = j\}$. Clearly, the union of $S'_1, \ldots, S'_k$ is equal to $V(Y_0)$. Therefore, fix $\ell \in [k]$ and an S'_ℓ-rooted model $\mathcal{M}_0 = (B_x \mid x \in V(X_0))$ of X_0 in Y_0 with $u' \in B_u$ since u is a root of X_0 and u' is a root of Y_0. For every $x \in V(X_0)$, let

$$D_x = B_x \cup \bigcup_{v \in B_x} B_{v,u_1}.$$

换言之，对于每个 $v \in B$，我们将包含 v 的 $\mathcal{M}_v$ 的分支集添加到 $\mathcal{M}_0$ 中的每个 B。令 $\mathcal{D} = (D_x \mid x \in V(X_0))$。考虑

$$\mathcal{M} = \mathcal{D} \cup \bigcup_{v \in S'_\ell} (\mathcal{M}_v - \{B_{v,u_1}\}).$$

注意，$\mathcal{M}$ 是 Y中的 S_ℓ-根模型。此外，$\mathcal{M}$ 是Y中 X 的某个超图的模型。最后，回顾一下，$u' \in V(Y_0)$是 Y和 $u' \in B_u \subseteq D_u$的一个根。至此，该断言得证。$\Diamond$

由于对每一个 $X \in \mathbf{T}(\mathcal{X})$，存在 $X' \in \mathbf{T}(\mathcal{X})$ 连通，使得 X作为子图包含于X'，断言 6.3 蕴含 $\mathbf{T}(\mathcal{X})$ 具有着色消去性质，从而完成该引理的证明。□

类 $\mathcal{R}_0$、$\mathcal{S}_0$、$\mathcal{S}_1$和 $\mathcal{S}_2$ 按定义在不交并运算下封闭。若图类$\mathcal{X}$在取并运算下封闭，则同样根据定义，类 $\mathbf{T}(\mathcal{X})$ 也在不交并运算下封闭。特别地，我们得到，对于每个非负整数 $\mathcal{R}_t$ 和 $\mathcal{S}_t$，它们在不交并运算下封闭。为了应用引理，我们还需要证明这些类在取子式运算下封闭。这由引理 6.1得出。6.4。

引理 6.4. 设 $\mathcal{X}$ 为一个图类。若 $\mathcal{X}$ 对取子式封闭，则 $\mathbf{T}(\mathcal{X})$ 对取子式封闭。

Proof. Assume that $\mathcal{X}$ is closed under taking minors. Let H and G be graphs such that H is a minor of G. Suppose that $G \in \mathbf{T}(\mathcal{X})$. We want to show that $H \in \mathbf{T}(\mathcal{X})$. Let $(B_u \mid u \in V(H))$ be a model of H in G. Consider a rooted forest decomposition $\big(F, (W_x \mid x \in V(F))\big)$ of G witnessing the fact that $G \in \mathbf{T}(\mathcal{X})$. Then, for every $x \in V(F)$, let $W'_x = \{u \in V(H) \mid B_u \cap W_x \neq \emptyset\}$. We claim that $\big(F, (W'_x \mid x \in V(F))\big)$ is a rooted forest decomposition of H witnessing the fact that $H \in \mathbf{T}(\mathcal{X})$. Clearly, $\big(F, (W'_x \mid x \in V(F))\big)$ is a forest decomposition. For every root x of F, $|W'_x| \leqslant |W_x| \leqslant 1$. Moreover, for every $xy \in E(F)$, $|W'_x \cap W'_y| \leqslant |W_x \cap W_y| \leqslant 1$. Hence, it only remains to prove that for every $x \in V(F)$, if x is not a root of F, then the graph $H[W'_x - W'_{\mathrm{p}(F,x)}]$ lies in $\mathcal{X}$. Let x be a non-root vertex of F. Since the $(F, (W_x \mid x \in V(F)))$ has adhesion at most 1, the family $(B_u \cap W_x \mid u \in W'_x - W'_{\mathrm{p}(F,x)})$ is a model of $H[W'_x - W'_{\mathrm{p}(F,x)}]$ in $G[W_x - W_{\mathrm{p}(F,x)}]$. Since $G[W_x - W_{\mathrm{p}(F,x)}] \in \mathcal{X}$, and as $\mathcal{X}$ is closed under taking minors, we conclude that $H[W'_x - W'_{\mathrm{p}(F,x)}] \in \mathcal{X}$. □

引理 6.5. 对于每个非负整数 t，类 $\mathcal{R}_t$ 和 $\mathcal{S}_t$ 都具有着色消去性质。

证明。对每个非负整数 t，类 $\mathcal{R}_t$ 与 $\mathcal{S}_t$ 在不交并下封闭。此外，由引理6.4，它们在取子式下封闭。因此，由引理6.1，只需证明 $\mathcal{R}_0$，$\mathcal{S}_0$，$\mathcal{S}_1$ 与 $\mathcal{S}_2$ 具有着色消去性质。

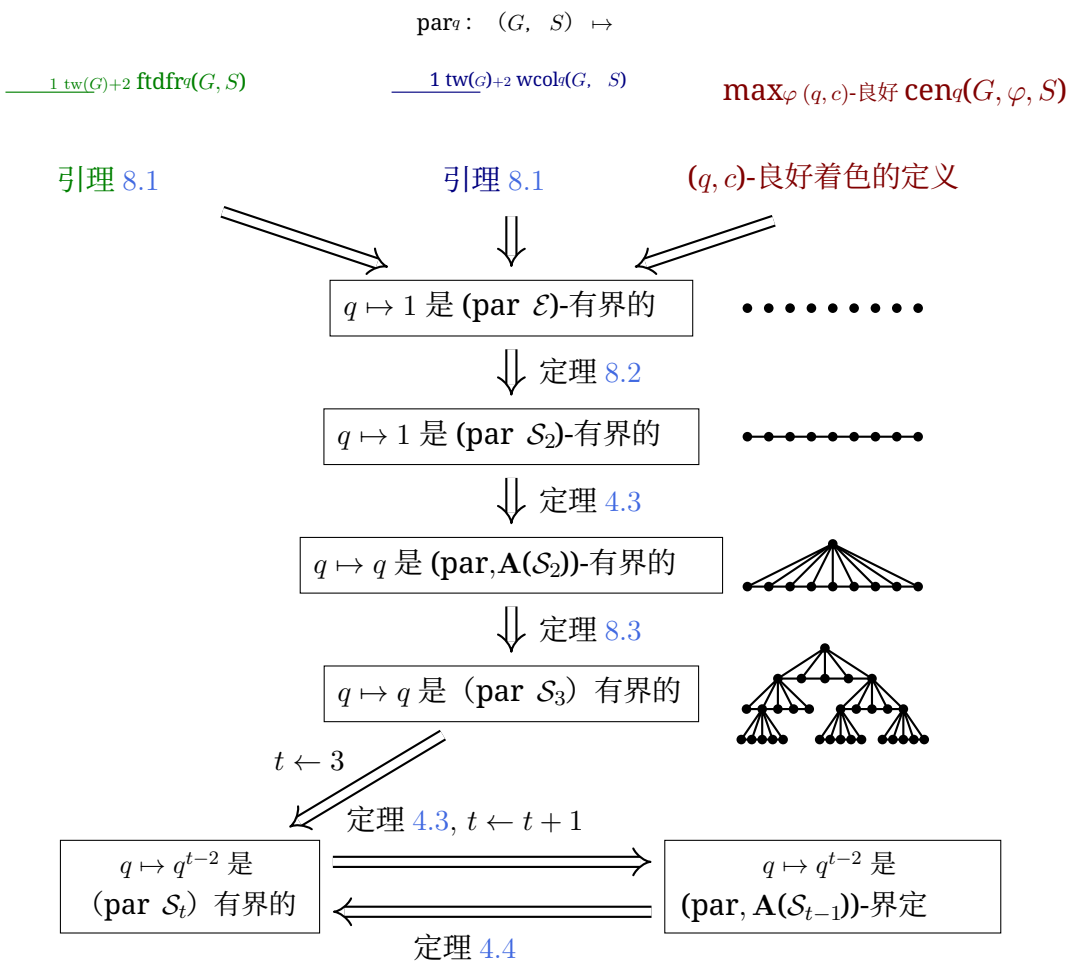


FIGURE 9. A roadmap for the proofs of Theorems 9.5, 10.2, and 11.14 (the first parts of the statements of Theorems 1.2 to 1.4). We show that, for a carefully chosen nice family of focused parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$, $q \mapsto q^{t-2}$ is (par $\mathcal{S}_t$)-bounding for every integer t with $t \geqslant 3$. The main part of the induction consists in applying Theorem 4.3 and Theorem 4.4. Since Theorem 4.4 can be applied to $\mathcal{X} = \mathcal{S}_{t-1}$ only if $t \geqslant 4$ (because $\mathcal{X}$ should be closed under leaf addition), the cases $t = 2$ and $t = 3$ are proved separately. Typical graphs in $\mathcal{E}, \mathcal{S}_2, \mathbf{A}(\mathcal{S}_2)$ $\mathcal{S}_3$ are depicted on the right side.

由于 $\mathcal{R}_0$ 与 $\mathcal{S}_0$ 仅由空图构成，类 $\mathcal{R}_0$ 与 $\mathcal{S}_0$ 具有着色消去性质。设 k 为正整数。

考虑 $X \in \mathcal{S}_1$。那么 X 由 $|V(X)|$ 个孤立顶点组成。令 Y 由 $k(|V(X)|-1)+1$ 个孤立顶点组成。对于每个 S_1、……、S_k，其并集为 $V(Y)$，都存在 $i \in [k]$，使得根据鸽巢原理，$|S_i| \geqslant |V(X)|$，由此可知 Y 包含一个以 S_i 为根的 X 模型。这证明了 $\mathcal{S}_1$ 具有着色消除性质。

考虑 $X \in \mathcal{S}_2$。不妨设 X 是连通的，因此 X 是一条路径，设其有 ℓ个顶点。令 Y为一条具有 $k(\ell-1)+1$ 个顶点的路径。令 S_1、……、S_k 为其并集为 $V(Y)$ 的集合。根据抽屉原理，存在 $i \in [k]$ 使得 $|S_i| \geqslant \ell$，由此可知，在 Y中存在一个以 S_i为根的 X 模型。这表明 $\mathcal{S}_2$ 具有着色消除性质，并完成证明。□

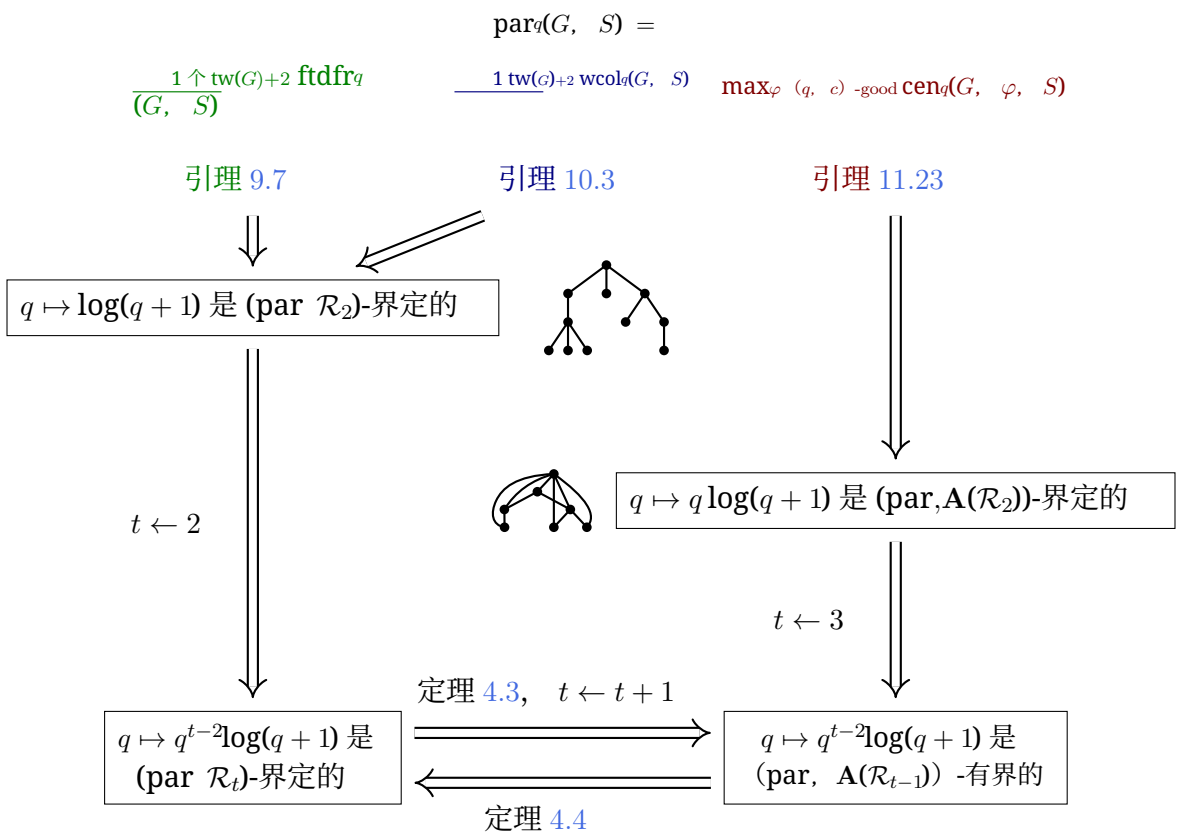


图 10。 定理9.8、 10.4和11.25（定理1.2 至 1.4陈述的第二部分）的证明路线图。

7. 抽象归纳步骤的证明

证明路线图见图 9和10。

7.1. **初步化简**。 设 $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一个聚焦参数族，设$g\colon \mathbb{N}_1 \to \mathbb{N}_1$，且 $\mathcal{X}$是一个图类。在若干情况下，为了证明 g是 (par $\mathcal{X}$)-有界的，我们首先证明一个较弱的性质；事实上，在某些温和的假设下，该性质蕴含所需结论，如引理 7.1所示。回顾第 4.4小节给出的定义。

若对于每个正整数 k、每个图 $X \in \mathcal{X}$，都存在非负整数 $\alpha(X)$ 和 $\beta(X)$，使得每个无 K_k-次要图 G都是 $(g, \mathrm{par}\ X, \alpha(X)\ \beta(X))$-良好的，则函数 g是弱 (par $\mathcal{X}$)-有界的。

此外，假设图 G 在上述定义以及 (par $\mathcal{X}$)-有界的定义中是连通的，也会很方便。为此，我们证明引理 7.2。类似地，在证明某个函数是 $(\mathrm{par}, \mathbf{T}(\mathcal{X}))$-有界时，假设 $X \in \mathbf{T}(\mathcal{X})$ 是连通的会很方便。这正是引理 7.3 和推论 7.4的目的。

引理 7.1。 设 $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一个良好的聚焦参数族， $\mathcal{X}$ 是一个图类，且 $g\colon \mathbb{N}_1 \to \mathbb{N}_1$。如果 g是 (par $\mathcal{E}$)-有界的，并且弱 (par $\mathcal{X}$)-有界，那么 g是(par $\mathcal{X}$)-有界的。

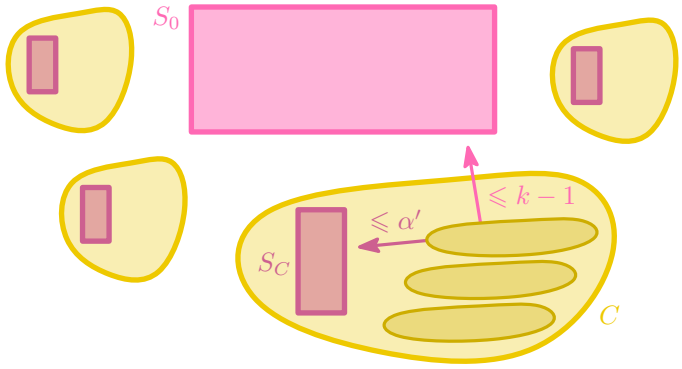


图 11。 引理 7.1 的证明示意图。

证明。 设 k为一个正整数。固定能够证明 g关于 k 弱（par $\mathcal{X}$）-有界的 $\alpha_0(\cdot)$ 和 $\beta_0(\cdot)$，并固定能够证明 g 关于 k为（par $\mathcal{E}$）-有界的 α' 和 $\beta'(\cdot)$。由于 $\mathcal{E}$ 中的图之间唯一的区别在于顶点数，我们将 $\beta'(\cdot)$视为一个以非负整数为自变量的函数，并对每个正整数 d设定 $\beta'(d) = \beta'(K_d)$。令

$$\alpha = \alpha' + (k-1) \quad \text{and} \quad \beta(X) = \beta_0(X) + \beta'\left(\binom{\alpha_0(X)}{k}(k-1)+1\right) \text{ for every } X \in \mathcal{X}.$$

We show that α and $\beta(\cdot)$ witness g being (par $\mathcal{X}$)-bounding for k. Let $X \in \mathcal{X}$ and let G be a K_k-minor-free graph We show that G is $(g, \text{par } X, \alpha, \beta(X))$-good. Let q be a positive integer and let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of X in G. Since $\alpha_0(\cdot)$ and $\beta_0(\cdot)$ witness g being weakly (par $\mathcal{X}$)-bounding for k, there exists $S_0 \subseteq V(G)$ such that

(g1-0) 对于每个 $F \in \mathcal{F}$， $S_0 \cap V(F) \neq \emptyset$；(g2-0) 对于 $G - S_0$ 的每个连通分量 C， $N_G($ $V(C))$ 与 $G - V(C)$ 的至多 $\alpha_0(X)$ 个连通分量相交；以及 (g3-0) $\text{par}_q(G, S_0) \leqslant \beta_0(X) \cdot g(q)$。

Let $\mathcal{C}$ be the family of all the components of $G - S_0$, and let $C \in \mathcal{C}$. Let $\mathcal{U}$ be the family of the vertex sets of all the components of $G - V(C)$ intersecting $N_G(V(C))$. By (g2-0), we have $|\mathcal{U}| \leqslant \alpha_0(X)$. Let $\mathcal{F}_C$ be the family of all the connected subgraphs H of C such that $|\{U \in \mathcal{U} \mid N_G(V(H)) \cap U \neq \emptyset\}| \geqslant k$. We claim that there are no $N = \binom{\alpha_0(X)}{k}(k-1)+1$ pairwise disjoint members of $\mathcal{F}_C$. Otherwise, by the pigeonhole principle, there exist $U_1, \ldots, U_k \in \mathcal{U}$ and $H_1, \ldots, H_k \in \mathcal{F}_C$ pairwise disjoint such that there is an edge between U_i and $V(H_j)$ in G for all $i, j \in [k]$. However, then $(U_i \cup V(H_i) \mid i \in [k])$ is a model of K_k in G, which is a contradiction proving that indeed there are no N pairwise disjoint members of $\mathcal{F}_C$.

回顾一下， α' 和 $\beta'(\cdot)$证明了 g 对 k是(par $\mathcal{E}$)-有界的。因此，对于每个 $C \in \mathcal{C}$，由于 C没有 K_N的 $\mathcal{F}_C$-富模型，即 $\mathcal{F}_C$中两两不相交的 N个成员，存在$S_C \subseteq V(C)$使得

(g1-C) 对每个 $F \in \mathcal{F}_C$， $S_C \cap V(F) \neq \emptyset$；(g2-C) 对 $C - S_C$的每个连通分量 C'， $N_G(V($ $C'))$至多与$C - V(C')$的 α' 个连通分量相交；以及(g3-C) $\text{par}_q(C, S_C) \leqslant \beta'(N) \cdot g(q)$。

我们设

$$S = S_0 \cup \bigcup_{C \in \mathcal{C}} S_C.$$

参见图11。只需证明

(g1*) $S \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}$ 均成立；

(g3⋆) $\mathrm{par}_q(G,S) \leqslant \beta(X) \cdot g(q)$。

It suffices to verify that G is $(g, \mathrm{par}\ X, \alpha, \beta)$-good. Since $S_0 \subseteq S$, by (g1-0), $S \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$, and so, (g1⋆) holds. For every component C' of $G-S$, there is a component $C \in \mathcal{C}$ such that $C' \subseteq C$. Since $C' \notin \mathcal{F}_C$ (by (g1-C)), $N_G(V(C'))$ intersects at most $k-1$ components of $G - V(C)$. Moreover, $N_G(V(C'))$ intersects at most α' components of $C - S_C$ (by (g2-C)). Therefore, $N_G(V(C'))$ intersects at most $\alpha' + (k-1) = \alpha$ components of $G - V(C')$. Thus, (g2⋆) holds. Finally,

$$\begin{aligned}
\mathrm{par}_q(G,S) &\leqslant \mathrm{par}_q(G,S_0) + \mathrm{par}_q\left(G - S_0, \textstyle\bigcup_{C\in\mathcal{C}} S_C\right) && \text{by (n3)}\\
&\leqslant \mathrm{par}_q(G,S_0) + \max_{C\in\mathcal{C}} \mathrm{par}_q(C,S_C) && \text{by (n1)}\\
&\leqslant \beta_0(X) \cdot g(q) + \beta'(N) \cdot g(q) && \text{by (g3-0) and (g3-C)}\\
&= \beta(X) \cdot g(q).
\end{aligned}$$

因此，(g3⋆) 成立。总之，我们得到 g 是 (par $\mathcal{X}$)-有界的，由此完成引理的证明。□

引理 7.2。设par 是一个良好的聚焦参数族，令 $g\colon \mathbb{N}_1 \to \mathbb{N}_1$ 是一个函数。令X 为一个图，令 α 和 β 为非负整数，并令 G 为一个图。若 G 的每个连通分量都是$(g, \mathrm{par}\ X, \alpha, \beta)$-良好的，则 G 是$(g, \mathrm{par}\ X, \alpha, \beta)$-良好的。

证明。 假设 G 的每个连通分支都是$(g, \mathrm{par}\ X, \alpha, \beta)$-好的。设 q 为一个正整数，并设 $\mathcal{F}$ 为 G 的一族连通子图，使得 G 没有 X的 $\mathcal{F}$-丰富模型。设$\mathcal{C}$ 为 G的所有连通分支构成的族。

令 C 为 G 的一个连通分量。由于 C 是 $(g, \mathrm{par}\ X, \alpha, \beta)$-良好的，存在 $S_C \subseteq V(C)$ 使得

(g1-C) $S_C \cap V(F) \neq \emptyset$对每一个 $F \in \mathcal{F}|_C$；(g2-C) 对每一个连通分量 C'（属于 $C - S_C$），$N_C(V(C'))$至多与 α 个连通分量（属于$C - V(C')$）相交；并且(g3-C) $\mathrm{par}_q(C, S_C) \leqslant \beta \cdot g(q)$。

我们设定 $S = \bigcup_{C\in\mathcal{C}} S_C$ 并证明了这一点。

(g1⋆) 对每个 $F \in \mathcal{F}$，$S \cap V(F) \neq \emptyset$；(g2⋆) 对于 $G - S$ 的每个分支 C'，$N_G(V(C'))$ 至多与 $G - V(C')$ 的 α 个分支相交；以及 (g3⋆) $\mathrm{par}_q(G, S) \leqslant \beta \cdot g(q)$。

对于每个 $F \in \mathcal{F}$，存在 $C \in \mathcal{C}$，使得 $V(F) \subseteq V(C)$，因此由 (g1-C)，$S_C \cap V(F) \neq \emptyset$，从而推出 $S \cap V(F) \neq \emptyset$。这给出了 (g1⋆)。对于 $G - S$ 的每个连通分支 C'，存在 $C \in \mathcal{C}$，使得 $V(C') \subseteq V(C)$。然后，由 (g2-C)，$N_G(V(C')) = N_C(V(C'))$ 至多与 $C - V(C')$ 的 α 个连通分支相交，因而也至多与 $G - V(C')$ 的同样多个连通分支相交。这证明了 (g2⋆)。最后，由 (n1) 和 (g3-C)，

$$\mathrm{par}_q(G,S) \leqslant \max_{C\in\mathcal{C}} \mathrm{par}_q(C,S_C) \leqslant \beta \cdot g(q).$$

由此得到 (g3⋆)。综合起来，我们得到 G 是 $(g, \mathrm{par}\ X, \alpha, \beta)$-好的。 □

设 $\mathcal{X}$ 为一个图类。我们将 $\mathsf{T}'(\mathcal{X})$ 定义为所有满足以下条件的图 $X \in \mathsf{T}(\mathcal{X})$：$X$ 是连通的，且 X 存在一个根树分解 $(T, (W_x \mid x \in V(T)))$，该分解见证了 $X \in \mathsf{T}(\mathcal{X})$，并且对于每个 $a \in V(T)$，都有 $W_a \neq \emptyset$。

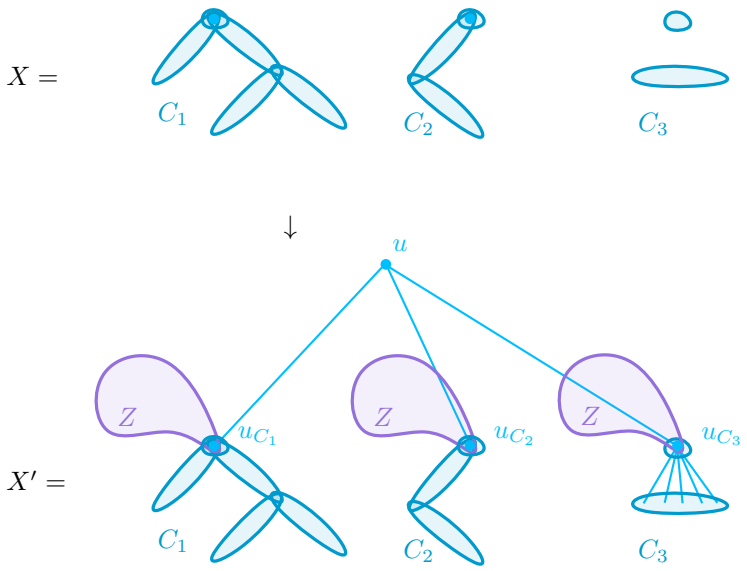


图 12. 引理证明中所用构造的示意图 7.3。图 X 有三个连通分量：C_1、C_2 和 C_3。在图的上半部分，我们描绘了一个森林分解，证明 $X \in \mathbf{T}(\mathcal{X})$。请注意，此处 $W_{C_3,\ r_{C_3}}$ 为空。

引理 7.3. 设 $\mathcal{X}$ 是一个包含非空图的图类。对于每个 $X \in \mathbf{T}(\mathcal{X})$，存在 $X' \in \mathbf{T}'(\mathcal{X})$，使得 X 是 X' 的子图。

Proof. Let Z be a nonnull graph in $\mathcal{X}$. Fix a vertex $z \in V(Z)$ and a graph $X \in \mathbf{T}(\mathcal{X})$. Assume that X and Z have disjoint vertex sets. Let $\mathcal{C}$ be the family of all components of X. Let $C \in \mathcal{C}$. We have $C \in \mathbf{T}(\mathcal{X})$. Since C is connected, we fix $\big(T_C, (W_{C,x} \mid x \in V(T_C))\big)$ a rooted tree decomposition of C witnessing the fact that $C \in \mathbf{T}(\mathcal{X})$, and let $r_C = \mathrm{root}(T_C)$. We can also assume without loss of generality that all bags except the one corresponding to the root are nonempty. Assume that the trees T_C for $C \in \mathcal{C}$ have pairwise disjoint vertex sets.

We illustrate the following construction in Figure 12. Let $C \in \mathcal{C}$. If $|W_{C,r_C}| = 1$, then we set u_C to be the unique vertex of W_{C,r_C} and we set $C' = C$. In this case, we also set $W_{C',x} = W_{C,x}$ for every $x \in V(T_C)$. Next, assume that $|W_{C,r_C}| = 0$. Let C' be obtained from C by adding a new vertex u_C and making it adjacent to all vertices in $\bigcup\{W_{C,x} \mid x \text{ and } r_C \text{ are adjacent in } T_C\}$. We set $W_{C',x} = W_{C,x}$ for every $x \in V(T_C) - \{r_C\}$ and $W_{C',r_C} = \{u_C\}$. In both cases C' is connected and $\big(T_C, (W_{C',x} \mid x \in V(T_C))\big)$ is a rooted tree decomposition witnessing that $C' \in \mathbf{T}(\mathcal{X})$.

对每个 $C \in \mathcal{C}$，令 C'' 由 C' 与 Z 的并通过识别 u_C 和 z得到。最后，令 X' 由每个 $C \in \mathcal{C}$ 对应的 C'' 的并得到，方法是添加一个新顶点 u，并使其与每个 C''中 Z 副本里的所有顶点相邻。根据构造，X' 是连通的。

We shall prove that $X' \in \mathbf{T}(\mathcal{X})$. Let T be obtained from the disjoint union of T_C for $C \in \mathcal{C}$ by adding a new vertex r adjacent to each r_C for $C \in \mathcal{C}$. We set $\mathrm{root}(T) = r$. Let $x \in V(T)$. If $x = r$, we set $W_r = \{u\}$. Let $C \in \mathcal{C}$ be such that $x \in V(T_C)$. If $x = r_C$, we set W_x to be the set of vertices in the copy of Z in C''. Finally, otherwise, we set $W_x = W_{C',x}$. It is easy to verify that $\big(T, (W_x \mid x \in V(T))\big)$ is a tree decomposition witnessing that $X' \in \mathbf{T}'(\mathcal{X})$. This ends the proof. □

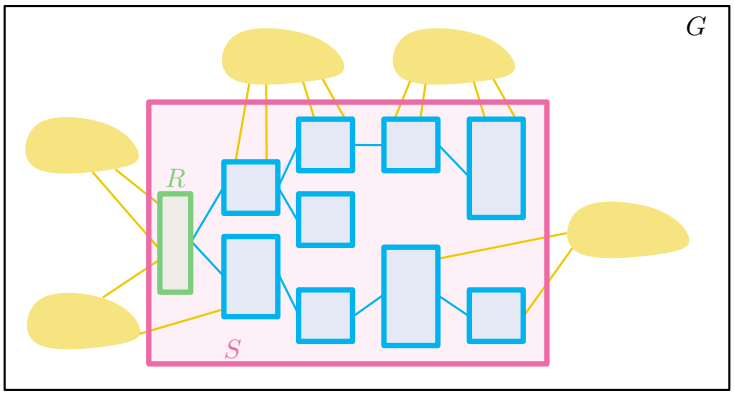


图 13。 断言 7.5 的示意图。

推论 7.4. 设 $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一族聚焦参数，设 $g\colon \mathbb{N}_1 \to \mathbb{N}_1$，并且设$\mathcal{X}$是一个包含非空图的图类。则， g是弱 $(\mathrm{par}, \mathbf{T}(\mathcal{X}))$-有界的当且仅当 g是弱 $(\mathrm{par}, \mathbf{T}'(\mathcal{X}))$-有界的。

7.2. **证明。**

Proof of Theorem 4.3. Without loss of generality, assume that $\mathcal{X}$ contains a nonnull graph. Let $\mathcal{X}' = \{X' \mid X' \text{ is a minor of } X \text{ for some } X \in \mathcal{X}\}$. Since $\mathcal{X}$ is closed under disjoint union, $\mathcal{X}'$ is also closed under disjoint union. Since $\mathcal{X}$ has the coloring elimination property, by Lemma 4.1, $\mathcal{X}'$ also has the coloring elimination property. Since $\mathbf{A}(\mathcal{X}) \subseteq \mathbf{A}(\mathcal{X}')$, if a function is $(\mathrm{par}, \mathbf{A}(\mathcal{X}'))$-bounding, then this function is also $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$-bounding. Therefore, again without loss of generality, we will assume that $\mathcal{X} = \mathcal{X}'$, and so, $\mathcal{X}$ is closed under taking minors. Since $\mathcal{X}$ has a nonnull graph, $K_1 \in \mathcal{X}$.

Assume that g is $(\mathrm{par}\ \mathcal{X})$-bounding. Let b and b' be as in (n4) for $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$. Let k be a positive integer. Fix α' and $\beta'(\cdot)$ witnessing that g is $(\mathrm{par}\ \mathcal{X})$-bounding for k. Let $X \in \mathbf{A}(\mathcal{X})$ and $z \in V(X)$ such that $X - z \in \mathcal{X}$. Note that $X' = K_1 \sqcup (X - z)$ is in $\mathcal{X}$. Let Y' witness the coloring elimination property of $\mathcal{X}$ for X' and α'.

Claim 7.5. Let G be a connected K_k-minor-free graph, let R be a nonempty set of at most α' vertices in G, let $\mathcal{F}$ be a family of connected subgraphs of G, and let q be a positive integer. If G has no $\mathcal{F}$-rich model of X, then there exist $S \subseteq V(G)$, a tree T rooted in $s \in V(T)$, and a tree partition $\big(T, (P_x \mid x \in V(T))\big)$ of (G, S) with $P_s = R$ such that

(a) 对每个 $F \in \mathcal{F}$， $S \cap V(F) \neq \emptyset$；(b) 对于 $G-S$ 的每个连通分量 C， $N_G(V(C))$ 至多与 $G-V(C)$ 的 $2\alpha'$ 个连通分量相交；以及 (c) 对每个 $x \in V(T)$，

$$\mathrm{par}_q(G_x, P_x) \leqslant \max\{\beta'(Y'), \alpha'\} \cdot g(q),$$

其中，对于每个 $x \in V(T)$， G_x 是 $U_x = \bigcup_{z \in V(\mathrm{Sub}(T,x))} P_z$与 $G - S$ 的所有在 U_x 中有邻点的连通分量的顶点集之并。

断言的证明。 该结论示于图 13。我们对$|V(G)|$进行归纳。若 $\mathcal{F}|_{G-R} = \emptyset$，则结论对 $S = R$ 和 $T = K_1$成立。因此，我们假设 $\mathcal{F}|_{G-R} \neq \emptyset$，特别是 $|V(G) - R| > 0$。

令 $\mathcal{F}'$ 为 $G - R$ 的所有连通子图 H 所构成的族，满足 $R \cap N_G(V(H)) \neq \emptyset$，并且$F \subseteq H$ 对某个 $F \in \mathcal{F}$ 成立。我们论证 $G - R$ 中不存在 Y' 的 $\mathcal{F}'$-富模型。假设

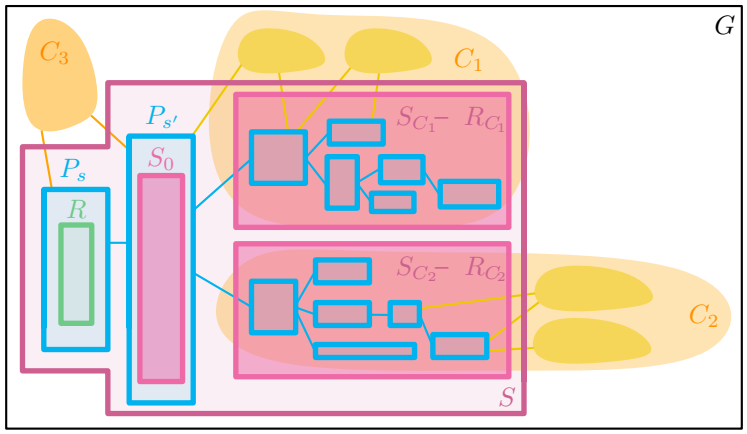


图 14。 断言证明的示意图7.5。在示意的情形中，C_1， $C_2 \in \mathcal{C}_1$ 且 $C_3 \notin \mathcal{C}_1$。

to the contrary that there is an $\mathcal{F}'$-rich model $(B_x \mid x \in V(Y'))$ of Y' in $G - R$. For each $u \in R$, let $S_u = \{x \in V(Y') \mid u \in N_G(B_x)\}$. Since the model is $\mathcal{F}'$-rich, $\bigcup_{u \in R} S_u = V(Y')$. Since Y' witnesses the coloring elimination property of $\mathcal{X}$ for X' and α', there exists $u \in R$ such that Y' contains an S_u-rooted model of X'. As a consequence, there is an $\mathcal{F}$-rich model $(D_x \mid x \in V(X'))$ of X' in $G - R$ such that every branch set D_x contains a neighbor of u in G. Recall that $X' = K_1 \sqcup (X - z)$. Say that y is the vertex of K_1 in X'. Let $D'_z = D_y \cup \{u\}$ and $D'_x = D_x$ for every $x \in V(X' - \{y\})$. It follows that $(D'_x \mid x \in V(X))$ is an $\mathcal{F}$-rich model of X in G, which is a contradiction. In turn, there is no $\mathcal{F}'$-rich model of Y' in $G - R$.

由于 α' 和 $\beta'(\cdot)$ 证明了 g 对 k 是（par $\mathcal{X}$）界定的，将其应用于 Y'、$G - R$ 和 $\mathcal{F}'$，我们得到一个集合 $S_0 \subseteq V(G - R)$，使得

(g1-0) 对每个 $F \in \mathcal{F}'$， $S_0 \cap V(F) \neq \emptyset$；(g2-0) 对 $(G-R)-S_0$ 的每个连通分量 C，$N_{G-R}(V(C))$ 至多与 $(G - R) - V(C)$ 的 α' 个连通分量相交；以及 (g3-0) $\mathrm{par}_q(G - R, S) \leqslant \beta'(Y') \cdot g(q)$。

由于 G 是连通的，且 $\mathcal{F}|_{G-R}$ 非空，因此 $G - R$ 有一个分量包含 $\mathcal{F}$中的一个元素，所以 $\mathcal{F}'$非空。由 (g1-0) 可得 $S_0 \neq \emptyset$。

Let $\mathcal{C}_1$ be the family of all components of $(G - R) - S_0$ with no neighbors in R. Consider $C \in \mathcal{C}_1$. Let A_C be the component of $G - R$ containing C, and let G_C be the graph obtained from A_C by contracting each component of $A_C - V(C)$ into a single vertex. Let R_C be the set of all vertices resulting from these contractions, that is $R_C = V(G_C) - V(C)$. Observe that R_C is nonempty since G is connected, and $|R_C| \leqslant \alpha'$ by (g2-0). Moreover, since $R \neq \emptyset$, $|V(G_C)| < |V(G)|$. Since G_C is a minor of G, G_C has no $\mathcal{F}|_C$-rich model of X. By the induction hypothesis applied to G_C, R_C, and $\mathcal{F}|_C$, there exist $S_C \subseteq V(G_C)$, a tree T_C rooted in $s_C \in V(T_C)$, and a tree partition $\big(T_C, (P_{C,x} \mid x \in V(T_C))\big)$ of (G_C, S_C) with $P_{C,s_C} = R_C$ such that

(a') $S_C \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}|_C$均成立；(b') 对于 $G_C - S_C$的每个分支 C'， $N_{G_C}(V(C'))$ 至多与$G_C - V(C')$的 $2\alpha'$个分支相交；以及(c') 对每个 $x \in V(T_C)$，

$$\mathrm{par}_q(G_{C,x}, P_{C,x}) \leqslant \max\{\beta'(Y'), \alpha'\} \cdot g(q)$$

其中，对于每个 $x \in V(T_C)$，$G_{C,x}$是$U_{C,x} = \bigcup_{z \in V(\mathrm{Sub}(T_C,x))} P_z$与 $G_C - S_C$ 的所有在$U_{C,x}$中有邻居的连通分量的顶点集的并。

我们令

$$S = R \cup S_0 \cup \bigcup_{C \in \mathcal{C}_1} (S_C - R_C).$$

Let T be obtained from the disjoint union of $\{T_C \mid C \in \mathcal{C}_1\}$ by identifying the vertices $\{s_C \mid C \in \mathcal{C}_1\}$ into a new vertex s' and by adding the root s adjacent only to s' in T. Let $P_s = R$, $P_{s'} = S_0$, and for each $C \in \mathcal{C}_1$, $x \in V(T_C - \{s_C\})$, let $P_x = P_{C,x}$. See Figure 14.
为了完成该断言的证明，我们论证 $\big(T,\ (P_x \mid x \in V(T))\big)$ 是 $(G,\ S)$ 的一个树划分，并且 (a)–(c) 成立。

Since for every $C \in \mathcal{C}_1$, $R \cap N_G(V(C)) = \emptyset$, every edge in $G[S]$ containing a vertex in R has another endpoint in $R \cup S_0 = P_s \cup P_{s'}$. Consider an edge vw in $G[S]$ such that $v \in S_0$ and $w \in S_C$ for some $C \in \mathcal{C}_1$. Since $\big(T_C, (P_{C,x} \mid x \in V(T_C))\big)$ is a tree partition of (G_C, S_C) with $P_{C,s_C} = R_C$ and $S_0 \subseteq V((G - R) - V(C))$, we conclude that $w \in P_x$ for some $x \in V(T_C)$ such that $s'x$ is an edge in T. Finally, for every edge vw of $G[S]$ with $v, w \notin R \cup S_0$, vw is an edge of $G[S_C - R_C]$ for some component $C \in \mathcal{C}_1$, and so $v \in P_{C,x}$ and $w \in P_{C,y}$ for adjacent or identical vertices x, y of T_C. Then $v \in P_x$ and $w \in P_y$. For every component C' of $G - S$, either $N_G(V(C')) \subseteq R \cup S_0 = P_s \cup P_{s'}$, or $C' \subseteq C$ for some $C \in \mathcal{C}_1$. In the latter case, C' is a component of $G_C - S_C$, and $N_G(R) \cap V(C) = \emptyset$. Since $\big(T_C, (P_{C,x} \mid x \in V(T_C))\big)$ is a tree partition of (G_C, S_C), there is $x, y \in V(T_C)$ such that $N_{G_C}(V(C')) \subseteq P_{C,x} \cup P_{C,y}$. If $x = s_C$, then $N_G(V(C')) \subseteq P_s \cup P_y$ and $sy \in E(T)$. If $y = s_C$, then $N_G(V(C')) \subseteq P_s \cup P_x$ and $sx \in E(T)$. Finally, if $x, y \neq s_C$, then $N_G(V(C')) \subseteq P_s \cup P_x$ and $xy \in E(T)$. This proves that $\big(T, (P_x \mid x \in V(T))\big)$ is a tree partition of (G, S).

Let $F \in \mathcal{F}$. If $(R \cup S_0) \cap V(F) \neq \emptyset$, then $S \cap V(F) \neq \emptyset$. Otherwise, $F \subseteq (G - R) - S_0$ and therefore, by (g1-0), we have $F \notin \mathcal{F}'$. In this case, there is a component $C \in \mathcal{C}_1$ such that $F \in \mathcal{F}|_C$. By (a'), $(S_C - R_C) \cap V(F) \neq \emptyset$ and so $S \cap V(F) \neq \emptyset$. This proves (a).

Consider a component C' of $G - S$. If $C' \subseteq C$ for some $C \in \mathcal{C}_1$, then by (b'), it follows that $N_{G_C}(V(C'))$ intersects at most $2\alpha'$ components of $G_C - V(C')$, and so $N_G(V(C'))$ intersects at most $2\alpha'$ components of $G - V(C')$. Otherwise, C' is a component of $(G - R) - S_0$ such that $N_G(R) \cap V(C') \neq \emptyset$. By (g2-0), $N_{G-R}(V(C'))$ intersects at most α' components of $(G - R) - V(C')$, and therefore, $N_G(V(C'))$ intersects at most $\alpha' + |R| \leqslant 2\alpha'$ components of $G - V(C')$. This proves (b).

最后，我们论证 (c)。令 $x \in V(T)$。若 $x = s$，$P_x = P_s = R$，其大小至多为 α'，因此由 (n2)可得断言成立。若 $x = s'$，则 $G_{s'}$ 是 $G - R$的若干连通分支之并，且$P_x = P_{s'} = S_0$，因此由 (n1) 和 (g3-0)，

$$\mathrm{par}_q(G_{s'}, S_0) \leqslant \mathrm{par}_q(G - R, S_0) \leqslant \beta'(Y')g(q).$$

如果 $x \in V(T_C - \{s_C\})$对于某个 $C \in \mathcal{C}_1$，我们有 $\mathrm{Sub}(T, x) = \mathrm{Sub}(T_C, x)$且 $G_x = G_{C,x}$。因此，所断言的不等式由 (c') 得出。至此，命题得证。◊

Let

$$\alpha = 2\alpha' \quad \text{and} \quad \beta(X) = 2b' \max\{\beta'(Y'), \alpha'\}.$$

为完成证明，我们论证每个无 K_k 次图 G 都是 $(q \mapsto qg(bq), \mathsf{par}\ X, \alpha, \beta(X))$-良好的。由于 X 是 $\mathbf{A}(\mathcal{X})$ 中任意一个图，这将表明 $q \mapsto qg(bq)$ 是 $(\mathsf{par}, \mathbf{A}(\mathcal{X}))$-有界的。

根据引理 7.2，只需证明每个连通的 K_k-minor-free 图 G 都是 $(q \mapsto qg(bq)$，抛 X，α，$\beta(X))$-良好的。设 G 是一个连通的 K_k-minor-free 图，设 q 是一个正整数，并设 $\mathcal{F}$ 是 G 的一个连通子图族，使得其中不存在 $\mathcal{F}$-rich 模型 X 于 G中。我们证明

(g1*) 对每个 $F \in \mathcal{F}$，$S \cap V(F) \neq \emptyset$；

(g2⋆) 对 $G - S$ 的每个连通分量 C， $N_G(V(C))$ 至多与 $G - V(C)$ 的 α 个连通分量相交；以及 (g3⋆) $\mathrm{par}_q(G, S) \leqslant \beta \cdot g(q)$。

根据断言 7.5，将其应用于 $V(G)$ 中任意一个单元素集 R，可得 $S \subseteq V(G)$、一棵以 $s \in V(T)$ 为根的树 T，以及 (G, S) 的树划分 $\big(T, (P_x \mid x \in V(T))\big)$，使得 (a)、(b) 和 (c) 均成立。

条目(a) 和(b) 分别蕴含(g1⋆) 和(g2⋆)。此外，(n4) 和(c) 蕴含(g3⋆)，因为

$$\mathrm{par}_q(G, S) \leqslant b'(q+1) \cdot \max_{x \in V(T)} \mathrm{par}_{bq}(G_x, P_x) \leqslant \beta(X) \cdot qg(bq).$$

综上，$q \mapsto qg(bq)$ 是 $(\mathsf{par}, \mathbf{A}(\mathcal{X}))$-界定的，由此完成证明。 □

Proof of Theorem 4.4. The statement is clear when $\mathcal{X}$ contains only the null graph, thus, we assume that $\mathcal{X}$ has a nonnull graph. Suppose that g is $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$-bounding. Since $\mathcal{X}$ is nonempty and closed under leaf addition, every edgeless graph is a subgraph of a graph in $\mathcal{X} \subseteq \mathbf{A}(\mathcal{X})$. Hence, since g is $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$-bounding, we deduce that g is $(\mathrm{par}\ \mathcal{E})$-bounding. Therefore, by Lemma 7.1 it suffices to show that g is weakly $(\mathrm{par}, \mathbf{T}(\mathcal{X}))$-bounding. Furthermore, by Corollary 7.4, it suffices to show that g is weakly $(\mathrm{par}, \mathbf{T}'(\mathcal{X}))$-bounding. Suppose that the assertion does not hold, i.e. g is not weakly $(\mathrm{par}, \mathbf{T}'(\mathcal{X}))$-bounding. This is witnessed by some graph $X \in \mathbf{T}'(\mathcal{X})$ and a positive integer k. There is a rooted tree decomposition $\big(T, (W_a \mid a \in V(T))\big)$ of X witnessing the fact that $X \in \mathbf{T}'(\mathcal{X})$ (i.e. such that $W_a \neq \emptyset$ for every $a \in V(T)$). We consider such a triplet $\big(k, X, (T, (W_a \mid a \in V(T)))\big)$ with T of minimum vertex-height.

It is easy to check that $T \neq K_1$, as otherwise X is the null graph or a single vertex graph. In both cases X can not be a counterexample. For every $Y \in \mathbf{T}'(\mathcal{X})$ which is not a counterexample, we denote by $\alpha_0(Y)$ and $\beta_0(Y)$ integers such that every K_k-minor-free graph G is $(g, (\mathrm{par}_q \mid q \in \mathbb{N}_1)\ Y, \alpha_0(Y)\ \beta_0(Y))$-good. Let α' and $\beta'(\cdot)$ witness g being $(\mathrm{par}, \mathbf{A}(\mathcal{X}))$-bounding for k.

In Figure 15, we illustrate the object constructed in the proof. We denote by r the root of T, and let $z \in V(X)$ be such that $W_r = \{z\}$. Let T_0 be the subtree of T induced by $\{a \in V(T) \mid z \in W_a\}$. Observe that for every $a \in V(T_0)$, $X[W_a - \{z\}] \in \mathcal{X}$. Since $\mathcal{X}$ is closed under disjoint union, this implies that $X[\bigcup_{a \in V(T_0)} W_a - \{z\}] \in \mathcal{X}$. Let

$$X_0 = X\left[\bigcup_{a \in V(T_0)} W_a\right].$$

令 X_1 由 X_0 得到，方法是为每个 $x \in V(X_0)$ 添加一个与 x 相邻的新顶点 ℓ_x。那么，由于 $\mathcal{X}$ 对添加叶顶点封闭且 $X_1 - \{z\} \in \mathcal{X}$，我们有 $X_1 \in \mathbf{A}(\mathcal{X})$。
对于每一个 $x \in V(X_0)$，令 $\mathcal{R}_x$ 为 $T - V(T_0)$ 的所有分支 R 的族，使得$x \in \bigcup_{a \in V(R)} W_a$，并且令

$$X_x = \begin{cases} X\left[\bigcup_{R \in \mathcal{R}_x, a \in V(R)} W_a\right] & \text{if } \mathcal{R}_x \neq \emptyset, \\ X[\{x\}] & \text{if } \mathcal{R}_x = \emptyset. \end{cases}$$

现在，令 X_2 通过将 $V(X_0)$ 中的所有顶点识别为单个顶点 z_2 而从 X 得到。根据证明 $X \in \mathbf{T}'(\mathcal{X})$ 的森林分解的定义，对于每个 $x \in V(X_0)$，图 X_2 包含 X_x 的一个副本，其中 x 充当 z_2 的角色。然后，令 X_3 通过将两个副本 $X_{2,\,1}$ 和 $X_{2,\,2}$ 的顶点 z_2 进行识别而从它们的并集得到。

现在我们构造一个根树分解 $\big(T', (W'_a \mid a \in V(T'))\big)$，以证明 X_3 的这一事实： $X_3 \in \mathbf{T}'(X)$。此外，T' 的顶点高度小于 T。令 T'_0 由 T 得到，方法是

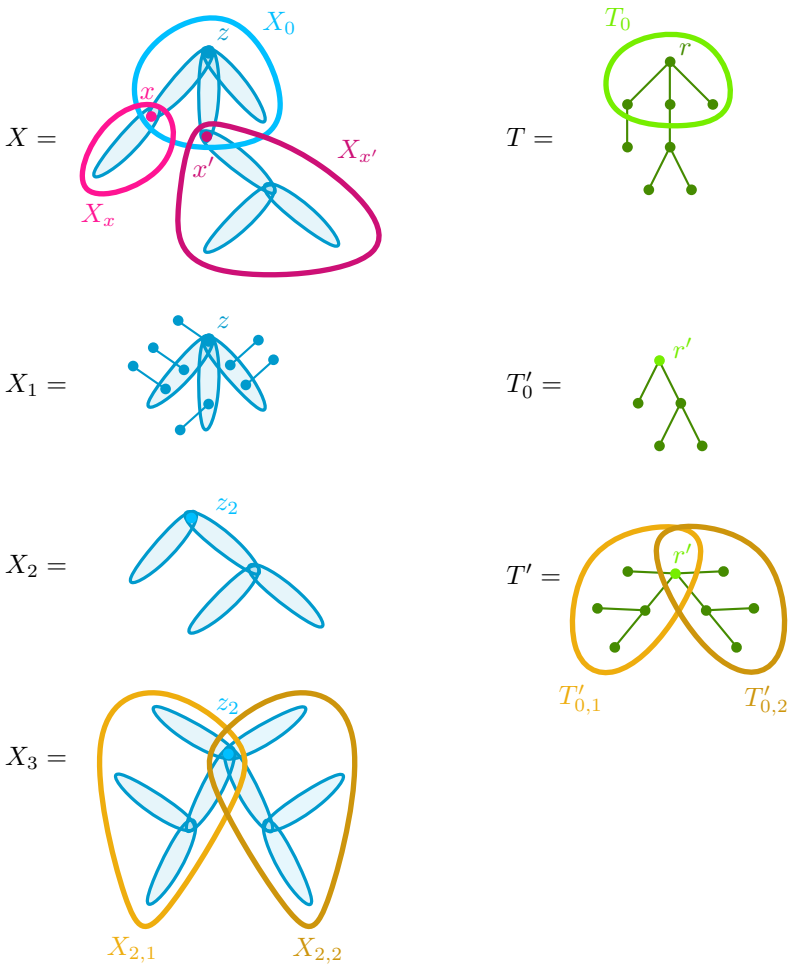


图 15. 定理证明中图的示意图4.4。

contracting T_0 into a single vertex r', and then let T' be obtained from the union of two copies $T'_{0,1}$ and $T'_{0,2}$ of T'_0 by identifying the vertices r' of both copies. Then, for every $a \in V(T'_{0,i})$ for each $i \in \{1,2\}$, let W'_a be obtained by adding z_2 of X_3 to the set of vertices playing roles of $W_a - V(X_0)$ in $X_{2,i}$. Then $\big(T', (W_a \mid a \in V(T'))\big)$ is a rooted tree decomposition of X_3 witnessing the fact that $X_3 \in \mathbf{T}'(\mathcal{X})$. We claim that the vertex-height of T' is smaller than the vertex-height of T. Indeed, for every neighbor s of r in T, since X is connected and $W_s \neq \emptyset$, we have $z \in W_s$, and so $s \in V(T_0)$. This proves that $N_T(r) \subseteq V(T_0)$, and so T' has smaller vertex-height than T. It follows that X_3 is not a counterexample and we can define

$$\alpha_0(X) = \alpha' + \alpha_0(X_3) \quad \text{and} \quad \beta_0(X) = \beta'(X_1) + \beta_0(X_3).$$

设 q 为一个正整数，设 G 为一个 K_k-minor-free 图，并设 $\mathcal{F}$ 是 G 的一个连通子图族，使得 G 中不存在 X 的 $\mathcal{F}$-rich model。设 $\mathcal{F}'$ 是 G 的所有连通子图 H 组成的族，使得 H 中存在 X_3 的 $\mathcal{F}|_H$-rich model。

We claim that there is no $\mathcal{F}'$-rich model of X_1 in G. Suppose for contradiction that there is an $\mathcal{F}'$-rich model $(A_x \mid x \in V(X_1))$ of X_1 in G. Let $x \in V(X_0)$. The following construction is illustrated in Figure 16. The graph $G[A_{\ell_x}]$ contains a member of $\mathcal{F}'$, and so, an $\mathcal{F}$-rich model $(B_{x,y} \mid y \in V(X_3))$ of X_3. Since $G[A_{\ell_x}]$ is connected, there is an $N(A_x)$–$(\bigcup_{y \in V(X_3)} B_{x,y})$ path P in $G[A_{\ell_x}]$. Let $y \in V(X_3)$ be such that P is an $N(A_x)$–$B_{x,y}$ path path, let $i \in [2]$ be such

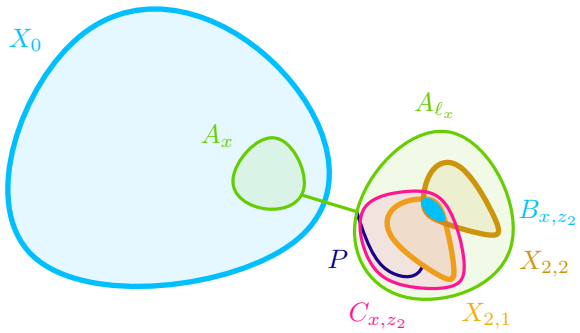


图 16. 构造过程的示意图： $\mathcal{F}$-富模型 $(E_y \mid y \in V(X))$，它是 X 在 G 中的模型，其证明见定理 4.4。

其中 $y \in X_{2,i}$，并令 $\{i,j\} = [2]$。对于每个 $y \in V(X_{2,j})$，令

$$C_{x,y} = \begin{cases} V(P) \cup \bigcup_{y' \in V(X_{2,i})} B_{x,y'} & \text{if } y = z_2, \\ C_{x,y} & \text{otherwise.} \end{cases}$$

It follows that $(C_{x,y} \mid y \in V(X_{2,j}))$ is a model of $X_{2,j}$ (which is a copy of X_2) in $G[A_{\ell_x}]$ such that C_{x,z_2} contains a vertex adjacent to A_x in G. By restricting this model to $(V(X_x) - \{x\}) \cup \{z_2\}$, and relabeling z_2 with x, we obtain a model $(D_{x,y} \mid y \in V(X_x))$ of X_x in $G[A_{\ell_x}]$ such that $D_{x,x}$ is adjacent to A_x in G. Finally, for every $y \in V(X)$, let

$$E_y = \begin{cases} A_y \cup D_{y,y} & \text{if } y \in V(X_0), \\ D_{x,y} & \text{if } y \notin V(X_0), \text{ for } x \in V(X_0) \text{ such that } y \in V(X_x). \end{cases}$$

于是，$(E_y \mid y \in V(X))$ 是 G 中 X 的一个 $\mathcal{F}$-丰富模型，矛盾。这证明了 G 在 G 中没有 X_1 的 $\mathcal{F}'$-丰富模型。

回顾一下，α' 和 β' 证明了 g 对 k 是 $(\mathsf{par}, \mathbf{A}(\mathcal{X}))$-有界的。由于 G 在 G 中没有 X_1 的 $\mathcal{F}'$-富模型，因此存在 $S_0 \subseteq V(G)$，使得

(g1-0) 对每个 $F \in \mathcal{F}'$，$S_0 \cap V(F) \neq \emptyset$；(g2-0) 对于 $G - S_0$ 的每个连通分支 C，$N_G(V(C))$ 至多与 $G - V(C)$ 的 α' 个连通分支相交；以及 (g3-0) $\mathsf{par}_q(G, S_0) \leqslant \beta'(X_1) \cdot g(q)$

由于对每个 $F \in \mathcal{F}'$，$S_0 \cap V(F) \neq \emptyset$，$G - S_0$ 没有 X_3 的 $\mathcal{F}|_{G-S_0}$-富模型。因此，由于 X_3 不是反例，存在 $S_1 \subseteq V(G - S_0)$，使得

(g1-1) 对每个 $F \in \mathcal{F}|_{G-S_0}$，$S_1 \cap V(F) \neq \emptyset$；(g2-1) 对于 $(G - S_0) - S_1$ 的每个连通分量 C，$N_{G-S_0}(V(C))$ 至多与 $(G - S_0) - V(C)$ 的 $\alpha_0(X_3)$ 个连通分量相交；并且 (g3-1) $\mathsf{par}_q(G - S_0, S_1) \leqslant \beta_0(X_3) \cdot g(q)$。

最后，令

$$S = S_0 \cup S_1.$$

现在我们来论证

(g1*) 对每个 $F \in \mathcal{F}|_G$，$S \cap V(F) \neq \emptyset$；(g2*) 对于 $G - S$ 的每个连通分量 C，$N_G(V(C))$ 至多与 $G - V(C)$ 的 $\alpha_0(X)$ 个连通分量相交；并且 (g3*) $\mathsf{par}_q(G, S) \leqslant \beta_0(X) \cdot g(q)$。

首先，注意到对每一个 $F \in \mathcal{F}$，要么 $S_0 \cap V(F) \neq \emptyset$ 由 (g1-0)，要么 $F \in \mathcal{F}|_{G-S_0}$ 从而 $S_1 \cap V(F) \neq \emptyset$ 由 (g1-1)。这证明了 (g1*)。

对于 $G - S$ 的每一个连通分支 C，由 (g2-1)， $N_G(V(C))$ 至多与 $(G - S_0) - V(C)$ 的 $\alpha_0(X_3)$ 个连通分支相交，并且至多与 $G - S_0$ 的 α' 个连通分支相交。因此， $N_G(V(C))$ 至多与 $G - V(C)$ 的 $\alpha' + \alpha_0(X_3) = \alpha_0(X)$ 个连通分支相交，从而证明了 (g2⋆)。

最后，

$$\begin{aligned}\mathrm{par}_q(G,S) &\leqslant \mathrm{par}_q(G,S_0) + \mathrm{par}_q(G - S_0, S_1) && \text{by (n3)}\\ &\leqslant \beta'(X_1)\cdot g(q) + \beta_0(X_3)\cdot g(q) && \text{by (g3-0) and (g3-1)}\\ &\leqslant \beta_0(X)\cdot g(q).\end{aligned}$$

这证明了 (g2⋆)。由此可知， X 不是 g 弱 $(\mathrm{par}, \mathbf{T}'(\mathcal{X}))$-有界的反例，这一矛盾完成了证明。□

8. Common arguments for base cases

Recall that $\mathcal{E}$ is the class of all edgeless graphs. We begin this section by showing that for certain families of focused parameters par, that the function $q \mapsto 1$ is (par $\mathcal{E}$)-bounding (Lemma 8.1). In Subsection 8.1, we study graphs with no $\mathcal{F}$-rich models of a fixed linear forest. Namely, we show that for each family of focused parameters par, if a function is (par $\mathcal{E}$)-bounding, then this function is also (par $\mathcal{S}_2$)-bounding (Theorem 8.2), and if a function is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-bounding, then it is also (par $\mathcal{S}_3$)-bounding (Theorem 8.3). In Subsection 8.2, we study graphs with no $\mathcal{F}$-rich models of a fixed forest. This will be used in Sections 9 and 10 to show that $q \mapsto \log q$ is (par $\mathcal{R}_2$)-bounding, and in Section 11 to show that $q \mapsto q \log q$ is (par $\mathcal{R}_3$)-bounding, for some families of focused parameters $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$.

引理 8.1。 设 $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一个聚焦参数族。对于每个图 G、每个 $S \subseteq V(G)$ 以及每个正整数 q，假设 $\mathrm{par}_q(G,S) \leqslant |S|$ 并定义

$$\mathrm{par}'_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\mathrm{par}_q(G,S).$$

那么，函数 $q \mapsto 1$ 是（平′, $\mathcal{E}$）-有界的，其中 par′ = （平′$_q$ | $q \in \mathbb{N}_1$）。

Proof. We set $\alpha = 2$ and for every $X \in \mathcal{E}$, we set $\beta(X) = \max\{0, 2|V(X)| - 3\}$. We claim that for every $X \in \mathcal{E}$, every graph G is $(q \mapsto 1, \mathrm{par}', X, \alpha, \beta(X))$-good. Let $X \in \mathcal{E}$. Let q be a positive integer and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of X, or in other words, G does not contain $|V(X)|$ pairwise disjoint members of $\mathcal{F}$. We have to prove that there exists $S \subseteq V(G)$ such that

(g1⋆) 对于每个 $F \in \mathcal{F}$， $S \cap V(F) \neq \emptyset$；(g2⋆) 对于 $G - S$ 的每个连通分量 C， $N_G(V(C))$ 与 $G - V(C)$ 的至多两个连通分量相交；以及 (g3⋆) $\mathrm{par}'_q(G,S) \leqslant \max\{0, 2|V(X)| - 3\}$。

If $|V(X)| \leqslant 1$, then $\mathcal{F} = \emptyset$, and so $S = \emptyset$ satisfies (g1⋆)–(g3⋆). Now assume $|V(X)| \geqslant 2$. Let $\mathcal{W}$ be a tree decomposition of G of width $\mathrm{tw}(G)$. By Lemma 3.2, there exists a natural tree decomposition of $\mathcal{W}' = \big(T, (W_x \mid x \in V(T))\big)$ of G of the same width as $\mathcal{W}$. Since G has no $|V(X)|$ pairwise disjoint members of $\mathcal{F}$, by Lemma 3.3, there exists $Z \subseteq V(T)$ of size at most $|V(X)| - 1$ such that if $S' = \bigcup_{x \in Z} W_x$, then $S' \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$. We root T arbitrarily and define

$$S = \bigcup_{x \in \mathrm{LCA}(T,Z)} W_x.$$

由于 $S' \subseteq S$，我们有 (g1⋆)。由引理 3.5，我们有 $|\mathrm{LCA}(T,Z)| \leqslant 2|V(X)| - 3$，并且对于每个分量 C 中的 $G - Z$， $N_G(V(C))$ 至多与 $G - V(C)$的两个分量相交。

因此，(g2⋆)成立。最后，

$$\mathrm{par}'_q(G,S)=\frac{1}{\mathrm{tw}(G)+2}\mathrm{par}_q(G,S)\leqslant\frac{|S|}{\mathrm{tw}(G)+2}\leqslant\frac{(2|V(X)|-3)(\mathrm{tw}(G)+1)}{\mathrm{tw}(G)+2}\leqslant 2|V(X)|-3.$$

因此，我们也有 (g3⋆)。证毕。 □

8.1. **排除线性森林**。回忆一下， $\mathcal{E}$ 是所有无边图构成的类，而 $\mathcal{S}_2$ 是所有线性森林构成的类。在以下证明中，我们用 P_ℓ 表示顶点集为$[\ell]$ 的路径，其中 i 和 $i+1$ 对每个 $i\in[\ell-1]$都相邻。以下定理的证明修改自 Paul、Protopapas 和 Thilikos 的一项论证 [29, 定理 3。22]。

定理 8.2。设 $(\mathrm{par}_q\mid q\in\mathbb{N}_1)$ 是一个良好的聚焦参数族，且 $g\colon\mathbb{N}_1\to\mathbb{N}_1$。若 g 是 (par $\mathcal{E}$)-界定的，则 g 是 (par $\mathcal{S}_2$)-界定的。

证明。 假设 g 是 (par $\mathcal{E}$)-有界的。根据引理 7.1，只需证明 g 是弱(par $\mathcal{S}_2$)-有界的。

令 k 为正整数。令 α' 和 $\beta'(\cdot)$ 证明 g 对 k 是 (par $\mathcal{E}$)-有界的。我们可以假设，当 $X=K_d\in\mathcal{E}$ 时， β' 以正整数 d 为参数。对于每个$X\in\mathcal{S}_2$，令

$$\alpha(X)=(|V(X)|-1)\alpha'\quad\text{and}\quad\beta(X)=(|V(X)|-1)\beta'(2).$$

我们证明 $\alpha(\cdot)$ 和 $\beta(\cdot)$ 证明 g 对于 k 是弱（par $\mathcal{S}_2$）有界的。

Since every $X\in\mathcal{S}_2$ is a subgraph of a path on $|V(X)|$ vertices, it suffices to prove that for every positive integer ℓ, every K_k-minor-free graph G is $(g,\mathrm{par}\ P_\ell,(\ell-1)\alpha',(\ell-1)\beta'(2))$-good. By Lemma 7.2, we may also assume that G is connected. The proof is by induction on ℓ. Let G be a connected K_k-minor-free graph, let q be a positive integer, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of P_ℓ. We claim that there exists $S\subseteq V(G)$ such that

(g1⋆) 对每个 $F\in\mathcal{F}$， $S\cap V(F)\neq\emptyset$；(g2⋆) 对 $G-S$ 的每个连通分量 C， $N_G(V(C))$ 至多与 $G-V(C)$ 的 $(\ell-1)\alpha'$ 个连通分量相交；以及 (g3⋆) $\mathrm{par}_q(G,S)\leqslant(\ell-1)\beta'(2)\cdot g(q)$。

如果 $\ell=1$，那么 $\mathcal{F}$ 为空，且该断言对于 $S=\emptyset$ 成立，因为 (n2)。因此，我们假设$\ell>1$。

Let $\mathcal{F}'$ be the family of all the connected subgraphs H of G such that H contains an $\mathcal{F}$-rich model of $P_{\ell-1}$. We claim that there are no two disjoint members of $\mathcal{F}'$ in G. Indeed, suppose to the contrary that $(A_1,\dots,A_{\ell-1})$ and $(B_1,\dots,B_{\ell-1})$ are two disjoint $\mathcal{F}$-rich models of $P_{\ell-1}$ in G. Since G is connected, there is a $(\bigcup_{k\in[\ell-1]}A_k)$–$(\bigcup_{k\in[\ell-1]}B_k)$ path Q in G. Let $i,j\in[\ell-1]$ be such that Q is a A_i–B_j path. By possibly reversing the orderings $(A_1,\dots,A_{\ell-1})$ and $(B_1,\dots,B_{\ell-1})$, we assume that $j\leqslant\frac{\ell-1}{2}\leqslant i$. Note that $(A_1,\dots,A_i,(V(Q)-A_i)\cup B_j,\dots,B_{\ell-1})$ is an $\mathcal{F}$-rich model of $P_{\ell'}$ in C, where $\ell'=i+\ell-1-(j-1)\geqslant\frac{\ell-1}{2}+\ell-\frac{\ell-1}{2}=\ell$. This contradicts the fact that G has no $\mathcal{F}$-rich model of P_ℓ. Therefore, there are no two disjoint members of $\mathcal{F}'$.

由于 α' 和 $\beta'(\cdot)$ 证明了 g 对 k 是 (par $\mathcal{E}$)-有界的，因此存在 $S_0\subseteq V(C)$，使得：(g1-0) 对每个 $F\in\mathcal{F}'$， $S_0\cap V(F)\neq\emptyset$；(g2-0) 对于 $G-S_0$ 的每个连通分支 C， $N_G(V(C))$ 至多与 $G-V(C)$ 的 α' 个连通分支相交；并且 (g3-0) $\mathrm{par}_q(G,S_0)\leqslant\beta'(2)\cdot g(q)$。

现在，由 (g1-0)，不存在 $\mathcal{F}$-富模型 $P_{\ell-1}$ 于 $G-S_0$。因此，根据归纳假设应用于 $\ell-1$， q，$G-S_0$，以及 $\mathcal{F}|_{G-S_0}$，存在 $S_1 \subseteq V(G)$ 使得

(g1-1) $S_1 \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}|_{G-S_0}$;

(g2-1) for every component C of $(G-S_0)-S_1$, $N_{G-S_0}(V(C))$ intersects at most $(\ell-2)\alpha'$ components of $(G-S_0)-V(C)$; and

(g3-1) $\mathrm{par}_q(G-S_0, S_1) \leqslant (\ell-2)\beta'(2)\cdot g(q)$.

现在，设

$$S = S_0 \cup S_1.$$

We now argue (g1*), (g2*), and (g3*). First, for every $F \in \mathcal{F}$, either $S_0 \cap V(F) \neq \emptyset$ or $F \in \mathcal{F}|_{G-S_0}$, and so, $S_1 \cap F \neq \emptyset$ by (g1-1). This proves (g1*). By (g2-0) and (g2-1), for every component C of $G-S$, $N_G(V(C))$ intersects at most $\alpha' + (\ell-2)\alpha' = (\ell-1)\alpha'$ components of $G-V(C)$. This gives (g2*). Finally,

$$\begin{aligned} \mathrm{par}_q(G,S) &\leqslant \mathrm{par}_q(G,S_0) + \mathrm{par}_q(G-S_0,S_1) && \text{作者：(n3)} \\ &\leqslant (\ell-1)\beta'(2)\cdot g(q) && \text{作者 (g3-0)和 (g3-1)。} \end{aligned}$$

这就证明了 (g3*)。 □

定理 8.3。 设 $(\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一个良好的聚焦参数族，且 $g\colon \mathbb{N}_1 \to \mathbb{N}_1$。若 g 是 $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-有界的，则 g 是 $(\mathrm{par}\ \mathcal{S}_3)$-有界的。

Proof. Suppose that g is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-bounding. Since $\mathcal{E} \subseteq \mathbf{A}(\mathcal{S}_2)$, g is also $(\mathrm{par}\ \mathcal{E})$-bounding. Therefore, by Lemma 7.1, it suffices to show that g is weakly $(\mathrm{par}\ \mathcal{S}_3)$-bounding. Recall that $\mathcal{S}_3 = \mathbf{T}(\mathcal{S}_2)$. By Corollary 7.4, it suffices to show that g is weakly $(\mathrm{par}, \mathbf{T}'(\mathcal{S}_2))$-bounding. Suppose that the assertion does not hold, i.e. g is not weakly $(\mathrm{par}, \mathbf{T}'(\mathcal{S}_2))$-bounding. This is witnessed by some graph $X \in \mathbf{T}'(\mathcal{S}_2)$ and a positive integer k. There is a rooted tree decomposition $\big(T, (W_a \mid a \in V(T))\big)$ of X witnessing the fact that $X \in \mathbf{T}'(\mathcal{S}_2)$ (i.e. such that $W_a \neq \emptyset$ for every $a \in V(T)$). We consider such a triplet $\big(k, X, (T, (W_a \mid a \in V(T)))\big)$ with T of minimum vertex-height.

我们将定义 $\alpha_0(X)$ 和 $\beta_0(X)$，使得每个 K_k-minor-free 图 G 都是 $(g, (\mathrm{par}_q \mid q \in \mathbb{N}_1)\ X,\ \alpha_0(X)\ \beta_0(X))$-良好的，从而得到矛盾并完成证明。

对于每个并非反例的 $Y \in \mathbf{T}'(\mathcal{X})$，我们记 $\alpha_0(Y)$ 和 $\beta_0(Y)$ 为满足以下条件的整数：每个 K_k-minor-free 图 G 都是 $(g, (\mathrm{par}_q \mid q \in \mathbb{N}_1)\ Y,\ \alpha_0(Y)\ \beta_0(Y))$-良好的。令 α' 和 $\beta'(\cdot)$ 证明 g 对 k 是 $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-有界的。

如果 $T = K_1$，那么 $X \in \mathbf{A}(\mathcal{S}_2)$，因此只需令 $\alpha_0(X) = \alpha'$ 和 $\beta_0(X) = \beta'(X)$。于是，我们可以假设 T 的顶点高度 h 至少为 2。

令 T_0 为 T 的子树，该子树由 T 中到根的距离小于 h 的顶点诱导。令 $X_0 = X[\bigcup_{a\in V(T_0)} W_a]$。对于每个 $x \in V(X_0)$，令

$$X_x = X\left[\bigcup_{a\in V(T)-V(T_0), x\in W_a} W_a\right].$$

参见图 17 的上半部分。对于每个 $x \in V(X_0)$，$X_x \in \mathbf{A}(\mathcal{S}_2)$ 且若 $V(X_x) \neq \emptyset$，则 x在 3 中 X_0 的度数至多为 ℓ，其值为 $|V(X_x)|-1$ 在所有 $x \in V(X_0)$ 中的最大值。

Let

$$\alpha_0(X) = \alpha_0(X_0) + \alpha' \quad \text{and} \quad \beta_0(X) = \beta_0(X_0) + \beta'(K_1 \oplus P_{4\ell}).$$

令 q 为正整数，令 G 为一个不含 K_k minor 的图，令 $\mathcal{F}$ 为 G 的一个连通子图族，使得 G 不含 X 的 $\mathcal{F}$- 富模型。

令 $\mathcal{F}'$为 G 的所有连通子图 H 所组成的族，其中 H包含 $K_1 \oplus P_{4\ell}$ 的一个 $\mathcal{F}$-富模型。我们断言， G中不存在 X_0 的 $\mathcal{F}'$-富模型。假设.....

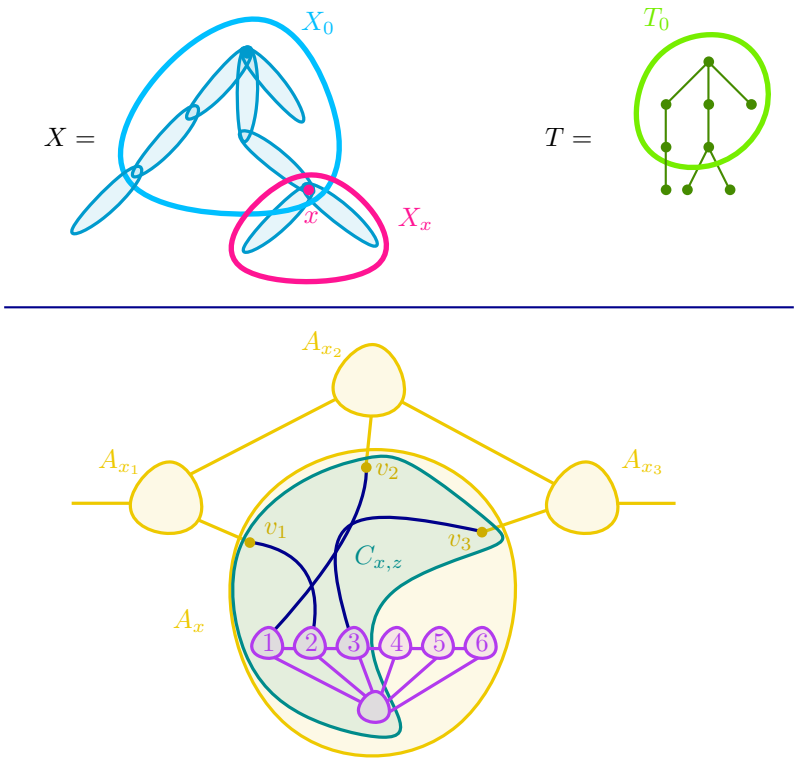


图 17. 定理 8.3 证明的示意图。在图的上半部分，我们描绘了一般设置。在图的下半部分，我们说明了 $\mathcal{F}$-丰富模型 $(E_x \mid x \in V(X))$ 的构造，即 X 在 G 中的模型。在此情形下，$y_1 = 2$、$y_2 = 1$和 $y_3 = 3$。此外，$I = \{4, 5, 6\}$。我们得到 $K_1 \oplus P_3$ 在 $G[A_x]$ 中的一个模型，使得对应于 K_1 的顶点的分支集与 A_{x_1}、A_{x_2}和 A_{x_3}相邻。

contrary otherwise, i.e. let $(A_x \mid x \in V(X_0))$ be an $\mathcal{F}'$-rich model of X_0 in G. The following constuction is illustrated in Figure 17. Let $x \in V(X_0)$ such that $V(X_x) \neq \emptyset$. The subgraph $G[A_x]$ contains an $\mathcal{F}$-rich model $(B_{x,y} \mid y \in V(K_1 \oplus P_{4\ell}))$ of $K_1 \oplus P_{4\ell}$. Let $x_1, \dots, x_d$ be the neighbors of x in X_0. Recall that $d \leqslant 3$. For every $i \in [d]$, A_x contains a neighbor v_i of A_{x_i}. Since $G[A_x]$ is connected, for every $i \in [d]$, there exists $y_i \in V(K_1 \oplus P_{4\ell})$ and a v_i–y_i path path Q_i in $G[A_x]$ internally disjoint from $\bigcup_{y \in V(K_1 \oplus P_{4\ell})} B_{x,y}$. Since $d \leqslant 3$, by the pigeonhole principle, there is an interval $I \subseteq [4\ell]$ of size ℓ such that $I \cap \{y_1, \dots, y_d\} = \emptyset$. Let z be the vertex of K_1 in $K_1 \oplus P_\ell$, where we label the vertices of P_ℓ by the elements of I. For every $y \in I \cup \{z\}$, let

$$C_{x,y} = \begin{cases} \bigcup_{i \in [d]} V(Q_i) \cup \bigcup_{y \in V(K_1 \oplus P_{4\ell}) - I} B_{x,y} & \text{if } y = z, \\ B_{x,y} & \text{otherwise.} \end{cases}$$

我们得到 $\mathcal{F}$-富模型 $(C_{x,\,y} \mid y \in V(K_1 \oplus P_\ell))$，其中 $K_1 \oplus P_\ell$位于 $G[A_x]$，并且对于每个 $i \in [d]$，$C_{x,\,z}$都包含 A_{x_i} 的一个邻点。现在，由于 $\ell \geqslant |V(X_x)| - 1$ 且 $X_x \in \mathbf{A}(\mathcal{S}_2)$，$K_1 \oplus P_\ell$包含 X_x 作为子图。因此，我们得到 $\mathcal{F}$-富模型 $(D_{x,\,y} \mid y \in V(X_x))$，其中 X_x位于 $G[A_x]$，并且对于每个 $y \in N_X(x)$，$D_{x,\,x}$ 都包含 A_y 的一个邻点。最后，对于每个$x \in V(X)$，令

$$E_x = \begin{cases} D_{x',x} & \text{if } x \notin V(X_0) \text{ and there exists } x' \in V(X_0) \text{ is such that } x \in V(X_{x'}), \\ D_{x,x} & \text{if } x \in V(X_0) \text{ and } V(X_x) \neq \emptyset, \\ A_x & \text{if } x \in V(X_0) \text{ and } V(X_x) = \emptyset. \end{cases}$$

那么，$(E_x \mid x \in V(X))$ 是 G 中 X 的 $\mathcal{F}$-富模型，这与 G 有一个 X_0 的 $\mathcal{F}'$-富模型相矛盾。这证明了 G 中不存在 X_0 的 $\mathcal{F}'$-富模型。

由于 $(T_0,(W_a \mid a \in V(T_0)))$ 是 X_0 的树分解，证明了$X_0 \in \mathbf{T}'(\mathcal{S}_2)$，且因为 T_0 的顶点高度小于 T，我们可以应用归纳法。因此，存在 $S_0 \subseteq V(G)$，使得

(g1-0) $S_0 \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}'$；(g2-0) 对于 $G - S$ 的每个连通分量 C， $N_G(V(C))$ 与 $G - V(C)$ 的至多 $\alpha_0(X_0)$ 个连通分量相交；并且 (g3-0) $\mathrm{par}_q(G, S) \leqslant \beta_0(X_0) \cdot g(q)$ 】【。

根据 (g1-0)，不存在 $\mathcal{F}$-丰富的 $K_1 \oplus P_{4\ell}$ 模型属于 $G - S_0$。由于 α' 和 $\beta'(\cdot)$ 见证了 g 是 $(\mathrm{par},\mathbf{A}(\mathcal{S}_2))$-有界的，对于 k 和 $K_1 \oplus P_{4\ell} \in \mathbf{A}(\mathcal{S}_2)$，存在 $S_1 \subseteq V(G - S_0)$ 使得

(g1-1) $S_1 \cap V(F) \neq \emptyset$对于每一个 $F \in \mathcal{F}|_{G-S_0}$；(g2-1) 对于$(G-S_0)-S_1$的每一个连通分支 C， $N_{G-S_0}(V(C))$至多与$(G - S_0) - V(C)$的 α' 个连通分支相交；并且(g3-1) $\mathrm{par}_q(G - S_0, S_1) \leqslant \beta'(K_1 \oplus P_{4\ell}) \cdot g(q)$。

Let

$$S = S_0 \cup S_1.$$

我们现在论证

(g1⋆) $S \cap V(F) \neq \emptyset$对于每一个 $F \in \mathcal{F}|_G$；(g2⋆) 对于 $G - S$的每一个连通分支 C， $N_G(V(C))$至多与$G - V(C)$的 $\alpha_0(X)$个连通分支相交；并且(g3⋆) $\mathrm{par}_q(G, S) \leqslant \beta_0(X) \cdot g(q)$。

首先，对每个 $F \in \mathcal{F}$，要么 $S_0 \cap V(F) \neq \emptyset$，要么 $F \in \mathcal{F}|_{G-S_0}$，因此 $S_1 \cap V(F) \neq \emptyset$由 (g1-1) 得出。在这两种情况下， $S \cap V(F) \neq \emptyset$。这就证明了 (g1⋆)。

Second, for every component G of $G-S$, there is a component C' of $G-S_0$ containing C. Since $N_G(V(C'))$ intersects at most $\alpha_0(X_0)$ components of $G - V(C')$ by (g2-0), and $N_{G-S_0}(V(C))$ intersects at most α' components of $(G - S_0) - V(C)$ by (g2-1), we conclude that $N_G(V(C))$ intersects at most $\alpha_0(X_0) + \alpha' = \alpha_0(X)$ components of $G - V(C)$. This proves (g2⋆).

最后，

$$\begin{aligned} \mathrm{par}_q(G, S) &\leqslant \mathrm{par}_q(G, S_0) + \mathrm{par}_q(G - S_0, S_1) && \text{by (n3)} \\ &\leqslant (\beta_0(X_0) + \beta'(K_1 \oplus P_{4\ell})) \cdot g(q) && \text{by (g3-0) and (g3-1)} \\ &= \beta_0(X) \cdot g(q). \end{aligned}$$

这表明(g3⋆)，从而完成证明。 □

根据定理 8.2 和 8.3，要证明某个函数是 $(\mathrm{par}\ \mathcal{S}_t)$-有界的，只需证明其是 $(\mathrm{par}\ \mathcal{E})$-有界的。

推论 8.4. 设 $\mathrm{par} = (\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 是一个良好的聚焦参数族，且$g\colon \mathbb{N}_1 \to \mathbb{N}_1$是一个多项式函数。若 g是$(\mathrm{par}\ \mathcal{E})$-有界的，则对于每个正整数$t$，函数 $q \mapsto g(q) \cdot q^{t-2}$ 都是$(\mathrm{par}\ \mathcal{S}_t)$-有界的。

Proof. Assume that g is $(\mathrm{par}\ \mathcal{E})$-bounding. The proof is by induction on t. Recall that $\mathcal{S}_1 = \mathcal{E}$ and assume that $t \geqslant 2$. Since g is $(\mathrm{par}\ \mathcal{E})$-bounding, by Theorem 8.2, g is also $(\mathrm{par}\ \mathcal{S}_2)$-bounding. This shows the inductive claim for $t = 2$. Next, suppose that $t \geqslant 3$. By Observation 3.1, $\mathcal{S}_{t-1}$ is closed under disjoint union, and by Lemma 6.5, $\mathcal{S}_{t-1}$ has the coloring elimination property. Theorem 4.3 implies that $q \mapsto g(q) \cdot q$ is $(\mathrm{par}, \mathbf{A}(\mathcal{S}_2))$-bounding. Then, Theorem 8.3 implies that $q \mapsto g(q) \cdot q$ is $(\mathrm{par}\ \mathcal{S}_3)$-bounding. This shows the inductive

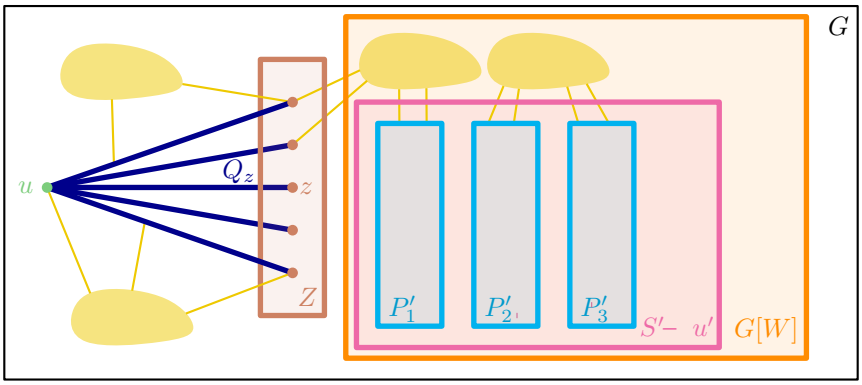


图 18。 引理证明示意图 8.5。

关于 $t=3$。最后，假设 $t\geqslant 4$。特别地， $\mathcal{S}_{t-1}$ 对添加叶节点运算封闭（再次参见观察 3.1 ）。根据归纳法， $q\mapsto g(q)\cdot q^{t-3}$ 是 (par $\mathcal{S}_{t-1}$)-有界的。因此，由定理 4.2， $q\mapsto g(q)\cdot q^{t-2}$ 是 (par $\mathcal{S}_t$)-有界的，证毕。□

8.2. **排除森林**。 在本节中，我们将展示不含给定森林的$\mathcal{F}$-丰富模型的图所具有的结构性质。本小节证明的主要灵感来源于 Dujmović、Hickingbotham、Joret、Micek、Morin 和 Wood 的论文 [5]。

对于所有正整数 h 和 d，我们用$F_{h,\,d}$表示顶点高度为 h 的（有根）完全 d 叉树。特别地，$F_{2,\,d}$ 是有 d 个叶子的星。对于每个森林 X，都存在正整数 h 和 d，使得 X 是 $F_{h,\,d}$ 的子图。

Lemma 8.5. *Let d be a positive integer. Let G be a connected graph, let $\mathcal{D}=\big(T,(W_x\mid x\in V(T))\big)$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{2,d}$. For every $u\in V(G)$, there is a set $S\subseteq V(G)$ and a path partition $(P_0,\dots,P_\ell)$ of (G,S) with $P_0=\{u\}$ such that*

(a) $S\cap V(F)\neq\emptyset$ 对每个 $F\in\mathcal{F}$ 都成立，且(b) 对每个 $i\in[\ell]$， P_i 都包含在 $\mathcal{D}$至多 d 个袋的并集中。

Proof. We illustrate some objects appearing in the proof in Figure 18. We proceed by induction on $|V(G)|$. Let $u\in V(G)$. If $\mathcal{F}|_{G-\{u\}}$ is empty, then it suffices to take $\ell=0$ and $P_0=\{u\}$. In particular, this is the case for $|V(G)|=1$. Therefore, assume $\mathcal{F}|_{G-\{u\}}\neq\emptyset$ and $|V(G)|>1$. Let $\mathcal{F}_0$ be the family of all the connected subgraphs A of $G-\{u\}$ such that A contains some member of $\mathcal{F}$ and $V(A)\cap N_G(u)\neq\emptyset$. Since $\mathcal{F}|_{G-\{u\}}\neq\emptyset$ and G is connected, $\mathcal{F}_0$ is nonempty. 注意，任意一组两两不相交的 $d+1$ $A_1,\dots,A_{d+1}\in\mathcal{F}_0$ 都会在 G 中形成一个 $\mathcal{F}$-丰富的 $F_{2,d}$ 模型。事实上，只需令 $\{u\}\cup A_{d+1}$ 作为 $F_{2,d}$ 中对应根节点的分支集，并令 $A_1,\dots,A_d$ 作为 $F_{2,d}$ 中其余 d 个顶点的分支集即可。因此， $\mathcal{F}_0$ 中两两不相交的成员至多有 d 个。于是，由引理 3.3 应用于 $G-\{u\}$ 和 $\mathcal{F}_0$，存在一个集合 $Z\subseteq V\left(G-\{u\}\right)$ ，它包含于 $\mathcal{D}$ 至多 d 个包的并集中，且对每个 $F\in\mathcal{F}_0$ 都有 $Z\cap V\left(F\right)\neq\emptyset$。假设 Z 关于这一性质是包含意义下极小的。注意，由于 $\mathcal{F}_0\neq\emptyset$， Z 非空。

Let $\mathcal{C}_0$ be the family of all the components C of $G-(\{u\}\cup Z)$ such that $N_G(u)\cap V(C)=\emptyset$. Let $W=\bigcup_{C\in\mathcal{C}_0}V(C)$. Let $z\in Z$. By the minimality of Z, there exists $A_z\in\mathcal{F}_0$ with $V(A_z)\cap(Z-z)=\emptyset$. We have $A_z-\{z\}\notin\mathcal{F}_0$, thus, $z\in V(A_z)$. Since $A_z\in\mathcal{F}_0$, $V(A_z)\cap N_G(u)\neq\emptyset$, and so, there is a u–z path Q_z in $G[\{u\}\cup A_z]\subseteq G-(Z-z)$. For every component $C\in\mathcal{C}_0$, we have $V(C)\cap V(Q_z)=\emptyset$, hence, $W\cap V(Q_z)=\emptyset$.

令 $Q = \bigcup_{z \in Z} V(Q_z)$，并令 G' 表示将 Q 收缩为单个顶点 u' 后由 $G[W \cup Q]$ 得到的图。注意，G' 是 G 的一个 minor，且 $V(G') = \{u'\} \cup W$。此外，由于 $Z \neq \emptyset$，我们有 $|V(G')| < |V(G)|$。

对于每个 $x \in V(T)$，令

$$W'_x = \begin{cases} W_x & \text{if } W_x \subseteq W \\ (W_x \cap W) \cup \{u'\} & \text{otherwise.} \end{cases}$$

It follows that $\mathcal{D}' = \big(T, (W'_x \mid x \in V(T))\big)$ is a tree decomposition of G'. By the induction hypothesis applied on $G', \mathcal{D}', u', \mathcal{F}|_{G[W]}$, there is a set $S' \subseteq V(G')$ and a path partition $(P'_0, \ldots, P'_{\ell'})$ of (G', S') with $P'_0 = \{u'\}$ such that

(a') 对于每个 $F \in \mathcal{F}|_{G[W]}$，$S' \cap V(F) \neq \emptyset$，且 (b') 对于每个 $i \in [\ell']$，P'_i 都包含在 $\mathcal{D}'$ 中至多 d 个包的并集中。

Let $S = \{u\} \cup Z \cup (S' - \{u'\})$, let $P_0 = \{u\}$, let $P_1 = Z$, and let $P_i = P'_{i-1}$ for every $i \in \{2, \ldots, \ell' + 1\}$ – we set $\ell = \ell' + 1$. We claim that $(P_0, \ldots, P_\ell)$ is a path partition of $G[S]$ satisfying (a)–(b), which will complete the proof of the lemma.

Let $i, j \in \{0, \ldots, \ell\}$ with $i < j$ and assume that there is an edge incident to a vertex in P_i and a vertex in P_j in G. If $i \geqslant 2$, then $P_i \subseteq P'_{i-1}$ and $P_j = P'_{j-1}$, which implies $|i - j| \leqslant 1$ since $(P'_0, \ldots, P'_{\ell'})$ is a path partition of $G'[S']$. Otherwise, $i \in \{0, 1\}$. If $i = 0$, then $j = 1$ since u has no neighbors in W. If $i = 1$, then $j = 2$ since $N_{G'}(u') \subseteq P'_1$. For every component C of $G - S$, either $C \cap W = \emptyset$ and so $N(V(C)) \subseteq \{u\} \cup Z = P_0 \cup P_1$, or $V(C) \subseteq W$, and so C is a component of $G' - S'$. It follows that there exists $i, j \in \{0, \ldots, \ell'\}$ with $|i - j| \leqslant 1$ such that $N_{G'}(V(C)) \subseteq P'_i \cup P'_j$ and so $N_G(V(C)) \subseteq P_{i+1} \cup P_{j+1}$. It follows that $(P_0, \ldots, P_\ell)$ is a path partition of (G, S).

Let $F \in \mathcal{F}$. If $(\{u\} \cup Z) \cap V(F) \neq \emptyset$, then $S \cap V(F) \neq \emptyset$. Otherwise, $F \subseteq C$ for some component of $G - (\{u\} \cup Z)$, and in particular, $C \notin \mathcal{F}_0$. In this case, $N_G(u) \cap V(C) \neq \emptyset$, hence, $C \in \mathcal{C}_0$, thus, $F \in \mathcal{F}|_{G[W]}$, and finally, $S' \cap V(F) \neq \emptyset$ by (a'). This proves (a).

最后，P_0 包含在 $\mathcal{D}$ 的一个袋中，P_1 包含在至多 d 个 $\mathcal{D}$ 的袋中，这是由 Z 的定义得出的；并且对于每个 $i \in \{2, \ldots, \ell\}$，$P_i$ 包含在至多 d 个 $\mathcal{D}$ 的袋中，这是由 (b') 得出的。因此，(b) 成立，从而完成了该引理的证明。□

为了将引理 8.5 推广到不存在 $F_{h,d}$ 的 $\mathcal{F}$-丰富模型（其中 $h > 2$）的图，我们需要以下显然的性质。

引理 8.6. 设 h 和 d 为正整数。设 G 为连通图。如果 $F_{h,d+1}$ 在 G中有一个模型 $(B_x \mid x \in V(F_{h,d+1}))$，那么对每个 $u \in V(G)$，G 中都存在 $F_{h,d}$ 的一个模型$(B'_x \mid x \in V(F_{h,d}))$，使得

(a) $u \in B'_s$，其中 s 是 F_h 的根，d 且(b) 对于每个 $x \in V(F_{h,d})$，存在某个 $y \in V(F_{h,d+1})$，使得 $B_y \subseteq B'_x$。

Proof. See Figure 19. In this proof, we treat $F_{h,d}$ as a subgraph of $F_{h,d+1}$ with the same root s. Suppose that there is a model $(B_x \mid x \in V(F_{h,d+1}))$ of $F_{h,d+1}$ in G. Since G is connected, we can assume that $\bigcup_{x \in V(F_{h,d+1})} B_x = V(G)$. Let $u \in V(G)$. There is a subtree T' of $F_{h,d+1}$ rooted in a child of s such that $u \in \bigcup_{x \in V(T') \cup \{s\}} B_x$. Define $B'_s = \bigcup_{x \in V(T') \cup \{s\}} B_x$ and $B'_x = B_x$ for every $x \in V(F_{h,d+1}) - (\{s\} \cup V(T'))$. The collection $(B'_x \mid x \in V(F_{h,d}))$ is a model of $F_{h,d}$ in G satisfying (a) and (b). □

引理 8.7. 设 h 和 d 为正整数。设 G 为一个图，$\mathcal{D} = \big(T$为 G的一个树分解，$(W_x \mid x \in V(T))\big)$，且 $\mathcal{F}$ 为 G 的一个连通子图族，满足 G

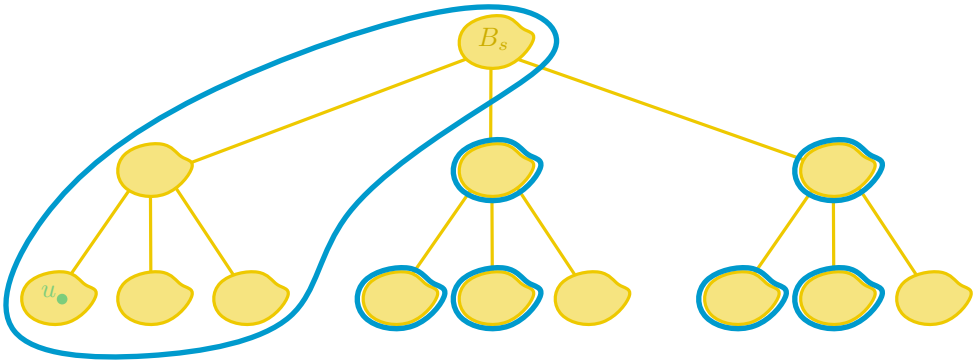


图19. 引理证明的示意图 8.6。用黄色描绘 $F_{3,3}$的一个模型。用蓝色描绘$F_{3,2}$ 的一个模型，使得 u 位于对应于根的分支集中。

没有 $\mathcal{F}$-富模型 $F_{h+1,d}$。存在两两不相交的 $S_1,\dots,S_h \subseteq V(G)$，并且对于每个$a \in [h]$，存在一个路径划分 $(P_{a,0},\dots,P_{a,\ell_a})$，将 $(G-(S_1\cup\cdots\cup S_{a-1})\ S_a)$ 划分为若干路径，使得对于 $S=\bigcup_{a\in[h]} S_a$，

（a） $S\cap V(F)\neq\emptyset$ 对于每个 $F\in\mathcal{F}$，以及（b）对于每个 $a\in[h]$，且对于每个 $i\in[\ell_a]$，$P_{a,\,i}$ 都包含在 $\mathcal{D}$ 至多 $d+h-1$ 个包的并集中。

证明。我们对 h进行归纳。对于 $h=1$，结论由引理8.5得出。接下来，假设 $h>1$，且结论对 $h-1$成立。

令 $\mathcal{F}'$为 G的所有连通子图 H构成的族，其中包含一个 $\mathcal{F}$-富模型$F_{h,d+1}$。我们断言， G中不存在 $\mathcal{F}'$-富模型$F_{2,d}$ 。

Suppose for a contradiction that there is an $\mathcal{F}'$-rich model $\big(B_x \mid x\in V(F_{2,d})\big)$ of $F_{2,d}$ in G. We will find an $\mathcal{F}$-rich model of $F_{h,d}$ in G. We denote by c the center of the star $F_{2,d}$ and by $x_1,\dots,x_d$ the leaves of $F_{2,d}$. We denote by r the root of $F_{h,d+1}$. Let $i\in[d]$. Since $(B_x \mid x\in V(F_{2,d}))$ is $\mathcal{F}'$-rich, B_{x_i} contains a neighbor u_i of B_c, and $G[B_{x_i}]$ has an $\mathcal{F}|_{G[B_{x_i}]}$-rich model $(B^i_x \mid x\in V(F_{h,d+1}))$ of $F_{h,d+1}$. By Lemma 8.6, there is an $\mathcal{F}$-rich model $\mathcal{M}_i$ of $F_{h,d}$ in $G[B_{x_i}]$ whose branch set of the root contains u_i. It follows that the union of the models $\mathcal{M}_1,\dots,\mathcal{M}_d$, together with B_c for the branch set of the root, yields an $\mathcal{F}$-rich model of $F_{h+1,d}$ in G, a contradiction. This proves that there is no $\mathcal{F}'$-rich model of $F_{2,d}$ in G.

因此，根据引理 8.5 应用于任意的 u，存在一个集合 $S_1\subseteq V(G)$ 以及一个路径划分 $(P_0,\dots\dots,P_\ell)$ 将 $G[S_1]$ 划分为，使得

8.5.(a) $S_1\cap V(F)\neq\emptyset$对每个 $F\in\mathcal{F}'$和

8.5。（b）对于每个 $i\in[\ell]$，P_i 都包含在至多 d 个 $\mathcal{D}$的袋的并集中。

By 8.5.(a), $G-S_1$ has no $\mathcal{F}$-rich model of $F_{h,d+1}$. Therefore, by the induction hypothesis applied to $G-S_1$ with the tree decomposition $\mathcal{D}|_{G-S_1}$, there exist pairwise disjoint $S'_1,\dots,S'_{h-1}\subseteq V(G-S_1)$, and for every $a\in[h-1]$ there is a path partition $(P'_{a,0},\dots,P'_{a,\ell_a})$ of $((G-S_1)-(S'_1\cup\cdots\cup S'_{a-1})\ S'_a)$ such that for $S'=\bigcup_{a\in[h-1]} S'_a$,

(a') $S'\cap V(F)\neq\emptyset$对每个 $F\in\mathcal{F}|_{G-S_1}$均成立；(b') 对每个 $a\in[h-1]$及每个 $j\in[\ell_a]$，$P_{a,\,j}$ 均包含于至多 $(d+1)+(h-1)-1=d+h-1$ 个 $\mathcal{D}|_{G-S_1}$的并集中。

Now, for every $a\in[h-1]$, let $S_{a+1}=S'_a$, and let $(P_{a+1,0},\dots,P_{a+1,\ell_{a+1}})=(P'_{a,0},\dots,P'_{a,\ell_a})$. Finally, let $(P_{1,0},\dots,P_{1,\ell_1})=(P_0,\dots,P_\ell)$. Then, $S_1,\dots,S_h$ is as claimed, which concludes the proof of the lemma. □

Lemma 8.8. *Let h and d be positive integers. Let G be a graph, let $\mathcal{D} = \big(T, (W_x \mid x \in V(T))\big)$ be a tree decomposition of G, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{h+1,d}$. There exist pairwise disjoint $S_1, \ldots, S_{h+1} \subseteq V(G)$, and for every $a \in [h+1]$ there exists a path partition $(P_{a,0}, \ldots, P_{a,\ell_a})$ of $(G - (S_1 \cup \cdots \cup S_{a-1})\ S_a)$ such that for $S = \bigcup_{a\in[h+1]} S_a$,*

(a) $S \cap V(F) \neq \emptyset$ 对每个 $F \in \mathcal{F}$ 均成立；(b) 对于 $G - S$ 的每个连通分量 C，$N_G(V(C))$ 与 $G - S$ 的至多两个连通分量相交；并且(c) 对于每个 $a \in [h+1]$ 和每个 $i \in [\ell_a]$，$P_{a,i}$ 包含在 $\mathcal{D}$ 的至多 $4h(d+h-1)$ 个包的并集中。

证明。 我们对 $|V(G)|$ 进行归纳。

首先，假设 G 不连通，令 $\mathcal{C}$ 为 G 的所有连通分量构成的族，并令 $C \in \mathcal{C}$。由归纳假设，存在两两不相交的 $S_{C,1}, \ldots, S_{C,h+1} \subseteq V(C)$，且对每个 $a \in [h+1]$，都存在 $(P_{C,a0}, \ldots, P_{C,a,\ell_{C,a}})$ 的路径划分，其中 $(C - (S_{C,1} \cup \cdots \cup S_{C,a-1})\ S_{C,a})$ 使得对于 $S_C = \bigcup_{a\in[h+1]} S_{C,a}$，

(a-C) $S_C \cap V(F) \neq \emptyset$ 对每个 $F \in \mathcal{F}|_C$ 均成立；(b-C) 对于 $C - S_C$ 的每个连通分量 C'，$N_C(V(C'))$ 至多与$C - S_C$的两个连通分量相交；并且 (c-C) 对于每个 $a \in [h+1]$ 以及每个 $i \in [\ell_{C,a}]$，$P_{C,a,i}$均包含于 $\mathcal{D}|_C$ 至多$4h(d+h-1)$ 个袋的并集中。

于是，对于每个 $a \in [h+1]$，令

$$S_a = \bigcup_{C\in\mathcal{C}} S_{C,a},$$

并令 $(P_{a,0}, \ldots, P_{a,\ell_a})$ 为所有 $C \in \mathcal{C}$ 上的 $(P_{C,a,0}, \ldots, P_{C,a,\ell_{C,a}})$ 的串接。这些对象满足引理的结论。

Next, suppose that G is connected. By Lemma 3.2, we can assume that $\mathcal{D}$ is a natural tree decomposition. By Lemma 8.7, there exist pairwise disjoint $S_1, \ldots, S_h \subseteq V(G)$, and for every $a \in [h]$ there exists a path partition $(P_{a,0}, \ldots, P_{a,\ell_a})$ of $(G - (S_1 \cup \cdots \cup S_{a-1})\ S_a)$ such that for $S' = \bigcup_{a\in[h]} S_a$,

8.7.(a) $S' \cap V(F) \neq \emptyset$ 对于每个 $F \in \mathcal{F}$ 都成立，并且8.7.(b) 对于每个 $a \in [h]$，对于每个 $i \in [\ell_a]$，$P_{a,i}$都包含在 $\mathcal{D}$ 的至多 $d+h-1$ 个包的并集中。

Let C' be a component of $G - S'$. By 8.7.(b), $N_G(V(C'))$ is included in the union of at most $2h(d+h-1)$ bags of $\mathcal{D}$. By Lemma 3.5, there is a set $Z_{C'} \subseteq V(G)$ containing $N_G(V(C'))$ which is included in the union of at most $4h(d+h-1)$ bags of $\mathcal{D}$ such that for every component C of $G - Z_{C'}$, $N_G(V(C))$ intersects at most two components of $G - V(C)$.

Let

$$S_{h+1} = \bigcup_{C'\in\mathcal{C}'} (Z_{C'} \cap V(C'))$$

其中，$\mathcal{C}'$ 是 $G - S'$ 的所有连通分支组成的族，并令 $S = \bigcup_{a\in[h+1]} S_a$。

First, since $S' \subseteq S$, $S \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$, which proves (a). Second, for every component C of $G - S$, there exists $C' \in \mathcal{C}'$ such that $C \subseteq C'$, and so C is a component of $G - Z_{C'}$, which implies that $N_G(V(C))$ intersects at most two components of $G - V(C)$, and so (b) holds. Finally, for an arbitrary ordering $C_1, \ldots, C_{\ell_{h+1}}$ of $\mathcal{C}'$, $(Z_{C_1} \cap V(C_1)\ \ldots, Z_{C_{\ell_{h+1}}} \cap V(C_{\ell_{h+1}}))$ is a path partition of $(G - S', S_{h+1})$, and for every $i \in [\ell_{h+1}]$, $Z_{C_i} \cap V(C_i)$ is included

在至多 $4h(d+h-1)$ 个 $\mathcal{D}$的并集中。这证明了 (c) 并完成了引理的证明。□

9. 分数树深度脆弱率

在本节中，我们证明定理 1.4。在小节 9.2 中，我们证明包含树宽的界；在小节 9.3 中，我们证明一般界。证明遵循我们所建立的抽象框架。特别地，证明采用归纳法，归纳步骤可由定理 4.2 轻松推出。请回顾图 9 和 10 中给出的证明路线图。另请注意，我们将采用一种分数树深度脆弱率的替代概率定义，该定义已在小节 5.3 中引入。

如引言中所述，已有结果表明，对于 $\mathcal{O}(q^{t-1})$， qth fractionaltreedepth fragility rates of K_t-minor-free graphs 的界为 9.1。在第 9.1 小节中，我们给出这一结果的证明，同时也借此展示我们的方法。

定理 9.1. 设 t 是满足 $t \geqslant 2$ 的整数。存在一个整数 c，使得对于每个K_t-小图自由图 G 以及每个正整数 q，

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1}.$$

9.1. **K_t-minor-free 图。** 如前所述，在本小节中，我们证明定理 9.1。我们首先陈述证明的前两个要素。

定理 9.2 ([9, 定理 28])。 对于每个正整数 w，存在一个常数 $c_{9.2}(w)$，使得对于每个图 G 满足 $\mathrm{tw}(G) \leqslant w$ 以及每个正整数 q

$$\mathrm{ftdfr}_q(G) \leqslant c_{9.2}(w) \cdot q^w.$$

定理 9.3 ([25, 定理 4])**.** 对所有正整数 t, w 满足 $t \geqslant 2$，每个 K_t-无小图 G满足 $\mathrm{tw}(G) \leqslant w$存在一个顶点划分 $\mathcal{P}$使得 $G/\mathcal{P}$的树宽至多为 $t-2$，且 $\mathcal{P}$ 的每一部分至多有 $w+1$ 个元素。[8]

Theorems 9.2 and 9.3 imply that for every K_t-minor-free graph G of bounded treewidth, we have $\mathrm{ftdfr}_q(G) = \mathcal{O}(q^{t-2})$ as we show below. Let t and w be positive integers, let G be a K_t-minor-free with $\mathrm{tw}(G) \leqslant w$ and let $\mathcal{P}$ be a partition of $V(G)$ as in Theorem 9.3. Since $\mathrm{tw}(G/\mathcal{P}) \leqslant t-2$, we have $\mathrm{ftdfr}_q(G/\mathcal{P}) = \mathcal{O}(q^{t-2})$ by Theorem 9.2. In particular, there exists a random variable Z' over subsets of $\mathcal{P}$ such that $\mathrm{td}(G/\mathcal{P} - \mathsf{Z}') = \mathcal{O}(q^{t-2})$, and Z' is equipped with a q-thin probability distribution. Let $\mathsf{Z} = \bigcup \mathsf{Z}'$. Since every element of $\mathcal{P}$ has at most $w+1$ vertices of G, $\mathrm{td}(G - \mathsf{Z}) \leqslant (w+1) \cdot \mathrm{td}(G/\mathcal{P} - \mathsf{Z}')$. It follows that Z witnesses that $\mathrm{ftdfr}_q(G) = \mathcal{O}((w+1) \cdot q^{t-2})$.

将这一推理推广到一般的 K_t-无子式情形，需要以下技术引理；它在定理 1.4 的证明中也会用到。

引理 9.4。 对于每个正整数 t，存在一个正整数 $c_{9.4}(t)$，使得对于每个 K_t-无子式图 G 和每个正整数 q，都存在一个取值于 $V(G)$ 的子集的随机变量 Y，满足 $\mathrm{tw}(G - \mathsf{Y}) \leqslant c_{9.4}(t) \cdot q$，且 Y 配备了一个 q-薄概率分布。

证明。令 t 为一个正整数。由定理3.6，可得一个整数 $c_{\mathrm{LRS}}(t)$，使得每个K_t-次要无关图都存在一个宽度至多为 $c_{\mathrm{LRS}}(t)$的分层 RS 分解。我们令

[8]换言之，存在一个树宽至多为 $t-2$ 的图 A，使得 $G \subseteq A \boxtimes K_{w+1}$，其中 $\boxtimes$ 表示强积。

$c_{9.4}(t) = 2c_{\mathrm{LRS}}(t)$. Let G be a K_t-minor-free graph. Therefore, G admits a layered RS-decomposition $(T, \mathcal{W}, \mathcal{A}, \mathcal{D}, \mathcal{L})$ of width at most $c_{\mathrm{LRS}}(t)$. Let q be a positive integer. We root T at an arbitrary vertex s. Let $\mathcal{W} = \big(T, (W_x \mid x \in V(T))\big)$, let $\mathcal{A} = \big(A_x \mid x \in V(T)\big)$, let $\mathcal{D} = \big(\mathcal{D}_x \mid x \in V(T)\big)$, and let $\mathcal{L} = \big(\mathcal{L}_x \mid x \in V(T)\big)$, where $\mathcal{L}_x = (L_{x,i} \mid i \in \mathbb{N})$ and $\mathcal{D}_x = \big(T_x, (D_{x,z} \mid z \in V(T_x))\big)$ for every $x \in V(T)$. For every vertex $u \in V(G)$, let s_u be the root of the subtree of T induced by $\{x \in V(T) \mid u \in W_x\}$. For every $x \in V(T)$, let $\mathsf{i}(x)$ be a random variable over $\{0, \ldots, q-1\}$ with the uniform distribution. Let Y be a the random variable defined by

$$\mathsf{Y} = \big\{u \in V(G) \mid u \in \bigcup_{k \in \mathbb{N}} L_{s_u, \mathsf{i}(s_u)+kq}\big\}.$$

首先，注意对于每个 $u \in V(G)$，

$$\Pr[u \in \mathsf{Y}] = \begin{cases} 0 & \text{if } u \in A_{s_u}, \\ \frac{1}{q} & \text{otherwise,} \end{cases}$$

因此，Y 的分布是 q-稀疏的。

接下来，我们界定 $\mathsf{tw}(G - \mathsf{Y})$。设 $x \in V(T)$。令 U_x 在 $x \neq s$时定义为 $W_x \cap W_{\mathrm{p}(T,x)}$，否则定义为$\emptyset$。注意，$|U_x| \leqslant c_{\mathrm{LRS}}(t)$。对于躯干$G, \mathcal{W}(W_x) - A_x - U_x - \mathsf{Y}$ 的每个连通分量 C，对于每个 $z \in V(T_x)$，$V(C)$ 与 $\mathcal{L}_x$ 的至多 $q-1$ 个层相交，因此

$$|D_{x,z} \cap V(C)| \leqslant c_{\mathrm{LRS}}(t) \cdot (q-1).$$

因此，正如 $\big(T_x, (D_{x,z} \cap V(C) \mid z \in V(T_x))\big)$ 所示，$\mathsf{tw}(C) \leqslant c_{\mathrm{LRS}}(t) \cdot (q-1) - 1$。因此，$\mathsf{tw}(\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x - U_x - \mathsf{Y}) \leqslant c_{\mathrm{LRS}}(t) \cdot (q-1) - 1$，于是

$$\begin{aligned} \mathrm{tw}(\mathrm{torso}_{G,\mathcal{W}}(W_x) - \mathsf{Y}) &\leqslant |A_x| + |U_x| + c_{\mathrm{LRS}}(T) \cdot (q-1) - 1 \\ &\leqslant c_{\mathrm{LRS}}(t) + c_{\mathrm{LRS}}(t) + c_{\mathrm{LRS}}(T) \cdot (q-1) - 1 \\ &\leqslant c_{\mathrm{LRS}}(t) \cdot (q+1) - 1 \leqslant 2c_{\mathrm{LRS}}(t) \cdot q. \end{aligned}$$

根据观察结果 3.7 应用于 $G - \mathsf{Y}$ 及其树分解 $\big(T, (W_x - \mathsf{Y} \mid x \in V(T))\big)$，这意味着 $\mathrm{tw}(G - \mathsf{Y}) \leqslant c_{9.4}(t) \cdot q$ 并完成了证明。□

Proof of Theorem 9.1. Let t be an integer with $t \geqslant 2$, let $c_{9.4}(\cdot)$ and $c_{9.2}(\cdot)$ be as in Lemma 9.4 and Theorem 9.2, respectively. We set $c = 2^t \cdot c_{9.4}(t) \cdot c_{9.2}(t-2)$. Let G be a K_t-minor-free graph and let q be a positive integer. By Lemma 9.4, there exists a random variable Y over subsets of $V(G)$ such that $\mathrm{tw}(G - \mathsf{Y}) \leqslant c_{9.4}(t) \cdot 2q$, and Y is equipped with a $2q$-thin probability distribution. By Theorem 9.3, $G - \mathsf{Y}$ admits a vertex-partition $\mathcal{P}_{\mathsf{Y}}$ such that $(G - \mathsf{Y})/\mathcal{P}_{\mathsf{Y}}$ has treewidth at most $t-2$ and each part of $\mathcal{P}_{\mathsf{Y}}$ has at most $c_{9.4}(t) \cdot 2q + 1$ elements. Furthermore, by Theorem 9.2, $\mathrm{ftdfr}_{2q}((G - \mathsf{Y})/\mathcal{P}_{\mathsf{Y}}) \leqslant c_{9.2}(t-2) \cdot (2q)^{t-2}$, and so, there exists a random variable Z' over subsets of $\mathcal{P}_{\mathsf{Y}}$ such that $\mathrm{td}((G - \mathsf{Y})/\mathcal{P}_{\mathsf{Y}} - \mathsf{Z}') \leqslant c_{9.2}(t-2) \cdot (2q)^{t-2}$, and Z' is equipped with a $2q$-thin probability distribution. Let

$$\mathsf{Z} = \mathsf{Y} \cup \bigcup \mathsf{Z}'.$$

对于每个 $u \in V(G)$，如果 $\mathsf{P}_u \in \mathcal{P}_{\mathsf{Y}}$ 满足 $u \in \mathsf{P}_u$，那么

$$\Pr[u \in \mathsf{Z}] \leqslant \Pr[u \in \mathsf{Y}] + \Pr[\mathsf{P}_u \in \mathsf{Z}'] \leqslant \frac{1}{2q} + \frac{1}{2q} = \frac{1}{q}.$$

因此，Z 的概率分布是 q-稀疏的。最后，

$$\mathrm{td}(G - \mathsf{Z}) \leqslant (c_{9.4}(t) \cdot 2q + 1) \cdot (c_{9.2}(t-2) \cdot (2q)^{t-2}) \leqslant c \cdot q^{t-1}.$$

这证明了 $\mathsf{ftdfr}_q(G) \leqslant c \cdot q^{t-1}$。□

9.2.**有界树宽的情形**。定理9.5 和9.8 推出了定理1.4中涉及树宽的界。在引理9.7中，我们考虑排除森林的$\mathcal{F}$-富模型的有界树宽图；这给出了定理9.8中的一个必要基例。

定理 9.5. 设 t 为满足 $t \geqslant 2$的整数。对于每个 $X \in \mathcal{S}_t$，存在一个整数 c，使得对于每个正整数 q 以及每个 X-无 *minor* 图 G，

$$\operatorname{ftdfr}_q(G) \leqslant c \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2}.$$

证明。 对于每个图 G、每个 $S \subseteq V(G)$ 以及每个正整数 q，令 t

$$\operatorname{par}_q(G,S) = \frac{1}{\operatorname{tw}(G)+2} \operatorname{ftdfr}_q(G,S)$$

由引理 5.16，聚焦参数族 $\operatorname{par} = (\operatorname{par}_q \mid q \in \mathbb{N}_1)$ 是良好的。由引理 8.1，$q \mapsto 1$ 是 $(\operatorname{par}\ \mathcal{E})$-有界的。因此，由推论 8.4，$q \mapsto q^{t-2}$ 是 $(\operatorname{par}\ \mathcal{S}_t)$-有界的。由引理 4.5，这意味着对于每个 $X \in \mathcal{S}_t$，都存在一个常数 $\beta = \beta(X)$，使得对于每个正整数 q，有

$$\operatorname{ftdfr}_q(G) = \operatorname{ftdfr}_q(G, V(G)) \leqslant \beta \cdot (\operatorname{tw}(G)+2) \cdot q^{t-2} \leqslant 2\beta \cdot (\operatorname{tw}(G)+1) \cdot q^{t-2}. \qquad \square$$

引理 9.6. 设 G 是一个图，并设 $S \subseteq V(G)$。对于每一个路径划分$(P_1, \ldots, P_\ell)$ 的(G, S)，

$$\operatorname{ftd}(G,S) \leqslant \lceil \log(\ell+1) \rceil \cdot \max_{i \in [\ell]} |P_i|.$$

Proof. Let $(P_1, \ldots, P_\ell)$ be a path partition of (G,S), and let $N = \max_{i\in[\ell]} |P_i|$. It is enough to show that $\operatorname{ftd}(G,S) \leqslant k \cdot N$ if $\ell = 2^k - 1$. We proceed by induction on k. When $k = 1$, the statement is clear. Next, suppose that $k > 1$. Let $S_0 = P_{2^{k-1}}$. Note that S_0 intersects every path in G between $\bigcup_{i\in[2^{k-1}-1]} P_i$ and $\bigcup_{i\in\{2^{k-1}+1,\ldots,2^k\}} P_i$. Thus, there exists a partition (V_1, V_2) of $V(G - S_0)$ such that $\bigcup_{i\in[2^{k-1}-1]} P_i \subseteq V_1$, $\bigcup_{i\in\{2^{k-1}+1,\ldots,2^k-1\}} P_i \subseteq V_2$, and there is no edge between V_1 and V_2 in G. We obtain,

$$\begin{aligned}
\operatorname{ftd}(G,S) &\leqslant \operatorname{ftd}(G,S_0) + \operatorname{ftd}(G-S_0, S-S_0) && \text{by Lemma 5.14}\\
&= \max\{\operatorname{ftd}((G-S_0)[V_1], S\cap V_1), \operatorname{ftd}((G-S_0)[V_2], S\cap V_2)\} && \text{by Observation 5.13}\\
&\leqslant |S_0| + (k-1)\cdot N && \text{by induction}\\
&\leqslant k \cdot N. && \square
\end{aligned}$$

引理 9.7. 对于每个正整数 q、每个图 G 以及每个 $S \subseteq V(G)$，令

$$\operatorname{par}_q(G,S) = \frac{1}{\operatorname{tw}(G)+2} \operatorname{ftdfr}_q(G,S).$$

函数 $q \mapsto \log(q+1)$ 对于 $(\operatorname{par}\ \mathcal{R}_2)$ 是 $\operatorname{par} = (\operatorname{par}_q \mid q \in \mathbb{N}_1)$ 有界的。

证明。 回想一下， $\mathcal{R}_2$ 是所有森林组成的类，并令 $X \in \mathcal{R}_2$。设 h 和 d 为正整数，使得 $X \subseteq F_{h+1,d}$ 并令

$$\beta(X) = 8(h+1)h(d+h-1).$$

We will prove that every graph is $(q \mapsto \log(q+1), \operatorname{par}\ X, 2, \beta(X))$-good. Let G be a graph and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of X. Let $\mathcal{D}$ be a tree decomposition of G of width $\operatorname{tw}(G)$. Since G does not contain an $\mathcal{F}$-rich model of X, by Lemma 8.8, there exist pairwise disjoint sets $S_1, \ldots, S_{h+1} \subseteq V(G)$, such that for every $a \in [h+1]$, there exists a path partition $(P_{a,0}, \ldots, P_{a,\ell_a})$ of $(G - (S_1 \cup \cdots \cup S_{a-1})\ S_a)$ such that for $S = \bigcup_{a\in[h+1]} S_a$,

8.8.(a) $S \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}$;

8.8.(b) 对于 C 的每个连通分量，$N_G(V(C))$至多与$G - S$的两个连通分量相交；并且
8.8.(c) 对每个 $a \in [h+1]$，对于每个 $i \in [\ell_a]$，$P_{a,\,i}$ 都包含在 $\mathcal{D}$的至多$4h(d+h-1)$个包的并集中。

令 q 为正整数。我们将证明，
(g1⋆) $S \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}$；(g2⋆) 对每个连通分支 C（属于 $G - S$），$N_G(V(C))$与至多两个连通分支相交，这些连通分支属于$G - V(C)$；且(g3⋆) $\mathrm{par}_q(G,S) \leqslant \beta(X) \cdot \log(q+1)$。

Cle早些时候，8.8.(a) 和8.8.(b) 分别推出 (g1⋆) 和 (g2⋆)。剩下需要证明的是 (g3⋆).
对于每个 $a \in [h+1]$，令 $\mathsf{i}(a)$ 为一个取值于 $\{0,\ldots,q-1\}$ 的随机变量，且服从均匀分布。现在，令

$$\mathsf{Y} = \bigcup \{P_{a,i} \mid a \in [h+1], i \in \{0,\ldots,\ell_a\}, i \equiv \mathsf{i}(a) \bmod q\}.$$

首先，注意到对于每个 $u \in S$，$\Pr[u \in \mathsf{Y}] = \frac{1}{q}$，因此 Y 的分布是 q-薄的。此外，对于每个 $a \in [h+1]$，由于 $G - (S_1 \cup \cdots \cup S_{a-1}) - \mathsf{Y}$有一个包含 $q-1$个部分的路径划分，根据引理 9.6和8.8.(c)。

$$\begin{aligned}\mathrm{ftd}(G - (S_1 \cup \cdots \cup S_{a-1}) - \mathsf{Y}, S_a - \mathsf{Y}) &\leqslant \lceil \log(q-1)+1 \rceil \cdot 4h(d+h-1) \cdot (\mathrm{tw}(G)+1) \\ &\leqslant 8h(d+h-1) \cdot (\mathrm{tw}(G)+1) \cdot \log(q+1)\end{aligned}$$

反复应用引理 5.14，可得
$\mathrm{ftd}(G - \mathsf{Y}\ S - \mathsf{Y}) \leqslant 8(h+1)h(d+h-1) \cdot (\mathrm{tw}(G)+1) \cdot \log(q+1) = c \cdot (\mathrm{tw}(G)+1) \cdot \log(q+1)$。因此，

$$\mathrm{ftdfr}_q(G,S) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot \log(q+1),$$

由此完成了该引理的证明。□

定理 9.8. 设 t 是一个满足 $t \geqslant 2$的整数。对于每个 $X \in \mathcal{R}_t$，都存在一个整数 c，使得对于每个满足 $q \geqslant 2$ 的整数 q 以及每个 X-*minor-free* 图 G，

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q.$$

证明。对于每个图 G、每个 $S \subseteq V(G)$ 以及每个正整数 q，令 t

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2} \mathrm{ftdfr}_q(G,S)$$

By Lemma 5.16, the family of focused parameters $\mathrm{par} = (\mathrm{par}_q \mid q \in \mathbb{N}_1)$ is nice. We show by induction on t (integers with $t \geqslant 2$) that $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}\ \mathcal{R}_t)$-bounding. When $t = 2$, this is given by Lemma 9.7. Next, suppose that $t \geqslant 3$. By Observation 3.1, $\mathcal{R}_{t-1}$ is closed under disjoint union and leaf addition, and by Lemma 6.5, $\mathcal{R}_{t-1}$ has the coloring elimination property. By induction, $q \mapsto q^{t-3}\log(q+1)$ is $(\mathrm{par}\ \mathcal{R}_{t-1})$-bounding. Therefore, by Theorem 4.2, $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}\ \mathcal{R}_t)$-bounding, as desired. By Lemma 4.5, this implies that for every $X \in \mathcal{R}_t$, there exists a constant $\beta = \beta(X)$ such that for every integer q with $q \geqslant 2$, we have

$\mathrm{ftdfr}_q(G) = \mathrm{ftdfr}_q(G, V(G)) \leqslant \beta \cdot (\mathrm{tw}(G)+2) \cdot q^{t-2}\log(q+1) \leqslant 4\beta \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2}\log q.$ □

9.3. **The general case.** In this subsection, we conclude discussing fractional treedepth fragility rates by proving the remaining bounds in Theorem 1.4. In fact, these bounds (Theorems 9.9 and 9.10) follow easily from the bounds in terms of treewidth (Theorems 9.5 and 9.8, respectively) and Lemma 9.4. Note that the proofs of Theorems 9.9 and 9.10 are almost the same.

定理 9.9. 设 t 是满足 $t \geqslant 2$ 的整数，并设 $X \in \mathcal{S}_t$。存在整数 c，使得对每个正整数 q 以及每个 X-无子式图 G，

$$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1}.$$

证明。设 G 是一个 X-无小图，并设 q 是一个正整数。根据应用于 $2q$的引理9.4，存在一个定义在 $V(G)$的子集上的随机变量Y_1，使得$\mathrm{tw}(G-\mathsf{Y}_1) \leqslant c_{9.4}(|V(X)|)\cdot 2q$，并且$\mathsf{Y}_1$配备有一个 $2q$-薄概率分布。根据定理9.5，存在一个正整数 c_0，使得

$$\mathrm{ftdfr}_{2q}(G-\mathsf{Y}_1) \leqslant c_0 \cdot (\mathrm{tw}(G-\mathsf{Y}_1)+1)\cdot (2q)^{t-2}.$$

设 Y_2 是定义在 $V(G-\mathsf{Y}_1)$ 的子集上的随机变量，见证了这一不等式，并令 $\mathsf{Y} = \mathsf{Y}_1 \cup \mathsf{Y}_2$。我们有，

$$\begin{aligned}\mathrm{td}(G-\mathsf{Y}) &\leqslant \mathrm{ftdfr}_{2q}(G-\mathsf{Y}_1)\\ &\leqslant 2^{t-2}c_0\cdot(\mathrm{tw}(G-\mathsf{Y}_1)+1)\cdot q^{t-2}\\ &\leqslant 2^{t-2}c_0\cdot(c_{9.4}(|V(X)|)\cdot 2q+1)\cdot q^{t-2} \leqslant c\cdot q^{t-1}\end{aligned}$$

对于 $c = 2^{t-1}c_0\cdot(c_{9.4}(|V(X)|)+1)$，此外，对于每个 $u \in V(G)$，

$$\Pr[u\in\mathsf{Y}] \leqslant \Pr[u\in\mathsf{Y}_1]+\Pr[u\in\mathsf{Y}_2] \leqslant \frac{1}{2q}+\frac{1}{2q} = \frac{1}{q}$$

由于 Y_2 配备了一个 $2q$-薄概率分布。因此，$\mathrm{ftdfr}_q(G) \leqslant c\cdot q^{t-1}$。□

定理 9.10. 设 t 为正整数，并设 $X \in \mathcal{R}_t$。存在整数 c，使得对每一个满足 $q \geqslant 2$ 的整数 q，以及对每一个 X-*minor-free*图 G，

$$\mathrm{ftdfr}_q(G) \leqslant c\cdot q^{t-1}\log q.$$

证明。令 G 为一个 X-minor-free 图，并令 q 为正整数。根据应用于 $2q$ 的引理 9.4，存在一个取值为 $V(G)$ 子集的随机变量 Y_1，使得 $\mathrm{tw}(G-\mathsf{Y}_1) \leqslant c_{9.4}(|V(X)|)\cdot 2q$，且 Y_1 配备了一个 $2q$-薄概率分布。根据定理 9.8，存在一个正整数 c_0 使得

$$\begin{aligned}\mathrm{ftdfr}_{2q}(G-\mathsf{Y}_1) &\leqslant c_0\cdot(\mathrm{tw}(G-\mathsf{Y}_1)+1)\cdot(2q)^{t-2}\log(2q)\\ &\leqslant 2^{t-1}c_0\cdot(\mathrm{tw}(G-\mathsf{Y}_1)+1)\cdot q^{t-2}\log q\end{aligned}$$

设 \(\mathsf{Y}_2\) 为 \(\{v2\}(\{v3\}Y\{v4\})\) 的子集上的一个随机变量，用以见证该不等式，并设 \(Y \{v5\}Y\{v6\}Y\{v7\}\)。我们有，

$$\begin{aligned}\mathrm{td}(G-\mathsf{Y}) &\leqslant \mathrm{ftdfr}_{2q}(G-\mathsf{Y}_1)\\ &\leqslant 2^{t-1}c_0\cdot(\mathrm{tw}(G-\mathsf{Y}_1)+1)\cdot q^{t-2}\log q\\ &\leqslant 2^{t-1}c_0\cdot(c_{9.4}(|V(X)|)\cdot 2q+1)\cdot q^{t-2}\log q \leqslant c\cdot q^{t-1}\log q\end{aligned}$$

对于 $c = 2^t c_0\cdot(c_{9.4}(|V(X)|)+1)$，此外，对于每个 $u \in V(G)$，

$$\Pr[u\in\mathsf{Y}] \leqslant \Pr[u\in\mathsf{Y}_1]+\Pr[u\in\mathsf{Y}_2] \leqslant \frac{1}{2q}+\frac{1}{2q} = \frac{1}{q}$$

因为 Y_2 配备了一个 $2q$-薄概率分布。因此，$\mathrm{ftdfr}_q(G) \leqslant c \cdot q^{t-1} \log q$。□

10. 弱着色数

在本节中，我们证明定理1.3。在小节10.1中，我们证明涉及树宽的界；在小节10.2中，我们证明其余的界。请回顾图9和10中的路线图。

在开始证明之前，我们再陈述一个关于弱着色数的著名结果。树具有线性的弱着色数，见观察 5.4。对于路径，Joret 和 Micek [26]证明了一个对数级上界。我们将在后文图 22 的图注中简要介绍该证明。

引理 10.1 ([26, 定理 1])。对于每个正整数 q 以及每条路径 P，弱着色数$_q(P) \leqslant 2 + \lceil \text{对数}\, q \rceil$。

10.1. **有界树宽情形**。定理 10.2 和 10.4给出了定理 1.3中涉及树宽的界。在引理 10.3中，我们考虑不含森林的$\mathcal{F}$-丰富模型的有界树宽图，这为定理 10.4提供了必要的基例。

定理 10.2. 设 t 为正整数。对于每个 $X \in \mathcal{S}_t$，都存在一个整数 c，使得对于每个满足 $q \geqslant 2$ 的整数 q，以及每个无 X 次子式的图 G，

$$\mathrm{wcol}_q(G) \leqslant c \cdot (\mathrm{tw}(G) + 1) \cdot q^{t-2}.$$

证明。对于每个图 G、每个 $S \subseteq V(G)$，以及每个正整数 q，令

$$\mathrm{par}_q(G, S) = \frac{1}{\mathrm{tw}(G) + 2} \mathrm{wcol}_q(G, S)$$

By Lemma 5.7, the family of focused parameters $\mathrm{par} = (\mathrm{par}_q \mid q \in \mathbb{N}_1)$ is nice. By Lemma 8.1, $q \mapsto 1$ is $(\mathrm{par}\ \mathcal{E})$-bounding. Therefore, by Corollary 8.4, $q \mapsto q^{t-2}$ is $(\mathrm{par}\ \mathcal{S}_t)$-bounding. By Lemma 4.5, this implies that for every $X \in \mathcal{S}_t$, there exists a constant $\beta = \beta(X)$ such that for every positive integer q, we have

$$\mathrm{wcol}_q(G) = \mathrm{wcol}_q(G, V(G)) \leqslant \beta \cdot (\mathrm{tw}(G) + 2) \cdot q^{t-2} \leqslant 2\beta \cdot (\mathrm{tw}(G) + 1) \cdot q^{t-2}. \qquad \square$$

引理 10.3。对于每个正整数 q、每个图 G 以及每个 $S \subseteq V(G)$，令

$$\mathrm{par}_q(G, S) = \frac{1}{\mathrm{tw}(G) + 2} \mathrm{wcol}_q(G, S).$$

函数 $q \mapsto \log(q+1)$ 是 $(\mathrm{par}\ \mathcal{R}_2)$ 对 $\mathrm{par} = (\mathrm{par}_q \mid q \in \mathbb{N}_1)$ 的界。

证明。回忆一下，$\mathcal{R}_2$ 是所有森林构成的类，并令 $X \in \mathcal{R}_2$。设 h 和 d 为正整数，使得 $X \subseteq F_{h+1,d}$ 并令

$$\beta(X) = 24(h+1)h(d+h-1).$$

We will prove that every graph is $(q \mapsto \log(q+1), \mathrm{par}\ X, 2, \beta(X))$-good. Let G be a graph and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of X. Let $\mathcal{D}$ be a tree decomposition of G of width $\mathrm{tw}(G)$. Since G does not contain an $\mathcal{F}$-rich model of X, by Lemma 8.8, there exist pairwise disjoint sets $S_1, \ldots, S_{h+1} \subseteq V(G)$, such that for every $a \in [h+1]$, there exists a path partition $(P_{a,0}, \ldots, P_{a,\ell_a})$ of $(G - (S_1 \cup \cdots \cup S_{a-1})\ S_a)$ such that for $S = \bigcup_{a \in [h+1]} S_a$,

8.8.(a) $S \cap V(F) \neq \emptyset$ 对每个 $F \in \mathcal{F}$；

8.8.(b) 对于 $G - S$ 的每个连通分支 C， $N_G(V(C))$ 至多与$G - S$的两个连通分支相交；并且8.8.(c) 对于每个 $a \in [h+1]$，对于每个 $i \in [\ell_a]$， $P_{a,\,i}$ 包含在 $\mathcal{D}$ 至多$4h(d+h-1)$ 个袋的并集中。

设 q 为正整数。我们将证明：
(g1⋆) 对每个 $F \in \mathcal{F}$， $S \cap V(F) \neq \emptyset$；(g2⋆) 对每个分量 C， $N_G(V(C))$ 与 $G - S$ 的至多两个分量相交；以及 (g3⋆) $\mathrm{par}_q(G,S) \leqslant \beta(X) \cdot \log(q+1)$。

Cle早期，8.8.(a) 和8.8.(b) 意味着(g1⋆) 和(g2⋆)，分别。还需证明(g3 ⋆).

Let $a \in [h+1]$. For convenience, we set $P_{a,\ell_a+1} = \emptyset$. Consider the path Q with $V(Q) = \{0,\dots,\ell_a+1\}$ where two numbers are connected by an edge whenever they are consecutive. Let $\sigma' = (i_0,\dots,i_{\ell_a})$ be an ordering of $\{0,\dots,\ell_a+1\}$ given by Lemma 10.1, that is such that $\mathrm{wcol}_q(Q,\sigma') \leqslant 2 + \lceil \log q \rceil \leqslant 3\log(q+1)$. For each $i \in \{0,\dots,\ell_a+1\}$, let σ_i be an arbitrary ordering of P_i. Let σ be the concatenation of $\sigma_{i_0} \dots \sigma_{i_{\ell_a+1}}$ in this order.

令 $u \in V(G-(S_1 \cup \cdots \cup S_{a-1}))$，并令 $W = \mathsf{WReach}_r[G-(S_1 \cup \cdots \cup S_{a-1})\ S_a,\ \sigma,\ u]$。我们论证 rǎng

$$|W| \leqslant 24h(d+h-1)\cdot(\mathrm{tw}(G)+1)\cdot\log(q+1).$$

Suppose $W \neq \emptyset$. Let $j_u \in V(T)$ be such that if $u \in S_a$, then $u \in P_{j_u}$, and otherwise, $j_u \in \{0,\dots,\ell\}$ is the least value such that P_{j_u} intersects $N_G(C)$, where C is the component of $G-S$ containing u. Let $A = \mathrm{WReach}_q[Q,\sigma',j_u] \cup \mathrm{WReach}_q[Q,\sigma',j_u+1]$. In particular, $|A| \leqslant 2\cdot\mathrm{wcol}_q(Q,\sigma') \leqslant 2\cdot 3\log(q+1)$. Since $(P_{a,0},\dots,P_{a,\ell_a+1})$ is a path partition of $(G-(S_1 \cup \cdots \cup S_{a-1})\ S_a)$，我们有 $W \subseteq \bigcup_{j\in A} P_j$. 根据 8.8.(c)，对于每个 $j \in \{0,\dots,\ell+1\}$， P_j都包含在至多 d 个 $\mathcal{D}$的袋的并集中，因此 $|P_j| \leqslant 2h(d+h-1)(\mathrm{tw}(G)+1)$。由此可得

$$|W| \leqslant |A|\cdot 4h(d+h-1)(\mathrm{tw}(G)+1) \leqslant 24h(d+h-1)(\mathrm{tw}(G)+1)\log(q+1).$$

特别地，

$$\mathrm{wcol}_q(G-(S_1 \cup \cdots \cup S_{a-1}), S_a) \leqslant 24h(d+h-1)(\mathrm{tw}(G)+1)\log q.$$

最后，反复应用观察 5.3，得到

$$\begin{aligned}\mathrm{wcol}_q(G,S) &\leqslant \sum_{a\in[h+1]} \mathrm{wcol}_q(G-(S_1 \cup \cdots \cup S_{a-1}), S_a)\\ &\leqslant 24(h+1)h(d+h-1)(\mathrm{tw}(G)+1)\log(q+1) = \beta(X)\cdot(\mathrm{tw}(G)+1)\cdot\log(q+1).\end{aligned}$$

这表明(g3⋆)，并完成了引理的证明。 □

定理 10.4. 设 t 为正整数。对于每个 $X \in \mathcal{R}_t$，存在一个整数 c，使得对于每个满足 $q \geqslant 2$ 的整数 q 以及每个 X-*minor-free* 图 G，

$$\mathrm{wcol}_q(G) \leqslant c\cdot(\mathrm{tw}(G)+1)\cdot q^{t-2}\log q.$$

证明。 对于每个图 G、每个 $S \subseteq V(G)$ 以及每个正整数 q，令

$$\mathrm{par}_q(G,S) = \frac{1}{\mathrm{tw}(G)+2}\,\mathrm{wcol}_q(G,S)$$

By Lemma 5.16, the family of focused parameters $\mathrm{par} = (\mathrm{par}_q \mid q \in \mathbb{N}_1)$ is nice. We show by induction on t (integers with $t \geqslant 2$) that $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}\ \mathcal{R}_t)$-bounding. When $t=2$, this is given by Lemma 10.3. Next, suppose that $t \geqslant 3$. By Observation 3.1, $\mathcal{R}_{t-1}$ is closed under disjoint union and leaf addition, and by Lemma 6.5, $\mathcal{R}_{t-1}$ has the coloring elimination property. By induction, $q \mapsto q^{t-3}\log(q+1)$ is $(\mathrm{par}\ \mathcal{R}_{t-1})$-bounding. Therefore, by Theorem 4.2, $q \mapsto q^{t-2}\log(q+1)$ is $(\mathrm{par}\ \mathcal{R}_t)$-bounding, as desired. By Lemma 4.5, this

意味着对于每个 $X \in \mathcal{R}_t$，都存在一个常数 $\beta = \beta(X)$，使得对于每个满足 $q \geqslant 2$ 的整数 q，我们都有

$$\mathrm{wcol}_q(G) = \mathrm{wcol}_q(G, V(G)) \leqslant \beta \cdot (\mathrm{tw}(G)+2) \cdot q^{t-2} \log(q+1) \leqslant 4\beta \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q. \ \square$$

10.2. **The general case.** In this subsection, we prove the remaining bounds in Theorem 1.3, i.e. Theorems 10.9 and 10.14. Since the proofs follow the general framework that we developed, in particular, since the induction step follows from Theorem 4.2, most of this subsection is devoted to the base cases. In Lemmas 10.10 and 10.11 we study graphs excluding $\mathcal{F}$-rich models of a star, and in Lemma 10.12 we study graphs excluding $\mathcal{F}$-rich models of general trees. Note that these involved base cases are necessary for the proof of Theorem 10.14 in contrast to Theorem 10.9.

A key ingredient in the proofs of the bounds in Theorem 1.3 involving treewidth is the Helly property of graphs of bounded treewidth: Lemma 3.3. Such a Helly property is false for K_k-minor-free graphs.[9] Therefore, we consider a relaxed statement, where instead of bounding the size of a hitting set (see Lemma 3.3.(2)), we restrict its structure. Namely, we require it to be the union of a small number of geodesics, see Corollary 10.8.

Before stating this generalized Helly property, let us pause to discuss geodesics in the context of weak coloring numbers. Let G be a graph. For all $u, v \in V(G)$, a *u–v geodesic* in G is an u–v path in G of minimum length. A *geodesic* in G is a path in G that is a u–v geodesic for some $u, v \in V(G)$. The following folklore statement makes geodesics a fundamental tool in bounding weak coloring numbers.

引理 10.5 ([4, 引理 23])。 设 G 是一个图，且 q 是一个非负整数。对于 Q 中的每条测地线 G 以及每个顶点 $v \in V(G)$，

$$|\{u \in V(Q) \mid \mathrm{dist}_G(u, v) \leqslant q\}| \leqslant 2q+1.$$

结合观察 5.3与引理 10.5，我们得到如下结论。

观察 10.6. 设 G 为一个图，设 $S \subseteq V(G)$，设 ℓ 为正整数，并设 $Q_1, \ldots, Q_\ell$ 为 G 中的测地线。对于每个非负整数 q，我们有

$$\mathrm{wcol}_q(G, S \cup V(Q_1) \cup \cdots \cup V(Q_\ell)) \leqslant \mathrm{wcol}_q(G, S) + \ell(2q+1).$$

下一个引理由 Dujmović 等人证明。 [4, 引理 21]。其证明依赖于 Robertson 和 Seymour 的图小结构定理。我们利用该引理，为 K_k-无小图类得到所述的广义 Helly 性质：推论10.8。

引理 10.7 ([4, 引理 21])。 对于每个正整数 k，存在一个正整数$c_{10.7}(k)$，使得对于每个正整数 d，对于每个 K_k-无 *minor* 图 G，对于每个由 G 的连通子图组成的族

(1) *there are d pairwise vertex-disjoint subgraphs in $\mathcal{F}$; or*
(2) *there exists $A \subseteq V(G)$ with $|A| \leqslant (d-1)c_{10.7}(k)$, and there exists a subgraph X of G which is the union of at most $(d-1)^2 c_{10.7}(k)$ geodesics in $G - A$, such that for every $F \in \mathcal{F}$ we have $(V(X) \cup A) \cap V(F) \neq \emptyset$.*

推论 10.8. 对于所有正整数 k 和 d，都存在一个正整数 $c_{10.8}(k, d)$，使得对于每个连通的 K_k-次级无关图 G，以及每个由以下图的连通子图组成的族 $\mathcal{F}$

[9]考虑 $n \times n$ 平面网格 G 以及族 $\mathcal{F}$，该族由网格中一行与一列的所有并集构成。 $\mathcal{F}$的装填数为1，但不存在少于 n 个元素的 $\mathcal{F}$的击中集。此外，每个平面网格都是 K_5-无小图的。

G，要么其中存在 d 个两两顶点不相交的子图，要么存在 $S \subseteq V(G)$，使得

(a) $S \cap V(F) \neq \emptyset$ 对于每个 $F \in \mathcal{F}$;(b) $G[S]$是连通的;
以及(c) $\mathrm{wcol}_q(G,S) \leqslant c_{10.8}(k,d) \cdot q$ 对于每个正整数 q。

Proof. Let $c_{10.8}(k,d) = 12(d-1)^2 c_{10.7}(k)$. Let G be a K_k-minor-free graph and let $\mathcal{F}$ be a family of connected subgraphs of G. Suppose that there are no d pairwise disjoint members of $\mathcal{F}$, and hence, Lemma 10.7.(2) holds, yielding $A \subseteq V(G)$ and a subgraph X of G such that $|A| \leqslant (d-1)c_{10.7}(k)$ and X is the union of at most $(d-1)^2 c_{10.7}(k)$ geodesics in $G - A$. Note that $G[A \cup V(X)]$ has at most $|A| + (d-1)^2 c_{10.7}(k)$ components. Let $Q_1, \ldots, Q_\ell$ be a family of at most $(d-1)c_{10.7}(k) + (d-1)^2 c_{10.7}(k) - 1$ geodesics in G such that the set

$$S = A \cup V(X) \cup \bigcup_{i \in [\ell]} V(Q_i)$$

在 G 中诱导一个连通子图。特别地，$\ell \leqslant 2(d-1)^2 c_{10.7}(k)$。对于每个正整数 q，根据观察 10.6 且由于 $\mathrm{wcol}_q(G, A \cup V(X)) \leqslant |A \cup V(X)|$，，

$$\begin{aligned}
\mathrm{wcol}_q(G,S) &\leqslant \mathrm{wcol}_q(G, A \cup V(X)) + \ell \cdot (2q+1) \\
&\leqslant |A| + (d-1)^2 c_{10.7}(k) \cdot (2q+1) + \ell \cdot (2q+1) \\
&\leqslant (d-1)c_{10.7}(k) + (d-1)^2 c_{10.7}(k)(2r+1) + 2(d-1)^2 c_{10.7}(k) \cdot (2q+1) \\
&\leqslant 4(d-1)^2 c_{10.7}(k) \cdot (2q+1) \leqslant c_{10.8}(k,d) \cdot q.
\end{aligned}$$

□

定理 10.9。设 t 为正整数。对于每个 $X \in \mathcal{S}_t$，都存在一个整数 c，使得对于每个满足 $q \geqslant 2$ 的整数 q，以及每个不含 *X-minor* 的图 G，

$$\mathrm{wcol}_q(G) \leqslant c \cdot q^{t-1}.$$

Proof. By Lemma 5.6, the family of focused parameters $\mathrm{wcol} = (\mathrm{wcol}_q \mid q \in \mathbb{N}_1)$ is nice. By Corollary 10.8 and Observation 5.1, $q \mapsto q$ is $(\mathrm{wcol}\ \mathcal{E})$-bounding. Therefore, by Corollary 8.4, $q \mapsto q^{t-1}$ is $(\mathrm{wcol}\ \mathcal{S}_t)$-bounding. By Lemma 4.5, this implies that for every $X \in \mathcal{S}_t$, there exists a constant $\beta = \beta(X)$ such that for every positive integer q, we have

$$\mathrm{wcol}_q(G) = \mathrm{wcol}_q(G, V(G)) \leqslant \beta \cdot q^{t-1}.$$

□

Lemma 10.10. *Let k and d be positive integers. Let G be a connected K_k-minor-free graph, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{2,d}$. For every nonempty $U \subseteq V(G)$ such that $G[U]$ is connected, there is a path decomposition $(W_0, \ldots, W_\ell)$ of G with $\ell \geqslant 1$ and sets $R_2, \ldots, R_\ell \subseteq V(G)$ such that for $S = U \cup \bigcup_{i \in \{2,\ldots,\ell\}} (W_{i-1} \cap W_i)$,*

(a) $W_0 = U$;
(b) $S \cap V(F) \neq \emptyset$ *for every* $F \in \mathcal{F}$;
(c) $G[S]$ *is connected;*
(d) $G[R_i]$ *is connected for every* $i \in \{2, \ldots, \ell\}$;
(e) $W_{i-1} \cap W_i \subseteq R_i \subseteq \bigcup_{j \in \{0,\ldots,i-1\}} W_j$ *for every* $i \in \{2, \ldots, \ell\}$;
(f) W_i *and* W_{i+2} *are disjoint for every* $i \in \{0, \ldots, \ell-2\}$; *and*
(g) $\mathrm{wcol}_q(G, R_i) \leqslant (c_{10.8}(k, d+1)+3) \cdot q$ *for every* $i \in \{2, \ldots, \ell\}$ *and for every positive integer* q.

证明。引理的陈述示意于图 20中。证明中定义的一些对象展示于图 21中。我们对 $|V(G)| - |U|$归纳。令 $U \subseteq V(G)$ 为非空集合，且满足 $G[U]$连通。若 $\mathcal{F}|_{G-U} = \emptyset$，则只需取 $W_0 = W_1 = U$，

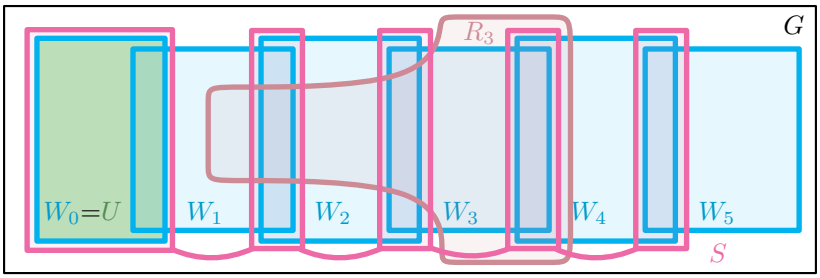


图 20。我们说明引理 10.10的陈述。绿色区域是给定集合 U，它应当变为 W_0。集合 R_3 必须满足 $W_3 \cap W_4 \subseteq R_3 \subseteq W_0 \cup W_1 \cup W_2 \cup W_3 \cup W_4$，并且我们希望 $\mathrm{wcol}_r(G, R_3)$ 较小。集合S 是连通的。为便于阅读，我们没有画出所有集合 R_i。

$\ell = 1$. In particular, this is the case for $U = V(G)$. Therefore, assume $|U| < |V(G)|$ and $\mathcal{F}|_{G-U} \neq \emptyset$. Let $\mathcal{F}_0$ be the family of all the connected subgraphs A of $G - U$ such that A contains a member of $\mathcal{F}$ and $V(A) \cap N_G(U) \neq \emptyset$. We argue that $\mathcal{F}_0 \neq \emptyset$. Since $\mathcal{F}|_{G-U} \neq \emptyset$, there is a component C of $G-U$ containing a member of $\mathcal{F}$. Since G is connected, $V(C) \cap N_G(U) \neq \emptyset$ and so $C \in \mathcal{F}_0$.

Observe that any collection of $d+1$ pairwise disjoint $A_1, \ldots, A_{d+1} \in \mathcal{F}_0$ yields an $\mathcal{F}$-rich model of $F_{2,d}$. Indeed, it suffices to take $U \cup A_{d+1}$ as the branch set corresponding to the root of $F_{2,d}$ and $A_1, \ldots, A_d$ as the branch sets of the remaining d vertices of $F_{2,d}$. Therefore, there are no $d+1$ pairwise disjoint members of $\mathcal{F}_0$, and thus, by Lemma 10.8 applied to G and $\mathcal{F}_0$, there exists a set $S_0 \subseteq V(G)$ such that

10.8。 (a) $S_0 \cap V(F) \neq \emptyset$ 对于每个 $F \in \mathcal{F}_0$; 10.8。 (b) $G[S_0]$ 是连通的; 并且10.8。 (c) 弱着色数$q(G, S_0) \leqslant c_{10.8}(k, d+1) \cdot q$对于每个正整数 q。

由于 $\mathcal{F}_0 \neq \emptyset$，我们有 $S_0 - U \neq \emptyset$。令 Q 为 U–S_0 中的一条测地线，G（可能只是一个单顶点路径），并令 $S_1 = S_0 \cup V(Q)$。注意，根据 10.8.(b)，$G[S_1]$ 是连通的。

Let $\mathcal{C}_0$ be the family of all the components C of $G - U - S_1$ such that $N_G(U) \cap V(C) = \emptyset$. Let $U' = V(G) - \bigcup_{C \in \mathcal{C}_0} V(C)$. Observe that $|U'| > |U|$ since $S_0 - U \neq \emptyset$ and U' contains $U \cup S_0$. Let $\mathcal{F}' = \{F \in \mathcal{F} \mid V(F) \cap U' = \emptyset\}$. By the induction hypothesis applied to G, $\mathcal{F}'$ and U', there is a path decomposition $(W'_0, \ldots, W'_{\ell'})$ of G and sets $R'_2, \ldots, R'_{\ell'} \subseteq V(G)$ such that for $S' = U' \cup \bigcup_{i \in \{2,\ldots,\ell'\}} (W'_{i-1} \cap W'_i)$,

(a') $W'_0 = U'$;
(b') $S' \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}'$;
(c') $G[S']$ is connected;
(d') $G[R'_i]$ is connected for every $i \in \{2, \ldots, \ell'\}$;
(e') $W'_{i-1} \cap W'_i \subseteq R'_i \subseteq \bigcup_{j \in \{0,\ldots,i-1\}} W'_j$ for every $i \in \{2, \ldots, \ell'\}$;
(f') W'_i and W'_{i+2} are disjoint for every $i \in \{0, \ldots, \ell' - 2\}$; and
(g') $\mathrm{wcol}_q(G, R'_i) \leqslant (c_{10.8}(k, d+1) + 3) \cdot q$ for every $i \in \{2, \ldots, \ell\}$ and for every positive integer q.

令 $\ell = \ell' + 1$，$W_0 = U$，$W_1 = U'$，$W_2 = (W'_1 - U') \cup (S_1 - U)$，$W_i = W'_{i-1}$对每个 $i \in \{3, \ldots, \ell\}$，$R_2 = S_1$，并且 $R_i = R'_{i-1}$对每个 $i \in \{3, \ldots, \ell\}$。注意到(a) 由构造可知成立。我们声称$(W_0, \ldots, W_\ell)$ 是 G 的一个路径分解，并且(b)–(g) 成立，由此完成引理的证明。

Let $u \in V(G)$. We claim that $I = \{i \in \{0, \ldots, \ell\} \mid u \in W_i\}$ is an interval. Since $(W'_0, \ldots, W'_{\ell'})$ is a path decomposition of G, $I' = \{i \in \{0, \ldots, \ell'\} \mid u \in W'_i\}$ is an interval. If $u \notin U' = W'_0$, then $I = \{i \in \{2, \ldots, \ell\} \mid u \in W'_{i-1}\} = \{i + 1 \mid i \in I'\}$, which is an interval too. Next, suppose

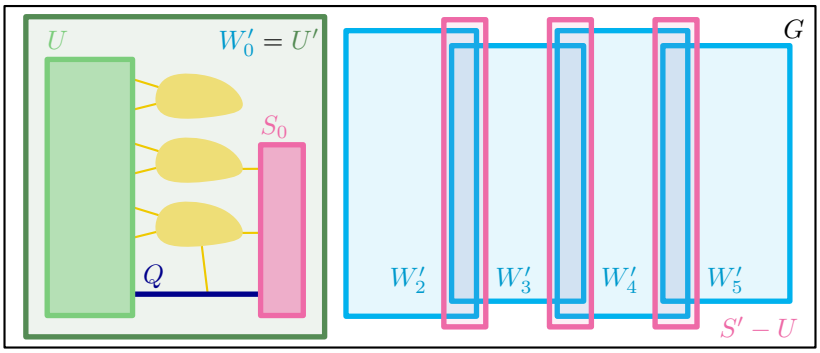


图 21。 引理中所考虑对象的示意图10.10。注意， S_0 可能与 U相交。

that $u \in U'$, and so $0 \in I'$. If $u \notin S_1 - U$, then $u \notin W_2$ and $u \notin W'_i$ for every $i \in \{2, \ldots, \ell'\}$ by (a') and (f'). Hence $I = \{0, 1\}$ if $u \in U$, and $I = \{1\}$ otherwise, which is an interval in both cases. If $u \in S_1 - U$, then $u \notin U = W_0$, and so $I = \{1\} \cup \{i + 1 \mid i \in I' - \{0\}\}$, which is an interval. This proves that I is an interval.

Let uv be an edge of G. We claim that there exists $i \in \{0, \ldots, \ell\}$ such that $u, v \in W_i$. If there exists $i' \in \{2, \ldots, \ell'\}$ such that $u, v \in W'_{i'}$, then $u, v \in W'_{i'} = W_{i'-1}$ and we are done. Next, suppose that u and v are not both in W'_i for every $i \in \{2, \ldots, \ell\}$. Since $(W'_0, \ldots, W'_{\ell'})$ is a path decomposition of G, there exists $i' \in \{0, 1\}$ such that $u, v \in W'_{i'}$. If $i' = 0$, then $u, v \in W'_0 = U' = W_1$. Now suppose that u and v are not both in $W'_0 = U'$; in particular, $i' = 1$. Without loss of generality, assume that $v \notin U'$. It follows that $v \in W'_1 - U' \subseteq W_2$. Let C be the component of v in $G - U - S_1$. Since $v \notin U'$, C belongs to $\mathcal{C}_0$, and so $N_G(V(C)) \cap U = \emptyset$. It follows that $N_G(V(C)) \cap U = \emptyset$, and so, $u \in S_1 - U$. Therefore, $u \in W_2$, which concludes the claim. Furthermore, we obtained that $(W_0, \ldots, W_\ell)$ is a path decomposition of G.

We now prove (b). Consider $F \in \mathcal{F}$. If F intersects U, then $S \cap V(F) \neq \emptyset$ since $U \subseteq S$. If F intersects $S_1 - U$, then F intersects $W_1 \cap W_2 \subseteq S$. Now suppose that F is disjoint from $U \cup S_1$. Let C be the component containing F in $G - U - S_1$. Since C is disjoint from $S_0 \subseteq S_1$, by 10.8.(a), C is not a member of $\mathcal{F}_0$. This implies that $N_G(U) \cap V(C) = \emptyset$, and thus, $C \in \mathcal{C}_0$. In particular, F is disjoint from U', and so, $F \in \mathcal{F}'$. By (b'), $S' \cap V(F) \neq \emptyset$, hence, there exists $i \in \{2, \ldots, \ell'\}$ such that $V(F)$ intersects $W'_{i-1} \cap W'_i$. It follows that $W'_{i-1} \cap W'_i = W_i \cap W_{i+1}$ and so $S \cap V(F) \neq \emptyset$. This proves (b).

让我们暂停一下，强调一个可由该构造直接得出的简单观察， (a') ，以及 (f') ：

对于每个 $i \in \{3, \ldots, \ell\}$，我们都有 $W_{i-1} \cap W_i = W'_{i-2} \cap W'_{i-1}$ $(\star)$ 。

By (c'), $S' = U' \cup \bigcup_{i \in \{2, \ldots, \ell'\}} (W'_{i-1} \cap W'_i)$ induces a connected subgraph of G. In particular, every component of $G[S'] - U'$ has a neighbor in $N_G(V(G) - U') \subseteq S_1$. Since $G[U \cup S_1]$ is connected, it follows that $(S' - U') \cup U \cup S_1$ induces a connected subgraph of G. However, $S = (S' - U') \cup U \cup S_1$ by $(\star)$, which yields (c).

对于每个 $i \in \{3, \ldots, \ell\}$， $R_i = R'_{i-1}$根据 G由 (d')诱导出连通子图，而 $R_2 = S_1$根据定义， G 诱导出连通子图，因此 (d) 得证。

For the proof of (e), first, observe that by construction, $\bigcup_{j \in \{0, \ldots, i-1\}} W_j = \bigcup_{j \in \{0, \ldots, i-2\}} W'_j$ for every $i \in \{2, \ldots, \ell\}$. In particular, it follows that $R_i = R'_{i-1} \subseteq \bigcup_{j \in \{0, \ldots, i-2\}} W'_j = \bigcup_{j \in \{0, \ldots, i-1\}} W_j$ for every $i \in \{3, \ldots, \ell\}$ by (e'). Moreover, $R_2 = S_1 \subseteq W_1$. It remains to show that $W_{i-1} \cap W_i \subseteq R_i$ for every $i \in \{2, \ldots, \ell'\}$. For $i = 2$, $W_1 \cap W_2 = S_1 - U \subseteq R_2$. For $i \in \{3, \ldots, \ell\}$, $W_{i-1} \cap W_i = W'_{i-2} \cap W'_{i-1} \subseteq R'_{i-1} = R'_i$ by $(\star)$ and (e'). This gives (e).

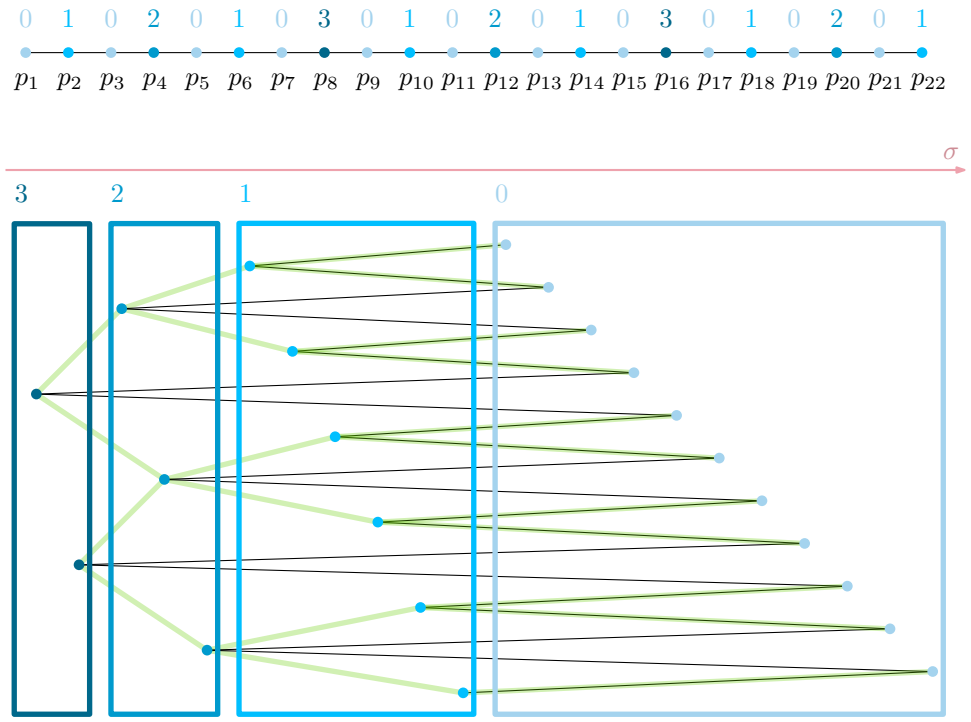


FIGURE 22. Consider the path $P = p_1 \cdots p_{22}$. Let $r = 7$ and $s = 3 = \lceil \log q \rceil$. We claim that $\mathrm{wcol}_q(P) \leqslant 2 + s$. We mark every eighth vertex with 3 (this corresponds to the set I_3 in the proof of Lemma 10.11). Then, we mark each fourth unmarked vertex with 2 (the set $I_2 - I_3$), every second unmarked vertex with 1 (the set $I_1 - I_2$), and finally all remaining vertices with 0 (the set $I_0 - I_1$). Next, we construct an ordering on the vertices $p_1, \ldots, p_{22}$. First, preserving the original ordering in the path, we put the vertices marked with 3, then similarly the ones marked with 2, with 1, and with 0. It suffices to argue that for every $u \in V(P)$, we have $|\mathrm{WReach}_q[P, \sigma, u]| \leqslant 5$. We build an auxiliary tree (depicted in green in the figure), where for every $i \in \{3, 2, 1\}$, we connect every vertex marked with i with the closest vertices in the path marked with $i - 1$. It is not hard to see that $\mathrm{WReach}_q[P, \sigma, u]$ consists only of the ancestors of u in the auxiliary tree.

For every $i \in \{3, \ldots, \ell - 2\}$, $W_i \cap W_{i+2} = W'_{i-1} \cap W'_{i+1} = \emptyset$ by (f'). Moreover, $W'_3 = W_4$ is disjoint from $S_1 - U \subseteq W'_0$ by (f'). Hence $W_2 \cap W_4 = W'_1 \cap W'_3 = \emptyset$ by (f'). Similarly, $W'_2 = W_3$ is disjoint from $U' = W'_0$ by (f'). Hence $W_1 \cap W_3 = W'_0 \cap W'_2 = \emptyset$. Finally, $W_0 \cap W_2 = \emptyset$ by construction, and so, (f) holds.

还需证明(g)。首先，对于每个 $i \in \{3, \ldots, \ell\}$，$R_i = R'_{i-1}$ 因此 $\mathrm{wcol}_q(G, R_i) \leqslant (c_{10.8}(k, d+1) + 3) \cdot q$ 对每个正整数 q 由(g')可得。此外，$R_2 = S_1 = V(Q) \cup S_0$。因此，根据10.8.(c)和观察10.6，

$$\mathrm{wcol}_q(G, R_2) \leqslant c_{10.8}(k, d+1) \cdot q + (2q + 1) \leqslant (c_{10.8}(k, d+1) + 3) \cdot q$$

对每个正整数 q。这表明(g) 成立，从而完成了引理的证明。□

下一个引理的证明沿用了证明以下结论时采用的思路：$\mathrm{wcol}_q(P) \leqslant 2 + \lceil \log q \rceil$，对于每条路径 P 和每个正整数 q [26，定理 1]。参见图 22。

引理 10.11. 设 k 和 d 为正整数。对于每个连通的 K_k-无小图图 G，以及 G的每个由连通子图组成的族 $\mathcal{F}$，如果 G 没有 F_2的 $\mathcal{F}$-丰富模型，d，那么存在一个集合 $S \subseteq V(G)$ 使得

(a) $S \cap V(F) \neq \emptyset$对于每个 $F \in \mathcal{F}$;(b) $G[S]$是连通的；并且(c) $\operatorname{wcol}_q(G,S) \leqslant 5(c_{10.8}(k,d+1)+3)\cdot q\log q$对于每个整数 q满足 $q \geqslant 2$。

Proof. Let q be an integer with $q \geqslant 2$, let G be a connected K_k-minor-free graph, let $\mathcal{F}$ be a family of connected subgraphs of G, and suppose that G has no $\mathcal{F}$-rich model of $F_{2,d}$. Let U be an arbitrary singleton of a vertex in G. Lemma 10.10 applied to G, $\mathcal{F}$, and U gives a path decomposition $(W_0,\dots,W_\ell)$ and sets $R_2,\dots,R_\ell \subseteq V(G)$ such that for $S' = U \cup \bigcup_{i\in\{2,\dots,\ell\}}(W_{i-1}\cap W_i)$,

10.10.(a) $W_0 = U$;
10.10.(b) $S' \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$;
10.10.(c) $G[S']$ is connected;
10.10.(d) $G[R_i]$ is connected for every $i \in \{1,\dots,\ell\}$;
10.10.(e) $W_{i-1}\cap W_i \subseteq R_i \subseteq \bigcup_{j\in\{0,\dots,i-1\}} W_j$ for every $i \in \{2,\dots,\ell\}$;
10.10.(f) W_i and W_{i+2} are disjoint for every $i \in \{0,\dots,\ell-2\}$; and
10.10.(g) $\operatorname{wcol}_q(G,R_i) \leqslant (c_{10.8}(k,d+1)+3)\cdot q$ for every $i \in \{2,\dots,\ell\}$.

为方便起见，我们设定 $R_1 = U$。

Let $s = \lceil \log(q+1)\rceil$. For every $i \in \{0,\dots,s\}$, let $I_i = \{i \in \{1,\dots,\ell\} \mid j = 0 \mod 2^i\}$. We construct recursively families $\{R'_j\}_{j\in\{1,\dots,\ell\}}$ and $\{S_i\}_{i\in\{0,\dots,s\}}$ of subsets of $V(G)$ and a family $\{\sigma_j\}_{j\in\{1,\dots,\ell\}}$ such that σ_j is an ordering of R'_j for every $j \in \{1,\dots,\ell\}$. For every $j \in I_s$, let

$$R'_j = R_j - \bigcup_{a\in\{0,\dots,j-2^s-1\}} W_a$$

and let $S_s = \bigcup_{j\in I_s} R'_j$. Let $j \in I_s$. If $j < 2\cdot 2^s$, then $j = 2^s$ and $R'_j = R_j$, and so by 10.10.(g), $\operatorname{wcol}_q(G,R'_j) \leqslant (c_{10.8}(k,d+1)+3)\cdot q$. Now assume that $j \geqslant 2\cdot 2^s$. Since $(W_0,\dots,W_\ell)$ is a path decomposition of G, $W_{j-2^s-1}\cap W_{j-2^s}$ separates $\bigcup_{a\in\{0,\dots,j-2^s-1\}} W_a$ and $\bigcup_{a\in\{j-2^s,\dots,\ell\}} W_a$ in G. Since $W_{j-2^s-1}\cap W_{j-2^s} \subseteq R'_{j-2^s}$ (by 10.10.(e)), by Observation 5.1, we obtain

$$\begin{aligned}\operatorname{wcol}_q\Big(G - \bigcup_{a\in\{1,\dots,j-2^s\}\cap I_s} R'_a, R'_j\Big) &= \operatorname{wcol}_q\left(G - (W_{j-2^s-1}\cap W_{j-2^s}), R'_j\right)\\ &= \operatorname{wcol}_q\Big(G - \bigcup_{a\in\{0,\dots,j-2^s-1\}} W_a, R'_j\Big).\end{aligned}$$

最后，

$$\begin{aligned}\operatorname{wcol}_q\Big(G - \bigcup_{a\in\{0,\dots,j-2^s-1\}} W_a, R'_j\Big) &\leqslant \operatorname{wcol}_q(G,R_j) && \text{by Observation 5.2}\\ &\leqslant (c_{10.8}(k,d+1)+3)\cdot q && \text{by 10.10.(g).}\end{aligned}$$

令 σ_j 为 R'_j 的一个序，使得

$$\operatorname{wcol}_q\Big(G - \bigcup_{a\in\{1,\dots,j-2^s\}\cap I_s} R'_a, R'_j, \sigma_j\Big) \leqslant (c_{10.8}(k,d+1)+3)\cdot q.$$

接下来，令 $i \in \{0,\dots,s-1\}$，并假设 S_{i+1}已定义。现在，对于每个 $j \in I_i - I_{i+1}$，令

$$R'_j = \Big(R_j - \bigcup_{a\in\{0,\dots,j-2^i-1\}} W_a\Big) - S_{i+1},$$

并令 $S_i = \bigcup_{j\in I_i} R'_j$。注意到 $S_{i+1} \subseteq S_i$。还需注意，对每个 $j \in I_i$，$W_{j-1}\cap W_j \subseteq R'_j$由10.10.(f)可知。令 $j \in I_i - I_{i+1}$。我们有 $j - 2^i \in I_{i+1}$，因此 $W_{j-2^i-1}\cap W_{j-2^i} \subseteq$

$R'_{j-2^i} \subseteq S_{i+1}$。由于 $(W_0, \ldots\ldots, W_\ell)$ 是 G的路径分解，$W_{j-2^i-1} \cap W_{j-2^i}$ 将 $\bigcup_{a\in\{0,\ldots,j-2^i-1\}} W_a$ 与 $\bigcup_{a\in\{j-2^i,\ldots,\ell\}} W_a$ 在 G中分离。由观察 5.1 可得

$$\begin{aligned}\operatorname{wcol}_q(G - S_{i+1}, R'_j) &= \operatorname{wcol}_q\left(G - (W_{j-2^i-1} \cap W_{j-2^i}), R'_j\right)\\ &= \operatorname{wcol}_q\left(G - \textstyle\bigcup_{a\in\{0,\ldots,j-2^i-1\}} W_a, R'_j\right).\end{aligned}$$

此外，

$$\begin{aligned}\operatorname{wcol}_q\left(G - \textstyle\bigcup_{a\in\{0,\ldots,j-2^i-1\}} W_a, R'_j\right) &\leqslant \operatorname{wcol}_q(G, R_j) && \text{by Observation 5.2}\\ &\leqslant (c_{10.8}(k, d+1)+3)\cdot q && \text{by 10.10.(g).}\end{aligned}$$

设 σ_j 是 R'_j 的一个排序，使得

$$\operatorname{wcol}_q(G - S_{i+1}, R'_j, \sigma_j) \leqslant (c_{10.8}(k, d+1)+3)\cdot q.$$

我们定义 $S = S_0$。现在，只需证明 (a)–(c) 成立。由于 $S' \subseteq S$，(a) 由 10.10.(b) 得证。

Recall that $G[S']$ is connected by 10.10.(c). Next, let C be a component of $G[R'_j]$ for some fixed $j \in I_s$. If $V(C) \cap (W_{j-1} \cap W_j) \neq \emptyset$, then $V(C) \cap S' \neq \emptyset$, and so, $G[S' \cup V(C)]$ is connected. Thus, assume that $V(C) \cap (W_{j-1} \cap W_j) = \emptyset$. However, $W_{j-1} \cap W_j \subseteq R'_j$, hence, C has a neighbor in W_{j-2^s-1}, in particular, in $W_{j-2^s-1} \cap W_{j-2^s} \subseteq S'$. Hence, again $G[S' \cup V(C)]$ is connected. In particular, we have just proved that $G[S' \cup S_s]$ is connected. Next, suppose that $G[S' \cup S_{i+1}]$ is connected for some $i \in \{0, \ldots, s-1\}$. Let C be a component of $G[R'_j]$ for some fixed $j \in I_i$. If $V(C) \cap (W_{j-1} \cap W_j) \neq \emptyset$, then $V(C) \cap S' \neq \emptyset$, and so, $G[S' \cup V(C)]$ is connected. Thus, assume that $V(C) \cap (W_{j-1} \cap W_j) = \emptyset$. However, $W_{j-1} \cap W_j \subseteq R'_j$, hence, C has a neighbor in $W_{j-2^i-1} \cup S_{i+1}$, in particular, in $(W_{j-2^i-1} \cap W_{j-2^i}) \cup S_{i+1} \subseteq S' \cup S_{i+1}$. Hence, $G[S' \cup S_{i+1} \cup V(C)]$ is connected. Finally, $G[S' \cup S_0] = G[S]$ is connected, which yields (b).

集合 $\{R'_j\}_{j\in\{2,\ldots,\ell\}}$ 两两不相交，且它们构成 S的一个划分。设 σ 是 S的一个排序，使得

(i) σ extends σ_j, for every $j \in \{1, \ldots, \ell\}$;
(ii) for every $j, j' \in I_s$ with $j < j'$, for all $u \in R'_j$ and $v \in R'_{j'}$, $u <_\sigma v$; and
(iii) for every $i \in \{0, \ldots, s-1\}$, for all $u \in S_{i+1}$ and $v \in S_i - S_{i+1}$, $u <_\sigma v$.

For convenience, let $R'_0 = \emptyset$ and $W_j = R'_j = \emptyset$ 对于满足 $j > \ell$ 的每个整数 j。

现在证明(c)。令 $u \in V(G)$。我们将证明 $|\operatorname{WReach}_q[G, S, \sigma, u]| \leqslant 5(c_{10.8}(k, d+1)+3)\cdot q \log q$。令 $j_u \in \{0, \ldots, \ell\}$ 为满足$u \in W_{j_u}$的最小值。我们断言

$$|\operatorname{WReach}_q[G, S, \sigma, u] \cap S_s| \leqslant 2(c_{10.8}(k, d+1)+3)\cdot q.$$

设 $\alpha = \max\{0\} \cup \{a \in I_s \mid a \leqslant j_u\}$，并设 $\beta = \alpha + 2^s$。因此，若 $\beta \leqslant \ell$，则 $\beta \in I_s$。接下来，我们论证

$$\operatorname{WReach}_q[G, S, \sigma, u] \cap S_s \subseteq R'_\alpha \cup R'_\beta.$$

Suppose to the contrary that there is a vertex $v \in \operatorname{WReach}_q[G, S, \sigma, u] \cap S_s$ with $v \notin R'_\alpha \cup R'_\beta$. Let $\gamma \in I_s - \{\alpha, \beta\}$ be such that $v \in R'_\gamma$. Then either $\gamma < \alpha$, or $\gamma > \beta$. First assume that $\gamma < \alpha$. Since $R'_\gamma \subseteq \bigcup_{a\in\{0,\ldots,\gamma-1\}} W_a$ and because $(W_0, \ldots, W_\ell)$ is a path decomposition of G, every u–v path in G intersects $W_{a-1} \cap W_a$ for each $a \in \{\gamma, \ldots, j_u\}$. Since $(W_{a-1} \cap W_a)_{a\in\{1,\ldots,\ell\}}$ are pairwise disjoint, we deduce that $\operatorname{dist}_G(u, v) \geqslant j_u - \gamma \geqslant \alpha - \gamma \geqslant 2^s > q$, which contradicts the fact that $v \in \operatorname{WReach}_r[G, S, \sigma, u]$. Finally, assume $\gamma > \beta$. Note that $\gamma \leqslant \ell$ since $R'_\gamma \neq \emptyset$ as $v \in R'_\gamma$. Since $R'_\gamma \subseteq \bigcup_{a\in\{\gamma-2^s,\ldots,\gamma-1\}} W_a$, and because $(W_0, \ldots, W_\ell)$ is a path decomposition of G, every u–v path in G intersects $W_{\beta-1} \cap W_\beta$. However, for every $w \in W_{\beta-1} \cap W_\beta$, we have $w <_\sigma v$, thus, $v \notin \operatorname{WReach}_q[G, S, \sigma, u]$, which is a contradiction. We obtain that $\operatorname{WReach}_q[G, S, \sigma, u] \cap S_s \subseteq R'_\alpha \cup R'_\beta$.

对于每个 $\varepsilon \in \{\alpha, \beta\}$，根据 σ 的定义，我们有

$$\begin{aligned}\mathrm{WReach}_q[G,S,\sigma,u] \cap R'_\varepsilon &\subseteq \mathrm{WReach}_q\left[G - \bigcup_{a\in I_s\cap\{1,\dots,\alpha-1\}} R'_a, R'_\varepsilon, \sigma_\varepsilon, u\right] \\ &\subseteq \mathrm{WReach}_q\left[G - \bigcup_{a\in\{1,\dots,j_u-2^s\}\cap I_s} R'_a, R'_\varepsilon, \sigma_\varepsilon, u\right]\end{aligned}$$

因此，

$$|\mathrm{WReach}_q[G,S,\sigma,u] \cap R'_\varepsilon| \leqslant (c_{10.8}(k,d+1)+3)\cdot q.$$

特别地，

$$\begin{aligned}|\mathrm{WReach}_q[G,S,\sigma,u] \cap S_s| &\leqslant |\mathrm{WReach}_q[G,S,\sigma,u] \cap (R'_\alpha \cup R'_\beta)| \\ &\leqslant 2(c_{10.8}(k,d+1)+3)\cdot q.\end{aligned}$$

接下来，令 $i \in \{0,\dots,s-1\}$。我们断言

$$|\mathrm{WReach}_q[G,S,\sigma,u] \cap (S_i - S_{i+1})| \leqslant (c_{10.8}(k,d+1)+3)\cdot q.$$

由于 S_{i+1} 的每个顶点在 σ 中都先于 S_i 的每个顶点，我们有

$$\mathrm{WReac}\ \ \mathrm{h}_q[G,S,\sigma,u] \cap (S_i - S_{i+1}) \subseteq \mathrm{WReach}_q[G - S_{i+1}, S - S_{i+1}, \sigma|_{S-S_{i+1}}, u] \cap (S_i - S_{i+1}).$$

Let $\alpha = \max\{a \in I_{i+1} \mid a \leqslant j_u\}$ and $\beta = \alpha + 2^{i+1}$. Let C be the component of u in $G - S_{i+1}$. Since $W_{\alpha-1} \cap W_\alpha, W_{\beta-1} \cap W_\beta \subseteq S_{i+1}$, and because $(W_0,\dots,W_\ell)$ is a path decomposition of G, $V(C) \cap S \subseteq \bigcup_{a\in\{\alpha,\dots,\beta-1\}} W_a$. We deduce that

$$\mathrm{WReach}_q[G - S_{i+1}, S - S_{i+1}, \sigma|_{S-S_{i+1}}, u] \cap (S_i - S_{i+1}) \subseteq \bigcup_{a\in\{\alpha,\dots,\beta-1\}} W_a.$$

Since $\{\alpha+1,\dots,\beta-1\}$ 中 $I_i - I_{i+1}$ 的唯一成员是 $\gamma = \alpha + 2^i$，事实上我们有

$$\mathrm{WReach}_q[G - S_{i+1}, S - S_{i+1}, \sigma|_{S-S_{i+1}}, u] \cap (S_i - S_{i+1}) \subseteq R'_\gamma,$$

由此我们推得

$$\begin{aligned}|\mathrm{WReach}_q[G - S_{i+1}, S - S_{i+1}, \sigma|_{S-S_{i+1}}, u] \cap (S_i - S_{i+1})| &\leqslant \mathrm{wcol}_q(G - S_{i+1}, R'_\gamma, \sigma_\gamma) \\ &\leqslant (c_{10.8}(k,d+1)+3)\cdot q.\end{aligned}$$

为方便起见，令 $S_{s+1} = \emptyset$。由于 $S = S_0$，可得。

$$\begin{aligned}|\mathrm{WReach}_q[G,S,\sigma,u]| &\leqslant \sum_{i\in\{0,\dots,s\}} |\mathrm{WReach}_q[G,S,\sigma,u] \cap (S_i - S_{i+1})| \\ &\leqslant (s+2)\cdot(c_{10.8}(k,d+1)+3)\cdot q \\ &\leqslant 5(c_{10.8}(k,d+1)+3)\cdot q\log q.\end{aligned}$$

□

引理 10.12. 设 k、 h 和 d 为满足 $h \geqslant 2$的正整数。存在整数$c_{10.12}(h、d、k)$，使得对于每个连通的 K_k-无小图图 G，以及 G的每个连通子图族 $\mathcal{F}$，若 G 不含F_h的 $\mathcal{F}$-富模型，$_d$，则存在集合 $S \subseteq V(G)$，使得

(a) $S \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}$ 都成立；(b) $G[S]$ 是连通的；并且(c) $\mathrm{wcol}_q(G,S) \leqslant c_{10.12}(h,d,k)\cdot q\log q$对每个满足 $q \geqslant 2$ 的整数 q 都成立。

证明。我们对 h 作归纳。对于 $h = 2$，由引理 10.11 可得该结论，令$c_{10.12}(1,d,k) = 5(c_{10.8}(k,d+1)+3)$。接下来，假设 $h > 2$，并且 $c_{10.12}(h-1,d,k)$ 是 $h-1$ 满足断言的一个见证。令 $c_{10.12}(h,d,k) = 5(c_{10.8}(k,d+1)+3)+3+c_{10.12}(h-1,d+1,k)$。

设 q 为一个满足 $q \geqslant 2$ 的整数，设 G 为一个连通的 K_k-无小图，且设 $\mathcal{F}$ 为 G 的连通子图族。假设 G 不含 F_h 的 $\mathcal{F}$-富模型，$_d$。设 $\mathcal{F}'$ 为 G 的所有连通子图 H 构成的族，且这些子图满足：H包含一个 $\mathcal{F}|_H$-富模型 of

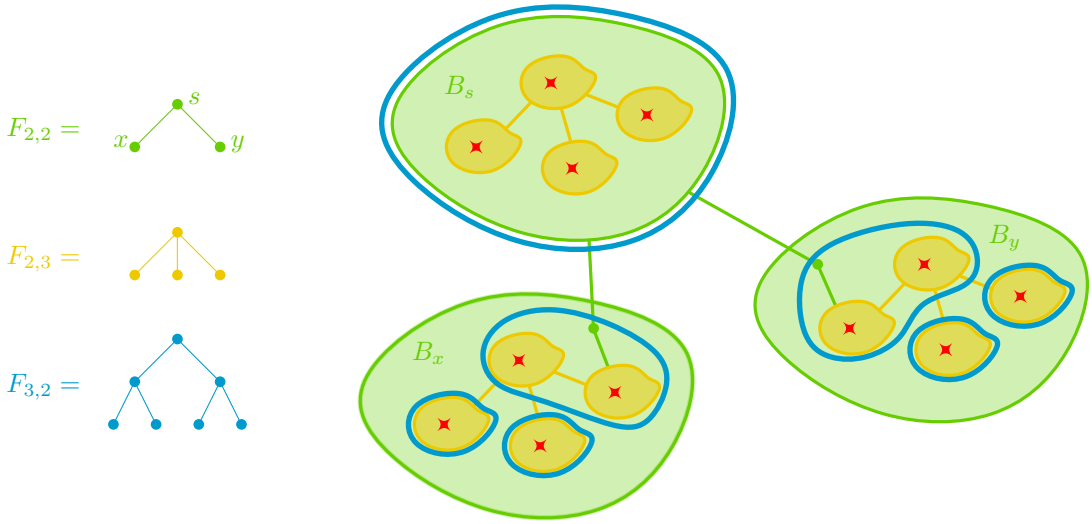


FIGURE 23. We provide an example of the construction of an $\mathcal{F}$-rich model of $F_{h,d}$ in G assuming that there is an $\mathcal{F}'$-rich model of $F_{2,d}$ in G in the case where $h = 3$ and $d = 2$. In green, we depict an $\mathcal{F}'$-rich model of $F_{2,d} = F_{2,2}$ in the graph. Each branch set contains an $\mathcal{F}$-rich model of $F_{h-1,d+1} = F_{2,3}$. We depict these models in yellow and the red stars are the elements of $\mathcal{F}$. The obtained model of $F_{h,d} = F_{3,2}$ we depict in blue. Note that this model is $\mathcal{F}$-rich.

$F_{h-1,d+1}$. We claim that there is no $\mathcal{F}'$-rich model of $F_{2,d}$ in G. Suppose to the contrary that $\big(B_x \mid x \in V(F_{2,d})\big)$ is such a model. Let s be the root of $F_{2,d}$ and let s' be the root of $F_{h-1,d}$. For every $x \in V(F_{2,d}) - \{s\}$, by Lemma 8.6, there is an $\mathcal{F}$-rich model $\big(C_y \mid y \in V(F_{h-1,d})\big)$ of $F_{h-1,d}$ in $G[B_x]$ such that $C_{s'}$ contains a vertex of $N_G(B_s) \cap B_x$. The union of these models together with B_s yields an $\mathcal{F}$-rich model of $F_{h,d}$ in G, which is a contradiction. See Figure 23.

由于 G 没有 $\mathcal{F}'$-丰富模型 $F_{2,\ d}$，由引理 10.11，存在集合 $S_0 \subseteq V(G)$ 使得 t
10.11.(a) 对每个 $F \in \mathcal{F}'$, $S_0 \cap V(F) \neq \emptyset$;10.11.(b) $G[S_0]$ 是连
通的；并且10.11.(c) $\mathrm{wcol}_q(G, S_0) \leqslant 5(c_{10.8}(k, d+1) + 3) \cdot q$
$\log q$.

令 C为 $G - S_0$的一个分支。根据10.11.(a)， $C \notin \mathcal{F}'$，因此， C没有 $\mathcal{F}|_C$-富模型$F_{h-1,\ d+1}$。于是，根据归纳假设，存在一个集合 $S_C \subseteq V(C)$，使得
(a') 对每个 $F \in \mathcal{F}|_C$， $S_C \cap V(F) \neq \emptyset$；(b') $C[S_C]$ 是
连通的；以及 (c') $\mathrm{wcol}_q(C,\ S_C) \leqslant c_{10.12}(h-1,\ d+1,$
$k) \cdot q\log\ q$。

令 Q_C 为 G中的一条 S_C–$N_G(S_0)$ 地理路径。特别地， Q_C 是 C中的一条地理路径。令 $\mathcal{C}$ 为 $G - S_0$ 的分支族，并令

$$S = S_0 \cup \bigcup_{C \in \mathcal{C}} (S_C \cup V(Q_C)).$$

参见图 24示例。我们声称 (a)–(c) 成立。

设 $F \in \mathcal{F}$。若 $S_0 \cap V(F) = \emptyset$，则 $F \subseteq C$对某个连通分量 C属于 $G - S_0$。特别地，$F \in \mathcal{F}|_C$，因此由 (a')， $S_C \cap V(F) \neq \emptyset$，这就证明了 (a)。图 $G[S]$按构造是连通的， (b') 以及10.11.(b)，由此得到 (b)。下面的不等式序列

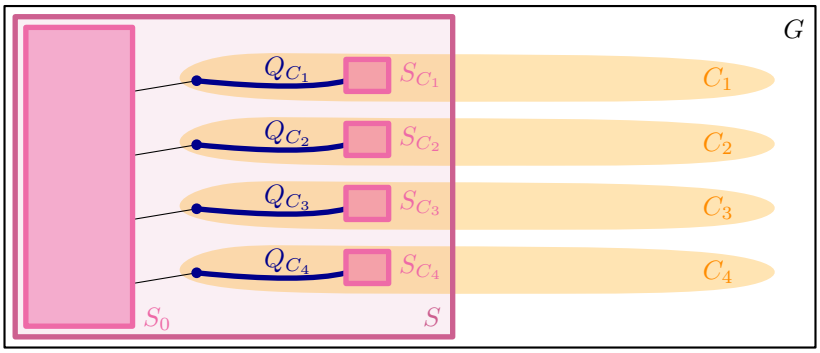


图 24。 集合 S 在引理 10.12的证明中的构造示意图。

完成了 (c) 及引理的证明:

$$\begin{aligned}
\operatorname{wcol}_q(G,S) &\leqslant \operatorname{wcol}_q(G,S_0) + \operatorname{wcol}_q\left(G-S_0, \bigcup_{C\in\mathcal{C}} (S_C \cup V(Q_C))\right) && \text{by Observation 5.3}\\
&\leqslant \operatorname{wcol}_q(G,S_0) + \max_{C\in\mathcal{C}} \operatorname{wcol}_q(C, S_C\cup V(Q_C)) && \text{by Observation 5.1}\\
&\leqslant \operatorname{wcol}_q(G,S_0) + \max_{C\in\mathcal{C}} \operatorname{wcol}_q(C,S_C) + (2q+1) && \text{by Observation 10.6}\\
&\leqslant (5(c_{10.8}(k,d+1)+3) + c_{10.12}(h-1,d+1,k)+3)\cdot q \text{对数 } q \text{依据 10.11.(c) 以及 (c')}\\
&= c_{10.12}(h,d,k)\cdot q \text{对数 } q. && \square
\end{aligned}$$

推论 10.13。 函数 $q \mapsto q\log(q+1)$ 是聚焦参数族 $\operatorname{wcol} = (\operatorname{wcol}_q \mid q\in\mathbb{N}_1)$的 $(\operatorname{wcol}\ \mathcal{R}_2)$界。

证明。 回忆一下， $\mathcal{R}_2$ 是所有森林组成的类。令 k 为正整数，并令 $X\in\mathcal{R}_2$。令 k' 为一个整数，使得所有 K_k-无小图的图 G 都是 k'-退化的，换言之，$\operatorname{wcol}_1(G)\leqslant k'$。我们选取整数 h 和 d，使得$X\subseteq F_{h,d}$，并令

$$\beta(k,X) = \max\{k', c_{10.12}(h,d,k)\}.$$

我们证明，每个不含 K_k 次小图的图都是 $(q\mapsto q\log(q+1), \operatorname{wcol}\ X, 1, \beta(k,X))$-良好的。设 G 为一个不含 K_k 次小图的图，并设 $\mathcal{F}$ 为 G 的一个连通子图族，使得 G 不含 X 的 $\mathcal{F}$-富模型（因此也不含 $F_{h,d}$ 的模型）。

设 C 是 G 的一个连通分量。根据引理 10.12，存在 $S_C\subseteq V(C)$ 使得
10.12.(a) $S_C\cap V(F)\neq\emptyset$对于每个 $F\in\mathcal{F}|_C$；10.12.(b) $C[S_C]$ 是连通的；并且
10.12.(c) 弱着色数$_q(C,S_C)\leqslant c_{10.12}(h,d,k)\cdot q$对数 q对于每个满足 q的整数 $q\geqslant 2$。

令 $\mathcal{C}$ 为 G的连通分量集合，我们定义

$$S = \bigcup_{C\in\mathcal{C}} S_C.$$

令 q 为正整数。我们将证明:
(g1*) 对每个 $F\in\mathcal{F}$， $S\cap V(F)\neq\emptyset$；(g2*) 对于 $G-S$ 的每个连通分量 C'， $N_G(V(C'))$ 至多与 $G-V(C')$ 的一个连通分量相交；以及 (g3*) $\operatorname{par}_q(G,S)\leqslant\beta(k,X)\cdot\log(q+1)$。

显然， 10.12.(a) 和10.12.(b) 分别推出(g1*) 和(g2*)。最后， 10.12.(c)和观察5.1推出(g3*)。$\square$

定理 10.14。设 t 为正整数。对于每个 $X \in \mathcal{R}_t$，存在一个整数 c，使得对于每个满足 $q \geqslant 2$ 的整数 q，以及每个 *X-minor-free* 图 G，

$$\mathrm{wcol}_q(G) \leqslant c \cdot q^{t-1} \log q.$$

Proof. By Lemma 5.6, the family of focused parameters $\mathrm{wcol} = (\mathrm{wcol}_q \mid q \in \mathbb{N}_1)$ is nice. We show by induction on t (integers with $t \geqslant 2$) that $q \mapsto q^{t-1}\log(q+1)$ is (wcol $\mathcal{R}_t$)-bounding. When $t = 2$, this is given by Corollary 10.13. Next, suppose that $t \geqslant 3$. By Observation 3.1, $\mathcal{R}_{t-1}$ is closed under disjoint union and leaf addition, and by Lemma 6.5, $\mathcal{R}_{t-1}$ has the coloring elimination property. By induction, $q \mapsto q^{t-2}\log(q+1)$ is (wcol $\mathcal{R}_{t-1}$)-bounding. Therefore, by Theorem 4.2, $q \mapsto q^{t-1}\log(q+1)$ is (wcol $\mathcal{R}_t$)-bounding. This completes the inductive proof that for every integer t with $t \geqslant 2$, $q \mapsto q^{t-1}\log(q+1)$ is (wcol $\mathcal{R}_t$)-bounding, as desired. By Lemma 4.5, this implies that for every $X \in \mathcal{R}_t$, there exists a constant $\beta = \beta(X)$ such that for every integer q with $q \geqslant 2$, we have

$$\mathrm{wcol}_q(G) = \mathrm{wcol}_q(G, V(G)) \leqslant \beta \cdot q^{t-1}\log(q+1) \leqslant 2\beta \cdot q^{t-1}\log q. \qquad \square$$

11. 中心色数

In this section, we prove Theorem 1.2. First, we show the existence of suitable precolorings of K_t-minor-free graphs in Subsection 11.1. Then, in Subsection 11.2, we follow the roadmaps depicted in Figures 9 and 10 to prove Theorem 1.2. Most of the work lies in the base case corresponding to excluding a graph in $\mathbf{A}(\mathcal{R}_2)$, and which is inspired by the proof of Dębski, Felsner, Micek, and Schröder [10] that outerplanar graphs have q-centered chromatic numbers in $\mathcal{O}(q \log q)$. We set our base case to excluding a graph in $\mathbf{A}(\mathcal{R}_2)$ instead of $\mathcal{R}_2$ (as we did in Sections 9 and 10) because graphs excluding a forest as a minor do not have qth centered chromatic numbers in $\mathcal{O}(\log q)$, but in $\Theta(q)$. This particularity of centered chromatic numbers is highlighted in the case $(t, s) = (2, 3)$ of Corollary 1.5.

We remark that one of the conference versions of this paper [22] contains a direct proof of the special case of K_t-minor-free graphs (i.e. Theorem 1.1). This proof avoids the abstract framework. Its main building blocks are analogous to Theorem 9.1, where we bound the fractional treedepth fragility rates for K_t-minor-free graphs. Namely, a bound for centered chromatic numbers of bounded treewidth due to Pilipczuk and Siebertz [31], and extending the idea of Illingworth, Scott, and Wood [25, Theorem 4]. Additionally, we need the material of Subsection 11.1.

我们先给出一些简单的组合命题，并提供证明以求完整。设 T 是一棵树，且 $\mathcal{Q}$ 是 T 的一组连通子图，其顶点集划分 $V(T)$。对于每个 $X \subseteq V(T)$，令 $\mathcal{Q}(X) = \{Q \in \mathcal{Q} \mid V(Q) \cap X \neq \emptyset\}$。

引理 11.1。设 T 为一棵树，$\mathcal{Q}$ 为 T 的一组连通子图，其顶点集划分了 $V(T)$。设 $X, Y \subseteq V(T)$ 满足 $X \subseteq Y$，且 $\mathrm{LCA}(T, X) = X$。若 $\mathcal{Q}(X) = \mathcal{Q}(Y)$，则 $\mathcal{Q}(Y) = \mathcal{Q}(\mathrm{LCA}(T, Y))$。

Proof. Suppose that $\mathcal{Q}(X) = \mathcal{Q}(Y)$. Since $Y \subseteq \mathrm{LCA}(T, Y)$, we have $\mathcal{Q}(Y) \subseteq \mathcal{Q}(\mathrm{LCA}(T, Y))$. Thus, it suffices to prove that $\mathcal{Q}(\mathrm{LCA}(T, Y)) \subseteq \mathcal{Q}(Y)$. Consider $Q \in \mathcal{Q}(\mathrm{LCA}(T, Y))$. There exist $y_1, y_2 \in Y$ such that $\mathrm{lca}(T, y_1, y_2) \in Q$. Let $Q_1, Q_2 \in \mathcal{Q}$ be such that $y_1 \in V(Q_1)$, $y_2 \in V(Q_2)$. In particular, $Q_1, Q_2 \in \mathcal{Q}(Y) = \mathcal{Q}(X)$. Hence, there exist $x_1 \in V(Q_1) \cap X$ and $x_2 \in V(Q_2) \cap X$.

如果 Q_1 和 Q_2 的根在 T中不存在祖先—后代关系，那么对于所有 $q_1 \in V(Q_1)$ 和 $q_2 \in V(Q_2)$，都有 $\mathsf{lca}(T, q_1, q_2) = \mathsf{lca}(T, \mathsf{root}(Q_1), \mathsf{root}(Q_2))$。特别是，

$$\mathrm{lca}(T, x_1, x_2) = \mathrm{lca}(T, \mathrm{root}(Q_1), \mathrm{root}(Q_2)) = \mathrm{lca}(T, y_1, y_2) \in V(Q).$$

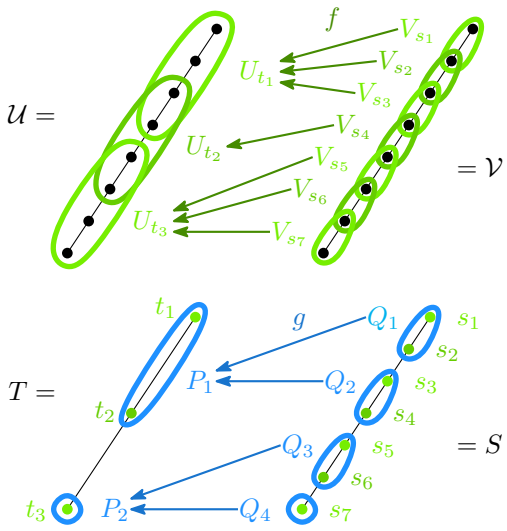


图 25. 在这个例子中，G 是一条 8 个顶点的路径。$\mathcal{P} = \{P_1, P_2\}$ 和 $\mathcal{Q} = \{Q_1, Q_2, Q_3, Q_4\}$。可以验证，$(\mathcal{V}, \mathcal{Q})$ 精化了 $(\mathcal{U}, \mathcal{P})$，其见证为 f 和 g。注意，f 是定义在 $V(S)$ 上的函数，但为便于阅读，我们将其描绘为作用于 $\mathcal{V}$ 的包上。

注意到 $\mathsf{lca}(T, x_1, x_2) \in \mathsf{LCA}(T, X) = X$，因此，$V(Q) \cap X \neq \emptyset$，于是，$Q \in \mathcal{Q}(X) = \mathcal{Q}(Y)$。

Therefore, we can assume that one of the roots say $\mathrm{root}(Q_1)$, is an ancestor of the other $\mathrm{root}(Q_2)$ in S. It follows that for all $q_1 \in V(Q_1)$ and $q_2 \in V(Q_2)$, $\mathrm{lca}(T, q_1, q_2)$ is either equal to $\mathrm{root}(Q_1)$ or is a descendant of $\mathrm{root}(Q_1)$. In both cases, $\mathrm{lca}(T, q_1, q_2)$ lies in the path from q_1 to $\mathrm{root}(Q_1)$ in S. Since Q_1 is connected, we obtain that $\mathrm{lca}(T, q_1, q_2) \in V(Q_1)$. In particular, $\mathrm{lca}(T, x_1, x_2) \in V(Q_1)$, and so, $Q_1 = Q$ implying $Q \in \mathcal{Q}(Y)$, which ends the proof. □

引理 11.2。设 G 是一个图，$\mathcal{W} = \big(T \text{ 是 } G \text{ 的一个树分解 } W_x \mid x \in V(T)\big)\big)$，且 $x \in V(T)$。如果 H 是 G 的一个连通子图，且与 W_x 相交，那么 $V(H) \cap W_x$ 在躯于$_{G,\mathcal{W}}(W_x)$ 中诱导一个连通子图。

Proof. Let H be a connected subgraph of G intersecting W_x, and suppose to the contrary that the subgraph H' of $\mathrm{torso}_{G,\mathcal{W}}(W_x)$ induced by $V(H) \cap W_x$ is not connected. Let $u, v \in V(H')$ be vertices in distinct components of H such that the distance between u and v is minimal in H. Note that the internal vertices of a shortest path between u and v in H do not lie in W_x as otherwise we obtain a pair of vertices of H' in distinct components that are closer in H. By the properties of tree decompositions, such a shortest path has all its vertices in $\bigcup_{z \in V(T_{y|x})} W_z$ for some $y \in N_T(x)$, and so $u, v \in W_x \cap W_y$. It follows that u and v are adjacent in $\mathrm{torso}_{G,\mathcal{W}}(W_x)$, and so, in H', which contradicts the assumption and completes the proof. □

我们还需要引理3.2的一个更明确的版本。事实上，在证明过程中的某处，我们希望假设所考虑的树分解是自然的，同时仍保留某些性质。为此，我们引入以下定义，并证明引理11.3。

设 G 是一个图。若 $\mathcal{U} = \big(T,\ \text{且 } (U_t \mid t \in V(T))\big)$ 是 G 的树分解，$\mathcal{P}$ 是 T 的一组连通子图，其顶点集划分了 $V(T)$，则称有序对 $(\mathcal{U}, \mathcal{P})$ 是 G 的一个正规对。考虑 G 的两个正规对 $(\mathcal{U}, \mathcal{P})$ 和 $(\mathcal{V}, \mathcal{Q})$，其中 $\mathcal{U} = \big(T,\ (U_t \mid t \in V(T))\big)$，且 $\mathcal{V} = \big(S,\ (V_s \mid s \in V(S))\big)$。若存在 $f\colon V(S) \to V(T)$ 以及 $g\colon \mathcal{Q} \to \mathcal{P}$，使得 ..... 则称 $(\mathcal{V}, \mathcal{Q})$ 精化了 $(\mathcal{U}, \mathcal{P})$，或称其为 $(\mathcal{U}, \mathcal{P})$ 的一个精化：

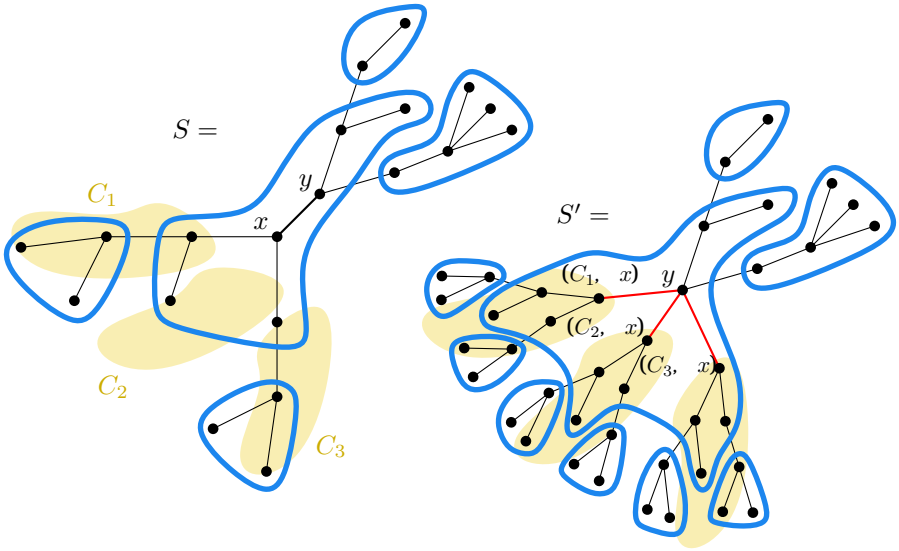


图 26。 引理11.3的证明：从 S' 构造 S，以及从 $\mathcal{Q}'$ 构造$\mathcal{Q}$。 S 的蓝色划分为 $\mathcal{Q}$，而 S' 的蓝色划分为 $\mathcal{Q}'$。

(r1) 对每个 $s \in V(S)$,

$$V_s \subseteq U_{f(s)};$$

对于每个 $Q \in \mathcal{Q}$, **(r2)**

$$f(V(Q)) \subseteq V(g(Q)).$$

参见图中的示例25。

引理 11.3. 设 G 是一个连通图，并设$(\mathcal{U}, \mathcal{P})$ 是 G的一个正规对。存在$(\mathcal{V}, \mathcal{Q})$ 作为$(\mathcal{U}, \mathcal{P})$ 的一个加细，使得 $\mathcal{V}$是自然的。

证明。 对于每个正整数 i，对于 G 的每个树分解 $\mathcal{W}$，我们用 $n_i(\mathcal{W})$ 表示 $\mathcal{W}$ 中大小为 i 的袋的数量。于是令 $n(\mathcal{W}) = (n_{|V(G)|}(\mathcal{W}) \dots, n_0(\mathcal{W}))$.
由于细化关系是自反的，每个 G 的正规对都有一个细化。考虑按字典序使 $n(\mathcal{V})$ 最小的 $(\mathcal{U}, \mathcal{P})$ 的细化 $(\mathcal{V}, \mathcal{Q})$。我们声称$\mathcal{V}$ 是自然的。

令 $\mathcal{V} = \big(S, (V_s \mid s \in V(S))\big)$。反设存在 $x, y \in V(S)$，使得$xy \in E(S)$ 且 $G\left[\bigcup_{s \in V^{(S_{x|y})}} V_s\right]$ 不连通，并令 $\mathcal{C}$为其连通分支族。
为导出矛盾，我们的策略是找到 $(\mathcal{V}', \mathcal{Q}')$ 对 $(\mathcal{V}, \mathcal{Q})$ 的一个细化，使得$n(\mathcal{V}') < n(\mathcal{V})$ 在字典序下成立。对于每个 $C \in \mathcal{C}$，令 S_C 为 $S_{x|y}$ 的一个副本，其顶点集为 $\{(C, s) \mid s \in V(S_{x|y})\}$，边集为 $\{(C, s_1)(C, s_2) \mid s_1 s_2 \in E(S_{x|y})\}$。令 S'为这样一棵树：从 $S_{y|x}$ 与所有 $C \in \mathcal{C}$ 上的 S_C 的不交并出发，对每个 $C \in \mathcal{C}$添加边 $(C, x)y$。参见图 26。令 $f\colon V(S') \to V(S)$ 满足：对于每个 $z \in V(S')$,

$$f(z) = \begin{cases} s & \text{if } z = s \text{ for some } s \in V(S_{y|x}), \\ s & \text{if } z = (C, s) \text{ for some } s \in V(S_{x|y}) \text{ and } C \in \mathcal{C}. \end{cases}$$

此外，对于每个 $z \in V(S')$，令

$$V'_z = \begin{cases} V_s & \text{if } z = s \text{ for some } s \in V(S_{y|x}), \\ V_s \cap V(C) & \text{if } z = (C,s) \text{ for some } s \in V(S_{x|y}) \text{ and } C \in \mathcal{C}. \end{cases}$$

注意，由构造可知，对每个 $z \in V(S')$，$V'_z \subseteq V_{f(z)}$。此外，$\mathcal{V}' = \left(S', (V'_z \mid z \in V(S'))\right)$ 是 G 的一个树分解。

为完成这一构造，我们指定一组 $\mathcal{Q}'$ 子树，其顶点集划分 $V(S')$。同时，我们定义 g: $\mathcal{Q}' \to \mathcal{Q}$。首先，对于每个满足 $Q \cap V(S_{y|x}) \neq \emptyset$ 的 $Q \in \mathcal{Q}$，令

$$Q' = S'\left[(V(Q) \cap V(S_{y|x})) \cup \bigcup_{C \in \mathcal{C}} \{(C,z) \mid z \in V(Q) \cap V(S_{x|y})\}\right] \text{ and } g(Q') = Q.$$

接下来，对于每个带有 $Q \subseteq V(S_{x|y})$ 的 $Q \in \mathcal{Q}$，对每个 $C \in \mathcal{C}$，令

$$Q_C = S'\left[\{(C,z) \mid z \in Q\}\right] \text{ and } g(Q_C) = Q.$$

最后，令

$$\mathcal{Q}' = \left\{Q' \mid Q \in \mathcal{Q} \text{ with } V(Q) \cap V(S_{y|x}) \neq \emptyset\right\} \cup \bigcup_{C \in \mathcal{C}} \left\{Q_C \mid V(Q) \subseteq V(S_{x|y})\right\}.$$

由此可知，$\mathcal{Q}'$ 是 S' 的一族子树，其顶点集构成 $V(S')$的一个划分，并且对每个 $Q' \in \mathcal{Q}'$，$f(V(Q')) = Q = g(Q)$。换言之，(r2)成立。注意，(r1)也成立，因此，$(\mathcal{V}', \mathcal{Q}')$细化$(\mathcal{V}, \mathcal{Q})$。此外，由于细化关系具有传递性，$(\mathcal{V}', \mathcal{Q}')$细化$(\mathcal{U}, \mathcal{P})$。

To conclude the proof, we show that $n(\mathcal{V}')$ is less than $n(\mathcal{V})$ in the lexicographic order, which leads to a contradiction. For every $s \in V(S_{x|y})$, we have $\sum_{C \in \mathcal{C}} |V'_{(C,s)}| = |V_s|$. Thus, either $|V'_{(C,s)}| < |V_s|$ for every $C \in \mathcal{C}$ or there is exactly one $C \in \mathcal{C}$ with $|V'_{(C,s)}| = |V_s|$ and $|V'_{(D,s)}| = 0$ for all $D \in \mathcal{C} - \{C\}$. Moreover, for every $s \in V(S_{y|x})$, we have $V_s = V'_s$. Since G is connected, V_x intersects every component in $\mathcal{C}$. Also, note that we supposed $|\mathcal{C}| \geqslant 2$. Consider the maximum integer i_0 such that there exists $s \in V(S_{x|y})$ with $|V_s| = i_0$ and V_s intersects at least two components in $\mathcal{C}$. From the construction, we obtain that $n_i(\mathcal{V}') = n_i(\mathcal{V})$ for every integer i with $i > i_0$ and $n_{i_0}(\mathcal{V}') < n_{i_0}(\mathcal{V})$. We conclude that $n(\mathcal{V}')$ is less than $n(\mathcal{V})$ in the lexicographic order. □

11.1. **良好着色：Helly 性质的推广**。与弱着色数的情形类似，我们需要一个适用于 3.3 中引理所述 Helly 性质的变体，用于 K_t-无小图的图。再次地，我们不再像引理 3.3。(2)）那样限制击中集的大小，而是要求它具有“良好的结构”。在这种情况下，要表达这种结构则更为复杂。

设 c 为正整数。若着色 φ 满足 G 是 (q,c)-良好着色，则对于 G的每个子图 G_0，对于 $\mathcal{F}$中由 G_0的连通子图构成的每个族 d，对于每个正整数 $d+1$，若 $\mathcal{F}$中不存在 $Z \subseteq V$两两不相交的成员，则存在 G_0 使得

(qc1) $Z \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$;
(qc2) for every component C of $G_0 - Z$, $N_{G_0}(V(C))$ intersects at most two components of $G_0 - V(C)$; and
(qc3) for every $P \subseteq Z$, there exists $R \subseteq Z$ with
  (i) $P \subseteq R$,
  (ii) $\mathrm{cen}_p(G_0, \varphi, R) \leqslant c \cdot d$, and
  (iii) $|R - P| \leqslant c \cdot d$.

注意，(qc3) 应用于 $P = Z$ 可得

(qc4) $\mathrm{cen}_q(G_0, \varphi, Z) \leqslant c \cdot d$。
此外，对于所有图 G 与 H，其中 H 是 G的子图，如果 φ 是一个（q, c）-好的，属于 G，则 $\varphi|_{V(H)}$ 是一个（q, c）-好着色，属于 H。这蕴含如下直接的观察。

观察 11.4。设 q 和 c 为正整数，并设 $G_1, \ldots, G_\ell$ 为两两顶点不相交的图。假设对每个 $i \in [\ell]$，都存在一个 G_i 的 (q, c)-良好着色 φ_i。设 G 为 $G_1, \ldots, G_\ell$ 的不交并，并设 φ 为满足对每个 $i \in [\ell]$ 都有 $\varphi|_{V(G_i)} = \varphi_i$ 的着色。那么，φ 是 (q, c)-良好的。

在小节 5.2中，我们基于预染色图，考虑了一个构建与中心色数相关的聚焦参数族的抽象框架。我们将把该框架应用于预染色为 (q, c)-良好染色的图。

引理 11.5. 设 c 为正整数。设 $\mathcal{U}$ 为无限图，且对每个正整数 q，设 φ_q 为 $\mathcal{U}$的一种着色，使得对 $\mathcal{U}$的每个有限子图 G，$\varphi_q|_{V(G)}$ 是 G的一个(q, c)-好着色。对每个正整数 q，对每个图 G，以及对每个 $S \subseteq V(G)$，设

$$\mathrm{par}_q(G, S) = \begin{cases} \mathrm{cen}_q(G, \varphi_q|_{V(G)}, S) & \text{if } V(G) \subseteq V(\mathcal{U}) \text{ and } E(G) \subseteq E(\mathcal{U}), \\ 0 & \text{otherwise.} \end{cases}$$

那么，$q \mapsto 1$ 是（$\mathcal{E}$）-有界的，其中$\mathrm{par} = (_q \mid q \in \mathbb{N}_1)$ 是一族聚焦参数。

Proof. Let $X = \overline{K_d} \in \mathcal{E}$. For each graph G for which it does not hold that $V(G) \subseteq V(\mathcal{U})$ and $E(G) \subseteq E(\mathcal{U})$, choosing the empty set as S witnesses that G is $(q \mapsto 1, \mathrm{par}\ X, 1, 0)$-good. Let G be a graph for which $V(G) \subseteq V(\mathcal{U})$ and $E(G) \subseteq E(\mathcal{U})$. We claim that G is $(q \mapsto 1, \mathrm{par}\ X, 2, cd)$-good. Let q be a positive integer and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of X, i.e. there are no d pairwise disjoint members of $\mathcal{F}$. By assumption, $\varphi_q|_{V(G)}$ is (q, c)-good. It follows that there exists $S \subseteq V(G)$ such that

(qc1') $S \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}$均成立；(qc2') 对于 $G - S$ 的每个连通分支 C，$N_G(V(C))$ 至多与 $G - V(C)$ 的两个连通分支相交；以及 (qc4') $\mathrm{cen}_q(G, \varphi_q|_{V(G)} S) \leqslant c \cdot d$。

条目 (qc4') 意味着 $\mathrm{par}_q(G, S) \leqslant cd$，这表明确实 G 是 $(q \mapsto 1, \mathrm{par}\ X, 2, cd)$- 良好的，因此，$q \mapsto 1$ 是 $(\mathrm{par}\ \mathcal{E})$-有界的。□

A simple example of $(q, 2)$-good coloring of a graph G using $\mathrm{tw}(G) + 1$ colors is constructed in Lemma 11.6 below. We will use it in the proof of bounds in Theorem 1.2 involving treewidth. Next, relying on layered RS-decompositions (see Theorem 3.6), we construct in Lemma 11.7 a (q, c)-good coloring using $q + 1$ colors for each K_t-minor-free. This is an essential step in the proof of the bounds in Theorem 1.2.

引理 11.6。设 q 为正整数，且 G 为一个图。存在一种 q（q，2）-良好着色，为 G 使用 $\mathrm{tw}(G) + 1$ 种颜色。

Proof. By Lemma 3.2, we may fix a natural tree decomposition $\big(T, (W_x \mid x \in V(T))\big)$ of G of width $\mathrm{tw}(G)$. Let $\varphi\colon V(G) \to [\mathrm{tw}(G) + 1]$ be a proper coloring of the graph $\big(V(G) \bigcup_{x \in V(T)} \binom{W_x}{2}\big)$, which is possible since this graph has treewidth $\mathrm{tw}(G)$ and chromatic number at most treewidth plus 1. Observe that for every $x \in V(T)$, $\varphi|_{W_x}$ is injective.

We now show that φ is a $(q,2)$-good coloring of G. Let G_0 be a subgraph of G, let d be a positive integer, and let $\mathcal{F}$ be a family of connected subgraphs of G_0 with no $d+1$ pairwise disjoint members. By Lemma 3.3 and Lemma 3.5, there is a set $X \subseteq V(T)$ of size at most $2d-1$ such that for $Z = \bigcup_{x\in X} V(T)$,

(i) $Z \cap V(F) \neq \emptyset$ 对于每个 $F \in \mathcal{F}$，因此 (qc1) 成立；且 (ii) 对于 C 的每个连通分量 $G_0 - Z$，$N_{G_0}(V(C))$ 与 $G_0 - V(C)$ 的至多两个连通分量相交，因此 (qc2) 成立。

现在，令 $\psi\colon Z \to X$，使得对于每个 $z \in Z$ 都有 $z \in W_{\psi(x)}$。注意，$\varphi|_Z \times \psi$ 是单射。因此，对于每个与 Z 相交的 G 的连通子图 H，$V(H) \cap Z$ 中的任意顶点都是 $V(H) \cap Z$ 的一个 $(\varphi|_Z \times \psi)$-中心。

最后，对于每个 $P \subseteq Z$，$\mathrm{cen}_q(G_0, \varphi, P) \leqslant |\psi(P)| \leqslant |X| \leqslant 2d$。这表明 (qc3) 并证明 φ 是 $(q, 2)$-优良着色 G。□

本小节剩余部分将用于证明以下结论。

引理 11.7。对于每个正整数 t，都存在一个正整数 $c_{11.7}(t)$，使得对于每个无 K_t *minor* 的图 G 以及每个正整数 q，G 都存在一种 $(q, c_{11.7}(t))$-良好染色，且使用 $q+1$ 种颜色。

The proof of Lemma 11.7 is rather technical, however, the general idea of the proof is relatively simple to describe. For purposes of this short discussion, consider a simpler variant of good colorings where we drop (qc2), which is required in the real definition for very technical reasons. Observe that in the definition of good colorings, a subgraph G_0 occurs only in (qc2), hence, if we ignore this item, we can also assume that $G_0 = G$. We say that a coloring is a *simple* (q,c)-good coloring if it admits the relaxed definition above.

For graphs of bounded treewidth, there is a straightforward way of constructing (simple) good colorings, as shown in Lemma 11.6. A true inspiration for the definition of good colorings comes from the case of graphs of bounded layered treewidth. Let G be a graph, let q be a positive integer, and let $\mathrm{ltw}(G)$ be witnessed by $(\mathcal{W}, \mathcal{L})$, where $\mathcal{W} = \big(T, (W_x \mid x \in V(T))\big)$ and $\mathcal{L} = (L_i \mid i \in \mathbb{N})$. For all $i \in \mathbb{N}$ and $v \in L_i$, we define $\varphi(v) = i \bmod (q+1)$. We claim that φ is a simple $(q, \mathrm{ltw}(G))$-good coloring of G. Let $\mathcal{F}$ be a family of connected subgraphs of G with no $d+1$ pairwise disjoint members. Let Z be the union of at most d bags of $\mathcal{W}$ that is a hitting set of $\mathcal{F}$ in G (given by Lemma 3.3). In particular, $|Z \cap L_i| \leqslant \mathrm{ltw}(G) \cdot d$ for every $i \in \mathbb{N}$. Let $\psi\colon Z \to [\mathrm{ltw}(G) \cdot d]$ be any coloring of Z which is injective on $Z \cap L_i$ for every $i \in \mathbb{N}$. It remains to verify (qc4). Let H be a connected subgraph of G with $V(H) \cap Z \neq \emptyset$. Since H is connected and $\mathcal{L}$ is a layering of G, $V(H)$ intersects a set of consecutive layers in $\mathcal{L}$. If this set has at least $q+1$ elements, then $|\varphi(V(H))| > q$, and otherwise any element of $V(H) \cap Z$ is a $(\varphi \times \psi)$-center of $V(H) \cap Z$.

为了将这一思想推广到一般的 K_t-无子式图类，我们使用分层 RS 分解（参见第3节）。

Given a layered RS-decomposition $(T, \mathcal{W}, \mathcal{A}, \mathcal{D}, \mathcal{L})$ of a graph G of bounded width, the strategy to find a simple good coloring of G is the following. Let q be a positive integer. The intention is to mimic the proof for the bounded layered treewidth case. That is, ignoring the overlapping between the bags of $\mathcal{W}$, we set $\varphi(v) \equiv i \bmod (q+1)$ for every $i \in \mathbb{N}$ and every $v \in L_{x,i}$. We claim that φ is a simple (q,c)-good coloring, where c depends only on the width of the given decomposition. Let $\mathcal{F}$ be a family of connected subgraphs of G with no $d+1$ pairwise disjoint members. We glue all tree decompositions in $\mathcal{D}$ in a natural way to obtain a tree decomposition $\mathcal{U}$ of G. The vertices of A_x do not occur in bags of tree decompositions of $\mathcal{D}$, but we can add them to all bags of $\mathcal{U}$ corresponding to W_x. The important property of $\mathcal{U}$ is that (lrs5) is preserved. Next, we apply Lemma 3.3 to $\mathcal{F}$ and $\mathcal{U}$ obtaining a hitting set Z of

$\mathcal{F}$ in G, which is the union of at most d bags of $\mathcal{U}$. Note that there are also at most d bags of $\mathcal{W}$, whose union contains Z. Say that these bags correspond to the vertices in $X \subseteq V(T)$. The next step is to "disconnect" elements of Z in different bags of $\mathcal{W}$. To this end, we root T arbitrarily and we define B as the union of $\bigcup_{x\in X} A_x$ and the union of adhesions between W_x and the bag corresponding to the parent of x in $\mathcal{W}$ for all $x \in X$. In particular, the size of B is bounded. Finally, we define a coloring ψ of Z so that ψ is injective on B and ψ is injective on $L_{x,i} \cap Z$ but the colors used are disjoint from $\psi(B)$. Checking that ψ witnesses φ being a simple (q, c)-good coloring of G is very similar to the bounded layered treewidth case.

引理 *11.7*的证明。 设 t 为正整数。设 $c_{\mathrm{LRS}}(t)$ 为定理3.6中的常数。令

$$c_{11.7}(t) = c_{\mathrm{LRS}}(t)\,(12c_{\mathrm{LRS}}(t) + 10)\,.$$

设 G 为一个不含 K_t次子式的图，且 q 为正整数。根据定理 3.6， G 存在一个宽度至多为 $c_{\mathrm{LRS}}(t)$ 的分层 RS 分解 $(T, \mathcal{W}, \mathcal{A}, \mathcal{D}, \mathcal{L})$。令 $\mathcal{W} = (W_x \mid x \in V(T))$,$\mathcal{A} = (A_x \mid x \in V(T))$, $\mathcal{D} = \big((T_x, (D_{x,z} \mid z \in V(T_x))) \mid x \in V(T)\big)$, 以及 $\mathcal{L} = ((L_{x,i} \mid i \in \mathbb{N}) \mid x \in V(T))$。我们将 T以任意顶点 $r \in V(T)$ 为根。

我们将染色 $\varphi\colon V(G) \to \{0, \dots, q\}$ 定义如下。对于每个 $v \in V(G)$， 令

$$\varphi(v) = \begin{cases} 0 & \text{if } v \in A_r, \\ i \bmod (q+1) & \text{if } v \in W_r - A_r \text{ and } v \in L_{r,i}, \\ 0 & \text{if } v \in A_x - W_{\mathrm{p}(T,x)} \text{ for } x \in V(T) - \{r\}, \\ i \bmod (q+1) & \text{if } v \in W_x - (A_x \cup W_{\mathrm{p}(T,x)}) \text{ and } v \in L_{x,i} \text{ for } x \in V(T) - \{r\}. \end{cases}$$

It remains to show that φ 是q， $c_{11.7}(t)$- G的良好着色。固定 G的一个子图 G_0 。

Now, we reduce to the case where G_0 is connected. Indeed, if G_0 is not connected, consider the family $\mathcal{C}$ of all the components of G_0. Let d be a nonnegative integer and let $\mathcal{F}$ be a family of connected subgraphs of G_0 such that there are no $d+1$ disjoint members of $\mathcal{F}$. For every $C \in \mathcal{C}$, let d_C be the smallest integer such that there are no $d_C + 1$ disjoint members of $\mathcal{F}_C = \{F \in \mathcal{F} \mid F \subseteq C\}$. Clearly $d_C \leqslant d$. Assuming we can prove the result when G_0 is connected, we apply it for C and $\mathcal{F}_C$ and obtain a set $Z_C \subseteq V(C)$ and $\psi_C\colon Z_C \to [c \cdot d]$ such that (qc1)–(qc2) holds. Now take $Z = \bigcup_{C\in\mathcal{C}} Z_C$ and let ψ be defined by $\psi(u) = \psi_C(u)$ for every $C \in \mathcal{C}$ and every $u \in Z_C$. Recall that each $F \in \mathcal{F}$ is connected, so $F \subseteq C$ for some $C \in \mathcal{C}$, and therefore $Z_C \cap V(F) \neq \emptyset$. Thus, (qc1) holds. Since every connected subgraph H of G_0 is a subgraph of C for some $C \in \mathcal{C}$, (qc4) holds. Finally, (qc2) holds because for every component C' of $G - Z$, there exists $C \in \mathcal{C}$ such that $C' \subseteq C$.

因此，从现在起，我们假设 G_0 是连通的。我们首先构造一个正规对$(\mathcal{U}, \mathcal{P})$，其中 $\mathcal{U}$是 G_0的一个树分解，它由沿着 T的边对 $\mathcal{D}$中的树分解进行"粘合"得到。该构造见图27。

对于所有的 $x, y \in V(T)$， 若 $xy \in E(T)$， 则 $W_x \cap W_y$在 $\mathrm{torso}_{G,\mathcal{W}}(W_x)$ 和 $\mathrm{torso}_{G,\mathcal{W}}(W_y)$ 中都诱导出一个团，因此，存在 $z_{xy} \in V(T_x)$ 和 $z_{yx} \in V(T_y)$， 使得 $W_x \cap W_y = D_{x,z_{xy}} \cap D_{y,z_{yx}}$。令 T_0 为如下定义的树

$$V(T_0) = \{(x, z) \mid x \in V(T), z \in V(T_x)\} \text{ and}$$
$$E(T_0) = \{(x, z_1)(x, z_2) \mid x \in V(T), z_1z_2 \in E(T_x)\} \cup \{(x, z_{xy})(y, z_{yx}) \mid xy \in E(T)\}.$$

In other words, T_0 is obtained from the disjoint union of T_x over $x \in V(T)$ by adding the edges between z_{xy} and z_{yx} for adjacent vertices x and y in T. Next, let $U_{(x,z)} = (D_{x,z} \cup A_x) \cap V(G_0)$ for every $(x, z) \in V(T_0)$. We claim that $\mathcal{U} = \big(T_0, (U_{(x,z)} \mid (x, z) \in V(T_0))\big)$ is a tree decomposition of G_0. For every $v \in V(G_0)$, $T_0[\{(x, z) \in V(T_0) \mid v \in U_{(x,z)}\}]$ is isomorphic to the connected subgraph of T_0 formed by replacing every vertex x of $T(v) = T[\{x \in V(T) \mid v \in W_x\}]$ with

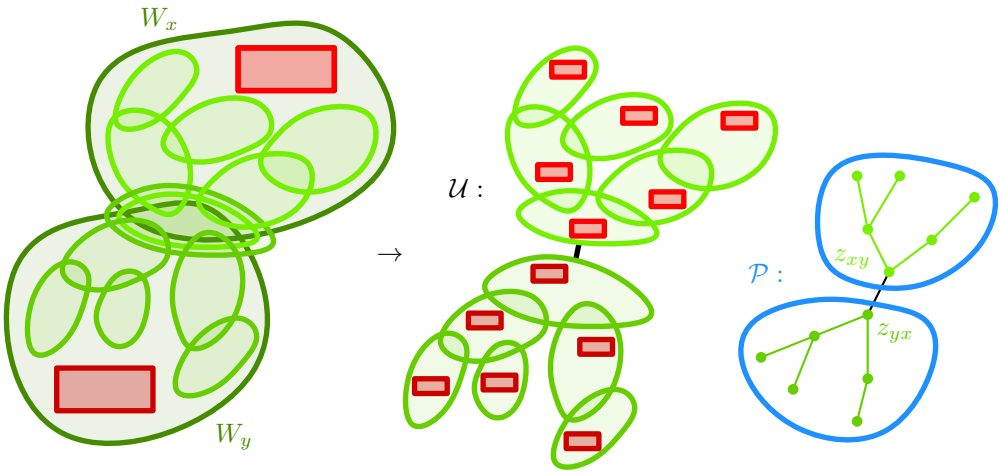


图 27. 树分解 $\mathcal{D}_x$ 和 $\mathcal{D}_y$ 以一种自然的方式“粘合”在一起。此外，我们还将所有对应的顶点（ A_x 和 A_y的成员）加入树分解 $\mathcal{U}$中对应部分的每个袋。划分 $\mathcal{P}$指明给定顶点来自 $\{T_x \mid x \in V(T)\}$ 中的哪一个。

$T_x[\{z \in V(T_x) \mid v \in D_{x,z}\}]$ when $v \notin A_x$ or with T_x when $v \in A_x$, and every edge xy of $T(v)$ with $z_{xy}z_{yx}$. For every edge $uv \in E(G_0)$, there exists $x \in V(T)$ such that $u, v \in W_x$. If $u, v \in W_x - A_x$, then there exists $z \in V(T_x)$ such that $u, v \in D_{x,z}$ and thus $u, v \in U_{(x,z)}$. If $u \in A_x$ and $v \in W_x - A_x$, then for any $z \in V(T_x)$ with $v \in D_{x,z}$ we have $v \in U_{(x,z)}$ and $u \in A_x \cap V(G_0) \subseteq U_{(x,z)}$. If $u, v \in A_x$, then for every $z \in V(T_x)$, we have $u, v \in U_{(x,z)}$. Thus, indeed $\mathcal{U}$ is a tree decomposition of G_0.

注意， $\mathcal{P} = \{T_0[\{x\} \times V(T_x)] \mid x \in V(T)\}$ 是 T_0 的一组子树，其顶点集构成了 $V(T_0)$的一个划分。因此， $(\mathcal{U},\ \mathcal{P})$ 是 G_0的一个正规对。

By Lemma 11.3, there exists $\mathcal{V} = \big(S, (V_y \mid y \in V(S))\big)$ and $\mathcal{Q}$ such that $\mathcal{V}$ is natural and $(\mathcal{V}, \mathcal{Q})$ is a normal pair of G refining $(\mathcal{U}, \mathcal{P})$. Among all such pairs $(\mathcal{V}, \mathcal{Q})$, we take one with $|\mathcal{Q}|$ minimum. Let $f\colon V(S) \to V(T_0)$ and $g\colon \mathcal{Q} \to \mathcal{P}$ witness the refinement relation. For every $Q \in \mathcal{Q}$, let $x(Q) \in V(T)$ be such that $g(Q) = T_0[\{x(Q)\} \times V(T_{x(Q)})]$. See Figure 28.

在接下来的两个断言中，我们证明新的树分解 $\mathcal{V}$ 在某种意义上保持了小粘连（仅来自 $\mathcal{W}$的那些）以及与 $\mathcal{L}$中分层的各层的小交。在第一个断言中，我们利用了$(\mathcal{V}, \mathcal{Q})$的极小性。

断言 11.8. 设 $y,y' \in V(S)$与 $Q, Q' \in \mathcal{Q}$满足 $y \in V(Q)$、 $y' \in V(Q')$以及 $Q \neq Q'$。则

$$|V_y \cap V_{y'}| \leqslant c_{\mathrm{LRS}}(t).$$

Proof of the claim. By properties of tree decompositions, there is an edge zz' in S such that $V_y \cap V_{y'} \subseteq V_z \cap V_{z'}$. Therefore, without loss of generality, we assume that yy' is an edge in S. We argue that $g(Q) \neq g(Q')$. Suppose to the contrary that $g(Q) = g(Q')$. Since yy' is an edge in S, the subgraphs Q and Q' are adjacent in S, and thus, $Q'' = S[V(Q) \cup V(Q')]$ is a connected subgraph of S. Observe that $(\mathcal{V}, \mathcal{Q} - \{Q, Q'\} \cup \{Q''\})$ is a normal pair which refines $(\mathcal{U}, \mathcal{P})$ as witnessed by f and g' defined by $g'(P) = g(P)$ for every $P \in \mathcal{Q} - \{Q, Q'\}$ and $g'(Q'') = g(Q) = g(Q')$. However, this contradicts the minimality of $|\mathcal{Q}|$. We obtain that indeed $g(Q) \neq g(Q')$, and so, $x(Q) \neq x(Q')$. In particular,

$$|V_y \cap V_z| \leqslant |U_{(x(Q),f(y))} \cap U_{(x(Q'),f(y'))}| \leqslant |W_{x(Q)} \cap W_{x(Q')}| \leqslant c_{\mathrm{LRS}}(t),$$

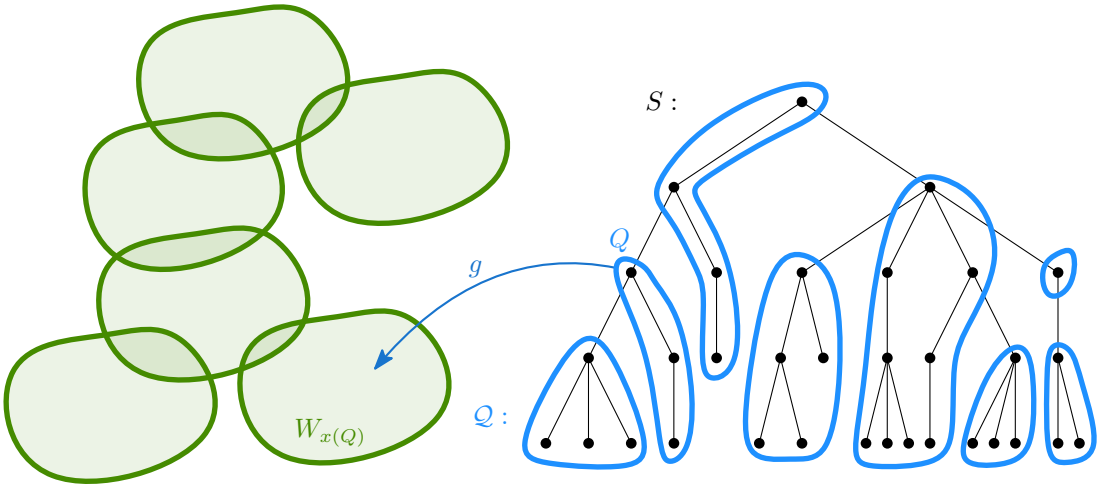


图 28. 直观地说，树分解 $\mathcal{V}$ 中对应于 Q 中顶点的部分来自 $\mathcal{D}_{x(Q)}$。

其中，第一个不等式由(r1)得出，第二个不等式由树分解的性质得出，最后一个不等式由(lrs1)得出。 ◊

断言 11.9. 设 $x \in V(T)$， $y \in V(S)$，且设 $i \in \mathbb{N}$。那么

$$|V_y \cap L_{x,i}| \leqslant c_{\mathrm{LRS}}(t).$$

该断言的证明。 由于 $(\mathcal{V}, \mathcal{Q})$ 细化了 $(\mathcal{U}, \mathcal{P})$，我们有 $V_y \subseteq U_{f(y)}$。令 $f(y) = (x', z)$，其中 $x' \in V(T)$ 且 $z \in V(T_{x'})$。因此，$U_{f(y)} = U_{(x',z)} = (D_{x',z} \cup A_{x'}) \cap V(G_0)$。回忆 $A_x \cap L_{x,i} = \emptyset$。因此，在 $x = x'$ 的情况下，

$$|V_y \cap L_{x,i}| \leqslant |U_{(x,z)} \cap L_{x,i}| \leqslant |(D_{x,z} \cup A_x) \cap L_{x,i}| \leqslant |D_{x,z} \cap L_{x,i}| \leqslant c_{\mathrm{LRS}}(t),$$

其中最后一个不等式由(lrs5)得出。最后，在 $x \neq x'$ 的情况下，我们有 $U_{(x',z)} \subseteq W_{x'}$ 和 $L_{x,i} \subseteq W_x$，因此，

$$|V_y \cap L_{x,i}| \leqslant |U_{(x',z)} \cap L_{x,i}| \leqslant |W_{x'} \cap W_x| \leqslant c_{\mathrm{LRS}}(t),$$

其中最后一个不等式由 (lrs1) 得出。 ◊

我们将 S 以任意顶点 $\mathrm{root}(S)$ 为根。对于每个 $X \subseteq V(S)$，我们定义

$$\mathcal{Q}(X) = \{Q \in \mathcal{Q} \mid V(Q) \cap X \neq \emptyset\}.$$

令 d 为一个正整数，并设 $\mathcal{F}$ 是 G_0 的一个连通子图族，其中不存在 $d+1$ 个两两不相交的 $\mathcal{F}$中的成员。在证明的其余部分，我们构造 Z，使其满足 (qc1)–(qc3)。

由引理 3.3，存在 $X_0 \subseteq V(S)$，其大小至多为 d，且 $\bigcup_{y \in X_0} V_y$ 与 $\mathcal{F}$中的每个成员都相交。令

$$X_1 = \mathrm{LCA}(S, X_0).$$

由引理3.4，

$$|\mathcal{Q}(X_1)| \leqslant |X_1| \leqslant 2d - 1. \tag{1}$$

集合 $\bigcup_{y \in X_1} V_y$ 是 $\mathcal{F}$ 的一个击中集，因此是作为 Z 的理想候选。不过，我们仍需定义 ψ，最终目标是在这里重复我们在有界分层树宽情形中采用的着色思路。由于 $\mathcal{L}_x$ 是 $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$ 的一个分层，我们需要考虑

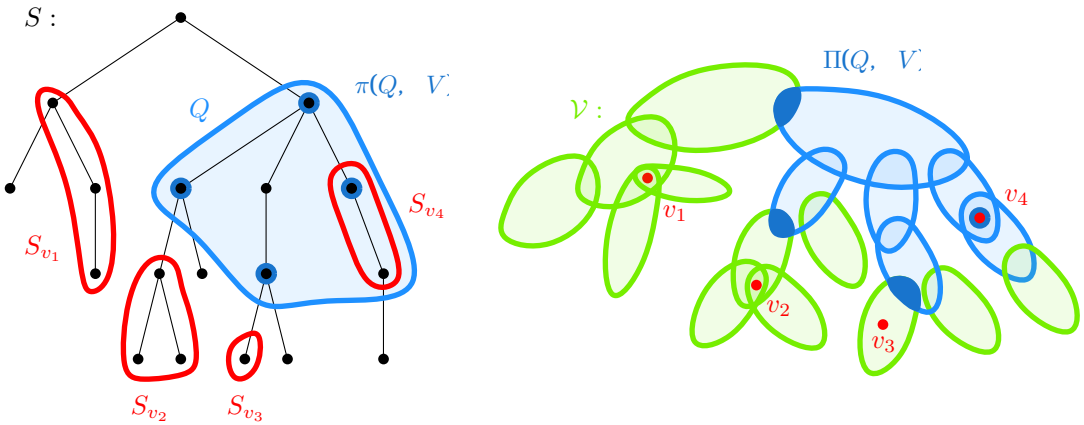


图 29。 这里， $V = \{v_1, v_2, v_3, v_4\}$。注意，$\bigcup_{y \in V(Q)} V_y$ 与 V 之间由 $\Pi(Q, V)$ 在 G_0 中分隔开，正如我们稍后在断言 11.10 中所示。

构造最终 Z 时，将 A_x 中的顶点考虑在内。此外，着色 φ 可能与 $\mathcal{W}$ 的粘合处上的分层不兼容，因此我们还需要考虑其中的一些顶点。为此，我们会向 Z 中添加一些顶点，最终使用一组单独的颜色对它们进行单射着色。由于我们使用 $\mathcal{V}$ 而非 $\mathcal{U}$，需要引入以下投影概念。另请参见图 29 中的示意图。

Let $Q \in \mathcal{Q}$ and $v \in V(G_0)$. Let $S_v = S[\{z \in V(S) \mid v \in V_z\}]$. Thus, Q and S_v are two subtrees of S. We define projections $\pi(Q, v)$ and $\Pi(Q, v)$ as follows. If Q and S_v share a vertex, then choose such a vertex z arbitrarily and set $\pi(Q, v) = z$ and $\Pi(Q, v) = \{v\}$. If $V(Q)$ and $V(S_v)$ are disjoint, then consider the shortest $V(Q)$–$V(S_v)$ path path in S. Let z be the endpoint of that path in Q and let y be the vertex adjacent to z in that path. Then set $\pi(Q, v) = z$ and $\Pi(Q, v) = V_y \cap V_z$. The definition can be naturally extended to subsets of $V(G_0)$. For each $V \subseteq V(G_0)$, let

$$\pi(Q, V) = \{\pi(Q, v) \mid v \in V\} \quad \text{and} \quad \Pi(Q, V) = \bigcup_{v \in V} \Pi(Q, v).$$

注意，根据命题 11.8，

$$|\Pi(Q, V)| \leqslant c_{\mathrm{LRS}}(t) \cdot |V|. \tag{2}$$

这些对象的关键性质如下。请再次参见图 29 。

命题 11.10。 设 $Q \in \mathcal{Q}$，且设 $V \subseteq V(G_0)$。 $G_0 - \Pi(Q, V)$ 的任意连通子图 H，只要与 $\bigcup_{y \in V(Q)} V_y$ 相交，就与 V 不相交。

证明。 只需注意到，根据树分解的性质和构造，$\Pi(Q, V)$ 与 $\bigcup_{y \in V(Q)} V_y$ 和 V 之间的每条路径都相交。因此，若 G_0 的连通子图 H 有一个顶点既是 $\bigcup_{y \in V(Q)} V_y$ 又是 V，那么它就有一个顶点属于 $\Pi(Q, V)$。□

Let

$$Y_Q = \begin{cases} \pi\left(Q, A_{x(Q)}\right) & \text{if } x(Q) = r, \\ \pi\left(Q, A_{x(Q)} \cup \left(W_{x(Q)} \cap W_{\mathrm{p}(T, x(Q))}\right)\right) & \text{if } x(Q) \neq r. \end{cases}$$

回顾一下， $|A_{x(Q)}| \leqslant c_{\mathrm{LRS}}(t)$ 由 (lrs2)可知，且若 $x(Q) \neq r$，则 $|W_{x(Q)} \cap W_{\mathrm{p}(T, x(Q))}| \leqslant c_{\mathrm{LRS}}(t)$ 由 (lrs1)可知。因此，我们有

$$\begin{aligned} |Y_Q| &\leqslant |A_{x(Q)}| && \leqslant c_{\mathrm{LRS}}(t) && \text{if } x(Q) = r, \\ |Y_Q| &\leqslant |A_{x(Q)}| + |W_{x(Q)} \cap W_{\mathrm{p}(T, x(Q))}| && \leqslant 2c_{\mathrm{LRS}}(t) && \text{if } x(Q) \neq r. \end{aligned} \tag{3}$$

此外，根据定义， $Y_Q \subseteq V(Q)$。

Let

$$X_2 = X_1 \cup \{\mathrm{root}(Q) \mid Q \in \mathcal{Q}(X_1)\} \cup \bigcup \{Y_Q \mid Q \in \mathcal{Q}(X_1)\}$$

and

$$X_3 = \mathrm{LCA}(S, X_2).$$

注意到

$$\begin{aligned} |X_3| &\leqslant 2|X_2| - 1 && \text{by Lemm}^{\text{一个}}_{3.4} \\ &\leqslant 2(|X_1| + |\mathcal{Q}(X_1)| + |\mathcal{Q}(X_1)| \cdot 2c_{\mathrm{LRS}}(t)) - 1 && \text{by (3)} \\ &\leqslant 2((2d-1) + (2d-1) + (2d-1) \cdot 2c_{\mathrm{LRS}}(t)) - 1 && \text{by (1)} \\ &\leqslant (8c_{\mathrm{LRS}}(t) + 8)d. \end{aligned} \tag{4}$$

回顾引理 11.1，对于所有满足 $X \subseteq Y$ 且 $\mathrm{LCA}(S, X) = X$ 的 $X, Y \subseteq V(S)$，若 $\mathcal{Q}(X) = \mathcal{Q}(Y)$，则 $\mathcal{Q}(Y) = \mathcal{Q}(\mathrm{LCA}(S, Y))$。由于根据构造， $\mathcal{Q}(X_1) = \mathcal{Q}(X_2)$，我们可以将 $X = X_1$ 和 $Y = X_2$ 代入上述结论，得到

$$\mathcal{Q}(X_1) = \mathcal{Q}(X_2) = \mathcal{Q}(X_3).$$

最后，作为该断言见证的集合 Z 为

$$Z = \bigcup_{y \in X_3} V_y.$$

接下来，考虑 $Q \in \mathcal{Q}(X_1)$，并令

$$B_Q = \begin{cases} \Pi\left(Q, A_{x(Q)}\right) & \text{if } x(Q) = r, \\ \Pi\left(Q, A_{x(Q)} \cup \left(W_{x(Q)} \cap W_{\mathrm{p}(T, x(Q))}\right)\right) & \text{if } x(Q) \neq r. \end{cases}$$

由于对每个 $v \in V(G_0)$， $\Pi(Q, v) \subseteq V_{\pi(Q,v)}$ 和 $Y_Q \subseteq X_2 \subseteq X_3$，我们有

$$B_Q \subseteq \bigcup_{y \in Y_Q} V_y \subseteq \bigcup_{y \in X_2} V_y \subseteq Z.$$

回顾一下， $|A_{x(Q)}| \leqslant c_{\mathrm{LRS}}(t)$ 根据 (lrs2)，并且如果 $x(Q) \neq r$，那么 $|W_{x(Q)} \cap W_{\mathrm{p}(T,x(Q))}| \leqslant c_{\mathrm{LRS}}(t)$根据 (lrs1)。因此，根据(2)，我们有

$$\begin{aligned} |B_Q| &\leqslant |A_{x(Q)}| \cdot c_{\mathrm{LRS}}(t) && \leqslant (c_{\mathrm{LRS}}(t))^2 && \text{if } x(Q) = r, \\ |B_Q| &\leqslant \left|A_{x(Q)} \cup (W_{x(Q)} \cap W_{\mathrm{p}(T,x(Q))})\right| \cdot c_{\mathrm{LRS}}(t) && \leqslant 2(c_{\mathrm{LRS}}(t))^2 && \text{if } x(Q) \neq r. \end{aligned} \tag{5}$$

Let

$$B = \left(\bigcup_{Q \in \mathcal{Q}(X_1), \mathrm{root}(S) \notin V(Q)} V_{\mathrm{root}(Q)} \cap V_{\mathrm{p}(S, \mathrm{root}(Q))} \right) \cup \left(\bigcup_{Q \in \mathcal{Q}(X_1)} B_Q \right).$$

回顾一下，对于每个 $Q \in \mathcal{Q}(X_1)$，其根为 $(S) \notin V(Q)$，根为 $(Q) \in X_2$，因此 $V_{\mathrm{root}(Q)} \cap V_{\mathrm{p}(S,\mathrm{root}(Q))} \subseteq V_{\mathrm{root}(Q)} \subseteq Z$。此外，对于每个 $Q \in \mathcal{Q}(X_1)$， $B_Q \subseteq Z$。因此，

$$B \subseteq Z.$$

另请注意，

$$\begin{aligned} |B| &\leqslant |\mathcal{Q}(X_1)| \cdot c_{\mathrm{LRS}}(t) + |\mathcal{Q}(X_1)| \cdot 2(c_{\mathrm{LRS}}(t))^2 && \text{by Claim 11.8 and (5)} \\ &\leqslant (2d-1) \cdot c_{\mathrm{LRS}}(t) + (2d-1) \cdot 2(c_{\mathrm{LRS}}(t))^2 && \text{by (1)} \\ &\leqslant c_{\mathrm{LRS}}(t)(4c_{\mathrm{LRS}}(t) + 2)d. \end{aligned} \tag{6}$$

断言 11.11：对于所有互不相同的 Q_1、$Q_2 \in \mathcal{Q}(X_1)$，在 $G_0 - B$ 中，$\bigcup_{y\in V(Q_1)} V_y - B$ 与 $\bigcup_{y\in V(Q_2)} V_y - B$ 之间不存在路径。特别地，$\bigcup_{y\in V(Q_1)} V_y - B$ 与 $\bigcup_{y\in V(Q_2)} V_y - B$ 不相交。

Proof of the claim. Let Q_1, Q_2 be distinct members of $\mathcal{Q}(X_1)$. Recall that Q_1, Q_2 are disjoint subtrees of S. Note that there exists $i \in \{1,2\}$ such that $\mathrm{root}(Q_i) \neq \mathrm{root}(S)$ and every path in S between Q_1 and Q_2 goes through the edge $\mathrm{root}(Q_i)\,\mathrm{p}(S, \mathrm{root}(Q_i))$ of S. Hence by properties of tree decompositions, $V_{\mathrm{root}(Q_i)} \cap V_{\mathrm{p}(S,\mathrm{root}(Q_i))}$ intersects every path between $\bigcup_{y\in V(Q_1)} V_y$ and $\bigcup_{y\in V(Q_2)} V_y$ in G_0. Since $V_{\mathrm{root}(Q_i)} \cap V_{\mathrm{p}(S,\mathrm{root}(Q_i))} \subseteq B$, this proves the claim. ◊

As a consequence of Claim 11.11, $\{(\bigcup_{y\in V(Q)} V_y) \cap (Z - B) \mid Q \in \mathcal{Q}(X_1)\}$ is a family of pairwise disjoint sets covering $Z - B$ (because $\mathcal{Q}(X_1) = \mathcal{Q}(X_3)$). We want to refine this family with the layerings of the torsos. To this end, we need the following fact where we substantially use the fact that $(\mathcal{V}, \mathcal{Q})$ refines $(\mathcal{U}, \mathcal{P})$.

断言 11.12. 对于每个 $Q \in \mathcal{Q}(X_1)$ 和 $V \subseteq V(G_0)$，若 $\Pi(Q,V) \subseteq B_Q$，则我们有 ave

$$\bigcup_{y\in V(Q)} V_y - B \subseteq W_{x(Q)} - V.$$

断言的证明。设 $Q \in \mathcal{Q}(X_1)$，并设 $V \subseteq V(G_0)$ 满足 $\Pi(Q,V) \subseteq B_Q$。由于$\Pi(Q,V) \subseteq B_Q \subseteq B$，因此

$$\bigcup_{y\in V(Q)} V_y - B \subseteq \bigcup_{y\in V(Q)} V_y - B_Q \subseteq \bigcup_{y\in V(Q)} V_y - \Pi(Q,V).$$

由断言 11.10，$\bigcup_{y\in V(Q)} V_y - \Pi(Q,V)$ 与 V 不相交。因此，为了证明该断言，只需证明 $\bigcup_{y\in V(Q)} V_y \subseteq W_{x(Q)}$。回顾一下，$g(Q) = T_0[\{x(Q)\} \times T_{x(Q)}]$。我们有

$$\bigcup_{y\in V(Q)} V_y \subseteq \bigcup_{y\in V(Q)} U_{f(y)} \subseteq \bigcup_{z\in V(g(Q))} U_z = \bigcup_{z\in\{x(Q)\}\times T_{x(Q)}} U_z \subseteq W_{x(Q)}$$

其中，第一个包含关系由(r1)得出，第二个由 (r2)得出，最后一个则由 $\mathcal{U}$ 的构造得出。◊

由于 $\Pi(Q,\ A_{x(Q)}) \subseteq B_Q$ 对于每个 $Q \in \mathcal{Q}(X_1)$，根据断言 11.12，

$$(\bigcup_{y\in V(Q)} V_y - B) \cap A_{x(Q)} = \emptyset.$$

回忆一下，对于每个 $Q \in \mathcal{Q}(X_1)$，$(L_{x(Q),\ i} \mid i \in \mathbb{N})$ 都是 $\mathrm{torso}_{G,\ \mathcal{W}}(W_{x(Q)}) - A_{x(Q)}$ 的一个分层。因此，这个族

$$\mathcal{B} = \left\{(\bigcup_{y\in V(Q)} V_y) \cap L_{x(Q),i} \cap (Z - B) \mid Q \in \mathcal{Q}(X_1), i \in \mathbb{N}\right\}$$

是由两两不相交的集合组成、覆盖 $Z - B$ 的一个集合族。此外，$\mathcal{B}$ 的成员大小适中，即对于每个 $Q \in \mathcal{Q}(X_1)$ 和 $i \in \mathbb{N}$，

$$\begin{aligned}
\left|(\bigcup_{y\in V(Q)} V_y) \cap L_{x(Q),i} \cap (Z - B)\right| &\leqslant |L_{x(Q),i} \cap Z| \leqslant \sum_{y\in X_3} |L_{x(Q),i} \cap V_y| \\
&\leqslant \sum_{y\in X_3} |L_{x(Q),i} \cap U_{f(y)}| && \text{by (r1)} \\
&\leqslant c_{\mathrm{LRS}}(t) \cdot |X_3| && \text{by (lrs5)} \\
&\leqslant c_{\mathrm{LRS}}(t) \cdot \big(8c_{\mathrm{LRS}}(t) + 8\big)d && \text{by (4)}.
\end{aligned} \tag{7}$$

我们定义一种 ψ 的 Z 着色，至多使用 $c_{11.7}(t)$ 种颜色。另请参见图30。首先，我们对B 进行单射着色，然后对 $\mathcal{B}$ 中的每个成员也进行单射着色，同时避开 $\psi(B)$中的颜色。在

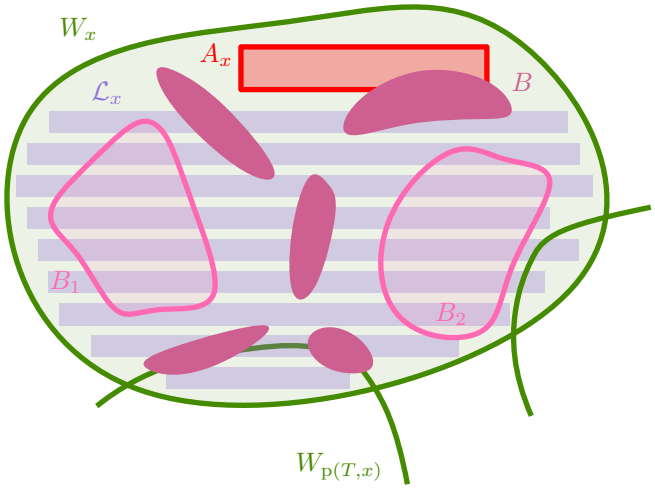


FIGURE 30. The set Z is the union of B and the members of $\mathcal{B}$. We color the vertices in B injectively. The role of the set B is to separate members of $\mathcal{B}$ so that if B is intersected by a connected subgraph H of G_0, we immediately get a center. In the figure, $B_1 = \bigcup_{y \in V(Q_1)} V_y - B$ and $B_2 = \bigcup_{y \in V(Q_2)} V_y - B$ for some $Q_1, Q_2 \in \mathcal{Q}(X_3)$. If a connected subgraph H of G_0, intersects both B_1 and B_2, it has to intersect B.

第一步中，我们至多使用 $c_{\mathrm{LRS}}(t)(4c_{\mathrm{LRS}}(t)+2)d$ 种颜色，根据(6) 进行着色；第二步中，我们至多使用 $c_{\mathrm{LRS}}(t) \cdot \big(8c_{\mathrm{LRS}}(t)+8\big)d$ 种颜色，根据(7)进行着色，因此 ψ 定义良好。
现在我们证明 Z 满足(qc1)–(qc3)。
回顾一下，选择 X_0 是为了使 $X_0 \subseteq X_3$ 和 $\bigcup_{y \in X_0} V_y$ 与 $\mathcal{F}$ 的每个成员都相交。因此，(qc1) 成立。根据引理 3.5 ，条目 (qc2) 成立，因为 $Z = \bigcup_{y \in \mathrm{LCA}(S,X_2)} V_y$，且 $\mathcal{V} = \big(S, (V_y \mid y \in V(S))\big)$ 是 G_0 的自然树分解。

It remains to prove (qc3). Let $P \subseteq Z$ and let $R = P \cup B$. First, $|R - P| \leqslant |B| \leqslant c_{\mathrm{LRS}}(t)(4c_{\mathrm{LRS}}(t)+2)d$. Consider a connected subgraph H of G_0 such that $V(H) \cap P \neq \emptyset$. If $V(H) \cap B \neq \emptyset$, then since every vertex of B has a unique color, any vertex in $V(H) \cap B$ is a $\psi|_R$-center of $V(H) \cap P$, and so, a $(\varphi \times \psi)$-center of $V(H) \cap Z$. Thus, we assume that $V(H) \cap B = \emptyset$.
由于 $V(H) \cap B = \emptyset$，由断言 11.11 可知，$V(H) \cap Z \subseteq \bigcup_{z \in V(Q)} V_z$ 对于某个 $Q \in \mathcal{Q}(X_1)$。

此外，由命题 11.12，

$$V(H) \cap Z \subseteq W_x - A_x \qquad \text{if } x = r,$$
$$V(H) \cap Z \subseteq W_x - (A_x \cup W_{\mathrm{p}(T,x)}) \qquad \text{if } x \neq r.$$

回顾一下，$(L_{x,i} \mid i \in \mathbb{N})$ 是 $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$ 的一个分层。此外，根据 φ 的定义，对每个 $u \in V(H) \cap Z$, $\varphi(u) = i \bmod (q+1)$，其中 $i \in \mathbb{N}$ 满足 $u \in L_{x,\,i}$。

Consider $u \in V(H) \cap R$. Let $i \in \mathbb{N}$ be such that $u \in L_{x,i}$. If u is a $(\varphi \times \psi)$-center in $V(H) \cap R$, then we are done. Assume now that there exists $u' \in V(H) \cap R$ distinct from u such that $\varphi(u) = \varphi(u')$ and $\psi(u) = \psi(u')$. Let $j \in \mathbb{N}$ be such that $u' \in L_{x,j}$. Without loss of generality, assume that $i \leqslant j$. By the definition of ψ, $j \neq i$, and by the definition of φ, $|j - i| > q$. By Lemma 11.2, $V(H) \cap W_x$ induces a connected subgraph of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$. Since $(L_{x,k} \mid k \in \mathbb{N})$ is a layering of $\mathrm{torso}_{G,\mathcal{W}}(W_x) - A_x$, it follows that $V(H)$ intersects $L_{x,k}$ for every $k \in \{i, \ldots, j-1\}$. We deduce that $|\varphi(V(H))| > q$. This proves that $\psi|_{V(H)}$ witnesses

这一事实，即

$$\mathrm{cen}_q(G,\varphi,R) \leqslant \big(c_{\mathrm{LRS}}(t)(4c_{\mathrm{LRS}}(t)+2)+c_{\mathrm{LRS}}(t)(8c_{\mathrm{LRS}}(t)+8)\big)\cdot d$$

因此，我们得到(qc3)，证明至此结束。 □

11.2. **The bounds.** In the case of centered chromatic numbers, we handle the bounds involving treewidth and the general bounds in Theorem 1.2 almost in the same way: the only distinction is applying either Lemma 11.6 or Lemma 11.7. Note that this differs from how we proved Theorem 1.3 and Theorem 1.4. Theorem 1.2 will follow from Theorems 11.14 and 11.25. Already developed material is almost enough to prove Theorem 11.14, where we exclude a graph from $\mathcal{S}_t$. To complete the proof, we need to define an appropriate family of focused parameters related to centered colorings, fitting the framework from Subsection 5.2. In order to prove Theorem 11.25 (excluding a graph from $\mathcal{R}_t$), we also need to inspect the base cases more carefully.

设 ℓ 为正整数。设 $(G_{\ell,i} \mid i\in\mathbb{N}_1)$ 为一组两两不同构的（有限） K_ℓ-无小图的图，它们的顶点集两两不交。设$(H_{\ell,i} \mid i\in\mathbb{N}_1)$ 为一组两两不同构的（有限）树宽小于 ℓ 的图，它们的顶点集两两不交。

我们将 $\mathcal{K}_\ell$ 定义为所有图 $G_{\ell、i}$ （其中 $i\in\mathbb{N}_1$）的不交并，并将 $\mathcal{T}_\ell$ 定义为所有图 $H_{\ell、i}$ （其中 $i\in\mathbb{N}_1$）的不交并。注意，图 $\mathcal{K}_\ell$ 和 $\mathcal{T}_\ell$ 是无限的。

Let q be a positive integer. For each $i\in\mathbb{N}_1$, we fix a $(q,c_{11.7}(\ell))$-good coloring $\varphi_{\ell,q,i}$ of $G_{\ell,i}$ using $q+1$ colors that exists by Lemma 11.7, and we fix a $(q,2)$-good coloring $\psi_{\ell,q,i}$ of $H_{\ell,i}$ using ℓ colors that exists by Lemma 11.6. We may assume that the colorings $\varphi_{\ell,q,i}$ for each $i\in\mathbb{N}_1$ use the same set of $q+1$ colors, and that the colorings $\psi_{\ell,q,i}$ for each $i\in\mathbb{N}_1$ use the same set of ℓ colors Let $\Phi_{\ell,q}$ be the coloring of $\mathcal{K}_\ell$ such that for every $i\in\mathbb{N}_1$, we have $\Phi_{\ell,q}|_{V(G_{\ell,i})}=\varphi_{\ell,q,i}$. Similarly, let $\Psi_{\ell,q}$ be the coloring of $\mathcal{T}_\ell$ such that for every $i\in\mathbb{N}_1$, we have $\Psi_{\ell,q}|_{V(H_{\ell,i})}=\psi_{\ell,q,i}$.

最后，我们定义 $\mathcal{T}$为 $\mathcal{T}_\ell$ （ $\ell\in\mathbb{N}_1$） Ψ_q 的不交并，并定义 $\mathcal{T}$为$\Psi_q|_{V(\mathcal{T}_\ell)}$ $=\Psi_{q的染色,\ \ell}$ 对每个 $\ell\in\mathbb{N}_1$均成立。观察11.4 蕴含以下命题。

观察 11.13。 设 ℓ 和 q 为正整数。对于 $\mathcal{K}_\ell$ 的每个有限子图 G ，$\Phi_{\ell,\ q}|_{V(G)}$ 是 G 的一个使用 $q+1$ 种颜色的 $(q,c_{11.7}(\ell))$-优良着色。类似地，对于 $\mathcal{T}_\ell$ 的每个有限子图 H， $\Psi_{\ell,\ q}|_{V(H)}$ 是 H 的一个使用 ℓ 种颜色的 $(q,2)$-优良着色。

对于每个正整数 q、每个图 G，以及每个 $S\subseteq V(G)$，令

$$\text{cen-k}_{\ell,q}(G,S)=\begin{cases}\mathrm{cen}_q(G,\Phi_{\ell,q}|_{V(G)},S) & \text{if } V(G)\subseteq V(\mathcal{K}_\ell) \text{ and } E(G)\subseteq E(\mathcal{K}_\ell),\\ 0 & \text{otherwise;}\end{cases}$$

对每个正整数 ℓ，并令

$$\text{cen-t}_q(G,S)=\begin{cases}\mathrm{cen}_q(G,\Psi_q|_{V(G)},S) & \text{if } V(G)\subseteq V(\mathcal{T}) \text{ and } E(G)\subseteq E(\mathcal{T}),\\ 0 & \text{otherwise.}\end{cases}$$

注意，cen-t 是定义良好的，因为 $\mathcal{T}$ 是 $\ell\in\mathbb{N}_1$ 的 $\mathcal{T}_\ell$ 不交并。此外，对于每个正整数 ℓ，族 $(\text{cen-k}_{\ell,\ q}\mid q\in\mathbb{N}_1)$ 和 $(\text{cen-t}_q\mid q\in\mathbb{N}_1)$ 都是聚焦参数族。我们强调，这些参数在图同构下并非不变。

定理 11.14。 设 t 是一个满足 $t\geqslant 2$的整数。对于每个 $X\in\mathcal{S}_t$，都存在一个整数 c，使得对于每个正整数 q 以及每个不含 X次子式的图 G，

$$\mathrm{cen}_q(G)\leqslant c\cdot(\mathrm{tw}(G)+1)\cdot q^{t-2}$$

and

$$\mathrm{cen}_q(G) \leqslant c \cdot q^{t-1}.$$

Proof. Let ℓ be a positive integer, let cen-k$_\ell$ $=$ (cen-k$_{\ell,q}$ $\mid$ $q \in \mathbb{N}_1$) and cen-t $=$ (cen-t$_{\ell,q}$ $\mid$ $q \in \mathbb{N}_1$), and let par $\in$ {cen-k$_\ell$, cen-t}. By Lemma 5.12, the family of focused parameters par is nice. By Observation 11.13 and Lemma 11.5 the function $q \mapsto 1$ is (par $\mathcal{E}$)-bounding. Therefore, by Corollary 8.4, $q \mapsto q^{t-2}$ is (par $\mathcal{S}_t$)-bounding. By Lemma 4.5, this implies that for every $X \in \mathcal{S}_t$, there exists a constant $\beta_{\mathrm{par}}(X)$ such that for every positive integer q and for every graph G, we have

$$\mathrm{par}_q(G, V(G)) \leqslant \beta_{\mathrm{par}}(X) \cdot q^{t-2}. \tag{8}$$

Let $X \in \mathcal{S}_t$, let q be a positive integer, and let G be an X-minor-free graph. In particular, G is $K_{|V(X)|}$-minor-free, hence, there exists $i \in \mathbb{N}_1$ such that G is isomorphic to $G_{|V(X)|,i}$, which is one of the building blocks of $\mathcal{K}_{|V(X)|}$. Recall that $\Phi_{|V(X)|,q}|_{V(G_{|V(X)|,i})} = \varphi_{|V(X)|,i,q}$ uses at most $q+1$ colors. It follows that for $\ell = |V(X)|$, we have

$$\begin{aligned}
\mathrm{cen}_q(G) = \mathrm{cen}_q(G_{\ell,i}) &\leqslant \mathrm{cen}_q(G_{\ell,i}, \Phi_{\ell,q}|_{V(G_{\ell,i})}, V(G_{\ell,i})) \cdot (q+1) && \text{by Lemma 5.8}\\
&= \text{cen-k}_\ell(G_{\ell,i}, V(G_{\ell,i})) \cdot (q+1)\\
&\leqslant \beta_{\text{cen-k}_\ell}(X) \cdot q^{t-2} \cdot (q+1) \leqslant (2\beta_{\text{cen-k}_\ell}(X)) \cdot q^{t-1} && \text{by (8)}.
\end{aligned}$$

这给出了定理陈述中的第二个界。对于第一个界，注意存在 $i \in \mathbb{N}_1$，使得 G 与 $H_{\mathrm{tw}(G)+1,i}$ 同构，而后者是 $\mathcal{T}_{\mathrm{tw}(G)+1}$ 的一个构件。回想一下，$\Psi_q|_{V(H_{\mathrm{tw}(G)+1,i})} = \Psi_{\mathrm{tw}(G)+1,i,q}$ 至多使用 $\mathsf{tw}(G)+1$ 种颜色。因此，对于 $\ell = \mathsf{tw}(G)+1$，我们有

$$\begin{aligned}
\mathrm{cen}_q(G) = \mathrm{cen}_q(H_{\ell,i}) &\leqslant \mathrm{cen}_q(H_{\ell,i}, \Psi_{\ell,q}|_{V(H_{\ell,i})}, V(H_{\ell,i})) \cdot \ell && \text{by Lemma 5.8}\\
&= \text{cen-t}(H_{\ell,i}, V(H_{\ell,i})) \cdot \ell\\
&\leqslant \beta_{\text{cen-t}}(X) \cdot q^{t-2} \cdot \ell && \text{by (8)}.\\
&= \beta_{\text{cen-t}}(X) \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2}.
\end{aligned}$$

至此，定理的证明结束。 □

In the reminder of this section, we show the bounds of Theorem 1.2 for the case $X \in \mathcal{R}_t$. The proof is similar to the one of Theorem 11.14, except for the base case, which corresponds to excluding an apex-forest. The base case is much more involved. The core of this proof is inspired by a result of Dębski, Felsner, Micek, and Schröder [10] which asserts that outerplanar graphs have q-centered chromatic number in $\mathcal{O}(q \log q)$. See Figure 31 for a sketch of their argument. Since outerplanar graphs exclude an apex-forest, namely $K_{2,3}$, this section can be seen as a generalization of their result. As in Section 8, we also use ideas from Dujmović et al.'s paper [5]. We start by proving structural properties for graphs with no $\mathcal{F}$-rich model of a given star (Lemma 11.15). Then, we extend it to graphs with no $\mathcal{F}$-rich model of a given forest (Lemmas 11.16 and 11.21), and finally to graphs with no $\mathcal{F}$-rich model of a given apex-forest (Lemma 11.22).

引理 11.15. 设 q、c 和 d 为正整数，G 为连通图，φ 为 G 的一个(q,c)-良好着色，$\mathcal{F}$ 为 G 的一族连通子图，且 G 不含 $F_{2,d}$ 的 $\mathcal{F}$- 丰富模型；设 $U \subseteq V(G)$ 满足 $G[U]$ 连通。则存在 $S \subseteq V(G)$、G 的一个路径划分$(P_0, \ldots, P_\ell)$，以及集合 $R_1, \ldots, R_\ell \subseteq V(G)$，使得

（a） $P_0 = U$；（b） $S \cap V(F) \neq \emptyset$ 对于每个 $F \in \mathcal{F}$；

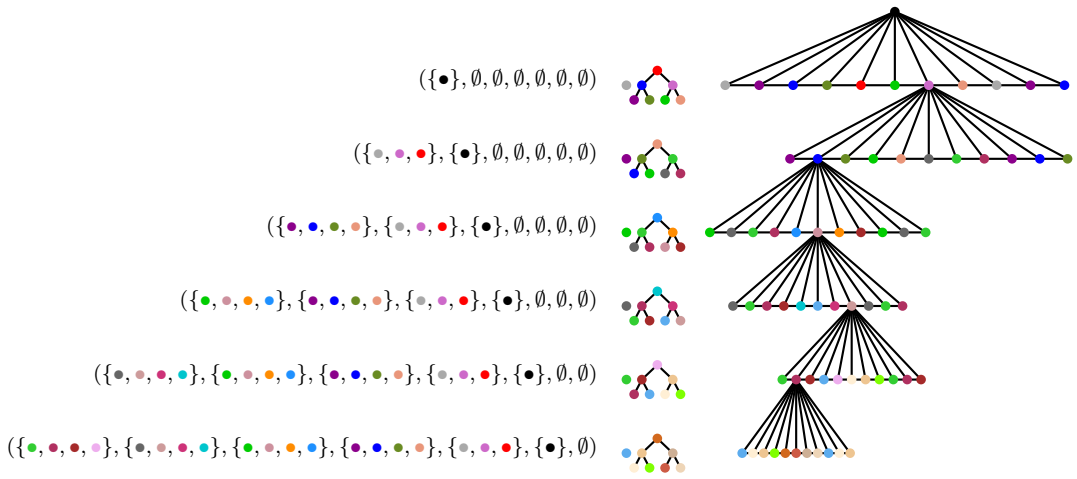


FIGURE 31. We present a sketch of the argument by Dębski et al. [10] showing that outerplanar graphs have q-centered chromatic number in $\mathcal{O}(q \log q)$. Assume $q = 2^k - 1$ for some $k \in \mathbb{N}$ ($k = 3$ and $q = 7$ in the picture). For simplicity, we consider a graph G which is a "tree of fans". Let Λ be a set of $q(k+2)+1 = \mathcal{O}(q \log q)$ colors. We will construct from top to bottom a q-centered coloring ψ of G using only colors in Λ. We want ψ to verify the following property: for every connected subgraph H of G using at most q colors, the highest path intersecting H in the tree of fans will contain a ψ-center of $V(H)$. We color the root of G with an arbitrary color in Λ. Now assume that a vertex u of G and all its ancestors are colored. We proceed with a coloring of the path P formed by the children of u in the tree. The coloring of layers above produces also some sets of forbidden colors to use on P. This is encoded as a sequence $(\Lambda_1, \ldots, \Lambda_q)$ of subsets of Λ, each of size at most $k+1$. The set Λ_a for $a \in [q]$ represents the colors forbidden because of the ath previous layer. To color P, we use $q+1$ colors in $\Lambda - \bigcup_{a \in [q]} \Lambda_a$. Among these colors, we choose one color to be special, and we arrange the $q = 2^k - 1$ other ones in a complete binary tree. Then, we color P periodically with a pattern of length $q+1$ starting with the special color, followed by the q colors in the auxiliary tree in the in-order. These auxiliary trees are depicted for some of the paths. Once P is colored, it remains to color the subgraphs under P. Let $v \in V(P)$, and let $\tilde{\Lambda}_v$ be the set containing the special color and the ancestors of the color of v in the auxiliary tree. Note that $|\tilde{\Lambda}_v| \leqslant k+1$. We apply the induction hypothesis to the subgraph under v with the sequence of sets of forbidden colors $(\tilde{\Lambda}_v, \Lambda_1, \ldots, \Lambda_{q-1})$. It remains to show that the obtained coloring ψ is q-centered. Let H be a connected subgraph of G using at most q colors, and let P be the highest path P in this tree structure intersecting $V(H)$. Then $V(P) \cap V(H)$ induces a subpath of P. If $V(H) \cap V(P)$ contains a vertex with the special color, then this vertex is a ψ-center of $V(H)$, since $V(H)$ uses at most q colors, and because the color of v is forbidden in every subtree below P in the next q layers. Otherwise, let $v \in V(H) \cap V(P)$ be such that the color $\psi(v)$ is the (unique) highest in the auxiliary tree. By construction, the color of v is forbidden in the next q layers of each subtree below P intersecting $V(H)$, and so v is a ψ-center of $V(H)$.

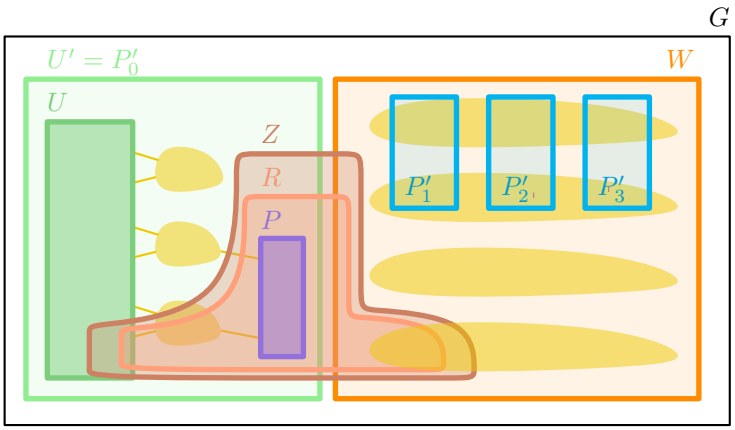


图 32。 引理证明中定义和使用的对象 11.15。

(c) 对于 $G - S$ 的每个分量 C， $N_G(V(C))$ 与 $G - V(C)$的至多三个分量相交；(d) $P_i \subseteq R_i$ 对于每个 $i \in [\ell]$ 都成立；(e) 中心$_q(G, \varphi, R_i) \leqslant c \cdot d$ 对于每个 $i \in [\ell]$ 都成立；以及 (f) $|R_i - P_i| \leqslant c \cdot d$ 对于每个 $i \in [\ell]$ 都成立。

Proof. We proceed by induction on $|V(G) - U|$. If every member of $\mathcal{F}$ intersects U, then the result holds for $S = P_0 = U$ and $\ell = 0$. Note that item (c) holds as $G[U]$ is connected. Now suppose that $\mathcal{F}|_{G-U} \neq \emptyset$, and so, $V(G) - U \neq \emptyset$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of $G - U$ such that $N_G(V(H)) \cap U \neq \emptyset$ and there exists $F \in \mathcal{F}$ such that $F \subseteq H$. Since G is connected, $\mathcal{F}'$ is nonempty. See an illustration of the proof in Figure 32.

There are no $d + 1$ pairwise disjoint members of $\mathcal{F}'$ as otherwise, if $H_1, \ldots, H_{d+1}$ are $d + 1$ pairwise disjoint members of $\mathcal{F}'$, then $(V(H_1) \ \ldots, V(H_d) \ U \cup V(H_{d+1}))$ is an $\mathcal{F}$-rich model of $F_{2,d}$ in G, which is a contradiction. Hence, since φ is a (q, c)-good coloring of G, there exists $Z \subseteq V(G)$ such that

(qc1') 对每个 $F \in \mathcal{F}'$， $Z \cap V(F) \neq \emptyset$；(qc2') 对于 $G - Z$ 的每个连通分量 C， $N_G(V(C))$ 至多与 $G - V(C)$ 的两个连通分量相交；并且

（qc3'）对于每个 $P \subseteq Z$，都存在 $R \subseteq Z$，使得 (i) $P \subseteq R$，(ii) cen$_q(G, \varphi, R) \leqslant c \cdot d$，以及 (iii) $|R - P| \leqslant c \cdot d$。

Let P 是满足以下性质的包含意义下极小的 $Z - U$ ：

(p1) 对每个 $F \in \mathcal{F}'$， $P \cap V(F) \neq \emptyset$；(p2) 对于 $G - U - P$的每个分支 C，若 $\mathcal{F}|_C = \emptyset$，则 $N_G(V(C))$与 $G - V(C)$至多三个分支相交。

To justify that P is well-defined, we show that $Z - U$ satisfies (p1) and (p2). Item (p1) for $P = Z - U$ follows directly from (qc1'). Let C be a component of $G - U - Z$. There is a component C' of $G - Z$ such that $C \subseteq C'$. By (qc2'), $N_G(V(C'))$ intersects at most two components of $G - V(C')$. These components are connected subgraphs of $G - V(C)$. Moreover, $N_G(V(C)) \subseteq N_G(V(C')) \cup U$ and $G[U]$ is connected by the assumption of the lemma. Altogether, $N_G(V(C))$ is contained in at most three connected subgraphs of $G - V(C)$, which are contained in at most three components of $G - V(C)$. This shows (p2) for $P = Z - U$ as claimed.

令 W为所有满足 C的连通分支的顶点集之并，其中 $G-U-P$满足 $\mathcal{F}|_C \neq \emptyset$。根据 (p1)，

$$N_G(W) \subseteq P. \tag{9}$$

We claim that for every $u \in N_G(W)$, there is a $\{u\}$–U path in G disjoint from W. By the minimality of P, either (p1) or (p2) is violated for $P - \{u\}$. First, we show that $P - \{u\}$ satisfies (p2). Let C be a component of $G - U - (P - \{u\})$. If $u \notin V(C)$, then C is a component of $G-U-P$, and so, by (p2) for P, if $\mathcal{F}|_C = \emptyset$, then $N_G(V(C))$ intersects at most three components of $G - V(C)$. Otherwise, we have $u \in V(C)$. Since $u \in N_G(W)$, it follows that there is a component C' of $G-U-P$, more precisely of $G[W]$, with $\mathcal{F}|_{C'} \neq \emptyset$ and $C' \subseteq C$. In particular, $\mathcal{F}|_C \neq \emptyset$. This yields (p2) for $P - \{u\}$. We conclude that (p1) is violated for $P - \{u\}$. That is, there exists $F \in \mathcal{F}'$ disjoint from $P - \{u\}$. Then, $u \in V(F)$. Also, since $F \in \mathcal{F}'$, $N_G(V(F)) \cap U \neq \emptyset$. Therefore, since F is connected, there is a $\{u\}$–$N_G(U)$ path Q in F. By (9) and the fact that $V(F) \cap P = \{u\}$, the path Q is disjoint from W. Finally, we can extend this path by one vertex in U to be a $\{u\}$–U path in G. This is the desired path.

Let $U' = V(G) - W$. We claim that $G[U']$ is connected. Consider a component C of $G[U']$. Since G is connected, C contains a vertex u of $N_G(W)$. By the previous claim, there exists a $\{u\}$–U path disjoint from W, i.e. contained in $G[U']$. Thus, every component of $G[U']$ contains a vertex of U. Since $G[U]$ is connected, we conclude that $G[U']$ is connected.

Since P hits all the members of $\mathcal{F}'$ (by (p1)) and since $\mathcal{F}'$ is nonempty, P is nonempty. Therefore, $|V(G)-U'| < |V(G)-U|$ because $P \subseteq U'-U$. Hence, by the induction hypothesis, there exists $S' \subseteq V(G)$, a path partition $(P'_0, \ldots, P'_{\ell'})$ of (G, S'), and sets $R'_1, \ldots, R'_{\ell'} \subseteq V(G)$ such that

(a') $P'_0 = U'$;
(b') $S' \cap V(F) \neq \emptyset$ for every $F \in \mathcal{F}$;
(c') for every component C of $G - S'$, $N_G(V(C))$ intersects at most three components of $G - V(C)$;
(d') $P'_i \subseteq R'_i$ for every $i \in [\ell']$;
(e') $\mathrm{cen}_q(G, \varphi, R'_i) \leqslant c \cdot d$ for every $i \in [\ell']$; and
(f') $|R'_i - P'_i| \leqslant c \cdot d$ for every $i \in [\ell']$.

令 $S = U \cup P \cup \bigcup_{i\in[\ell']} P'_i$。称 (qc3')，我们固定 $R \subseteq Z$，满足 $P \subseteq R$，$\mathrm{cen}_q(G, \varphi, R) \leqslant c \cdot d$，且 $|R - P| \leqslant c \cdot d$。我们令 $\ell = \ell' + 1$，并且

$$\begin{aligned} P_0 &= U, \\ P_1 &= P, \qquad R_1 = R, \\ P_i &= P'_{i-1}, \quad R_i = R'_{i-1} \quad \text{for every } i \in \{2, \ldots, \ell\}. \end{aligned}$$

为完成证明，我们证明 (a)–(f) 成立。根据构造，(a) 成立。

Let $F \in \mathcal{F}$. By (b'), $S' \cap V(F) \neq \emptyset$, and so, there exists $i \in \{0, \ldots, \ell'\}$ such that $P'_i \cap V(F)$. If $i \neq 0$, then $P'_i \subseteq S$, and thus, $S \cap V(F) \neq \emptyset$. If $i = 0$, then $U' \cap V(F) \neq \emptyset$. Recall that $U' = V(G) - W = U \cup P \cup \bigcup\{V(C) \mid C \in \mathcal{C}\}$ where $\mathcal{C}$ is the set of the components of $G-U-P$ such that $\mathcal{F}|_C = \emptyset$. Therefore, $(U \cup P) \cap V(F) \neq \emptyset$, and so, $S \cap V(F) \neq \emptyset$. This concludes (b).

设 C 是 $G - S$ 的一个连通分量。如果 $V(C) \cap U' = \emptyset$，那么 C 是 $G - S'$ 的一个连通分量，因此根据 (c')，$N_G(V(C))$ 至多与 $G - V(C)$ 的三个连通分量相交。否则，C 是 $G - U - P$ 的一个连通分量，且 $\mathcal{F}|_C = \emptyset$。因此，根据 (p2)，$N_G(V(C))$ 至多与 $G - V(C)$ 的三个连通分量相交。这就证明了 (c)。根据 (d') 和 (qc3').(i)、(e') 和 (qc3').(ii)，以及 (f') 和 (qc3').(iii)，分别可得 (d)、(e) 和 (f)。引理证毕。□

引理 11.16. 设 q、c 和 d 为满足 $q \geqslant 2$ 的正整数，并设 h 为非负整数，设 G 为一个图，设 φ 为 G 的一个 (q, c)-良好着色，并设 $\mathcal{F}$ 为一族连通

G 的子图，使得 G 不存在 $F_{h+1,d}$ 的 $\mathcal{F}$-丰富模型。存在一个集合 Λ，其至多含有 $12c($ $hd+\binom{h}{2})\cdot q\log q$ 种颜色，一个集合 $S\subseteq V(G)$，一个着色 $\lambda\colon S\to\Lambda$，以及一族 $(\Lambda_C \mid C$ $G-S$ 的分支）Λ 的子集，每个子集的大小至多为 $20c(hd+\binom{h}{2})\cdot\log q$，使得
(a) $S\cap V(F)\neq\emptyset$ 对每个 $F\in\mathcal{F}$ 均成立；(b) 对于 $G-S$ 的每个连通分量 C，$N_G(V(C))$ 与 $8c(hd+\binom{h}{2})\cdot\log q$ 个 $G-V(C)$的连通分量相交；并且

(c) 对于与 S 相交的 G 的每个连通子图 H，
(i) $|\varphi(V(H))|>q$，或(ii) $|\lambda(V(H)\cap S)|>q$，或(iii) 存在一个$(\varphi\times\lambda)$-中心 u，属于 $V(H)\cap S$，使得对于 $G-S$ 的每个与 $V(H)$ 相交的连通分量 C，均有 $\lambda(u)\in\Lambda_C$。

证明。我们对 $h+|V(G)|$进行归纳。当 $h=0$, $F_{h+1\text{时，}\ d}=K_1$ 因此 $\mathcal{F}$ 为空。因此，令 $S=\emptyset$ 和 $\Lambda=\emptyset$ 即得断言。接下来设 $h>0$。

First assume that G is not connected. Let $\mathcal{C}$ be the family of all the components of G. By the induction hypothesis, and by possibly relabeling the obtained color sets, there exists a set Λ of at most $20c(hd+\binom{h}{2})\cdot q\log q$ colors, such that for every $C\in\mathcal{C}$, there exists a set $S_C\subseteq V(C)$, a coloring $\lambda_C\colon S_C\to\Lambda$, and a family $(\Lambda_{C'} \mid C'$ component of $C-S_C)$ of subsets of Λ, each of size at most $12c(hd+\binom{h}{2})\cdot\log q$, such that
(a') 对每个 $F\in\mathcal{F}|_C$，$S_C\cap V(F)\neq\emptyset$；(b') 对于 $C-S_C$ 的每个连通分量 C'，$N_C(V($ $C'))$ 至多与 $C-V(C')$ 的 $\cdot\log q$ 个连通分量相交；并且

(c') 对于 C 的每一个与 S_C 相交的连通子图 H，
(i) $|\varphi(V(H))|>q$，或 (ii) $|\lambda_C(V(H)\cap S_C)|>q$，或 (iii) 存在一个（$\varphi\times\lambda_C$）中心 u，属于 $V(H)\cap S_C$，使得对于与 $V(H)$ 相交的每个连通分量 C'（属于$C-S_C$），都有 $\lambda_C(u)\in\Lambda_{C'}$。

Let
$$S=\bigcup_{C\in\mathcal{C}}S_C,$$
并设 $\lambda\colon S\to\Lambda$，使得对于每个 $C\in\mathcal{C}$ 都有 $\lambda|_{S_C}=\lambda_C$。那么，S、λ 和（$G-S$ 的 $\Lambda_{C'}$ $|C'$ 分量）满足引理的断言。现在假设 G 是连通的。
令 $\mathcal{F}'$ b e 所有包含 $F_{h,d+1}$ 的 $\mathcal{F}$-富集模型的 G 连通子图构成的族 .

We claim that there is no $\mathcal{F}'$-rich model of $F_{2,d}$ in G. Suppose for a contradiction that there is an $\mathcal{F}'$-rich model $\big(B_x \mid x\in V(F_{2,d})\big)$ of $F_{2,d}$ in G. We construct an $\mathcal{F}$-rich model of $F_{h+1,d}$ in G. Let z the center of the star $F_{2,d}$ and let $x_1,\dots,x_d$ be the leaves of this star. Let $i\in[d]$ and let $u_i\in B_{x_i}$ be a neighbor of some vertex of B_z in G. Since $\big(B_x \mid x\in V(F_{2,d})\big)$ is an $\mathcal{F}'$-rich model of $F_{2,d}$, the graph $G[B_{x_i}]$ has an $\mathcal{F}$-rich model $\big(B_{i,x} \mid x\in V(F_{h,d+1})\big)$ of $F_{h,d+1}$. Since $G[B_{x_i}]$ is connected, we can assume that $u_i\in B_{i,y_i}$ for some $y_i\in V(F_{h,d+1})$. By Lemma 8.6, there is an $\mathcal{F}$-rich model $\mathcal{M}_i$ of $F_{h,d}$ in $G[B_{x_i}]$ whose branch set of the root contains a neighbor u_i of B_z. It follows that the union of the models $\mathcal{M}_1,\dots,\mathcal{M}_d$, together with B_z as the branch set of the root, yields an $\mathcal{F}$-rich model of $F_{h+1,d}$ in G, which is a contradiction. This proves that there is no $\mathcal{F}'$-rich model of $F_{2,d}$ in G.

因此，由引理 11.15 应用于 U，即 G 中一个顶点的任意单元素集，存在$S'\subseteq V(G)$，一个路径划分 $(P_0,\dots,P_\ell)$，其所划分的是 (G,S')，以及集合 $R_1,\dots,R_\ell\subseteq V(G)$ 使得
11.15.(a) $P_0=U$;11.15.(b) $S'\cap V(F)\neq\emptyset$
对每个 $F\in\mathcal{F}'$;

11.15.(c) 对于每个分量， $N_G(V(C))$与 $G - V(C)$的交集至多包含三个分量，其中 C 属于 $G - S'$；11.15.(d) $P_i \subseteq R_i$ 对于每个 $i \in [\ell]$；11.15.(e) $\mathrm{cen}_q(G, \varphi, R_i) \leqslant c \cdot d$ 对于每个 $i \in [\ell]$；以及11.15.(f) $|R_i - P_i| \leqslant c \cdot d$。

我们设定 $R_0 = P_0 = U$。注意到 11.15.(d)–11.15.(f) 对于 $i = 0$也成立。

设 $i \in \{0, \ldots, \ell\}$。设 $\eta_i\colon\ R_i \to [cd]$ 满足以下条件 11.15.(e)，并设 $\psi_i\colon\ R_i \to [2cd]$ 满足 $\psi_i|_{P_i} = \eta_i|_{P_i}$，$\psi_i|_{R_i - P_i}$ 是单射，且 $\psi_i(u) \neq \psi_i(v)$ 对每个 $u \in P_i$ 和 $v \in R_i - P_i$均成立。根据11.15.(f)，这样的着色存在。

设 $\mathcal{C}$ 为 $G - S'$的所有连通分支构成的族。对于每个 $C \in \mathcal{C}$，固定一个指标 $i(C) \in [\ell]$，使得 $N_G(V(C)) \subseteq P_{i(C)-1} \cup P_{i(C)}$。接下来，对于每个 $i \in [\ell]$，令

$$W_i = P_{i-1} \cup P_i \cup \bigcup_{C \in \mathcal{C}, i(C) = i} V(C).$$

注意，$(W_1, \ldots, W_\ell)$ 是 G的路径分解。参见图 G的第一部分33。此外

$$W_{i-1} \cap W_{i+1} = \emptyset \qquad \text{for every } i \in [\ell - 1]. \tag{10}$$

对于每个 $u \in V(G)$，令

$$\mu(u) = \begin{cases} i \mod (q+1) & \text{if } u \in S', \text{ where } i \in \{0, \ldots, \ell\} \text{ is such that } u \in P_i, \\ 0 & \text{otherwise.} \end{cases}$$

令 $s = \lceil \log(3q) \rceil$。对于每个 $r \in \{0, \ldots, s\}$，令

$$I_r = \{i \in \{0, \ldots, \ell\} \mid i \equiv 0 \bmod 2^r\}.$$

对于每个 $r \in \{0, \ldots, s\}$ 和 $i \in I_r$，我们将构造一个集合 $R'_i \subseteq R_i$，使得 $(R'_i \mid i \in I_r)$ 两两不相交。该构造是递归的，从 $r = s$ 开始，一直向下到 $r = 0$。当所有 $i \in I_r$ 对应的 R'_i 都已定义后，我们令

$$S_r = \bigcup \{R'_i \mid i \in I_r\}.$$

我们保持以下不变量

$$P_i \subseteq S_r \text{ for every } i \in I_r, \tag{11}$$

$$P_i - S_{r+1} \subseteq R'_i \text{ for every } i \in I_r, \tag{12}$$

按照约定，$S_{s+1} = \emptyset$。图33展示了这一构造的示意图 .

我们从 $r = s$ 开始。对每一个 $i \in I_s$，令

$$R'_i = R_i \cap \bigcup \{W_j \mid j \in \{0, \ldots, \ell\} \cap \{i - q + 1, \ldots, i + q\}\}.$$

当 a，$b \in I_s$ 且 $a < b$时，我们有 $b - a \geqslant 2^s \geqslant 3q \geqslant 2q + 2$ 因为 $q \geqslant 2$。因此，$(b - q) - (a + q + 1) \geqslant 1$ 从而由 (10)，集合 R'_i 对 $i \in I_s$ 两两不交。还要注意$P_i \subseteq R_i \cap W_i \subseteq R'_i$ 对每个 $i \in I_s$，因此 (11) 成立。

接下来，令 $r < s$，并假设对 $i \in I_{r+1}$ 有 R'_i，且 S_{r+1} 已经构造完成。对每个$i \in I_r - I_{r+1}$，令

$$R'_i = (R_i - S_{r+1}) \cap \bigcup \{W_j \mid j \in \{0, \ldots, \ell\} \cap \{i - 2^r + 1, \ldots, i + 2^r\}\}.$$

When $a, b \in I_r - I_{r+1}$ and $a < b$, we have $b - a \geqslant 2^{r+1}$. Moreover, $R'_a \subseteq \bigcup \{W_j \mid j \in \{0, \ldots, \ell\}, j \leqslant a + 2^r\}$ and $R'_b \subseteq \bigcup \{W_j \mid j \in \{0, \ldots, \ell\}, j \geqslant b - 2^r + 1\}$. Hence, by (10) for $r + 1$, $R'_a \cap R'_b \subseteq W_{a+2^r} \cap W_{a+2^r+1} \subseteq P_{a+2^r} \subseteq S_{r+1}$, where the last inclusion follows from (11) and the fact that $a + 2^r \in I_{r+1}$. As R'_a and R'_b are both disjoint from S_{r+1}, we deduce that $R'_a \cap R'_b = \emptyset$. This proves that the sets R'_i for $i \in I_r$ are pairwise disjoint. For every

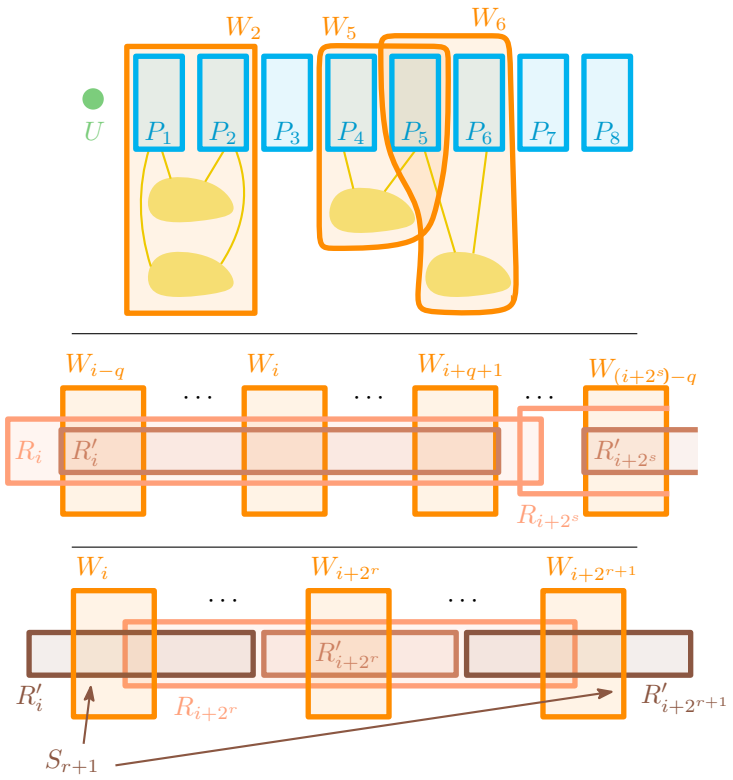


FIGURE 33. An illustration of objects in the proof of Lemma 11.16. In the first part, we show a few examples of what W_i looks like. In the second part, we show the idea behind the construction of the sets R'_i in the base case. Note that the sets R_i are not necessarily disjoint from each other, while the sets R'_i are. In the last part, we show the idea for constructing R'_{i+2^r} given that R'_i and $R'_{i+2^{r+1}}$ are constructed.

$i \in I_r - I_{r+1}$, $P_i - S_{r+1} \subseteq (R_i - S_{r+1}) \cap W_i \subseteq R'_i$ 因此 (12) 成立。此外, $P_i - S_{r+1} \subseteq S_r$。由于 $S_{r+1} \subseteq S_r$, (11) 成立。

接下来，我们如下定义 S_0 的一个着色 λ_0。为方便起见，令 $I_{s+1} = \emptyset$。对每个$r \in \{0, \dots, s\}$，对每个 $i \in I_r - I_{r+1}$，以及对每个 $u \in R'_i$，我们令

$$\lambda_0(u) = (r, \psi_i(u)).$$

Now that S_0 and λ_0 are defined, we decompose $G - S_0$. Since S' intersects every member of $\mathcal{F}'$ by 11.15.(b), and since $S' \subseteq S_0$, $G - S_0$ has no $\mathcal{F}$-rich model of $F_{h,d+1}$. Hence, by the induction hypothesis, there exists a set $\tilde{\Lambda}$ of at most $20c((h-1)(d+1) + \binom{h-1}{2}) \cdot q \log q$ colors, a set $\tilde{S} \subseteq V(G - S_0)$, a coloring $\tilde{\lambda}\colon \tilde{S} \to \tilde{\Lambda}$, and a family $(\tilde{\Lambda}_C \mid C \text{ component of } G - S_0)$ of subsets of $\tilde{\Lambda}$, each of size at most $12c((h-1)(d+1) + \binom{h-1}{2}) \cdot \log q$, such that

(a) $S \cap V(F) \neq \emptyset$ 对每个 $F \in \mathcal{F}|_{G-S_0}$;

(b) 对于 $G - S_0 - S$ 的每个连通分量 C, $N_{G-S_0}(V(C))$ 与 $G - S_0 - V(C)$ 的交集至多包含 $8c((h-1)(d+1) + \binom{h-1}{2}) \cdot \log$ 个连通分量；并且

(c) 对每个与 S 相交的 $G - S_0$ 的连通子图 H,

(i) $|\varphi(V(H))| > q$, 或

(ii) $|\lambda(V(H) \cap S)| > q$, 或

(iii) 存在一个 $(\varphi \times \lambda)$-中心 u of $V(H) \cap S$, 使得对于与 $V(H)$ 相交的 $G - S_0$ 的每个分支 C, 都有 $\lambda(u) \in \Lambda_C$。

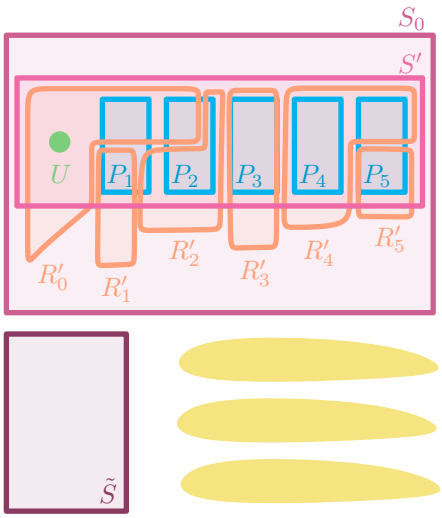


图 34。 引理 11.16证明中对象的示意图。记住， $S_0 = \bigcup_{i\in\{0,\dots,\ell\}} R'_i$。

~ ~在对 Λ 的元素重新标记后，我们可以假设 Λ 与 $\mu \times \lambda_0$ 的像不相交。最后，令

$$S = S_0 \cup \tilde{S}.$$

见图 34。此外，对于每个 $u \in S$，令

$$\lambda(u) = \begin{cases} (\mu(u), \lambda_0(u)) & \text{if } u \in S_0, \\ \tilde{\lambda}(u) & \text{if } u \in \tilde{S}. \end{cases}$$

令 Λ 为 λ的像。首先注意到， $s \leqslant 1 + \log(3q) \leqslant 1 + \log(4q) = 3 + \log q$，因为$q \geqslant 2$。因此，

$$\begin{aligned}
|\Lambda| &= |\lambda(S_0)| + \left|\lambda\big(\tilde{S}\big)\right| \\
&\leqslant |\mu(S_0)| \cdot |\lambda_0(S_0)| + \left|\tilde{\Lambda}\right| \\
&\leqslant (q+1) \cdot \big((s+1) \cdot 2cd\big) + 20c\left((h-1)(d+1) + \tbinom{h-1}{2}\right) \cdot q \log q \\
&\leqslant (q+1) \cdot \big((4+\log q) \cdot 2cd\big) + 20c\left((h-1)(d+1) + \tbinom{h-1}{2}\right) \cdot q \log q \\
&\leqslant 2q \cdot (5 \log q \cdot 2cd) + 20c\left((h-1)(d+1) + \tbinom{h-1}{2}\right) \cdot q \log q \\
&\leqslant 20cd \cdot q \log q + 20c\left((h-1)(d+1) + \tbinom{h-1}{2}\right) \cdot q \log q \\
&\leqslant 20c\left(hd + h - 1 + \tbinom{h-1}{2}\right) \cdot q \log q \\
&\leqslant 20c\left(hd + \tbinom{h}{2}\right) \cdot q \log q.
\end{aligned}$$

对于所有的 $i \in \{0,\dots,\ell\}$和 $r \in \{0,\dots,s\}$，令

$$\begin{aligned}
\mathrm{ext}(i,r,-) &= \max(\{j \in I_r \mid j < i\} \cup \{0\}) \text{ and} \\
\mathrm{ext}(i,r,+) &= \min(\{j \in I_r \mid j \geqslant i\} \cup \{\ell\}).
\end{aligned}$$

此外，对每一个 $i \in \{0,\dots,\ell\}$，令

$$\mathrm{Trace}(i) = \{\mathrm{ext}(i,r,\varepsilon) \mid r \in \{0,\dots,s\},\ \varepsilon \in \{-,+\}\}.$$

断言 11.17. 设 H 为 G 的连通子图，使得 $S_0 \cap V(H) \neq \emptyset$，并令 $r = \max\{t \in \{0,\dots,s\} \mid V(H) \cap S_t \neq \emptyset\}$。则要么

(1) $|\{i \in \{0,\dots,\ell\} \mid V(H) \cap P_i \neq \emptyset\}| > q$，或 (2) 存在唯一的 $j \in I_r$，使得 $V(H) \cap R'_j \neq \emptyset$，并且对于每个 $i \in [\ell]$，若 $V(H) \cap W_i \neq \emptyset$，则 $j \in \mathrm{Trace}(i)$。

Proof of the claim. Suppose that $|\{i \in \{0,\dots,\ell\} \mid V(H) \cap P_i \neq \emptyset\}| \leqslant q$. Recall that $S_r = \bigcup_{j \in I_r} R'_j$. We show that there is unique $j \in I_r$ such that $V(H) \cap R'_j \neq \emptyset$. To this end, let $j_1, j_2 \in I_r$ be such that $V(H)$ intersects both R'_{j_1} and R'_{j_2}, and $j_1 \leqslant j_2$. Suppose to the contrary that $j_1 < j_2$.

首先，考虑 $r < s$ 的情形。根据 r 的极大性，我们有 j_1，$j_2 \in I_r - I_{r+1}$，这意味着 $j_2 - j_1 \geqslant 2^{r+1}$，且 $j_1 + 2^r \in I_{r+1}$。然而，由于 $(W_1, \dots\dots, W_\ell)$ 是 G 的路径分解，且 $V(H)$ 与 $R'_{j_1} \subseteq \bigcup_{1 \leqslant i \leqslant j_1 + 2^r} W_i$ 和 $R'_{j_2} \subseteq \bigcup_{j_2 - 2^r + 1 \leqslant i \leqslant \ell} W_i$均相交，由 (10) 可推得 $V(H)$ 与.....相交

$$\left(\bigcup_{1 \leqslant i \leqslant j_1 + 2^r} W_i\right) \cap \left(\bigcup_{j_2 - 2^r + 1 \leqslant i \leqslant \ell} W_i\right) \subseteq \left(\bigcup_{1 \leqslant i \leqslant j_1 + 2^r} W_i\right) \cap \left(\bigcup_{j_1 + 2^r + 1 \leqslant i \leqslant \ell} W_i\right)$$
$$= W_{j_1 + 2^r} \cap W_{j_1 + 2^r + 1} = P_{j_1 + 2^r}.$$

但由于 $j_1 + 2^r \in I_{r+1}$，我们有 $P_{j_1 + 2^r} \subseteq S_{r+1}$ 根据 (11)，因此 $V(H)$ 与 S_{r+1}相交，这与 r的极大性矛盾。

Now suppose that $r = s$. Since $j_1, j_2 \in I_s$, we have $j_2 - j_1 \geqslant 2^s \geqslant 3q$. Again, since $(W_1, \dots, W_\ell)$ is a path decomposition of G, and since $V(H)$ intersects both $R'_{j_1} \subseteq \bigcup_{1 \leqslant i \leqslant j_1 + q} W_i$ and $R'_{j_2} \subseteq \bigcup_{j_2 - q + 1 \leqslant i \leqslant \ell} W_i$, we deduce that $V(H)$ intersects W_i for every integer i with $j_1 + q \leqslant i \leqslant j_2 - q + 1$. By the definition of $(W_1, \dots, W_\ell)$, and because $(P_0, \dots, P_\ell)$ is a path partition of (G, S'), it follows that $V(H)$ intersects P_i for every i with $j_1 + q \leqslant i \leqslant j_2 - q$, and so $|\{i \in \{0,\dots,\ell\} \mid V(H) \cap P_i \neq \emptyset\}| \geqslant (j_2 - q) - (j_1 + q) + 1 = (j_2 - j_1) - 2q + 1 \geqslant 3q - 2q + 1 = q + 1 > q$, a contradiction. This concludes the proof of $j_1 = j_2$.

In particular, we may fix the unique index $j \in I_r$ such that $V(H) \cap R'_j \neq \emptyset$. Next, we show that for every $i \in [\ell]$ such that $V(H) \cap W_i \neq \emptyset$, we have $j \in \{\mathrm{ext}(i, r, -), \mathrm{ext}(i, r, +)\}$. Since $\{\mathrm{ext}(i, r, -), \mathrm{ext}(i, r, +)\} \subseteq \mathrm{Trace}(i)$ for every $i \in [\ell]$, this will prove the lemma. Let $i \in [\ell]$ such that $V(H) \cap W_i \neq \emptyset$, and suppose for contradiction that $j \notin \{\mathrm{ext}(i, r, -), \mathrm{ext}(i, r, +)\}$. Note that since $(P_0, \dots, P_\ell)$ is a path partition of (G, S'), and by the definitions of $(W_1, \dots, W_\ell)$, $\mathrm{ext}(i, r, -)$, and $\mathrm{ext}(i, r, +)$, we have $P_{\mathrm{ext}(i,r,-)} \cup P_{\mathrm{ext}(i,r,+)}$ separates W_i and $\bigcup\{W_{i'} \mid i' \in [\ell],\ i' \leqslant \mathrm{ext}(i, r, -) \text{ or } i' > \mathrm{ext}(i, r, +)\}$. However, since $j \notin \{\mathrm{ext}(i, r, -), \mathrm{ext}(i, r, +)\}$, we have $R'_j \subseteq \bigcup\{W_{i'} \mid i' \in [\ell],\ i' \leqslant \mathrm{ext}(i, r, -) \text{ or } i' > \mathrm{ext}(i, r, +)\}$. Hence, since $V(H)$ intersects R'_j and H is connected, it follows that $V(H)$ intersects $P_{\mathrm{ext}(i,r,-)} \cup P_{\mathrm{ext}(i,r,+)}$. Then, since $V(H)$ is disjoint from S_{r+1}, and because $P_{i'} - S_{r+1} \subseteq R'_{i'}$ for every $i' \in I_r$ by (12), it follows that $V(H)$ intersects $R'_{\mathrm{ext}(i,r,-)} \cup R'_{\mathrm{ext}(i,r,+)}$. By the uniqueness of j we obtain that $j \in \{\mathrm{ext}(i, r, -), \mathrm{ext}(i, r, +)\}$, a contradiction. Since $\{\mathrm{ext}(i, r, -), \mathrm{ext}(i, r, +)\} \subseteq \mathrm{Trace}(i)$, this concludes the proof of the claim. ◊

对于每个 $i \in [\ell]$，令

$$\Lambda_i = \bigcup\{\lambda(R'_j) \mid j \in \mathrm{Trace}(i)\}.$$

断言 11.18。对于每个 $i \in [\ell]$，

$$|\Lambda_i| \leqslant cd(s+2) \leqslant 12cd \cdot \log q.$$

断言的证明。设 $j \in [\ell]$。注意

$$\begin{aligned}|\lambda(R'_j)| &\leqslant |\lambda(R'_j - P_j)| + |\lambda(P_j)| \\ &\leqslant |R'_j - P_j| + |\psi_j(P_j)| \\ &\leqslant |R_j - P_j| + cd \\ &\leqslant cd + cd \\ &= 2cd\end{aligned}$$

其中最后一个不等式由 11.15.(f) 得出。因此 $|\Lambda_i| \leqslant 2cd|\text{迹}(i)|$。

现在我们论证 $|\mathsf{Trace}(i)| \leqslant (s+2)$。首先，注意到 $|\mathsf{Trace}(i) \cap I_0| \leqslant 2$。对于每个 $r \in \{0,\dots,s-1\}$，$\{\mathsf{ext}(j,r,-), \mathsf{ext}(j,r,+)\}$ 与 $\{\mathsf{ext}(j,r+1,-), \mathsf{ext}(j,r+1,+)\}$ 相交。因此，对于每个 $r \in \{0,\dots,s\}$，我们有 $|\mathsf{Trace}(i) \cap \bigcup_{t\in\{0,\dots,r\}} I_t| \leqslant 2+r$。特别地，$|\mathsf{Trace}(i)| \leqslant s+2$。回忆一下，$s \leqslant 3+\log q$。因此，

$$\begin{aligned}|\Lambda_i| &\leqslant 2cd \cdot (s+2) \\ &\leqslant 2cd \cdot (5+\log q) \\ &\leqslant 12cd \cdot \log q.\end{aligned}$$

◊

断言 11.19. 对于与 S_0 相交的 G 的任意连通子图 H，以下条件之一成立，(1) $|\varphi(V(H))| > q$，或 (2) $|\lambda(V(H) \cap S_0)| > q$，或 (3) 存在 $V(H) \cap S_0$ 的一个 $(\varphi \times \lambda)$-中心 u，使得对于每个 $i \in [\ell]$，若 $W_i \cap V(H) \neq \emptyset$，则 $\lambda(u) \in \Lambda_i$。

断言的证明。设 H 是 G 的一个连通子图，且与 S_0相交，并且$|\varphi(V(H))| \leqslant q$且 $|\lambda(V(H) \cap S_0)| \leqslant q$。令 $r =$最大值$\{t \in \{0,\dots,s\} \mid V(H) \cap S_t \neq \emptyset\}$。根据断言 11.17，以下情况之一成立

（1）$|\{i \in \{0,\dots,\ell\} \mid V(H) \cap P_i \neq \emptyset\}| > q$，或者（2）存在唯一的 $j \in I_r$ 使得 $V(H) \cap R'_j \neq \emptyset$。此外，对于每个 $i \in [\ell]$，若$V(H) \cap W_i \neq \emptyset$，则 $j \in \mathsf{Trace}(i)$。

Note that, since $(P_0,\dots,P_\ell)$ is a path partition of (G,S'), the set $\{i \in \{0,\dots,\ell\} \mid V(H) \cap P_i \neq \emptyset\}$ is an interval of integers. By the definition of μ, this implies that $|\mu(V(H) \cap S')| \geqslant \min\{q+1, |\{i \in \{0,\dots,\ell\} \mid V(H) \cap P_i \neq \emptyset\}|\}$. Therefore, in the first case, $|\lambda(V(H) \cap S_0)| \geqslant |\mu(V(H) \cap S')| > q$, a contradiction. Hence there is a unique $j \in I_r$ such that $V(H) \cap R'_j \neq \emptyset$, and for every $i \in [\ell]$, if $V(H) \cap W_i \neq \emptyset$, then $j \in \mathrm{Trace}(i)$. We fix this index $j \in I_r$. We will find a $(\varphi \times \lambda)$-center u of $V(H) \cap S_0$ which belongs to R'_j. First, by the definition of ψ_j, there is a $(\varphi \times \eta_j)$-center u_0 of $V(H) \cap R_j$. It follows that u_0 is also a $(\psi_j \times \varphi)$-center of $V(H) \cap R_j$.

首先，假设 $u_0 \in R'_j$。我们断言，u_0 是 $V(H) \cap S_0$的一个（$\varphi \times \lambda$）-中心。事实上，对于任意满足 $(\varphi \times \lambda)(u') = (\varphi \times \lambda)(u_0)$ 的 $u' \in V(H) \cap S_0$，都有 $u' \in R'_{j'}$，其中某个 $j' \in I_r$。由 j 的唯一性可知，$j = j'$，因此 $u' \in R'_j$。由于 $(\varphi \times \psi_j)(u') = (\varphi \times \psi_j)(u_0)$，且 u_0 是 $V(H) \cap R'_j$的一个（$\varphi \times \psi_j$）-中心，我们推得 $u' = u_0$。这表明，u_0 是 $V(H) \cap S_0$的一个（$\varphi \times \lambda$）-中心。

Next, suppose that $u_0 \notin R'_j$. We claim that $V(H)$ is disjoint from P_j. First, suppose $r < s$. Then, since $\{j-2^r, j+2^r\} \cap \{0,\dots,\ell\} \subseteq I_{r+1}$, we have by (11) the inclusion $P_{j-2^r} \cup P_{j+2^r} \subseteq S_{r+1}$ (with the convention $P_i = \emptyset$ for every $i \notin \{0,\dots,\ell\}$). By the maximality of r, $V(H)$ is disjoint from S_{r+1}, and so $V(H)$ is disjoint from $P_{j-2^r} \cup P_{j+2^r}$. Since $(P_0,\dots,P_\ell)$ is a path partition of (G,S'), and by the definition of $(W_1,\dots,W_\ell)$, we deduce that $V(H) \subseteq \bigcup\{W_i \mid j-2^r+1 \leqslant i \leqslant j+2^r\}$. But then, $V(H) \cap R'_j = V(H) \cap R_j$, which contradicts the fact that $u_0 \in R_j - R'_j$. Hence $r = s$. Then, by the definition of R'_j, $u_0 \notin \bigcup\{W_i \mid j-q+1 \leqslant i \leqslant j+q\}$.

Hence, if $V(H)$ intersects P_j, because $(P_0, \dots, P_\ell)$ is a path partition of (G, S'), and by the definition of $(W_1, \dots, W_\ell)$, $V(H)$ intersects either P_i for each $i \in \{j-q, \dots, j\}$, or P_i for each $i \in \{j, \dots, j+q\}$. In both cases, we have $|\{i' \in [\ell] \mid P_{i'} \cap V(H) \neq \emptyset\}| > q$, and so $|\lambda(V(H) \cap S_0)| \geqslant |\mu(V(H) \cap S')| > q$, a contradiction. This proves that $V(H)$ is disjoint from P_j.

Then, let u be an arbitrary vertex in $R'_j \cap V(H)$. We claim that u is a $(\varphi \times \lambda)$-center of $V(H) \cap S_0$. Let $u' \in V(H) \cap S_0$ be such that $(\varphi \times \lambda)(u') = (\varphi \times \lambda)(u)$. Then, $u' \in S_r - S_{r+1}$, and so $u' \in R'_{j'}$ for some $j' \in I_r$. By the uniqueness of j, we deduce that $j' = j$ and so $u' \in R'_j$. Moreover, since $V(H)$ is disjoint from P_j, $u' \in R'_j - P_j$. Since $\psi_j(u') = \psi_j(u)$, and because $\psi_j|_{R_j - P_j}$ is injective, we deduce that $u = u'$. This proves that u is a $(\varphi \times \lambda)$-center of $V(H) \cap S_0$.

最后，对于每个满足 $V(H) \cap W_i \neq \emptyset$ 的 $i \in [\ell]$，我们有 $j \in \mathsf{Trace}(i)$ 和 $u \in R'_j$，因此 $\lambda(u) \in \lambda(R'_j)$。根据 Λ_i 的定义，我们推得 $\lambda(u) \in \Lambda_i$。这就证明了该断言。$\Diamond$

设 C 为 $G - S$ 的一个连通分支。设 i 为 $[\ell]$ 中唯一的指标，使得 W_i 包含 $V(C)$。我们设立

$$\Lambda_C = \Lambda_i \cup \tilde{\Lambda}_C.$$

为完成证明，我们证明 (a) – (c) 成立。

˜对于每个 $F \in \mathcal{F}$，如果 $S_0 \cap V(F) = \emptyset$，那么 $S \cap V(F) \neq \emptyset$ 由 (ã)。因此， $S \cap V(F) \neq \emptyset$，这证明了 (a)。

Next, we prove (b). Let C be a component of $G - S$. Let C' be the component of $G - S'$ containing C. First, by (b̃), $N_{G-S_0}(V(C)) = N_G(V(C)) \cap \tilde{S}$ intersects at most $8c((h-1)(d+1) + \binom{h-1}{2}) \cdot \log q$ component of $G - S_0 - V(C)$. Second, by 11.15.(c), $N_G(V(C'))$ intersects at most three components of $G - V(C')$. It is now enough to bound $|N_G(V(C)) \cap (S_0 - S')|$.

令 $r \in \{0, \dots, s\}$。考虑 $H = G[V(C) \cup (N_G(V(C)) \cap (S_r - S_{r+1}))]$ (约定$S_{s+1} = \emptyset$)。根据断言 11.17，由于 H至多与 $i' \in \{0, \dots, \ell\}$对应的两个 $P_{i'}$相交（即 $P_{i'-1}$ 和 $P_{i'}$），因此存在唯一的 $j \in I_r - I_{r+1}$ (约定 $I_{s+1} = \emptyset$)，使得 H 与 R'_j相交。因此，

$$|N_G(V(C)) \cap (S_r - (S_{r+1} \cup S'))| \leqslant |R'_j - S'| \leqslant |R'_j - P_j| \leqslant cd.$$

综上可得

$$|N_G(V(C)) \cap (S_0 - S')| \leqslant cd(s+1)$$

因此，与 $N_G(V(C))$ 相交的 $G - V(C)$ 的连通分支数至多为

$$\begin{aligned}
&3 + cd(s+1) + 8c\left((h-1)(d+1) + \tbinom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 3 + cd(4 + \log q) + 8c\left((h-1)(d+1) + \tbinom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 3 + cd \cdot 5 \log q + 8c\left((h-1)(d+1) + \tbinom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 8cd \cdot \log q + 8c\left((h-1)(d+1) + \tbinom{h-1}{2}\right) \cdot \log q \\
&\qquad \leqslant 8c\left(hd + \tbinom{h}{2}\right) \cdot \log q.
\end{aligned}$$

这证明了 (b)。

设 C 是 $G-S$ 的一个连通分量。根据断言 11.18,

$$\begin{aligned}|\Lambda_C| &\leqslant 12cd\cdot\log q + 12c\left((h-1)(d+1)+\tbinom{h-1}{2}\right)\cdot\log q\\ &\leqslant 12c\left(hd+\tbinom{h}{2}\right)\cdot\log q.\end{aligned}$$

We claim that (c) is satisfied. Let H be a connected subgraph of G intersecting S. If H is disjoint from S_0, then, by (c̃), one of (c̃).(i), (c̃).(ii), and (c̃).(iii) holds. If $|\varphi(V(H))|>q$ (i.e. (c̃).(i)), then (c).(i) holds. If $|\tilde\lambda(V(H)\cap\tilde S)|>q$ (i.e. (c̃).(ii)), then

$$|\lambda(V(H)\cap S)|\geqslant|\tilde\lambda(V(H)\cap S)|=|\tilde\lambda(V(H)\cap\tilde S)|>q,$$

and so, (c).(ii) holds. Finally, assume that there is a $(\varphi\times\tilde\lambda)$-center u of $V(H)\cap\tilde S$ such that for every component C of $G-S_0$ intersecting $V(H)$, $\tilde\lambda(u)\in\tilde\Lambda_C$ (i.e. (c̃).(iii)). It follows that u is a $(\varphi\times\lambda)$-center u of $V(H)\cap S$ such that for every component C of G intersecting $V(H)$, $\lambda(u)\in\Lambda_C$. This implies (c).(iii).

Next, suppose that $V(H)$ intersects S_0. By Claim 11.19, one of 11.19.(1), 11.19.(2), and 11.19.(3) holds. If $|\varphi(V(H))|>q$ (i.e. 11.19.(1)), then (c).(i) holds. If $|\lambda(V(H)\cap S)|>q$ (i.e. 11.19.(2)), then $|\lambda(V(H)\cap S)|\geqslant|\lambda(V(H)\cap S_0)|>q$, and so, (c).(ii) holds. Finally, assume that there is a $(\varphi\times\lambda)$-center u of $V(H)\cap S_0$ such that for every $i\in[\ell]$ with $W_i\cap V(H)\neq\emptyset$, we have $\lambda(u)\in\Lambda_i$ (i.e. 11.19.(3)). Let C be a component of $G-S$ intersecting $V(H)$. Let i be the unique index in $[\ell]$ such that C intersects W_i. Since λ uses the colors of Λ_i only on S_0, u is $(\varphi\times\lambda)$-center of $V(H)\cap S$ and $\lambda(u)\in\Lambda_i\subseteq\Lambda_C$. This implies (c).(iii) and concludes the proof of the lemma. □

引理 11.20 ([3, 命题 7.2。1])。对于每个整数 k 满足 $k\geqslant 3$，每个 K_k-次要无关图 G 至多有 $2^{k-3}|V(G)|-1$ 条边。

我们现在证明引理 11.16 的一个略加修改的版本，其中在 (b) 中，$G-V(C)$ 中与 $N_G(V(C))$ 相交的连通分支数目现在仅是 k 的函数，假设 G 是 K_k- 无子式的，对某个固定正整数 k。

Lemma 11.21. *Let q, c, d, and k be positive integers with $q\geqslant 2$ and $k\geqslant 3$. Let h be a nonnegative integer, let G be a K_k-minor-free graph, let φ be a (q,c)-good coloring of G, and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of $F_{h+1,d}$. There exists a set Λ of at most $28c^2(hd+\binom{h}{2})\cdot q\log q$ colors, a set $S\subseteq V(G)$, a function $\lambda\colon S\to\Lambda$, and a family $(\Lambda_C\mid C$ component of $G-S)$ of subsets of Λ, each of size at most $20c^2(hd+\binom{h}{2})\cdot\log q$ such that*

(a) $S\cap V(F)\neq\emptyset$ 对每个 $F\in\mathcal{F}$ 均成立；(b) 对于 $G-S$ 的每个连通分量 C，$N_G(V(C))$ 与 $G-V(C)$ 的连通分量相交的数量至多为 2^{k-1}；并且

(c) 对于与 S 相交的 G 的每个连通子图 H，

(1) $|\varphi(V(H))|>q$，或(2) $|\lambda(V(H)\cap S)|>q$，或(3) 存在一个$(\varphi\times\lambda)$-中心 u，属于 $V(H)\cap S$，使得对于与 $V(H)$相交的每个连通分量 C（$G-S$），都有 $\lambda(u)\in\Lambda_C$。

Proof. By Lemma 11.16, there exists a set Λ_0 of at most $20c(hd+\binom{h}{2})\cdot q\log q$ colors, a set $S_0\subseteq V(G)$, a function $\lambda_0\colon S_0\to\Lambda_0$, and a family $(\Lambda_{0,C}\mid C$ component of $G-S_0)$ of subsets of Λ_0, each of size at most $12c(hd+\binom{h}{2})\cdot\log q$, such that

11.16.(a) $S_0\cap V(F)\neq\emptyset$ 对每个 $F\in\mathcal{F}$；

11.16.(b) 对每个分支 C的 $G-S_0$， $N_G(V(C))$ 至多相交于 $8c(hd+\binom{h}{2})\cdot$对数个 q的分支 $G-V(C)$；并且

11.16.(c) 对于每个与 S相交的 H 的连通子图 G，

(i) $|\varphi(V(H))|>q$，或 (ii) $|\lambda_0(V(H)\cap S)|>q$，或 (iii) 存在一个 $(\varphi\times\lambda_0)$-中心 u，属于 $V(H)\cap S_0$，使得对于与 $V(H)$ 相交的每个分量 C（$G-S_0$），$\lambda_0(u)\in\Lambda_{0,C}$。

通过重新标记 Λ_0 中的元素，我们假设 Λ_0 与 $\mathbb{N}$ 不相交。

Let $\mathcal{C}$ be the family of all the components of $G-S_0$. Let $C\in\mathcal{C}$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of C such that $N_G(V(H))$ intersects at least 2^{k-2} components of $G-V(C)$. Let $N=\left\lfloor 8c(hd+\binom{h}{2})\cdot\log q\right\rfloor$. We claim that there are no $N+1$ pairwise disjoint members of $\mathcal{F}'$. Suppose for contradiction that $H_1,\dots,H_{N+1}$ are pairwise disjoint members of $\mathcal{F}'$. Let A be the minor of G obtained from G by contracting every component of $G-V(C)$ having a neighbor in $V(C)$ in to a single vertex, and every H_i into a single vertex. We denote by $u_1,\dots,u_m$ the vertices resulting the contractions of these components of $G-V(C)$, and by $v_1,\dots,v_{N+1}$ the vertices resulting from the contractions of $H_1,\dots,H_{N+1}$. By 11.16.(b), $m\leqslant N+1$. Then, A is a graph with $m+N+1\leqslant 2(N+1)$ vertices, and every v_i for $i\in[N+1]$ has at least 2^{k-2} neighbors in $\{u_1,\dots,u_m\}$. Hence $|E(A)|\geqslant 2^{k-2}(N+1)$. On the other hand, A is a minor of G and so is K_k-minor-free. By Lemma 11.20, this implies that $|E(A)|\leqslant 2^{k-3}|V(A)|-1<2^{k-2}(N+1)$. This contradiction proves that there are no $N+1$ pairwise disjoint members of $\mathcal{F}'$.

因此，由于 φ 是一个(q,c)-良好着色，存在一个集合 $Z_C\subseteq V(C)$以及一个着色 $\psi_C\colon Z_C\to[c\cdot N]$ 使得

(i) 对每个 $F\in\mathcal{F}'$， $Z_C\cap V(F)\neq\emptyset$；(ii) 对于 $C-Z$ 的每个连通分量 C'， $N_C(V(C'))$ 至多与 $C-V(C')$ 的两个连通分量相交；(iii) 对于 C 的每个连通子图 H，要么 $|\varphi(V(H))|>q$，要么存在一个 $V(H)$ 的 $(\varphi\times\psi_C)$-中心 $\cap Z$。

Let

$$S=S_0\cup\bigcup_{C\in\mathcal{C}}Z_C,$$

并且对每个 $u\in S$，令

$$\lambda(u)=\begin{cases}\lambda_0(u) & \text{if } u\in S_0,\\ \psi_C(u) & \text{otherwise, where } C \text{ is the unique member of } \mathcal{C} \text{ containing } u.\end{cases}$$

首先注意到 λ 至多使用 $20c(hd+\binom{h}{2})\cdot q\log q+c\cdot N\leqslant 28c^2(hd+\binom{h}{2})\cdot q\log q$ 种颜色。现在证明(a)–(c)。

首先， $S_0\subseteq S$ 因此 $S\cap V(F)\neq\emptyset$ 对每个 $F\in\mathcal{F}$ 由11.16.(a)。因此(a) 成立 .
设 C' 为 $G-S$的一个连通分支。设 C 为 $G-S_0$ 中包含 C'的连通分支。由于 C'与 Z_C不相交，由(i)， $N_G(V(C'))$至多与 $G-V(C)$的 $2^{k-2}-1$ 个连通分支相交。此外，由(ii)， $N_G(V(C'))$ 至多与 $C-V(C')$的两个连通分支相交，从而也至多与$G-V(C')$的两个连通分支相交。因此，$N_G(V(C'))$至多与 $G-V(C')$的 $2^{k-2}+1\leqslant 2^{k-1}$ 个连通分支相交。这就证明了(b)。

设 C 是 $G-S$ 的一个连通分量。设 C' 是 $G-S_0$ 中包含 C 的连通分量。设

$$\Lambda_C=\Lambda_{0,C'}\cup[c\cdot N].$$

注意,

$$\begin{aligned}|\Lambda_C| &\leqslant |\Lambda_{0,C'}| + c \cdot N \\ &\leqslant 12c\left(dh + \binom{h}{2}\right)\log q + 8c^2\left(dh + \binom{h}{2}\right)\log q \\ &\leqslant 20c^2\left(dh + \binom{h}{2}\right)\log q.\end{aligned}$$

Let H be a connected subgraph of G intersecting S such that $|\varphi(V(H))| \leqslant q$ and $|\lambda(V(H) \cap S)| \leqslant q$. Suppose that $C_1, \dots, C_m$ are the components of $G - S$ intersecting $V(H)$. For every $i \in [m]$, we denote by C_i' the component of $G - S_0$ containing $V(C_i)$. If H intersects S_0, then by 11.16.(c), there is a $(\varphi \times \lambda_0)$-center u of $V(H) \cap S_0$ with $\lambda_0(u) \in \bigcup_{i\in[m]} \Lambda_{0,C_i'}$, and so u is a $(\varphi \times \lambda)$-center of $V(H) \cap S$ with $\lambda(u) \in \bigcup_{i\in[m]} \Lambda_{C_i}$. Now suppose that H is disjoint from S_0. Then $C_1' = \cdots = C_m'$. By (iii), there is a $(\varphi \times \psi_{C_1'})$-center u of $V(H) \cap Z_{C_1'}$, and so u is a $(\varphi \times \lambda)$-center of $V(H) \cap S$ with $\lambda(u) \in [c \cdot N] \subseteq \Lambda_{C_i}$ for every $i \in [m]$. This shows (c), and concludes the proof of the lemma. □

接下来，我们迭代地应用引理 11.21，以对不含给定 apex-森林的 $\mathcal{F}$-丰富模型的图得到一种着色。在构造合适着色的同时，我们使用与断言 7.5 中相同的技术。引理 11.21 中的第 (c) 项允许我们在 $G - S$ 的一个连通分支 C 中重用大多数颜色，但大小为 $\mathcal{O}(\log q)$ 的集合 Λ_C 中的少数颜色除外。其想法是维持一个由尚不可重用的颜色构成的小“缓冲区”$(\Lambda_1, \dots, \Lambda_a)$。这里，对应于 $a \in [q]$ 的 Λ_a 是在当前归纳调用之前 a 步得到的、形如 Λ_C 的集合。我们将维持如下性质：若一个连通子图 H 与当前正在处理的连通分支相交，且其中心的颜色属于 Λ_a，则 H 已经包含至少 a 种不同的颜色。因此，当我们到达 $a = q + 1$ 时，便可以重用该集合中的颜色。

引理 *11.22*。设 c 和 k 为正整数， X 为森林。存在一个整数 $c_{11.22}(c, k, X)$，使得下述结论成立。设 q 为满足 $q \geqslant 2$的整数， Λ 为一个至少含有 $2c_{11.22}(c, k, X) \cdot q \log q$ 种颜色的集合。设 G 为一个连通的 K_k-无小图， φ 为 G的一个 (q, c)-良好着色，且 $\mathcal{F}$ 为 G 的一个连通子图族，使得 G 没有 $K_1 \oplus X$的 $\mathcal{F}$-富模型。对于每个非空集合 $U \subseteq V(G)$、 $G - U$的每个连通分量 C、每个 λ_0: $U \to \Lambda$，以及 Λ 的所有子集 Λ_1、……、 Λ_q，若对每个 $a \in [q]$ 都有 $|\Lambda_a| \leqslant c_{11.22}(c, k, X) \cdot \log q$，则……

(i) $N_G(V(C))$ 与 $G - V(C)$ 的交集至多包含 2^{k-1} 个连通分量，并且

（ii）对于 G 的每个同时与 U 和 V相交的连通子图 H （C）

(1) $|\varphi(V(H))| > q$, *or*
(2) $|\lambda_0(V(H) \cap U)| > q$, *or*
(3) *there exists a* $(\varphi \times \lambda_0)$*-center* u_H *of* $V(H) \cap U$ *and* $a \in [q]$ *such that* $\lambda_0(u_H) \in \Lambda_a$ *and* $|\lambda_0(V(H) \cap U) \cap (\bigcup_{b\in[a]} \Lambda_b)| \geqslant a$ *– we fix such a vertex* u_H*;*

那么，则存在 S，使得 $U \subseteq S \subseteq V(C) \cup U$，以及一个着色 λ: $S \to \Lambda$，它扩展了 λ_0，使得 (a) $S \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}|_C$; (b) 对于 $C - S$ 的每个连通分支 C'， $N_G(V(C'))$ 至多与 G –$V(C')$ 的 2^k 个连通分支相交; 以及(c) 对于 G 的每个同时与 S 和 $V(C)$相交的连通子图 H，

(1) $|\varphi(V(H))| > q$，或(2) $|\lambda(V(H) \cap S)| > q$，或(3) 存在一个$(\varphi \times \lambda)$-中心 u，属于 $V(H) \cap S$，而且，如果 $V(H) \cap U \neq \emptyset$，那么我们可以令 $u = u_H$。

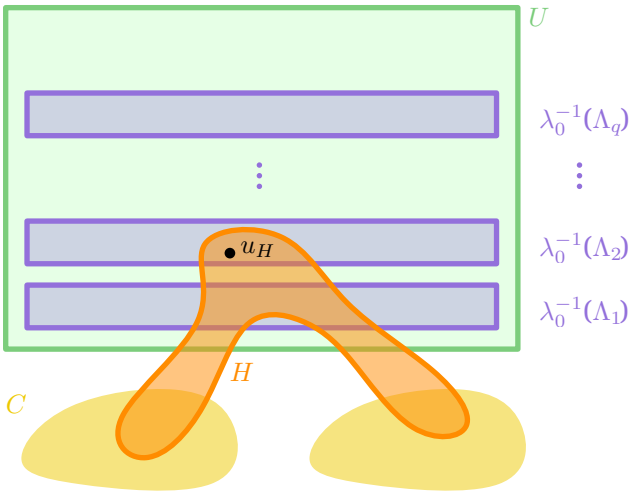


图 35. 假设 (ii)的示意图。(3) 见引理 11.22。直观地说，集合 Λ_a 对于 $a \in [q]$ 表示连通子图 H 同时与 U 和 $V(C)$相交时，其中心 u_H 可以取到的颜色，但代价是 $V(H)$至少使用 a 种颜色。

证明。见图35。根据引理6.5，所有森林构成的类具有着色消除性质。令 X' 为一个满足着色消除性质的森林，对应于 $X \sqcup K_1$；并令h 和 d 为正整数，满足 $X' \subseteq F_{h+1,\,d}$。令

$$c_{11.22}(c,k,X) = 28c^2\left(hd + \tbinom{h}{2}\right).$$

A tuple $(U, C, \lambda_0, (\Lambda_a \mid a \in [q])\ \mathcal{H}, (u_H \mid H \in \mathcal{H}))$ is an instance if $U \subseteq V(G)$ is a nonempty set, C is a component of $G - U$, $\lambda_0 \colon U \to \Lambda$, and $\Lambda_1, \ldots, \Lambda_q$ are subsets of Λ with $|\Lambda_a| \leqslant c_{11.22}(c,k,X) \cdot \log q$ for every $a \in [q]$, and these objects satisfy (i) and (ii); additionally, $\mathcal{H}$ is the family of all connected subgraphs H of G such that $V(H) \cap U \neq \emptyset$, $V(H) \cap V(C) \neq \emptyset$, $|\varphi(V(H))| \leqslant q$, and $|\lambda_0(V(H) \cap U)| \leqslant q$; and finally $(u_H \mid H \in \mathcal{H})$ witnesses item (ii).(3).

只需证明：给定一个实例，存在 S，使得 $U \subseteq S \subseteq V(C) \cup U$ 以及一个着色 $\lambda\colon S \to \Lambda$ 延拓 λ_0，使得 (a)–(c) 成立。我们通过对 $|V(C)|$进行归纳来证明这一陈述。

设 $(U,\ C,\ \lambda_0,\ (\Lambda_a \mid a \in [q])\ \mathcal{H},\ (u_H \mid H \in \mathcal{H}))$ 为一个实例。若 $\mathcal{F}|_C = \emptyset$，则只需令 $S = U$ 与 $\lambda = \lambda_0$，因此我们假设 $\mathcal{F}|_C \neq \emptyset$。证明中所考虑的一些对象如图36所示。

Let $A_1, \ldots, A_\ell$ be the components of $G - V(C)$. By assumption (i) for $\mathcal{I}$, we have $1 \leqslant \ell \leqslant 2^{k-1}$. Let $\mathcal{F}'$ be the family of all the connected subgraphs H of C such that H contains a member of $\mathcal{F}$ and intersects $N_G(U) = \bigcup_{i=1}^{\ell} N_G(V(A_i))$. We claim that there is no $\mathcal{F}'$-rich model of X' in C. Suppose to the contrary that C has an $\mathcal{F}'$-rich model $(D_x \mid x \in V(X'))$ of X'. For every $i \in [\ell]$, let S_i be the set of all $x \in V(X')$ such that D_x contains a vertex of $N_G(V(A_i))$. By the coloring elimination property, there exists $i \in [\ell]$ such that X' contains an S_i-rooted model of $X \sqcup K_1$. It follows that C contains an $\mathcal{F}$-rich model $(B_x \mid x \in V(X \sqcup K_1))$ of $X \sqcup K_1$ such that every branch set has a neighbor in A_i. Let x_0 be the vertex of K_1 in $X \sqcup K_1$. Then the model $(B_x \mid x \in V(X))$ together with $B_{x_0} \cup A_i$ is an $\mathcal{F}$-rich model of $K_1 \oplus X$ in G, contradicts one of the assumptions. This shows that there is no $\mathcal{F}'$-rich model of X' in C, as claimed.

Let $\tilde{\Lambda} = \Lambda - \bigcup_{a \in [q]} \Lambda_a$. Note that

$$|\tilde{\Lambda}| \geqslant 2c_{11.22}(c,k,X) \cdot q \log q - q \cdot c_{11.22}(c,k,X) \cdot \log q \geqslant c_{11.22}(c,k,X) \cdot q \log q.$$

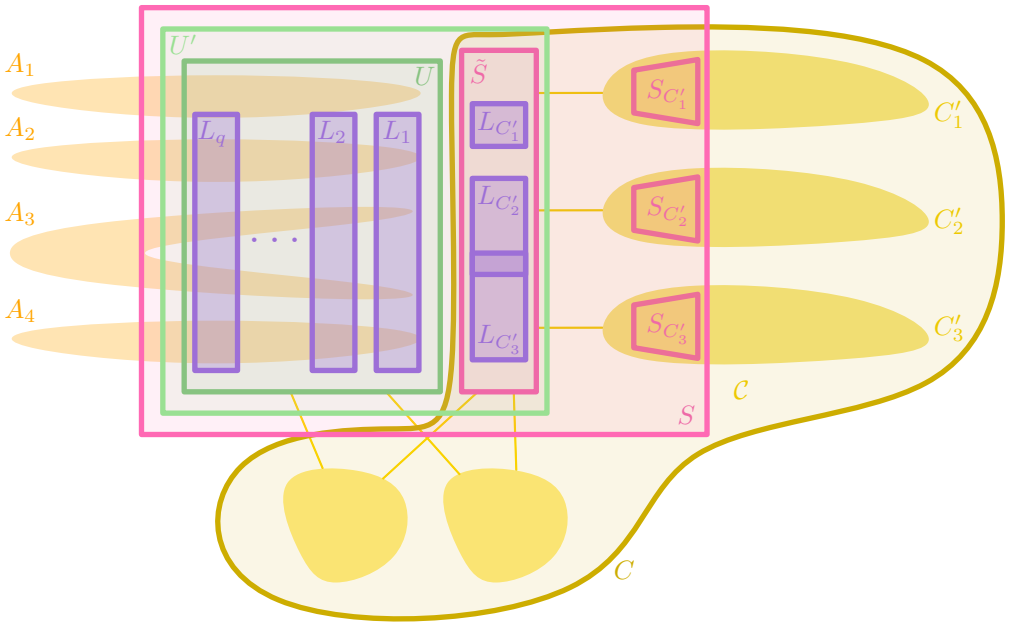


图 36. 引理 11.22 的证明中若干对象的示意图。这里，$L_a = \lambda_0^{-1}(\Lambda_a)$ 对每个 $a \in [q]$ 以及 $L_{C'_i} = \tilde{\lambda}^{-1}(\Lambda'_{C_i})$。此外，$\mathcal{C} = \{C'_1, C'_2, C'_3\}$。

根据引理11.21，存在一个集合 $S \subseteq V(C)$、一个函数 $\lambda\colon S \to \Lambda$，以及一个由($\Lambda_{C'}$ 对 C' 的连通分量 $C - \tilde{S}$) 的子集构成的族 $\tilde{\Lambda}$，其中每个子集的大小至多为 $20c^2(hd + \binom{h}{2}) \cdot \log q$，使得

11.21.(a) $S \cap V(F) \neq \emptyset$ 对于每个 $F \in \mathcal{F}'$；11.21.(b) 对于 C' 的每个连通分量 $C - \tilde{S}$，$N_C(V(C'))$ 至多与 2^{k-1} 的 $C - V(C')$ 个连通分量相交；并且

11.21.(c) 对于每个与 H 的 C 相交的连通子图 S，

(1) $|\varphi(V(H))| > q$，或 (2) $|\lambda(V(H) \cap S)| > q$，或

(3) 存在一个（$\varphi \times \lambda$）-中心 $\tilde{u}_H$，它属于 $V(H) \cap S$，使得对于 $C - S$的每个与 $V(H)$相交的连通分支 C'，$\lambda(\tilde{u}_H) \in \Lambda_{C'}$。

我们固定这样一个顶点 $\tilde{u}_H$。

由于 $\mathcal{F}|_C \neq \emptyset$ 和 G 是连通的，因此 S 非空。令

$$U' = U \cup \tilde{S},$$

并令每个 $u \in U'$ 定义为 $\lambda'_0\colon U' \to \Lambda$，其定义为

$$\lambda'_0(u) = \begin{cases} \lambda_0(u) & \text{if } u \in U, \\ \tilde{\lambda}(u) & \text{if } u \in \tilde{S}. \end{cases}$$

Let H be a connected subgraph of G intersecting U' with $|\varphi(V(H))| \leqslant q$ and $|\lambda'_0(V(H) \cap U')| \leqslant q$. In particular, $|\lambda_0(V(H) \cap U)| \leqslant q$ and $|\tilde{\lambda}(V(H) \cap \tilde{S})| \leqslant q$. If $V(H) \cap U \neq \emptyset$, then let $u'_H = u_H$, and if $V(H) \cap U = \emptyset$, then let $u'_H = \tilde{u}_H$.

Let $\mathcal{C}$ be the family of all the components of $C - \tilde{S}$ that contain a member of $\mathcal{F}$. Let $C' \in \mathcal{C}$, and let $\mathcal{H}'$ be the family of all the connected subgraphs H of G intersecting both $V(C')$ and U', and such that $|\varphi(V(H))| \leqslant q$ and $|\lambda'_0(V(H) \cap U'| \leqslant q$. Let $(\Lambda'_1, \ldots, \Lambda'_q) = (\tilde{\Lambda}_{C'}, \Lambda_1, \ldots, \Lambda_{q-1})$.

现在我们证明这个元组

$$\mathcal{I}' = \big(U', C', \lambda'_0, (\Lambda'_a \mid i \in [q]), \mathcal{H}', (u'_H \mid H \in \mathcal{H}')\big)$$

是一个实例。

˜由于 $C' \notin \mathcal{F}'$，我们有 $N_G(V(C')) \cap U = \emptyset$，从而 $N_G(V(C')) \subseteq S$。因此，由关于 $\mathcal{I}$ 的11.21.(b)， $N_G(V(C'))$至多与 $G - V(C')$的 2^{k-1} 个连通分支相交。这证明了关于 $\mathcal{I}'$的(i)。

Let $H \in \mathcal{H}'$. We assumed that (ii).(1) and (ii).(2) do not hold for H in the context of the instance $\mathcal{I}'$. Thus, the goal is to prove (ii).(3), and moreover, that it is witnessed by u'_H. Namely, we will show that u'_H is a $(\varphi \times \lambda'_0)$-center of $V(H) \cap U'$ and there exists $a \in [q]$ such that $\lambda'_0(u'_H) \in \Lambda'_a$ and $|\lambda'_0(V(H) \cap U') \cap (\bigcup_{b\in[a]} \Lambda'_b)| \geqslant a$.

First, suppose that $V(H)$ intersects U. Since $(u_H \mid H \in \mathcal{H}')$ witnesses (ii).(3) for $\mathcal{I}$, $u'_H = u_H$ is a $(\varphi \times \lambda_0)$-center of $V(H) \cap U$ such that for some $a \in [q]$, we have $\lambda_0(u'_H) \in \Lambda_a$. Since the colors used by λ_0 and $\tilde{\lambda}$ are pairwise distinct, u'_H is a $(\varphi \times \lambda'_0)$-center of $V(H) \cap U'$. Since $N_G(V(C')) \subseteq \tilde{S}$, the set $V(H)$ intersects $\tilde{S}$. Hence there is an edge vw in H with $v \in \tilde{S}$ and $w \in V(C')$. Since this edge is a connected subgraph of C intersecting both $\tilde{S}$ and $V(C')$, and as $q \geqslant 2$, by 11.21.(c), there is a $(\varphi \times \tilde{\lambda})$-center of $\{v, w\} \cap \tilde{S} = \{v\}$ and $\tilde{\lambda}(v) \in \tilde{\Lambda}_{C'}$. Therefore, $\lambda'_0(V(H) \cap U') \cap \tilde{\Lambda}_{C'}$ is nonempty. By the assumption on $\mathcal{I}$, $|\lambda_0(V(H) \cap U) \cap (\bigcup_{b\in[a]} \Lambda_b)| \geqslant a$. Since $\tilde{\Lambda}_{C'}$ is disjoint from $\bigcup_{b\in[a]} \Lambda_b$, we deduce that

$$|\lambda'_0(V(H) \cap U') \cap (\textstyle\bigcup_{b\in[a]} \Lambda_b \cup \tilde{\Lambda}_{C'})| \geqslant a + 1.$$

如果 $a = q$，那么我们有 $|\lambda'_0(V(H) \cap U')| > q$，矛盾。否则， $|\lambda'_0(V(H) \cap U') \cap (\bigcup_{b\in[a+1]} \Lambda'_b)| \geqslant a + 1$，如所需。

Next, suppose that $V(H)$ is disjoint from U. Then H is a connected subgraph of C. Hence, by 11.21.(c), and because $|\varphi(V(H))| \leqslant q$ and $|\tilde{\lambda}(V(H) \cap \tilde{S})| \leqslant q$, the vertex u'_H is a $(\varphi \times \tilde{\lambda})$-center of $V(H) \cap \tilde{S} = V(H) \cap U'$ such that $\lambda'_0(u'_H) \in \tilde{\Lambda}_{C'} = \Lambda'_1$. Clearly, $|\lambda'_0(V(H) \cap U) \cap (\bigcup_{b\in[1]} \Lambda'_b)| \geqslant 1$. This proves (ii) for $\mathcal{I}'$, and so $\mathcal{I}'$ is an instance.

˜由于 $\mathcal{F}' \neq \emptyset$，集合 S 非空，因此 $|V(C')| < |V(C)|$。所以我们可以对实例 $\mathcal{I}'$应用归纳假设。这将给出一个集合 $S_{C'}$，满足 $U' \subseteq S_{C'} \subseteq V(C') \cup U'$，以及一个着色 $\lambda_{C'}\colon S_{C'} \to \Lambda$，它扩展了 λ'_0，使得

(a') 对于每个 $F \in \mathcal{F}|_{C'}$，$S_{C'} \cap V(F) \neq \emptyset$；(b') 对于 $C' - S_{C'}$ 的每个连通分量 C''， $N_G(V(C''))$ 至多与 $G - V(C'')$ 的 2^k 个连通分量相交；并且

(c') 对于每个与 $V(C')$ 和 $S_{C'}$ 相交的连通子图 H G，

(1) $|\varphi(V(H))| > q$，或 (2) $|\lambda_{C'}(V(H) \cap S_{C'})| > q$，或 (3) 存在一个 $\varphi \times \lambda_{C'}$-中心 u，属于 $V(H) \cap S_{C'}$；此外，如果 $V(H) \cap U' \neq \emptyset$，那么我们可以令 $u = u'_H$。

Let

$$S = \bigcup_{C' \in \mathcal{C}} S_{C'}$$

并令 $\lambda\colon S \to \Lambda$ 定义如下。对于每个 $u \in S$，我们找到满足 $u \in S_{C'}$ 的 $C' \in \mathcal{C}$，并令

$$\lambda(u) = \lambda_{C'}(u).$$

这是定义良好的，因为着色 $\lambda_{C'}$ 在 $C' \in \mathcal{C}$ 上 U' 一致，因为它们都是 λ'_0的延拓。现在我们证明(a)–(c)成立。

令 $F \in \mathcal{F}|_C$。若 $U' \cap V(F) \neq \emptyset$则 $S \cap V(F) \neq \emptyset$。否则，存在 $C' \in \mathcal{C}$使得$F \in \mathcal{F}|_{C'}$。于是，由 (a')， $S_{C'} \cap V(F) \neq \emptyset$因此 $S \cap V(F) \neq \emptyset$。这就证明了 (a)。

˜设 C'' 是 $C - S$ 的一个连通分支。由于 C'' 与 S 不相交，因此要么 $N_G(V(C'')) \subseteq U \cup S$，要么存在 $C' \in \mathcal{C}$ 使得 $C'' \subseteq C'$。在第一种情况下， $N_G(V(C''))$ 至多与 2^{k-1}

个 $G - V(C)$ 的连通分支相交（由 (i)），并且至多与 $C - \tilde{S}$ 的 2^{k-1} 个连通分支相交（由 11.21.(b)）。这意味着 $N_G(V(C''))$ 至多与 $G - V(C'')$ 的 2^k 个连通分支相交。在第二种情况下，取 $C' \in \mathcal{C}$ 使得 $C'' \subseteq C'$。于是，由 (b')，$N_G(V(C''))$ 至多与$G - V(C'')$ 的 2^k 个连通分支相交。这就证明了 (b)。

Let H be a connected subgraph of G intersecting both $V(C)$ and S such that $|\varphi(V(H))| \leqslant q$ and $|\lambda(V(H) \cap S)| \leqslant q$. Let $C'_1, \ldots, C'_m$ be the components of $C - \tilde{S}$ intersecting $V(H)$ (with possibly $m = 0$). First, suppose that $V(H)$ intersects U. By (c').(3), the vertex $u'_H = u_H$ is a $(\varphi \times \lambda_0)$-center of $V(H) \cap S_{C'_i}$ for every $i \in [m]$. Therefore, u_H is a $(\varphi \times \lambda)$-center of $V(H) \cap S$. Now suppose that $V(H)$ is disjoint from U. Assume that $V(H)$ intersects $\tilde{S}$. By 11.21.(c), the vertex u'_H is a $(\varphi \times \tilde{\lambda})$-center u of $V(H) \cap S_{C'_i}$ for every $i \in [m]$, and so u'_H is a $(\varphi \times \lambda)$-center of $V(H) \cap S$. Finally, assume that $V(H)$ is disjoint from $\tilde{S} \cup U$. Then, there exists $C' \in \mathcal{C}$ such that $H \subseteq C'$. Therefore, by (c'), there is a $(\varphi \times \lambda_{C'})$-center u of $V(H) \cap S_{C'}$. By the definition of λ, and because $V(H) \cap S_{C'} = V(H) \cap S$, we conclude that u is a $(\varphi \times \lambda)$-center of $V(H) \cap S$. This shows (c) and concludes the proof of the lemma. □

Lemma 11.23. *Let c and k be positive integers. There exists a constant $\alpha_{11.23}(k)$ such that for every $X \in \mathbf{A}(\mathcal{R}_2)$, there exists a constant $c_{11.23}(c, k, X)$ such that the following holds. For every connected K_k-minor-free graph G, for every integer q with $q \geqslant 2$, for every (q, c)-good coloring φ of G, for every family $\mathcal{F}$ of connected subgraphs of G, if there is no $\mathcal{F}$-rich model of X in G, then there exists $S \subseteq V(G)$ such that*

(a) 对每个 $F \in \mathcal{F}$，$S \cap V(F) \neq \emptyset$；(b) 对于 $G - S$ 的每个分量 C，$N_G(V(C))$ 至多与 $G - V(C)$ 的 $\alpha_{11.23}(k)$ 个分量相交；以及 (c) $\mathrm{cen}_q(G, \varphi, S) \leqslant c_{11.23}(c, k, X) \cdot q \log q$。

Proof. We set $\alpha_{11.23}(k) = 2^k$. Let $X \in \mathbf{A}(\mathcal{R}_2)$. There exists $X' \in \mathcal{R}_2$ such that $X \subseteq K_1 \oplus X'$. Fix such a forest X', and let $c_{11.23}(c, k, X) = c_{11.22}(c, k, X')$. Let q be an integer with $q \geqslant 2$, let G be a connected K_k-minor-free graph, let $\mathcal{F}$ be a family of connected subgraphs of G such that there is no $\mathcal{F}$-rich model of X in G.

令 Λ 为一个包含 $c_{11.22}(c,\ k,\ X') \cdot q \log q$ 种颜色的集合，令 $y_0 \in \Lambda$，令 $\Lambda_1 = \{y_0\}$ 和 $\Lambda_2 = \cdots = \Lambda_q = \emptyset$。令 u 为 G 的任意一个顶点。令 $U = \{u\}$，令 $\lambda_0\colon U \to \Lambda$，其中 $\lambda_0(u) = y_0$。对于每个与 U 相交的 G 的连通子图 H，令 $u_H = u$。

设 $\mathcal{C}$ 为 $G - U$的所有连通分支构成的族，并设 $C \in \mathcal{C}$。根据引理 11.22，存在 $S_C \subseteq V(C) \cup U$ 包含 U，以及一个着色 $\lambda_C\colon S_C \to \Lambda$，它扩展了 λ_0，使得

11.22.(a) $S_C \cap V(F) \neq \emptyset$对于每个 $F \in \mathcal{F}$;11.22.(b) 对于每个连通分量 C'，属于 $C - S_C$，$N_G(V(C'))$ 至多与 2^k 的连通分量相交 $G - V(C')$；并且

11.22.(c) 对于每个与 H 相交的 G 连通子图 $V(C')$ 以及 S_C，或者

(1) $|\varphi(V(H))| > q$，或 (2) $|\lambda_C(V(H) \cap S)| > q$，或 (3) 存在一个 $\varphi \times \lambda_C$-中心 v，属于 $V(H) \cap S$；此外，如果 $V(H) \cap U \neq \emptyset$，那么我们可以令 $v = u$。

显然，11.22.(a) 和 11.22.(b) 分别推出 (a) 和 (b)。此外，对于每个 $C \in \mathcal{C}$ 和 $v \in S_C$，由 $\lambda(v) = \lambda_C(v)$ 定义的着色 $\lambda\colon \bigcup_{C \in \mathcal{C}} S_C \to \Lambda$ 见证了 $\mathrm{cen}_q(G, \varphi, \bigcup_{C \in \mathcal{C}} S_C) \leqslant c_{11.22}(c, k, X') \cdot q \log q = c_{11.23}(c, k, X) \cdot q \log q$。这证明了 (c)，并完成了证明。□

我们几乎可以完成定理 1.2 的证明了。总体思路与定理 11.14 的证明完全相同。回顾本小节开头引入的记号。我们在下面的陈述中概述基本情形。

推论 11.24。设 ℓ 为正整数，设 cen-k$_\ell$ $=$ (cen-k$_{\ell,q}$ $\mid$ $q \in \mathbb{N}_1$) 和 cen-t $=$(cen-t$_{\ell,q}$ $\mid$ $q \in \mathbb{N}_1$)，并设 par $\in$ {cen-k$_\ell$,cen-t}。函数 $q \mapsto q\log(q+1)$ 是 (par, $\mathbf{A}(\mathcal{R}_2)$)-有界的，适用于 par。

Proof. Let $X \in \mathbf{A}(\mathcal{R}_2)$. Let $\mathcal{U}$ be equal to $\mathcal{K}_\ell$ when par $=$ cen-k$_\ell$ and to $\mathcal{T}$ when par $=$ cen-t. Let $c = \max\{2, c_{11.7}$. For each graph G for which it does not hold that $V(G) \subseteq V(\mathcal{U})$ and $E(G) \subseteq E(\mathcal{U})$, choosing the empty set as S witnesses that G is even $(q \mapsto 1, \text{par } X, 1, 0)$-good. Let k be a positive integer and let k' be such that all K_k-minor-free graphs G have chromatic number at most k' or in other words, $\text{cen}_q(G) \leqslant k'$. Next, let

$$\beta(k, X) = \max\{k', c_{11.23}(c, k, X)\}$$

Let G be a graph for which $V(G) \subseteq V(\mathcal{U})$ and $E(G) \subseteq E(\mathcal{U})$. We claim that G is $(q \mapsto q\log(q+1), \text{par } X, \alpha_{11.23}(k)\ \beta(k, X)$-good. By Observation 5.9, we may assume without loss of generality that G is connected (see also the proof of Corollary 10.13). Let q be a positive integer and let $\mathcal{F}$ be a family of connected subgraphs of G such that G has no $\mathcal{F}$-rich model of X. Let Ξ_q be equal to $\Phi_{\ell,q}$ when par $=$ cen-k$_\ell$ and to Ψ_q when par $=$ cen-t. By assumption, $\Xi_q|_{V(G)}$ is (q, c)-good. Therefore, by Lemma 11.23, there exists $S \subseteq V(G)$ such that

11.23.(a) $S \cap V(F) \neq \emptyset$对每个 $F \in \mathcal{F}$ 都成立；11.23.(b) 对于 $G - S$ 的每个连通分量 C，$N_G(V(C))$ 至多与 $G - V(C)$ 的 $\alpha_{11.23}(k)$ 个连通分量相交；以及11.23.(c) $\text{cen}_q(G, \Xi_q|_{V(G)}\, S) \leqslant c_{11.23}(c,\ k,\ X) \cdot q\log(q+1)$。

这直接意味着

(g1*) $S \cap V(F) \neq \emptyset$对于每一个 $F \in \mathcal{F}$；(g2*) 对于 $G - S$的每一个分量 C，$N_G(V(C))$ 至多与$G - V(C)$的 $\alpha_{11.23}(k)$ 个分量相交；并且 (g3*) $\text{par}_q(G, S) \leqslant \beta(k, X) \cdot q\log(q+1)$。

因此，G 确实是 $(q \mapsto q\log(q+1), \text{par } X, \alpha_{11.23}(k)\ \beta(k, X)$-良好的，因此，$q \mapsto q\log(q+1)$ 是 par 的 (par,$\mathbf{A}(\mathcal{R}_2)$)-界。□

定理 11.25。设 t 为满足 $t \geqslant 3$的整数，并设 $X \in \mathcal{R}_t$。存在一个常数 c，使得对于每个 X-无子式图 G，以及每个满足 $q \geqslant 2$的整数 q，

$$\text{cen}_q(G) \leqslant c \cdot (\text{tw}(G) + 1) \cdot q^{t-2} \log q,$$

and

$$\text{cen}_q(G) \leqslant c \cdot q^{t-1} \log q.$$

Proof. Let ℓ be a positive integer, let cen-k$_\ell$ $=$ (cen-k$_{\ell,q}$ $\mid$ $q \in \mathbb{N}_1$) and cen-t $=$ (cen-t$_{\ell,q}$ $\mid$ $q \in \mathbb{N}_1$), and let par $\in$ {cen-k$_\ell$, cen-t}. By Lemma 5.12, the family of focused parameters par is nice. We show by induction on t (integers with $t \geqslant 3$) that $q \mapsto q^{t-2}\log(q+1)$ is (par $\mathcal{R}_t$)-bounding. Since $t \geqslant 3$, $\mathcal{R}_{t-1}$ is closed under disjoint union and leaf addition, and by Lemma 6.5, $\mathcal{R}_{t-1}$ has the coloring elimination property. By Corollary 11.24, the function $q \mapsto q\log(q+1)$ is (par, $\mathbf{A}(\mathcal{R}_2)$)-bounding, and so is (par $\mathcal{R}_3$)-bounding by Theorem 4.4. This gives the statement for $t = 3$, so suppose that $t \geqslant 4$. By induction, $q \mapsto q^{t-3}\log(q+1)$ is (par $\mathcal{R}_{t-1}$)-bounding. Therefore, by Theorem 4.2, $q \mapsto q^{t-2}\log(q+1)$ is (par $\mathcal{R}_t$)-bounding. This completes the inductive proof that for every integer t with $t \geqslant 3$, $q \mapsto q^{t-2}\log(q+1)$ is

（$\mathcal{R}_t$）-有界，如所需。由引理 4.5，这意味着对于每个 $X \in \mathcal{R}_t$，都存在一个常数 $\beta_{\mathrm{par}}(X)$，使得对于每个正整数 q 以及每个图 G，都有

$$\mathrm{par}_q(G, V(G)) \leqslant \beta_{\mathrm{par}}(X) \cdot q^{t-2} \log(q+1). \tag{13}$$

Let $X \in \mathcal{R}_t$, let q be an integer with $q \geqslant 2$, and let G be an X-minor-free graph. In particular, G is $K_{|V(X)|}$-minor-free, hence, there exists $i \in \mathbb{N}_1$ such that G is isomorphic to $G_{|V(X)|,i}$, which is one of the building blocks of $\mathcal{K}_{|V(X)|}$. Recall that $\Phi_{|V(X)|,q}|_{V(G_{|V(X)|,i})} = \varphi_{|V(X)|,i,q}$ uses at most $q+1$ colors. It follows that with $\ell = |V(X)|$, we have

$$\begin{aligned}
\mathrm{cen}_q(G) = \mathrm{cen}_q(G_{\ell,i}) &\leqslant \mathrm{cen}_q(G_{\ell,i}, \Phi_{\ell,q}|_{V(G_{\ell,i})}, V(G_{\ell,i})) \cdot (q+1) && \text{由引理5.8} \\
&= \text{cen-k}_\ell(G_{\ell,i}, V(G_{\ell,i})) \cdot (q+1) \\
&\leqslant \beta_{\text{cen-k}_\ell}(X) \cdot q^{t-2} \log(q+1) \cdot (q+1) && \text{by (13)}, \\
&\leqslant (4\beta_{\text{cen-k}_\ell}(X)) \cdot q^{t-1} \log q.
\end{aligned}$$

这给出了定理陈述中的第二个界。对于第一个界，注意到存在 $i \in \mathbb{N}_1$，使得 G 与 $H_{\mathrm{tw}(G)+1,\, i}$ 同构，而后者是 $\mathcal{T}_{\mathrm{tw}(G)+1}$ 的一个构件。回想一下，$\Psi_q|_{V(H_{\mathrm{tw}(G)+1,i})} = \psi_{\mathrm{tw}(G)+1,\, i,\, q}$ 至多使用 $\mathsf{tw}(G)+1$ 种颜色。因此，对于$\ell = \mathsf{tw}(G)+1$，我们有

$$\begin{aligned}
\mathrm{cen}_q(G) = \mathrm{cen}_q(H_{\ell,i}) &\leqslant \mathrm{cen}_q(H_{\ell,i}, \Psi_{\ell,q}|_{V(H_{\ell,i})}, V(H_{\ell,i})) \cdot \ell && \text{by Lemma 5.8} \\
&= \text{cen-t}(H_{\ell,i}, V(H_{\ell,i})) \cdot \ell \\
&\leqslant \beta_{\text{cen-t}}(X) \cdot q^{t-2} \log(q+1) \cdot \ell && \text{by (13)}. \\
&= 2\beta_{\text{cen-t}}(X) \cdot (\mathrm{tw}(G)+1) \cdot q^{t-2} \log q. && \square
\end{aligned}$$

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附录 A. 下界

A.1. **Weak coloring numbers.** In this section, we present a construction by Grohe, Kreutzer, and Siebertz [16], which was initially presented as a lower bound for weak coloring numbers of graph of treewidth t. This was later adapted to graph of simple treewidth at most t by Joret and Micek [26]. To show the link between these constructions and the classes $(\mathcal{R}_t)_{t\in\mathbb{N}_1}, (\mathcal{S}_t)_{t\in\mathbb{N}_1}$, we elected to provide them in the current appendix. The core of the argument is the following lemma.

引理 A.1。 设 $\mathcal{G}$ 为一个非空图类，并且对每个非负整数 q， $g(q) = \max_{G\in\mathcal{G}} \mathrm{wcol}_q(G)$ 。对每个非负整数 q，存在 $G \in \mathbf{T}(\mathcal{G})$ 使得

$$\mathrm{wcol}_q(G) \geqslant \sum_{i=0}^{q} g(i).$$

证明。 我们对 q 进行归纳。首先，假设 $q = 0$。接着考虑 $G \in \mathcal{G}$ ，使得 $\mathrm{wcol}_q(G) = g(0)$。由于 $\mathcal{G} \subseteq \mathbf{T}(\mathcal{G})$， G 符合要求。
接下来，假设 $q \geqslant 1$。根据归纳假设，存在 $G_0 \in \mathbf{T}(\mathcal{G})$，使得

$$\mathrm{wcol}_{q-1}(G_0) \geqslant \sum_{i=0}^{q-1} g(i).$$

令 $H \in \mathcal{G}$ 使得 $\mathrm{wcol}_q(H) = g(q)$。令 $k = \sum_{i=0}^{q} g(i)$。令 G 为一个图，它由 G_0 得到：对每个 $u \in V(G_0)$，添加 k 个 $H_{u,1}, \ldots, H_{u,k}$ 的互不相交副本，以及 u 与 $V(H_{u,i})$ 之间所有可能的边，对每个 $i \in [k]$ 均如此。

We claim that $G \in \mathbf{T}(\mathcal{G})$. Let $\big(T_0, (W_{0,x} \mid x \in V(T_0))\big)$ be a rooted forest decomposition of G_0 witnessing the fact that $G_0 \in \mathbf{T}(\mathcal{G})$. Then, for every $u \in V(G_0)$, choose $x_u \in V(T_0)$ such that $u \in W_{0,x_u}$. Let T be the rooted forest obtained from T_0 by adding, for every $u \in V(G_0)$ and for every $i \in [N]$, a leaf $x_{u,i}$ with parent x_u. Then, for every $x \in V(T)$, let

$$W_x = \begin{cases} W_{0,x} & \text{if } x \in V(T_0) \\ \{u\} \cup V(H_{u,i}) & \text{if } x = x_{u,i} \text{ for } u \in V(G_0),\ i \in [k]. \end{cases}$$

由此可知， $\big(T, (W_x \mid x \in V(T))\big)$ 是 G 的根森林分解，证明了 $G \in \mathbf{T}(\mathcal{G})$.

还需证明 $\mathrm{wcol}_q(G) \geqslant k$。令 σ 为 $V(G)$ 的一个排序。由于 $\mathrm{wcol}_{q-1}(G_0) \geqslant \sum_{i=0}^{q-1} g(i)$，因此存在 $v \in V(\{\mathrm{v13}),$ 使得

$$\left|\mathrm{WReach}_{q-1}\left[G_0, \sigma|_{V(G_0)}, v\right]\right| \geqslant \sum_{i=0}^{q-1} g(i).$$

If, for every $i \in [k]$, there is a vertex w in $V(H_{v,i})$ such that $w <_\sigma v$, then $|\,\mathrm{WReach}_q[G, \sigma, v]| \geqslant k$. Otherwise, there exists $i \in [k]$ such that every vertex w in $H_{v,i}$ is such that $w >_\sigma v$. Since $\mathrm{wcol}_q(H_{v,i}) = \mathrm{wcol}_q(H) = g(q)$, there exists $u \in V(H_{v,i})$ such that $\left|\mathrm{WReach}_q\left[H_{v,i}, \sigma|_{V(H_{v,i})}\ u\right]\right| \geqslant g(q)$. Then,

$$\mathrm{WReach}_{q-1}\left[G_0, \sigma|_{V(G_0)}, v\right] \cup \mathrm{WReach}_q\left[H_{v,i}, \sigma|_{V(H_{v,i})}, u\right] \subseteq \mathrm{WReach}_q\left[G, \sigma|_{V(G)}, u\right],$$

由此我们得出 $|\mathrm{WReach}_r[G, \sigma, u]| \geqslant \left(\sum_{i=0}^{q-1} g(i)\right) + g(q) = k$。

□

推论 A.2. 对于每个满足 $t \geqslant 1$ 的整数 t，

$$\max_{G \in \mathcal{R}_t} \mathrm{wcol}_q(G) = \Omega(q^{t-1}).$$

证明。 我们对 t 进行归纳，证明对于每个非负整数 q，

$$\max_{G \in \mathcal{R}_t} \mathrm{wcol}_q(G) \geqslant \binom{q+t-1}{t-1}.$$

对于 $t = 1$，回顾一下， $K_1 \in \mathcal{R}_1$。因此，对于所有非负整数 q，我们有 $\mathrm{wcol}_q(K_1) = 1 \geqslant \binom{q}{0}$，证毕。
对于 $t \geqslant 2$，根据归纳假设，对于每个非负整数 q，

$$\max_{G \in \mathcal{R}_{t-1}} \mathrm{wcol}_q(G) \geqslant \binom{q+t-2}{t-2}.$$

因此，根据引理 A.1，

$$\max_{G \in \mathcal{R}_t} \mathrm{wcol}_q(G) \geqslant \sum_{i=0}^{q} \binom{i+t-2}{t-2} = \binom{q+t-1}{t-1}.$$ □

推论 A.3。 对于满足 $t \geqslant 2$ 的每个正整数 t，

$$\max_{G \in \mathcal{S}_t} \mathrm{wcol}_q(G) = \Omega(q^{t-2} \log q).$$

证明。 我们对 t 进行归纳。对于 $t = 2$， $\mathcal{S}_2$ 包含所有路径，其第 q 弱着色数属于 $\Omega(\log q)$，参见 [26]。
对于 $t \geqslant 3$，根据归纳假设，存在 $c > 0$，使得对每一个满足$q \geqslant 2$的整数 q，

$$\max_{G \in \mathcal{S}_{t-1}} \mathrm{wcol}_q(G) \geqslant c \cdot q^{t-3} \log q.$$

因此，由引理 A.1，对每一个满足 $q \geqslant 4$ 的整数 q，

$$\begin{aligned}
\max_{G \in \mathcal{S}_t} \mathrm{wcol}_q(G) &\geqslant c \cdot \sum_{i=1}^{q} i^{t-3} \log i \\
&\geqslant c \cdot \sum_{q/2 \leqslant i \leqslant q} i^{t-3} \log i \\
&\geqslant c \cdot \left\lfloor \frac{q-1}{2} \right\rfloor (q/2)^{t-3} \log(q/2) \\
&\geqslant c \cdot \frac{q}{4} \cdot \frac{1}{2^{t-3}} \cdot q^{t-3} \cdot \frac{\log q}{2} \\
&= \frac{c}{2^t} \cdot q^{t-2} \log q.
\end{aligned}$$ □

A.2. **中心色数。** 本节中，我们证明 $\mathcal{R}_t$ 与 $\mathcal{S}_t$ 中图的中心色数的下界。这是对发表于 [10] 的证明按我们的记号所作的改写。

设 q 为正整数。设 G 为图。设 K 为颜色集。设 $\varphi\colon V(G) \to K$为一种着色，且 $\Xi\colon V(G) \to 2^{2^K}$ 为一个函数，它将 G 的每个顶点映射到 K中的一个颜色集族。二元组 (φ, Ξ) 称为广义 q-中心着色，如果对于每个连通

子图 H 属于 G，对于每个函数 $\xi \in \prod_{u\in V(H)} \Xi(u)$[10]，它将每个顶点 $u \in V(H)$ 映射到 $\Xi(u)$ 中的一个颜色集，要么

(i) $|\varphi(V(H)) \cup \bigcup \xi(V(H))| > q$ 或 (ii) 存在一个顶点 $u \in V(H)$，使得 $\varphi(u) \notin \varphi(V(H) - \{u\}) \cup \bigcup \xi(V(H))$。

(φ, Ξ) 使用的颜色数为整数 $|\varphi(V(G)) \cup \bigcup_{u\in V(G)} \bigcup \Xi(u)|$。注意，如果 $(\varphi, u \mapsto \{\emptyset\})$ 是 G 的广义 q-中心着色，则 φ 是 G 的 q-中心着色。

Let p be a nonnegative integer. A mapping $\Xi\colon V(G) \to 2^{2^K}$ is said to be *p-good* if for every $u \in V(G)$, $\emptyset \in \Xi(u)$, and for every subset of colors $L \subseteq K$, if there exists a set of colors $M \in \Xi(u)$ such that $L \subseteq M$, then there exists a set of colors $M' \in \Xi(u)$ such that $L \subseteq M'$ and $|M' - L| \leqslant p$. Note that a p-good mapping is also a $(p+1)$-good mapping and every image of a 0-good mapping is a family of sets that is closed under taking subsets.

现在，我们将 $\mathrm{cen}^{(k)}_{q\,p,}$ (G) 定义为满足下列条件的最小整数：ℓ 存在一个广义的q-中心着色 $(\varphi,\ \Xi)$ 对 G 使用至多 k 种颜色，且满足

(i) Ξ 是 p良好的，并且 (ii) 对于每个 $u \in V(G)$，存在 $L \in \Xi(u)$，使得 $|\varphi(V(G)) - L| \leqslant \ell$。

By convention, we set $\mathrm{cen}^{(k)}_{q,p}(G) = \infty$ if there is no such integer ℓ. Note that for every nonnegative integer p, $\mathrm{cen}^{(k)}_{q,p+1}(G) \leqslant \mathrm{cen}^{(k)}_{q,p}(G)$. Moreover, if φ is a q-centered coloring of G, then $(\varphi, u \mapsto \{\emptyset\})$ is a generalized q-centered coloring of G, and $u \mapsto \{\emptyset\}$ is 0-good. Therefore, for every positive integer k,

$$\mathrm{cen}_q(G) \geqslant \min\{k, \mathrm{cen}^{(k)}_{q,0}(G)\}.$$

引理 A.4. 设 k 与 q 为正整数。对每一个满足 $n \leqslant 2q$ 的正整数 n，

$$\mathrm{cen}^{(k)}_{4q,q}(P_n) \geqslant \lfloor \log n \rfloor.$$

证明。只需考虑某个非负整数 ℓ 的情形，即 $n = 2^\ell$。设 $(\varphi,\ \Xi)$ 是 P_n 的一个广义 $4q$-中心着色，至多使用 k 种颜色，且具有 Ξ q-良性质；令 N 为 $V(P_n)$ 中所有 φ-中心的颜色集合，即 $N = \{\alpha \mid \alpha \in \varphi(V(P_n))\ |\varphi^{-1}(\alpha)| = 1\}$。

我们会晃荡。1 通过对 ℓ 作归纳证明，存在 $u \in V(P_n)$，使得对每个 $L \in \Xi(u)$ ，

$$|N - L| \geqslant \ell.$$

当 $\ell = 0$ 时，这一点很明显。接下来，假设 $\ell \geqslant 1$。

令 Q_1, Q_2 分别为 P_n 中由前 $n/2 = 2^{\ell-1}$ 个顶点和后 $n/2 = 2^{\ell-1}$ 个顶点诱导的子路径。

令 $N_1 = N \cap \varphi(V(Q_1))$ 和 $N_2 = N \cap \varphi(V(Q_2))$。根据 N 的定义，N_1 和 N_2 不相交。

假设存在 $u_1 \in V(Q_1)$ 和 $L_1 \in \Xi(u_1)$，使得 $N_2 \subseteq L_1$；并且存在 $u_2 \in V(Q_2)$ 和$L_2 \in \Xi(u_2)$，使得 $N_1 \subseteq L_2$。由于 Ξ 是 q-良好的，因此存在 $M_1 \in \Xi(u_1)$ 和 $M_2 \in \Xi(u_2)$，使得 $N_2 \subseteq M_1$, $N_1 \subseteq M_2$, $|M_1 - N_2| \leqslant q$，且 $|M_2 - N_1| \leqslant q$。令 $\xi \in \prod_{u\in V(P_n)} \Xi(u)$ 定义为

$$\xi(u) = \begin{cases} M_1 & \text{if } u = u_1, \\ M_2 & \text{if } u = u_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

由于 (φ, Ξ) 是 $4q$-中心的，因此以下情形之一成立：

[10]~~给定一个~~集合 X 和一个集合族 $(Y_x \mid x \in X)$，我们用 $\prod_{x\in X} Y_x$ 表示所有满足以下条件的函数 f：$X \to \bigcup_{x\in X} Y_x$ 的集合：对于每个 $x \in X$，都有 $f(x) \in Y_x$。

(i) $|\varphi(V(P_n)) \cup \bigcup \xi(V(P_n))| > 4q$, but then $|V(P_n)| + |M_1 - N| + |M_2 - N| > 4q$, which implies $n > 4q - |M_1 - N_2| - |M_2 - N_1| = 2q$, a contradiction; or
(ii) there is a vertex u of $V(P_n)$ such that $\varphi(u) \notin \varphi(V(P_n) - \{u\}) \cup \bigcup \xi(V(P_n))$. In particular, u must be a φ-center of P_n, so $\varphi(u) \in N$. However, $\varphi(u) \in N = N_1 \cup N_2 \subseteq M_2 \cup M_1 \subseteq \bigcup \xi(V(P_n))$, a contradiction.

这证明存在 $i \in \{1,2\}$，使得对于每个 $u \in V(Q_i)$，对于每个 $L \in \Xi(u)$，$N_{3-i} - L \neq \emptyset$。现在我们对 Q_i、$\varphi|_{V(Q_i)}$ $\Xi|_{V(Q_i)}$应用归纳假设。因此，存在 $u \in V(Q_i)$，使得对于每个 $L \in \Xi(u)$，

$$|N_i - L| \geqslant \ell - 1.$$

由于 N_i 和 N_{3-i} 不相交，我们推得对于每个 $L \in \Xi(u)$，

$$|N - L| \geqslant |N_i - L| + |N_{3-i} - L| \geqslant \ell.$$

至此，归纳证明完成。

由于 Ξ 是 q-良好的，并且存在 $u \in V(P_n)$，使得对于每个 $L \in \Xi(u)$，$|\varphi(V(P_n)) - L| \geqslant |N - L| \geqslant \ell = \lfloor \log n \rfloor$，这也完成了引理的证明。□

引理 A.5。设 k 和 q 为正整数，设 $\mathcal{G}$ 为非空图类，并且对于每个非负整数 p，有$g(p) =$ 最大$G \in \mathcal{G}$ 居中$^{(k)}_{q\,p,}(G)$ 。对于满足 $0 \leqslant p \leqslant q$的每个整数 p，存在 $G \in \mathbf{T}(\mathcal{G})$ 使得

$$\mathrm{cen}^{(k)}_{q,p}(G) \geqslant \sum_{r=p}^{q} g(r).$$

证明。我们对 $q - p$ 进行归纳。当 $q = p$ 时，取 $G \in \mathcal{G}$ 使得 $\mathsf{cen}^{(k)}_{q\,p,}(G) = g(q)$。由于 $\mathcal{G} \subseteq \mathbf{T}(\mathcal{G})$，$G$ 正如所需。接下来，假设 $q - p \geqslant 1$。
根据归纳假设，存在 $G_0 \in \mathbf{T}(\mathcal{G})$，使得

$$\mathrm{cen}^{(k)}_{q,p+1}(G_0) \geqslant \sum_{r=p+1}^{q} g(r).$$

此外，根据 g 的定义，存在 $G' \in \mathcal{G}$ 使得

$$\mathrm{cen}^{(k)}_{q,p}(G') = g(p).$$

设 G 是由 G_0 按如下方式得到的图。对于每个 $u \in V(G_0)$，考虑 G' 的互不相交的副本$G'_{u,i}$，其中 $i \in [2^k + 1]$，并在 u 与 $\bigcup_{i=1}^{2^k+1} V$ 之间添加所有可能的边 $(G'_{u,i})$ 。

First, we show that $G \in \mathbf{T}(\mathcal{G})$. Let $\big(T_0, (W_{0,x} \mid x \in V(T_0))\big)$ be a rooted forest decomposition of G_0 witnessing the fact that $G_0 \in \mathbf{T}(\mathcal{G})$. Then, for every $u \in V(G_0)$, choose $x_u \in V(T_0)$ such that $u \in W_{0,x_u}$. Let T be the rooted forest obtained from T_0 by adding, for every $u \in V(G_0)$ and for every $i \in [2^k + 1]$, a leaf $x_{u,i}$ with parent a vertex x_u. Then, for every $x \in V(T)$, let

$$W_x = \begin{cases} W_{0,x} & \text{if } x \in V(T_0) \\ \{u\} \cup V(G'_{u,i}) & \text{if } x = x_{u,i} \text{ for } u \in V(G_0),\ i \in [2^k + 1]. \end{cases}$$

由此可得，$\big(T,\ (W_a \mid a \in V(T))\big)$ 是 G 的一个根森林分解，证明了 $G \in \mathbf{T}(\mathcal{G})$。

现在我们证明 $\mathsf{cen}^{(k)}_{q\,p,}(G) \geqslant \sum_{r=p}^{q} g(r)$。

设 (φ, Ξ) 是 G 的一个广义 q-中心着色，至多使用 k 种颜色，且满足 Ξ p-良好。令 $u \in V(G_0)$。根据抽屉原理，存在 $i, j \in [2^k + 1]$，使得 $\varphi(V(G'_{u,i})) =$

$\varphi(V(G'_{u,j}))$。不失一般性，假设 $i=1$ 且 $j=2$。由于 $\mathsf{cen}_{q\,p,}^{(k)}(G'_{u,1}) = g(p)$，并且 $(\varphi|_{V(G'_{u,1})}$ $\Xi|_{V(G'_{u,1})})$ 是 $G'_{u,1}$ 的一个广义 q-中心着色，因此我们定义

$$v(u) \in V(G'_{u,1}) \text{ such that for every } M \in \Xi(v(u)), |\varphi(V(G'_{u,1})) - M| \geqslant g(p).$$

Let

$$\Xi_0(u) = \{\emptyset\} \cup \big\{L \cup M \mid \varphi(v(u)) \in L \subseteq \varphi(V(G'_{u,1})), M \in \Xi(v(u))\big\} \text{ and}$$
$$\varphi_0 = \varphi|_{V(G_0)}.$$

断言 A.6. $(\varphi_0,\ \Xi_0)$ 是 G_0 的广义 q-中心着色。

断言的证明。设 H_0 为 G_0 的一个连通子图，并设$\xi_0 \in \prod_{u\in V(H_0)} \Xi_0(u)$。我们需要证明要么

(a) $|\varphi_0(V(H_0)) \cup \bigcup \xi_0(V(H_0))| > q$ 或者 (b) 存在一个顶点 $u \in V(H_0)$，使得 $\varphi_0(u)$ $\notin \varphi_0(V(H_0) - \{u\}) \cup \bigcup \xi_0(V(H_0))$。

考虑按照如下方式从 H_0 得到 H 的子图。令 $u \in V(H_0)$。若 $\xi_0(u) = \emptyset$，则令 $L(u) = M(u) = \emptyset$。若 $\xi_0(u) \neq \emptyset$，则固定 $L(u) \subseteq \varphi(V(G'_{u,1}))$，使得 $\varphi(v(u)) \in L(u)$ 且$M(u) \in \Xi(v(u))$，满足

$$\xi_0(u) = L(u) \cup M(u).$$

由于 $\varphi(V(G'_{u,\ 1})) = \varphi(V(G'_{u,\ 2}))$，存在 $V_1 \subseteq V(G'_{u,\ 1})$ 和 $V_2 \subseteq V(G'_{u,\ 2})$，使得$\varphi(V_1) = \varphi(V_2) = L(u)$，且 $v(u) \in V_1$。然后，将 $V_1 \cup V_2$ 中的顶点添加到 H 中，并对每个$w \in V_1 \cup V_2$添加边 uw。

我们定义 $\xi \in \prod_{x\in V\ (H)} \Xi(x)$ 如下。设

$$\xi(x) = \begin{cases} \emptyset & \text{if } x \in V(H_0), \\ M(u) & \text{if } x \notin V(H_0) \text{ and } x = v(u) \text{ for some } u \in V(H_0), \\ \emptyset & \text{otherwise.} \end{cases}$$

至此，H 和 ξ的定义完成。由于 H_0 是连通的，且所有添加的顶点都是 H_0中顶点的邻点，因此我们同样可以得出 H 是连通的。

注意到

$$\varphi(V(H) - V(H_0)) = \bigcup\{L(u) \mid u \in V(H_0)\},$$
$$\bigcup \xi(V(H)) = \bigcup\{M(u) \mid u \in V(H_0)\}, \text{ and}$$
$$\bigcup \xi_0(V(H_0)) = \bigcup\{L(u) \cup M(u) \mid u \in V(H_0)\}.$$

这三个方程意味着

$$\varphi_0(V(H_0)) \cup \bigcup \xi_0(V(H_0)) = \varphi(V(H)) \cup \bigcup \xi(V(H)).$$

由于 $(\varphi,\ \Xi)$ 是 G 的广义 q-中心着色，并且因为 H 是 G的连通子图，所以以下两者必有其一

(i) $|\varphi(V(H)) \cup \bigcup \xi(V(H))| > q$；或 (ii) 存在一个顶点 $u \in V(H)$，使得 $\varphi(u)$ $\notin \varphi(V(H) - \{u\}) \cup \bigcup \xi(V(H))$.

在前一种情况下，我们有 $|\varphi_0(V(H_0)) \cup \bigcup \xi_0(V(H_0))| = |\varphi(V(H)) \cup \bigcup \xi(V(H))| > q$，因此 (a) 成立。在后一种情况下，固定这样一个顶点 $u \in V(H)$。现在我们证明 $\varphi_0(u) \notin \varphi_0(V(H_0) - \{u\}) \cup \bigcup \xi_0(V(H_0))$。由于 $V(H)$ 中每个顶点的颜色在 $-V(H_0)$ 下至少重复出现两次

，我们得出 $u \in V(H_0)$。由于 $u \in V(H_0)$，

$$\begin{aligned}\varphi(V(H) - \{u\}) &= \varphi_0(V(H_0) - \{u\}) \cup \varphi(V(H) - V(H_0)) \\ &= \varphi_0(V(H_0) - \{u\}) \cup \bigcup\{L(v) \mid v \in V(H_0)\}.\end{aligned}$$

因此，我们有

$$\begin{aligned}\varphi(V(H) - \{u\}) \cup \bigcup \xi(V(H)) \\ &= \varphi_0(V(H_0) - \{u\}) \cup \bigcup\{L(v) \mid v \in V(H_0)\} \cup \bigcup\{M(v) \mid v \in V(H_0)\} \\ &= \varphi_0(V(H_0) - \{u\}) \cup \xi_0(V(H_0)).\end{aligned}$$

最后，利用 (ii)，我们得出

$$\varphi_0(u) = \varphi(u) \notin \varphi(V(H) - \{u\}) \cup \bigcup \xi(V(H)) = \varphi_0(V(H_0) - \{u\}) \cup \xi_0(V(H_0)),$$

因此 (b) 成立。这证明 (φ_0, Ξ_0) 是 q-中心着色 G_0。 ◊

断言A.7。 Ξ_0 是 $(p+1)$-良好的。

Proof of the claim. For every set of colors N, if $N \subseteq L \cup M$ for some $L \subseteq \varphi(V(G'_{u,1}))$ with $\varphi(v(u)) \in L$ and $M \in \Xi(v(u))$, then, because Ξ is p-good, there exists $M' \in \Xi(v(u))$ such that $N \cap M \subseteq M'$ and $|M' - (N \cap M)| \leqslant p$, and it follows that $M_0 = (L \cap N) \cup \{\varphi(v(u))\} \cup M'$ is a member of $\Xi_0(v(u))$ such that $N \subseteq M_0$ and $|M_0 - N| \leqslant p + 1$. This proves the claim. ◊

由于 $\mathrm{cen}_q^{(k)}{}_{,p+1}(G_0) \geqslant \sum_{r=p+1}^{q} g(r)$ 并且由断言 A.6 和 A.7，我们推出存在$u \in V(G_0)$ 使得对每一个 $M_0 \in \Xi_0(u)$,

$$|\varphi(V(G_0)) - M_0| \geqslant \sum_{r=p+1}^{q} g(r).$$

因此，对每一个 $M \in \Xi(v(u))$，将此不等式应用于 $M_0 = \varphi(V(G'_{u,1})) \cup M$，我们推出

$$|\varphi(V(G_0)) - (\varphi(V(G'_{u,1})) \cup M)| \geqslant \sum_{r=p+1}^{q} g(r).$$

此外，由 $v(u)$ 的定义，

$$|\varphi(V(G'_{u,1})) - M| \geqslant g(p)$$

由于颜色集 $\varphi(V(G_0)) - (\varphi(V(G'_{u,1})) \cup M)$ 与 $\varphi(V(G'_{u,1})) - M$ 互不相交，我们推出

$$|\varphi(V(G)) - M| \geqslant g(p) + \sum_{r=p+1}^{q} g(r) = \sum_{r=p}^{q} g(r).$$

这证明了 $\mathrm{cen}_{q\,p,}^{(k)}(G) \geqslant \sum_{r=p}^{q} g(r)$，并完成了该引理的证明。 □

推论 A.8. 对每一个正整数 t,

$$\max_{G \in \mathcal{R}_t} \mathrm{cen}_q(G) = \Omega(q^{t-1}).$$

证明。 设 q 和 t 为正整数。设 $k = \binom{q-1+t-1}{t-1} - 1$。

断言 A.9。对于每个正整数 s，以及每个满足 $p < q$ 的非负整数 p，

$$\max_{G \in \mathcal{R}_s} \mathrm{cen}_{q,p}^{(k)}(G) \geqslant \binom{q-1+s-1-p}{s-1}.$$

断言的证明。我们对 s 进行归纳。

First suppose $s = 1$. Note that $\binom{q-1+s-1-p}{s-1} = 1$. Consider $K_1 \in \mathcal{R}_1$, and let (φ, Ξ) be a generalized q-centered coloring of K_1 such that Ξ is p-good. We denote by u the unique vertex of K_1. Then, consider $L \in \Xi(u)$. If $\varphi(u) \in L$, then, since Ξ is p-good, there exists $M \in \Xi(u)$ such that $\varphi(u) \in M$ and $|M| \leqslant p + 1 \leqslant q$. But then, the subgraph K_1 equipped with $\xi\colon u \mapsto \{M\}$ uses at most q color and $\varphi(u) \in \xi(\{u\})$. This contradicts the fact that (φ, Ξ) is a generalized q-centered coloring of K_1. Therefore, $\varphi(u) \notin L$, and so $|\varphi(V(K_1)) - L| \geqslant 1$. This proves that $\mathrm{cen}^{(k)}_{q,p}(K_1) \geqslant 1$ and concludes the case $s = 1$.

现在，假设 $s > 1$。由 $\mathcal{R}_s = \mathbf{T}(\mathcal{R}_{s-1})$这一事实、引理A.5以及归纳假设，我们推得

$$\begin{aligned}\max_{G\in\mathcal{R}_s} \mathrm{cen}^{(k)}_{q,p}(G) &\geqslant \sum_{r=p}^{q} \max_{G\in\mathcal{R}_{s-1}} \mathrm{cen}^{(k)}_{q,r}(G)\\ &\geqslant \sum_{r=p}^{q-1} \max_{G\in\mathcal{R}_{s-1}} \mathrm{cen}^{(k)}_{q,r}(G)\\ &\geqslant \sum_{r=p}^{q-1} \binom{q-1+s-2-r}{s-2} = \binom{q-1+s-1-p}{s-1}.\end{aligned}$$

这就证明了归纳法。 ◊

By the previous claim, we consider a graph $G \in \mathcal{R}_t$ such that $\mathrm{cen}^{(k)}_{q,0}(G) \geqslant \binom{q-1+t-1}{t-1}$. Suppose by contradiction that G admits a q-centered coloring φ using at most k colors. Then $(\varphi, v \mapsto \{\emptyset\})$ is a generalized q-centered coloring of G and $v \mapsto \{\emptyset\}$ is 0-good. Therefore, there exists $u \in V(G)$, and $L \in \{\emptyset\}$ such that $|\varphi(V(G)) - L| = |\varphi(V(G))| \geqslant \binom{q-1+t-1}{t-1}$, contradicting the fact that φ uses at most k colors. Thus, every q-centered coloring of G uses at least $\binom{q-1+t-1}{t-1}$ colors. Since $\binom{q-1+t-1}{t-1} = \Omega(q^{t-1})$, this proves the corollary. □

推论 A.10。对于每个满足 $t \geqslant 2$ 的整数 t，

$$\max_{G\in\mathcal{S}_t} \mathrm{cen}_q(G) = \Omega(q^{t-2}\log q).$$

证明。令 q 和 t 为正整数。令 $k = \binom{q+t-2}{t-2}\lfloor \log q \rfloor - 1$。

断言 A.11。对于每个满足 $s \geqslant 2$ 的整数 s，以及每个满足 $p \leqslant q$ 的非负整数 p，

$$\max_{G\in\mathcal{S}_s} \mathrm{cen}^{(k)}_{4q,p}(G) \geqslant \binom{q+s-2-p}{s-2} \log q.$$

断言的证明。我们对 s 进行归纳。若 $s = 2$，则 $\binom{q+s-2-p}{s-2} = 1$，由引理 A.4 以及对每个非负整数 $p \leqslant q$ 都有 $\mathsf{cen}^{(k)}_{4q\,p,}(G) \geqslant \mathsf{cen}^{(k)}_{4q\,q,}(G)$ 这一事实，可得该不等式成立。接下来，假设 $s > 2$。

作者 Le 引理 A.5，因为 $\mathcal{S}_s = \mathbf{T}(\mathcal{S}_{s-1})$，再应用归纳假设，我们可以推出

$$\begin{aligned}\max_{G\in\mathcal{S}_s} \operatorname{cen}^{(k)}_{4q,p}(G) &\geqslant \sum_{r=p}^{4q} \max_{G\in\mathcal{S}_{s-1}} \operatorname{cen}^{(k)}_{4q,r}(G)\\ &\geqslant \sum_{r=p}^{q} \max_{G\in\mathcal{S}_{s-1}} \operatorname{cen}^{(k)}_{4q,r}(G)\\ &\geqslant \sum_{r=p}^{q} \binom{q+s-3-r}{s-3} \log q = \binom{q+s-2-p}{s-2} \log q.\end{aligned}$$

归纳证明至此结束。 ◊

By the previous claim, we consider a graph $G \in \mathcal{S}_t$ such that $\operatorname{cen}^{(k)}_{4q,0}(G) \geqslant \binom{q+t-2}{t-2} \log q$. Suppose by contradiction that G admits a $4q$-centered coloring φ using at most k colors. Then $(\varphi, v \mapsto \{\emptyset\})$ is a generalized $4q$-centered coloring of G and $v \mapsto \{\emptyset\}$ is 0-good. Therefore, there exist $u \in V(G)$ and $L \in \{\emptyset\}$ such that $|\varphi(V(G)) - L| = |\varphi(V(G))| \geqslant \binom{q+t-2}{t-2} \log q > k$, contradicting the fact that φ uses at most k colors. Therefore, every $4q$-centered coloring of G uses at least $\binom{q+t-1}{t-1} \lfloor \log q \rfloor$ colors. Since $\binom{\lfloor q/4 \rfloor + t-2}{t-2} \lfloor \log(\lfloor q/4 \rfloor) \rfloor = \Omega(q^{t-2} \log q)$, this proves the corollary. □

A.3. **分数树深脆弱率**。 在本节中，我们证明位于 $\mathcal{R}_t$ 与 $\mathcal{S}_t$ 中的图的分数树深脆弱率下界（见推论 A.19 与 A.20）。这本质上已由 Dvořák 与 Sereni [9], 证明，但我们选择给出完整证明（沿用相同方法），因为他们的陈述仅以树宽和简单树宽来表述。

对于每个图 G 以及每个函数$\mathtt{t}\colon V(G) \to \mathbb{N}$，我们递归地定义$\operatorname{td}(G, \mathtt{t})$如下 (i) $\operatorname{td}(G,\mathtt{t}) = 0$ 如果 G 是空图；(ii) $\operatorname{td}(G,\mathtt{t}) = \max\{\operatorname{td}(C,\mathtt{t}|_{V(C)}) \mid C$的连通分量，如果 $G\}$ 非空且不连通；以及 (iii) $\operatorname{td}(G,\mathtt{t}) = 1 + \min_{u\in V(G)} \max\{\mathtt{t}(u), \operatorname{td}(G-u,\mathtt{t}|_{V(G-u)})\}$ 否则。

一个直接的归纳表明，对每个图 G，对 G 的每个子图 G'，以及对每个 $\mathtt{t}\colon V(G) \to \mathbb{N}$，

$$\operatorname{td}(G,\mathtt{t}) \geqslant \operatorname{td}(G', \mathtt{t}|_{V(G')}).$$

首先，我们证明 $\operatorname{td}(\cdot,\cdot)$ 的另外几个性质。

引理 A.12. 设 G 是一个图，设 $t\colon V(G) \to \mathbb{N}$，设 $U \subseteq V(G)$ 且 $G[U]$ 连通，并设$u \in U$。设 G' 是由 G 将 U 中的顶点等同为单个顶点w而得到的图。设 $t'\colon V(G') \to \mathbb{N}$ 满足：对每个 $x \in V(G) - U$，有 $t'(w) \leqslant t(u)$ 且 $t'(x) \leqslant t(x)$。于是我们有

$$\operatorname{td}(G,\mathtt{t}) \geqslant \operatorname{td}(G', \mathtt{t}').$$

证明。 我们对 $|V(G)|$ 进行归纳。若 $V(G) = U$，则结论显然。接下来，假设 $V(G) - U \neq \emptyset$。

假设 G 不连通。考虑 G 和 G' 的所有连通分支组成的族 $\mathcal{C}$ 和 $\mathcal{C}'$。令 $C_0 \in \mathcal{C}$ 为包含 U 的 G 的连通分支。该连通分支对应于 G' 的一个连通分支 $C'_0 \in \mathcal{C}'$。根据归纳假设，$\operatorname{td}(C_0, \mathtt{t}|_{V(C_0)}) \geqslant \operatorname{td}(C'_0, \mathtt{t}'|_{V(C'_0)})$。此外，$\mathcal{C} - \{C_0\} = \mathcal{C}' - \{C'_0\}$。因此，

$$\operatorname{td}(G,\mathtt{t}) = \max_{C\in\mathcal{C}} \operatorname{td}(C, \mathtt{t}|_{V(C)}) \geqslant \max_{C'\in\mathcal{C}'} \operatorname{td}(C', \mathtt{t}'|_{V(C')}) = \operatorname{td}(G', \mathtt{t}').$$

现在，假设 G 连通。考虑满足 $\mathrm{td}(G, \mathtt{t}) = 1 + \max\{\mathtt{t}(z), \mathrm{td}(G - z, \mathtt{t}|_{V(G-z)})\}$ 的 $z \in V(G)$。

First, suppose that $z \notin U$. Then, by the induction hypothesis, $\mathrm{td}(G - z, \mathtt{t}|_{V(G-z)}) \geqslant \mathrm{td}(G' - z, \mathtt{t}'|_{V(G'-z)})$, and since $\mathrm{td}(G', \mathtt{t}') \leqslant 1 + \max\{\mathtt{t}'(z), \mathrm{td}(G' - z, \mathtt{t}'|_{V(G'-z)})\}$ and $\mathtt{t}'(z) \leqslant \mathtt{t}(z)$, we deduce that $\mathrm{td}(G, \mathtt{t}) \geqslant \mathrm{td}(G', \mathtt{t}')$.

Next, suppose that $z \in U$. Since $G' - w \subseteq G - z$ and $\mathtt{t}|_{V(G-U)}(v) \geqslant \mathtt{t}'|_{V(G'-w)}(v)$ for every $v \in V(G' - w)$, $\mathrm{td}(G' - w, \mathtt{t}'|_{V(G'-w)}) \leqslant \mathrm{td}(G - z, \mathtt{t}|_{V(G-z)})$. Moreover, if $z = u$, then $\mathtt{t}(z) \geqslant \mathtt{t}'(w)$, and if $z \neq u$, then $\mathrm{td}(G - z, \mathtt{t}|_{V(G-z)}) \geqslant 1 + \mathtt{t}(u) \geqslant 1 + \mathtt{t}'(w)$. In both cases, $1 + \mathtt{t}'(w) \leqslant \max\{1 + \mathtt{t}(z), \mathrm{td}(G - z, \mathtt{t}|_{G-z})\}$, and it follows that $\mathrm{td}(G, \mathtt{t}) = 1 + \max\{\mathtt{t}(z), \mathrm{td}(G - z, \mathtt{t}|_{V(G-z)})\} \geqslant 1 + \max\{\mathtt{t}'(w), \mathrm{td}(G' - w, \mathtt{t}'|_{V(G'-w)})\} \geqslant \mathrm{td}(G', \mathtt{t}')$. □

引理 *A.13.* 设 G 和 H 是图，且 $(B_x \mid x \in V(H))$ 是 G 中 H 的一个模型。设 $t\colon V(G) \to \mathbb{N}$ 且 $t'\colon V(H) \to \mathbb{N}$，使得对每个 $x \in V(H)$，$\max\{t(u) \mid u \in B_x\} \geqslant t'(x)$。我们有

$$\mathrm{td}(G, \mathtt{t}) \geqslant \mathrm{td}(H, \mathtt{t}').$$

Proof. Let $V(H) = \{x_1, \ldots, x_{|V(H)|}\}$. Let $G_0 = G$. For every $i \in [|V(H)|]$, let G_i be obtained from G_{i-1} by identifying the vertices in B_{x_i} into a single vertex x_i, let $u_i \in B_{x_i}$ be such that $\mathtt{t}'(x_i) \leqslant \mathtt{t}(u_i)$, and let $\mathtt{t}'_i\colon V(G_i) \to \mathbb{N}$ be such that $\mathtt{t}'_i(x_i) = \mathtt{t}'(x_i)$ and $\mathtt{t}'_i = \mathtt{t}(v)$ for every $v \in V(G_i) - B_{x_i}$. By applying iteratively Lemma A.12 for every $i \in [|V(H)|]$, we have

$$\mathrm{td}(G, \mathtt{t}) \geqslant \mathrm{td}(G_1, \mathtt{t}'_1) \geqslant \cdots \geqslant \mathrm{td}(G_{|V(H)|}, \mathtt{t}'_{|V(H)|}) = \mathrm{td}(H, \mathtt{t}').$$ □

Lemma A.14. *Let G_0 be a graph. Let $\mathtt{t}_0\colon V(G_0) \to \mathbb{N}$. Let $G_{u,i}$ be a family of pairwise disjoint graphs, all of them disjoint from G_0, for $u \in V(G_0)$ and $i \in \{1, 2\}$. For every $u \in V(G_0)$ and $i \in \{1, 2\}$, let $\mathtt{t}_{u,i}\colon V(G) \to \mathbb{N}$. Let G be a graph with vertex set $V(G_0) \cup \bigcup_{u \in V(G_0)\ i \in \{1,2\}} V(G_{u,i})$ and such that $E(G_0) \subseteq E(G)$, $E(G_{u,i}) \subseteq E(G)$, and $G[\{u\} \cup V(G_{u,i})]$ is connected for all $u \in V(G_0)$ and $i \in \{1, 2\}$. Let $\mathtt{t}\colon V(G_0) \to \mathbb{N}$ be such that $\mathtt{t}(v) \geqslant \mathtt{t}_{u,i}(v)$ for all $i \in \{1, 2\}$ and $v \in V(G_{u,i})$. If $\mathrm{td}(G_{u,i}, \mathtt{t}_{u,i}) \geqslant \mathtt{t}_0(u)$ for all $u \in V(G_0)$ and $i \in \{i, 2\}$, then*

$$\mathrm{td}(G, \mathtt{t}) \geqslant \mathrm{td}(G_0, \mathtt{t}_0).$$

证明。对于每个 $u \in V(G)$、每个 $i \in \{1, 2\}$ 以及每个 $v \in V(G_{u,i})$，将 $\mathtt{t}_{u,i}(v)$ 替换为 $\min\{\mathtt{t}_{u,i}(v), \mathtt{t}_0(u)\}$，可保持 $\mathrm{td}(G_{u,i}, \mathtt{t}_{u,i}) \geqslant \mathtt{t}_0(u)$，因此我们假设 $\mathtt{t}(v) = \mathtt{t}_{u,i}(v) \leqslant \mathtt{t}_0(u)$。

我们现在对 $|V(G)|$ 进行归纳。如果 G 是空图，那么 G_0 同样为空，因此 $\mathrm{td}(G, \mathtt{t}) = 0 = \mathrm{td}(G_0, \mathtt{t}_0)$。现在，假设 G 非空。

Suppose that G is not connected. By definition of G, G_0 is not connected. Let $\mathcal{C}_0$ be the family of all components of G_0 and let $C_0 \in \mathcal{C}_0$. Let $G' = G[\bigcup_{u \in V(C_0)}(\{u\} \cup V(G_{u,1}) \cup V(G_{u,2}))]$. Since G' is a subgraph of G, $\mathrm{td}(G, \mathtt{t}) \geqslant \mathrm{td}(G', \mathtt{t}|_{V(G')})$. Hence, by the induction hypothesis,

$$\mathrm{td}(G, \mathtt{t}) \geqslant \mathrm{td}(G', \mathtt{t}|_{V(G')}) \geqslant \mathrm{td}(C_0, \mathtt{t}_0|_{V(C_0)}).$$

因此，

$$\mathrm{td}(G, \mathtt{t}) \geqslant \max_{C_0 \in \mathcal{C}_0} \mathrm{td}(C_0, \mathtt{t}|_{V(C_0)}) = \mathrm{td}(G_0, \mathtt{t}_0).$$

至此，G 不连通的情形讨论完毕。

假设 G 是连通的。根据 G的定义，G_0 是连通的。考虑 $v \in V(G)$，使得

$$\mathrm{td}(G, \mathtt{t}) = 1 + \max\{\mathtt{t}(v), \mathrm{td}(G - v, \mathtt{t}|_{V(G-v)})\}.$$

Let $u \in V(G_0)$ and $i \in \{1,2\}$ such that $v \in V(G_{u,i}) \cup \{u\}$. Observe that $G_{u,3-i} \subseteq G - v$, and so $\mathrm{td}(G-v, \mathtt{t}|_{V(G-v)}) \geqslant \mathrm{td}(G_{u,3-i}, \mathtt{t}_{u,3-i}) \geqslant \mathtt{t}_0(u) \geqslant \mathtt{t}(v)$. Therefore, $\mathrm{td}(G,\mathtt{t}) = 1 + \mathrm{td}(G - v, \mathtt{t}|_{V(G-v)})$. Moreover, by the induction hypothesis applied to $G' = G - (\{u\} \cup V(G_{u,1}) \cup V(G_{u,2}))$, we have $\mathrm{td}(G', \mathtt{t}|_{V(G')}) \geqslant \mathrm{td}(G_0 - u, \mathtt{t}_0|_{V(G_0-u)})$. Since $G - v$ contains both $G_{u,3-i}$ and G' as subgraphs, we conclude that

$$\begin{aligned}\mathrm{td}(G-v, \mathtt{t}|_{V(G-v)}) &\geqslant \max\{\mathrm{td}(G_{u,3-i}, \mathtt{t}_{u,3-i}), \mathrm{td}(G', \mathtt{t}|_{V(G')})\}\\ &\geqslant \max\{\mathtt{t}_0(u), \mathrm{td}(G_0 - u, \mathtt{t}|_{V(G_0-u)})\}.\end{aligned}$$

由此可得

$$\begin{aligned}\mathrm{td}(G,\mathtt{t}) &= 1 + \mathrm{td}(G-v, \mathtt{t}|_{V(G-v)})\\ &\geqslant 1 + \max\{\mathtt{t}_0(u), \mathrm{td}(G_0 - u, \mathtt{t}|_{V(G_0-u)})\}\\ &\geqslant \mathrm{td}(G_0, \mathtt{t}_0) \qquad \text{因为 } G_0 \text{ 是连通的。}\end{aligned}$$

引理得证。 □

我们用 $\mathbb{Q}_{\geqslant 0}$ 表示非负有理数集。设 G 是一个图。设 $w\colon\ V(G) \to \mathbb{Q}_{\geqslant 0}$。对每个 $U \subseteq V(G)$，记 $w(U) = \sum_{u\in U} w(u)$。

设 q 是一个正整数。我们用$\mathrm{ftdfr}'_q(G)$ 表示满足下述条件的最大整数 k：存在 $w\colon\ V(G) \to \mathbb{Q}_{\geqslant 0}$，使得对每个 $\tau \in \mathbb{N}$，对每个$\mathtt{t}\colon\ V(G) \to \mathbb{N}$，对每个 $Y \subseteq V(G)$，若

(i) $w(Y) \leqslant \frac{1}{q} w(V(G))$, and
(ii) $w(\{u \in V(G) \mid \mathtt{t}(u) < \tau\}) \leqslant \frac{1}{2} w(V(G))$,

则存在互不相交的子集 $S_1, S_2 \subseteq V(G) - Y$使得

$$\mathrm{td}(G[S_i], \mathtt{t}|_{S_i}) \geqslant \tau + k$$

对于每个 $i \in \{1,2\}$。

现在，我们证明 ftdfr'_q 是 ftdfr_q的一个下界。

引理 A.15。对于每个图 G 以及每个正整数 q,

$$\mathrm{ftdfr}_q(G) \geqslant \mathrm{ftdfr}'_q(G).$$

证明。设 G 为一个图，设 q 为一个正整数，且 $k = \mathrm{ftdfr}'_q(G)$。设 Y 是定义在 $V(G)$ 的子集上的随机变量，并配备一个 q-稀疏分布。我们要证明存在 Y 的一个取值 Y，使得 $\mathrm{td}(G - Y) \geqslant \mathrm{ftdfr}'_q(G)$。

Let $w\colon V(G) \to \mathbb{Q}_{\geqslant 0}$ witness $\mathrm{ftdfr}'_q(G) \geqslant k$. The expected value of $w(\mathsf{Y})$ is at most $\frac{1}{q} w(V(G))$ since Y is equipped with a q-thin distribution. So, we fix $Y \subseteq V(G)$ of Y with $w(Y) \leqslant \frac{1}{q} w(V(G))$. Then, considering $\mathtt{t}\colon V(G) \to \mathbb{N}$ constant to 0 and $\tau = 0$, we have $S \subseteq V(G) - Y$ with $\mathrm{td}(G[S], \mathtt{t}|_S) \geqslant \mathrm{ftdfr}'_q(G)$. In particular, $\mathrm{td}(G-Y) = \mathrm{td}(G-Y, \mathtt{t}|_{V(G-Y)}) \geqslant \mathrm{td}(G[S], \mathtt{t}|_S) \geqslant k$. This proves that $\mathrm{ftdfr}_q(G) \geqslant k$. □

引理 A.16. 对于每个满足 $q \geqslant 5$ 的整数 q，

$$\mathrm{ftdfr}'_q(P_{q-1}) \geqslant \log(q) - 2.$$

Proof. Let q be a positive integer. Let $k = \lceil \log(1 + \lfloor (q-1)/4 \rfloor) \rceil$. Let $w\colon V(P_{q-1}) \to \mathbb{Q}_{\geqslant 0}$ be constant to 1. Let $\mathtt{t}\colon V(P_{q-1}) \to \mathbb{N}$, and let τ be a nonnegative integer such that $w(\{u \in V(P_{q-1}) \mid \mathtt{t}(u) < \tau\}) \leqslant \frac{q-1}{2}$. Every $Y \subseteq V(P_{q-1})$ such that $w(Y) \leqslant w(V(P_q))/q = \frac{q-1}{q}$, is empty. So, we only have to show that there exists disjoint subsets $S_1, S_2 \subseteq V(P_{q-1})$ such that $\mathrm{td}(G[S_i], \mathtt{t}|_{S_i}) \geqslant \tau + k$ for every $i \in \{1,2\}$.

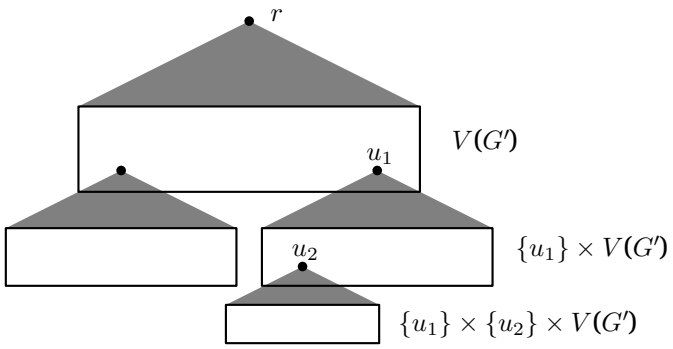


图 37。 G_h 定义示意图。

Since $w(\{u \in V(P_q) \mid \mathtt{t}(u) \geqslant \tau\}) \geqslant \frac{q-1}{2}$ and by the definition of w, we have that $|\{u \in V(P_{q-1}) \mid \mathtt{t}(u) \geqslant \tau\}| \geqslant \frac{q-1}{2}$. Consider disjoint subpaths Q_1, Q_2 of P_{q-1} such that $|\{v \in V(Q_i) \mid \mathtt{t}(u) \geqslant \tau\}| \geqslant \left\lfloor \frac{q-1}{4} \right\rfloor$ for each $i \in \{1,2\}$. Let $\mathtt{t}' \colon V(P_{\lfloor (q-1)/4 \rfloor}) \to \mathbb{N}$ be constant to τ. Thus, for every $i \in \{1,2\}$, Q_i contains a model $(B_x \mid x \in V(P_{\lfloor (q-1)/4 \rfloor}))$ such that $\mathtt{t}' \leqslant \max\{\mathtt{t}(u) \mid u \in B_x\}$ for every $x \in V(P_{\lfloor (q-1)/4 \rfloor})$. Therefore, by Lemma A.13, for each $i \in \{1,2\}$,

$$\mathrm{td}(Q_i, \mathtt{t}|_{V(Q_i)}) \geqslant \mathrm{td}(P_{\lfloor (q-1)/4 \rfloor}, \mathtt{t}').$$

接着，简单的归纳法表明，对于每个 $q \geqslant 5$,

$$\begin{aligned} \mathrm{td}(P_{\lfloor (q-1)/4 \rfloor}, \mathtt{t}') &\geqslant \tau + \mathrm{td}(P_{\lfloor (q-1)/4 \rfloor}) \\ &\geqslant \tau + \lceil \log(1 + \lfloor (q-1)/4 \rfloor) \rceil = \tau + k. \end{aligned}$$

由此，对于 $S_1 = V(Q_1)$ 和 $S_2 = V(Q_2)$，我们有 $\mathrm{td}(G[S_i], \mathtt{t}|_{S_i}) \geqslant \tau + k$ 对每个$i \in \{1,2\}$都成立。由于 $k = \lceil \log(1 + \lfloor (q-1)/4 \rfloor) \rceil \geqslant \log(q/4) = \log q - 2$，这就证明了该引理。□

引理 A.17。 设 $\mathcal{G}$ 是一个非空图类，并且对每个正整数 q，都有 $g(q) = \max_{G \in \mathcal{G}} \mathrm{ftdfr}'_q(G)$ 。对每个正整数 q，存在 $G \in \mathbf{T}(\mathcal{G})$，使得

$$\mathrm{ftdfr}'_{8q}(G) \geqslant q g(2q).$$

Proof. Let q be a positive integer, let G' be a graph in $\mathcal{G}$ such that $\mathrm{ftdfr}'_{2q}(G') \geqslant g(2q)$, and let $w' \colon V(G') \to \mathbb{Q}_{\geqslant 0}$ witness this fact. Observe that, by the definition of ftdfr', we have $\mathrm{ftdfr}'_{2q}(G' - v) = \mathrm{ftdfr}'_{2q}(G')$ for every $v \in V(G')$ with $w'(v) = 0$. Therefore, removing these vertices from v, we assume that w' takes only positive values. By possibly replacing w' by $\frac{1}{w'(V(G'))} \cdot w'$, we also assume $w'(V(G')) = 1$.

设 h 为正整数，并设 G_h 为由下式定义的图

$$V(G_h) = \bigcup_{i=0}^{h} V(G')^i$$

and

$$\begin{aligned} E(G_h) = \bigcup_{i=1}^{h} \Big(&\big\{(u_1, \ldots, u_{i-1})(u_1, \ldots, u_i) \mid (u_1, \ldots, u_i) \in V(G')^i\big\} \\ &\cup \big\{(u_1, \ldots, u_{i-1}, v)(u_1, \ldots, u_{i-1}, w) \mid (u_1, \ldots, u_{i-1}) \in V(G')^{i-1}, vw \in E(G')\big\} \Big). \end{aligned}$$

我们用 r 表示 G_h 在 $V(G')^0$中的唯一顶点。参见图37。

令 T_h 为定义如下的 $V(G_h)$ 上的树

$$E(T_h) = \bigcup_{i=1}^{h} \left\{ (u_1, \ldots, u_{i-1})(u_1, \ldots, u_i) \mid (u_1, \ldots, u_i) \in V(G')^i \right\}.$$

我们以 T_h 为根，位于 r 中。

回顾一下， $G' \in \mathcal{G}$。对于每个 $x \in V(T_h)$，令

$$W_x = \begin{cases} \{x\} \cup \{(x_1, \ldots, x_i, v) \mid v \in V(G')\} & \text{for } 1 \leqslant i \leqslant h-1 \text{ and } x = (x_1, \ldots, x_i), \\ \{x\} & \text{otherwise.} \end{cases}$$

对偶 ($\left(T_h, (W_x \mid x \in V(T_h))\right)$ 是 G 的一个树分解，这证明了 $G_h \in \mathbf{T}(\mathcal{G})$。

对于每个 u，令 $G_{h,u} = G_h[V(\mathsf{Sub}(T_h, u))]$。注意， $G_{h,u}$ 同构于 G_{h-i}，其中 i 是 T_h 中 u 的高度。

令 $w\colon V(G_h) \to \mathbb{Q}_{\geqslant 0}$ 定义如下：

$$w(u) = \begin{cases} 1 & \text{if } u = r, \\ \prod_{j=1}^{i} w'(u_j) & \text{if } u = (u_1, \ldots, u_i). \end{cases}$$

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(i) for every $u \in V(G_h)$, $w(u) = \max\{w(v) \mid v \in V(G_{h,u})\}$;
(ii) for every $i \in [h]$, $w(V(G')^i) = \sum_{(u_1,\ldots,u_i) \in V(G')^i} \prod_{j=1}^{i} w'(u_j) = (\sum_{v \in V(G')} w'(v))^i = 1$ so $w(V(G_h)) = h + 1$; and
(iii) for every $u \in V(G_h)$, $w(u) = w(N_{G_{h,u}}(u))$.

Claim A.18. Let p be a nonnegative integer, let h be a positive integer, let $s \in V(G_h)$, let $S \subseteq V(G_{h,s})$ such that $s \in S$ and $G_h[S]$ connected, let $\mathtt{t}\colon S \to \mathbb{N}$, and let $\tau \in \mathbb{N}$. If $w(\{u \in S \mid \mathtt{t}(u) \geqslant \tau\}) > 2(q+p) \cdot w(s)$, then there exist disjoint subsets $S_0, S_1, S_2 \subseteq S$ such that

(a) $s \in S_0$; (b) 对每个 $a \in \{0,1,2\}$， $G_h[S_a]$ 都是连通的；(c) 对每个 $a \in \{1,2\}$， S_0 和 S_a 在 G_h 中之间有一条边；以及 (d) 对每个 $a \in \{1,2\}$， $\mathsf{td}(G_h[S_a], \mathtt{t}|_{S_a}) \geqslant \tau + pg(2q)$。

证明 该断言。令 d 为 s 在 G中的高度。我们对 $p+(h-d)$进行归纳。 .
若 $d = h$，则 $S = \{s\}$，因此 $w(\{u \in S \mid \mathsf{t}(u) \geqslant \tau\}) \leqslant w(s)$。由此可知，该断言空泛成立。接下来，假设 $d < h$ and $w(\{u \in S \mid \mathsf{t}(u) \geqslant \tau\}) > 2(q+p) \cdot w(s)$。
令 $N = N_{G_{h,s}}(s)$，即 s 在 T_h 中的所有子节点集合，并令 $N_S = N \cap S$。对每个$v \in N_S$，令 $S_v = S \cap V(G_{h,v})$。由于 $G_h[S]$ 连通，对每个 $v \in N_S$， $G_h[S_v]$ 都是连通的。
若存在 $v \in N_S$使得 $w(\{u \in S_v \mid \mathsf{t}(u) \geqslant \tau\}) > 2(q+p) \cdot w(v)$，那么，由于 v 的高度大于 s的高度，我们可以对 S_v应用归纳假设。由此得到三个两两不相交的集合 S_0^0、 S_1^0、 $S_2^0 \subseteq S_v$，满足 (a)–(d)。于是 $(S_0, S_1, S_2) = (S_0^0 \cup \{s\}, S_1^0, S_2^0)$ 正是所需。

现在假设对每个 $v \in N_S$，

$$w(\{u \in S_v \mid \mathtt{t}(u) \geqslant \tau\}) \leqslant 2(q+p) \cdot w(v). \tag{14}$$

由此可得

$$\begin{aligned}2(q+p)\cdot w(s) &< w(\{u\in S \mid \mathtt{t}(u)\geqslant\tau\})\\ &= w(s)+\sum_{v\in N_S} w(\{v\in S_v \mid \mathtt{t}(u)\geqslant\tau\})\\ &\leqslant w(s)+\sum_{v\in N_S} 2(q+p)\cdot w(v) \qquad \text{by (14)}\\ &= w(s)+2(q+p)\cdot w(N_S).\end{aligned}$$

由于 $w(s)=w(N)$ 根据 (iii),

$$w(N_S) > \left(1-\frac{1}{2(q+p)}\right)\cdot w(N). \tag{15}$$

Let

$$L_S=\left\{v\in N_S \mid w(\{u\in S_v \mid \mathtt{t}(u)\geqslant\tau\})\leqslant 2(q+p-1)\cdot w(v)\right\} \text{ and}$$
$$M_S=\left\{v\in N_S \mid w(\{u\in S_v \mid \mathtt{t}(u)\geqslant\tau\})> 2(q+p-1)\cdot w(v)\right\}.$$

注意, L_S 和 M_S 构成了 N_S的一个划分。

利用(14), 我们可以推出

$$\begin{aligned}2(q+p)\cdot w(s) &< w(\{u\in S \mid \mathtt{t}(u)\geqslant\tau\})\\ &= w(s)+\sum_{v\in L_S} w(\{u\in S_v \mid \mathtt{t}(u)\geqslant\tau\})+\sum_{v\in M_S} w(\{u\in S_v \mid \mathtt{t}(u)\geqslant\tau\})\\ &\leqslant w(s)+2(q+p-1)\cdot w(L_S)+2(q+p)\cdot w(M_S)\\ &= w(s)++2(q+p-1)\cdot w(L_S)+2(q+p)\cdot (w(N_S)-w(L_S))\\ &= w(s)+2(q+p)\cdot w(N_S)-2w(L_S)\\ &\leqslant (2q+2p+1)\cdot w(s)-2w(L_S) \qquad \text{since } w(N_S)\leqslant w(N)=w(s) \text{ by (iii).}\end{aligned}$$

因此, 由于 $w(s)=w(N)$ 根据 (iii),

$$w(L_S) < \frac{1}{2}w(N). \tag{16}$$

回忆一下, $G[N]$与 G'同构, 并注意, 对于每个 $u\in N$, $w(u)=w(s)w'(u_{d+1})$, 其中 $u=(s_1,\ldots,s_d,u_{d+1})$。由于 w'证明了 $\mathrm{ftdfr}'_{2q}(G')\geqslant g(2q)$, $w|_N$证明了 $\mathrm{ftdfr}'_{2q}(G[N])\geqslant g(2q)$。由(15)可知, 集合 $Y=N-N_S$ 满足$w(Y)<\frac{1}{(q+p)}w(N)\leqslant\frac{1}{2q}w(N)$。

Let

$$\tau'=\tau+(p-1)g(2q)$$

以及由下式定义的 $\mathtt{t}'$: $N\to\mathbb{N}$

$$\mathtt{t}'(v)=\begin{cases}\tau' & \text{if } v\in M_S,\\ 0 & \text{otherwise.}\end{cases}$$

由 (16), $w(\{u\in N_S \mid \mathtt{t}'(u)<\tau'\})=w(L_S)<\frac{1}{2}w(N)$。根据 ftdfr′ 的定义, 取互不相交的子集 S'_1, $S'_2\subseteq N-Y=N_S$使得对每个 $a\in\{1,2\}$,

$$\mathrm{td}(G_h[S'_a],\mathtt{t}'|_{S'_a})\geqslant\tau'+g(2q)=\tau+pg(2q).$$

根据 td(·, ·) 的定义, 我们可以假设对每个 $a\in\{1,2\}$, $G_h[S'_a]$ 都是连通的.

若 $p=0$, 那么 $(S_0, S_1, S_2)=(\{s\}, S'_1, S'_2)$ 满足 (a)–(d) 即可得证。现在假设 $p\geqslant 1$。

设 $v \in M_S$。根据 M_S 的定义，$w(\{u \in S_v \mid \mathsf{t}(u) \geqslant \tau\}) > 2(q+p-1)w(v)$。由归纳假设，存在互不相交的子集 $S_{v,0}, S_{v,1}, S_{v,2} \subseteq S_v$，使得

(a') $v \in S_{v,0}$；(b') 对每个 $a \in \{0,1,2\}$，$G_h[S_{v,a}]$ 都是连通的；(c') 对每个 $a \in \{1,2\}$，在 G_h 中，$S_{v,0}$ 与 $S_{v,a}$ 之间有一条边；以及 (d') 对每个 $a \in \{1,2\}$，$\mathsf{td}(G_h[S_{v,a}], \mathsf{t}|_{S_{v,a}}) \geqslant \tau + (p-1)g(2q) = \tau'$。

Let $S_a = S'_a \cup \bigcup_{v \in S'_a \cap M_S} (S_{v,0} \cup S_{v,1} \cup S_{v,2})$ for every $a \in \{1,2\}$. Let $S_0 = \{s\}$ so (a) holds. By (a'), (b'), and the fact that S'_a is connected for every $a \in \{1,2\}$, (b) holds. Since $S'_a \subseteq N$ for every $a \in \{1,2\}$, (c) holds.

令 $a \in \{1,2\}$。令 H 为从 $G_h[S_a]$ 得到的图：对每个 $v \in S'_a$，将 $S_{v,0}$ 收缩为一个仍称为 v 的顶点。因此，$\mathsf{t}|_{V(H)}$ 定义良好。由引理 A.13，$\mathsf{td}(G_h[S_a], \mathsf{t}|_{S_a}) \geqslant \mathsf{td}(H, \mathsf{t}|_{V(H)})$。

令 $H_0 = H[S'_a]$，并且对于所有 $v \in V(H_0) = S'_a$ 和 $b \in \{1,2\}$，令

$$H_{v,b} = \begin{cases} H[S_{v,b}] & \text{if } v \in M_S, \\ \text{the null graph} & \text{otherwise.} \end{cases}$$

By (d'), we have $\mathrm{td}(H_{v,b}, \mathsf{t}|_{V(H_{u,b})}) \geqslant \mathsf{t}'(v)$ for all $v \in S'_a$ and $b \in \{1,2\}$. Therefore, by Lemma A.14, we have $\mathrm{td}(H, \mathsf{t}|_{V(H)}) \geqslant \mathrm{td}(G_h[S'_a], \mathsf{t}'|_{S'_a})$. Therefore, $\mathrm{td}(G_h[S_a], \mathsf{t}|_{S_a}) \geqslant \tau + pg(2q)$. This shows (d) and concludes the proof of the lemma. ◊

Now, 我们给出一个属于$\mathbf{T}(\mathcal{G})$的图，其 $\mathsf{ftdfr}'_{8q}(G) \geqslant qg(2q)$。我们取 $G = G_h$ for $h = 8q - 1$.

设 $\tau \in \mathbb{N}$，令 $\mathsf{t} : V(G) \to \mathbb{N}$，并设 $Y \subseteq V(G)$ 使得

(i) $w(Y) \leqslant \frac{1}{8q} w(V(G))$，以及 (ii) $w(\{u \in V(G) \mid \mathsf{t}(u) < \tau\}) \leqslant \frac{1}{2} w(V(G))$。

对于每个 $y \in Y \cup \{r\}$，令 S_y 为 y 在 $G_{h,y}[V(G_{h,y} - (Y \cup \{r\})) \cup \{y\}]$中的连通分支的顶点集。注意，$\bigcup_{y \in Y \cup \{r\}} S_y = V(G)$，因此

$$\frac{w(V(G))}{w(Y \cup \{r\})} \leqslant \sum_{y \in Y \cup \{r\}} \frac{w(y)}{w(Y \cup \{r\})} \cdot \frac{w(S_y)}{w(y)}.$$

因此，存在 $y \in Y \cup \{r\}$ 使得 $\frac{w(S_y)}{w(y)} \geqslant \frac{w(V(G))}{w(Y \cup \{r\})}$。固定这样一个 y 于 $Y \cup \{r\}$中。由 (ii)以及 $w(Y) \leqslant \frac{1}{8q} w(V(G))$，

$$w(S_y) \geqslant \frac{w(V(G))}{w(r) + w(Y)} \cdot w(y) \geqslant \frac{h+1}{1 + \frac{h+1}{8q}} \cdot w(y) = \frac{8q}{1 + \frac{8q}{8q}} \cdot w(y) = 4q \cdot w(y).$$

因此，由于命题 A.18 应用于 $p = q$，存在互不相交的 S_1，$S_2 \subseteq S_y - \{y\}$ 使得对于每个 $a \in \{1,2\}$

$$\mathrm{td}(G[S_a], \mathsf{t}|_{S_a}) \geqslant \tau + qg(2q).$$

那么，集合 S_1，S_2符合 ftdfr 的定义。由于 Y，t τ是任意的，这表明 $\mathsf{ftdfr}'_{8q}(G) \geqslant qg(2q)$。
□

推论 A.19. 对于每一个正整数 t，

$$\max_{G \in \mathcal{R}_t} \mathrm{ftdfr}_q(G) = \Omega(q^{t-1}).$$

证明. 设 t 为正整数。我们对 t 进行归纳，证明存在 $c>0$ ，使得对每一个满足 $q \geqslant 16$的整数 q ，

$$\max_{G\in\mathcal{R}_t} \operatorname{ftdfr}'_q(G) \geqslant c\cdot q^{t-1}.$$

对于 $t=1$，这对 $G=K_1$ 和 $c=1$显然成立。接下来，假设 $t\geqslant 2$。
根据归纳假设，存在 $c'>0$ 使得 $\max_{G\in\mathcal{R}_{t-1}}\operatorname{ftdfr}'_q(G) \geqslant c'\cdot q^{t-2}$对每个整数 q 满足 $q\geqslant 16$。
令 q 为满足 $q\geqslant 16$的整数。由于 $\mathcal{R}_t=\mathbf{T}(\mathcal{R}_{t-1})$，由引理 A.17，

$$\begin{aligned}\max_{G\in\mathcal{R}_t} \operatorname{ftdfr}'_q(G) &\geqslant \lfloor q/8\rfloor\cdot c'\cdot(2\lfloor q/8\rfloor)^{t-2}\\ &\geqslant c'\cdot q/16\cdot(q/8)^{t-2}\\ &= \frac{c'}{2^{3t-2}}\cdot q^{t-1}\\ &= c\cdot q^{t-1} \text{ for } c=\frac{c'}{2^{3t-2}}.\end{aligned}$$

归纳至此完成，由引理 A.15可得该推论。 □

推论 A.20. 对每个整数 t ，若满足 $t\geqslant 2$，则

$$\max_{G\in\mathcal{S}_t} \operatorname{ftdfr}_q(G) = \Omega(q^{t-2}\log q).$$

证明。 令 t 为满足 $t\geqslant 2$的整数。我们对 t 进行归纳，证明存在 $c>0$ 使得

$$\max_{G\in\mathcal{S}_t} \operatorname{ftdfr}'_q(G) \geqslant c\cdot q^{t-2}\log q$$

对于每个 整数 q 满足 $q\geqslant 16$。当 $t=2$ 时，这是引理 A.16 的推论。接下来
，假满足 $t\geqslant 3$。
根据归纳假设，存在 $c'>0$ 使得 $\max_{G\in\mathcal{R}_t}\operatorname{ftdfr}'_q(G) \geqslant c'\cdot q^{t-3}\log q$对每个整数 q 满足 $q\geqslant 16$。令 q 为满足 $q\geqslant 16$的整数。由于 $\mathcal{S}_t=\mathbf{T}(\mathcal{S}_{t-1})$，由引理 A.17，

$$\begin{aligned}\max_{G\in\mathcal{S}_t} \operatorname{ftdfr}'_q(G) &\geqslant \lfloor q/8\rfloor\cdot c'\cdot(2\lfloor q/8\rfloor)^{t-3}\log(2\lfloor q/8\rfloor)\\ &\geqslant c'\cdot q/16\cdot(q/8)^{t-3}\log(q/8)\\ &\geqslant c'\cdot q/16\cdot(q/8)^{t-3}\log(q)/4\\ &= \frac{c'}{2^{3t-3}}\cdot q^{t-2}\log q\\ &= c\cdot q^{t-2}\log q \text{ for } c=\frac{c'}{2^{3t-3}}.\end{aligned}$$

至此归纳证明完成，由引理 A.15可得该推论。 □

Appendix B. 2-treedepth and rooted 2-treedepth

本节将介绍一些关于根化 2-树深度的事实。具体而言，我们将证明它与 Huynh、Joret、Micek、Seweryn 和 Wollan 提出的 2-树深度密切相关 [24]。事实上，这两个参数至多相差一个乘法常数2，参见引理 B.4。此外，我们给出的界是最优的。图 2 的树深度 G，记作 $\operatorname{td}_2(\cdot)$，递归定义如下：

(t1) $\operatorname{td}_2(G)=0$ 如果 G 是空图；

(t2) $\mathrm{td}_2(G) = \max_B \mathrm{td}_2(B)$ 遍历所有块[11] B 中的 G，如果 G 包含多个块；以及 (t3) $\mathrm{td}_2(G) = 1 + \min^{u \in V(G)} \mathrm{td}_2(G-u)$ 如果 G 恰好包含一个块。

设 G 是一个图，且设 k 是一个非负整数。一个 阶数为 k 的 G 分离(A,B) 是 G 的两个子图，满足 $A \cup B = G$，$E(A \cap B) = \emptyset$，且 $|V(A \cap B)| = k$。

我们的新参数——根化 2-树深度，也可以用类似于 2-树深度的方式等价定义。为此，我们需要下面两个引理。

Le引理 B.1。设 G 是一个图。对于 G 的每个阶数至多为 A、B) 分离 e,

$$\mathrm{rtd}_2(G) \leqslant \max\{\mathrm{rtd}_2(A), |V(A) \cap V(B)| + \mathrm{rtd}_2(B - V(A))\}.$$

证明。考虑 G 的一个阶数至多为一的分离（A,B）。令

$$t = \max\left\{\mathrm{rtd}_2(A), \mathrm{rtd}_2\left(B - V(A)\right) + |V(A) \cap V(B)|\right\}.$$

假设 G 至少有一条边，因此 $t \geqslant 2$。

First consider the case where (A,B) is of order 0. By the definition of t, we have $A, B \in \mathcal{R}_t$. Let $\big(F_1, (W_{1,x} \mid x \in V(F_1))\big)$ and $\big(F_2, (W_{2,x} \mid x \in V(F_2))\big)$, be rooted forest decompositions of A and B witnessing the fact that $A, B \in \mathcal{R}_t = \mathbf{T}(\mathcal{R}_{t-1})$ respectively. Then $\big(F_1 \cup F_2, (W_x \mid x \in V(F_1 \cup F_2))\big)$, where $W_x = W_{i,x}$ for every $x \in V(F_i)$, for every $i \in \{1,2\}$, is a rooted forest decomposition of G witnessing the fact that $G \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$.

Next, suppose that (A,B) is of order 1. Let u be the unique vertex in $V(A) \cap V(B)$. By the definition of t, we have $A \in \mathcal{R}_t$ and $B - u \in \mathcal{R}_{t-1}$. Let $\big(F, (W_x \mid x \in V(F))\big)$ be a rooted forest decomposition of A witnessing the fact that $A \in \mathcal{R}_t = \mathbf{T}(\mathcal{R}_{t-1})$. There exists $y \in V(F)$ such that $u \in W_y$. Let F' be obtained from F by adding a fresh leaf z with parent y. Then, let

$$W'_x = \begin{cases} W_x & \text{if } x \in V(F), \\ V(B) & \text{if } x = z. \end{cases}$$

由此可得，$\big(F', (W'_x \mid x \in V(F'))\big)$ 是一个根森林分解，证明了$G \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$这一事实。这证明了该观察结果。□

引理 B.2. 设 G 是一个至少有两个顶点的图。G 存在一个阶数至多为一的分离 (A,B)，满足 $V(A) \neq \emptyset$ 且 $V(B) - V(A) \neq \emptyset$，使得

$$\mathrm{rtd}_2(G) \geqslant \max\{\mathrm{rtd}_2(A), |V(A) \cap V(B)| + \mathrm{rtd}_2(B - V(A))\}.$$

证明。令 $t = \mathrm{rtd}_2(G)$。令 $\big(T, (W_x \mid x \in V(T))\big)$ 为 G 的一个有根森林分解，它证明了 $G \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$。考虑这样一个森林分解，使得 $|V(T)|$ 最小。特别地，对于每个 $x \in V(T)$，都有 $W_x \neq \emptyset$；对于每个 $xy \in E(T)$，都有 $W_x \not\subseteq W_y$。

如果 T不连通，则考虑 T的一个连通分支 S。令$A = G[\bigcup_{x \in V(S)} W_x]$和$B = G - V(A)$。那么 (A,B) 是 G 的一个阶为 0 的分离，使得 $V(A) \neq \emptyset$和$V(B) - V(A) \neq \emptyset$。此外，$\big(S, (W_x \mid x \in V(S))\big)$ 证明了 $S \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$和 $\big(T - V(S), (W_x \mid x \in V(T - V(S)))\big)$ 证明了 $B \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$这一事实。

Next, suppose that T is connected. If T has only one vertex, then $V(G)$ has at most one vertex, a contradiction. Hence T has at least two vertices. Therefore, T has a leaf y. Let $A = G[\bigcup_{x \in V(T-y)} W_x]$ and $B = G[W_y]$. Since W_y is not included in $W_{\mathrm{p}(T,y)}$, $V(B) - V(A) \neq \emptyset$, and since $W_{\mathrm{p}(T,y)} \neq \emptyset$, $V(A) \neq \emptyset$. Then $B - V(A) \in \mathcal{R}_{t-1}$ by the choice of $\big(T, (W_x \mid x \in V(T))\big)$. Moreover, $\big(T - y, (W_x \mid x \in V(T-y))\big)$ witnesses the fact that $A \in \mathbf{T}(\mathcal{R}_{t-1}) = \mathcal{R}_t$. Since

[11]图的一个 块 是图的一个 G 的极大子图 G，它或者是 2-连通图、单条边或孤立顶点。

$|V(T)|$是极小的，且 T是连通的，$(T, (W_x \mid x \in V(T)))$根据定义，其粘合度恰好为一。因此，$|V(A) \cap V(B)| = \left|G[W_y \cap W_{\mathrm{p}(T,y)}]\right| = 1$。证明结束。□

引理B.1和B.2 给出了根化2-树深度的如下递归定义。对于每个图 G,

(r1) $\mathrm{rtd}_2(G) = 0$ if G is the null graph;
(r2) $\mathrm{rtd}_2(G) = 1$ if G is a one vertex graph; and otherwise
(r3) $\mathrm{rtd}_2(G)$ is the minimum of $\max\left\{\mathrm{rtd}_2(A), \mathrm{rtd}_2\left(B - V(A)\right) + |V(A) \cap V(B)|\right\}$ over all separations (A, B) of G of order at most one with $V(A) \neq \emptyset$ and $V(B) - V(A) \neq \emptyset$.

以下性质是前述定义的直接推论。对于每个图G,

(r4) $\mathrm{rtd}_2(G)$ is the maximum of $\mathrm{rtd}_2(C)$ over all components C of G when G is not connected;
(r5) $\mathrm{rtd}_2(G)$ is the minimum of $\mathrm{rtd}_2(G - v) + 1$ over all vertices v of G, when G consists of one block;
(r6) $\mathrm{rtd}_2(G)$ is the minimum of $\max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B - V(A)) + 1\}$ over all separations (A, B) of G of order one with $V(A) \cap V(B)$ consisting of a cut-vertex, when G is connected and consists of more than one block;
(r7) $\mathrm{rtd}_2(G) \leqslant 1 + \mathrm{rtd}_2(G - u)$ for every vertex u;

条件(r3) 定义中的这一条件事实上可以如下加强。

Lemma B.3. *For every graph G with at least two vertices, $\mathrm{rtd}_2(G)$ is the minimum of $\max\left\{\mathrm{rtd}_2(A), \mathrm{rtd}_2\left(B - V(A)\right) + |V(A) \cap V(B)|\right\}$ over all separations (A, B) of G of order at most one with $V(A) \neq \emptyset$ and $V(B) - V(A) \neq \emptyset$ such that B is a block.*

Proof. Indeed, first, assume that G is connected. Consider (A, B) a separation of G such that $\mathrm{rtd}_2(G) = \max\left\{\mathrm{rtd}_2(A), \mathrm{rtd}_2\left(B - V(A)\right) + |V(A) \cap V(B)|\right\}$, $V(A) \neq \emptyset$, and $V(B) - V(A) \neq \emptyset$. Since G is connected, (A, B) has order 1. Let (A', B') be a separation of G of order at most one with $V(A') \neq \emptyset$ and $V(B') - V(A') \neq \emptyset$ such that B' is a block and $A \subseteq A'$ and $B' \subseteq B$. By (r3), $\mathrm{rtd}_2(G) \leqslant \max\left\{\mathrm{rtd}_2(A') \ |V(A') \cap V(B')| + \mathrm{rtd}_2\left(B' - V(A')\right)\right\}$. To prove the reverse inequality, first observe that $\mathrm{rtd}_2(A') \leqslant \mathrm{rtd}_2(G)$ because $A' \subseteq G$ and $\mathrm{rtd}_2(B' - V(A')) \leqslant \mathrm{rtd}_2(B - V(A))$ because $B' - V(A') \subseteq B - V(A)$. Since $|V(A') \cap V(B')| \leqslant 1 = |V(A) \cap V(B)|$, $\mathrm{rtd}_2(B' - V(A')) + |V(A') \cap V(B')| \leqslant \mathrm{rtd}_2(B - V(A)) + |V(A) \cap V(B)| \leqslant \mathrm{rtd}_2(G)$. Therefore, connected graphs with at least two vertices satisfy the lemma.

Next, suppose that G is not connected. By (r4), $\mathrm{rtd}_2(G) = \mathrm{rtd}_2(C)$ for some component C of G. If C is a block, then $(G - V(C)\ C)$ is a separation witnessing that G satisfies the statement of the lemma. If C consists of more than one block, then C satisfies the lemma. Therefore, we consider a separation (A, B) of C such that $\mathrm{rtd}_2(C) = \max\left\{\mathrm{rtd}_2(A), \mathrm{rtd}_2\left(B - V(A)\right) + |V(A) \cap V(B)|\right\}$, $V(A) \neq \emptyset$, $V(B) - V(A) \neq \emptyset$, and B is a block. Either $\mathrm{rtd}_2(B - V(A)) + |V(A) \cap V(B)| = \mathrm{rtd}_2(C) = \mathrm{rtd}_2(G)$, or $\mathrm{rtd}_2(A) = \mathrm{rtd}_2(C)$ and $\mathrm{rtd}_2((G - V(C)) \cup A) = \max\{\mathrm{rtd}_2(G - V(C)), \mathrm{rtd}_2(A)\} = \mathrm{rtd}_2(C) = \mathrm{rtd}_2(G)$. Thus, $((G - V(C)) \cup A, B)$ witnesses that G satisfies the lemma. This concludes the proof. □

引理 B.4. 对于每个至少含有一条边的图 G，

$$\mathrm{td}_2(G) \leqslant \mathrm{rtd}_2(G) \leqslant 2\,\mathrm{td}_2(G) - 2.$$

证明。首先，我们证明对于每个图 G，$\mathrm{td}_2(G) \leqslant \mathrm{rtd}_2(G)$。我们对$|V(G)|$进行归纳。当 G 是空图时，有 $\mathrm{td}_2(G) = \mathrm{rtd}_2(G) = 0$；当 G 是单顶点图时，有 $\mathrm{td}_2(G) = \mathrm{rtd}_2(G) = 1$。因此，我们假设 $|V(G)| \geqslant 2$。如果 G 由一个

块组成，那么根据 (r5) 和归纳假设，

$$\mathrm{td}_2(G) = \min_{v\in V(G)} \mathrm{td}_2(G-v)+1 \leqslant \min_{v\in V(G)} \mathrm{rtd}_2(G-v)+1 = \mathrm{rtd}_2(G).$$

如果 G 由块 B_1、.....、B_k 组成，其中 $k>1$，那么根据归纳假设，

$$\mathrm{td}_2(G) = \max_{i\in[k]} \mathrm{td}_2(B_i) \leqslant \max_{i\in[k]} \mathrm{rtd}_2(B_i) \leqslant \mathrm{rtd}_2(G).$$

Now, we prove the other inequality for every graph G with at least one edge. We again proceed by induction on $|V(G)|$. If $\mathrm{td}_2(G)=2$, then G is a forest with at least one edge, and so as mentioned earlier $\mathrm{rtd}_2(G)=\mathrm{td}_2(G)=2$. Now assume that $\mathrm{td}_2(G)\geqslant 3$, and so in particular $|V(G)|\geqslant 3$, and that the result holds for smaller graphs. In particular, for every graph H with $|V(H)|<|V(G)|$, either H has no edge and so $\mathrm{rtd}_2(H)=\mathrm{td}_2(H)=1$, or $\mathrm{rtd}_2(H)\leqslant 2\,\mathrm{td}_2(H)-2$. In both cases, $\mathrm{rtd}_2(H)\leqslant \max\{1,2\,\mathrm{td}_2(H)-2\}$. By Lemma B.3, there is a separation (A,B) of G of order at most one such that $\mathrm{rtd}_2(G)=\max\{\mathrm{rtd}_2(A),\mathrm{rtd}_2(B-V(A))+|V(A)\cap V(B)|\}$, $V(B)-V(A)\neq\emptyset$, $V(A)\neq\emptyset$, and B is a block of G. If $|V(A)\cap V(B)|=0$, then $B-V(A)=B$ and so

$$\begin{aligned}\mathrm{rtd}_2(G) &= \max\{\mathrm{rtd}_2(A),\mathrm{rtd}_2(B)\}\\ &\leqslant \max\{\max\{1,2\,\mathrm{td}_2(A)-2\},\max\{1,2\,\mathrm{td}_2(B)-2\}\}\\ &\leqslant \max\{1,2\max\{\mathrm{td}_2(A),\mathrm{td}_2(B)\}-2\}\\ &\leqslant 2\,\mathrm{td}_2(G)-2.\end{aligned}$$

因此，我们假设 $|V(A)\cap V(B)|=1$ 且 $V(A)\cap V(B)=\{u\}$。存在 $v\in V(B)$，使得 $\mathrm{td}_2(B-v)=\mathrm{td}_2(B)-1$。于是，根据 (r7)，

$$\begin{aligned}\mathrm{rtd}_2(B-u) &\leqslant \mathrm{rtd}_2(B-v)+1\\ 3 &\leqslant \max\{1,2\,\mathrm{td}_2(B-v)-2\}+1\\ &\leqslant \max\{2,2\,\mathrm{td}_2(B-v)-1\}\\ &\leqslant \max\{2,2\,\mathrm{td}_2(B)-3\}.\end{aligned}$$

最后，由于 $\mathrm{td}_2(G)\geqslant 3$，

$$\begin{aligned}\mathrm{rtd}_2(G) &= \max\{\mathrm{rtd}_2(A),\mathrm{rtd}_2(B-u)+1\}\\ &\leqslant \max\{\max\{1,2\,\mathrm{td}_2(A)-2\},\max\{2,2\,\mathrm{td}_2(B)-3\}+1\}\\ &\leqslant \max\{3,2\max\{\mathrm{td}_2(A),\mathrm{td}_2(B)\}-2\}\\ &\leqslant \max\{3,2\,\mathrm{td}_2(G)-2\}\\ &\leqslant 2\,\mathrm{td}_2(G)-2.\end{aligned}$$

□

引理 B.4 中的界是紧的。事实上，对于每个正整数 n，都有 $\mathrm{td}_2(K_n)=\mathrm{rtd}_2(K_n)=n$，这表明第一个不等式是紧的。关于第二个不等式，见引理B.6，在此之前我们先给出一个简单的观察。

引理 B.5。 对于每个图 G，

$$\mathrm{rtd}_2(K_1\oplus G)=1+\mathrm{rtd}_2(G).$$

Proof. Let G be a graph and let s the vertex of K_1 in $K_1\oplus G$. By (r7), $\mathrm{rtd}_2(K_1\oplus G)\leqslant 1+\mathrm{rtd}_2(G)$. For the other inequality, we proceed by induction on $\mathrm{rtd}_2(G)$. The assertion is clear when G is the null graph, thus, assume that G is not the null graph. If G is not connected, then $\mathrm{rtd}_2(G)=\mathrm{rtd}_2(C)$ for some component C of G by (r4), and since $\mathrm{rtd}_2(K_1\oplus G)\geqslant\mathrm{rtd}_2(K_1\oplus C)$, it suffices to show $\mathrm{rtd}_2(K_1\oplus C)\geqslant 1+\mathrm{rtd}_2(C)$. Therefore, we assume that G is connected. Since $K_1\oplus G$ is also connected, there is a separation (A,B) of $K_1\oplus G$ of order one such that

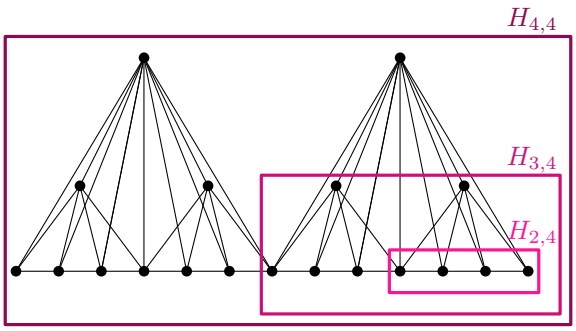


图 38。引理的证明 B.6 蕴含 $\mathrm{td}_2(H_{4,\,4}) \leqslant 4$ 且 $\mathrm{rtd}_2(H_{4,\,4}) \geqslant 6$。

$\mathrm{rtd}_2(G) = \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B - V(A)) + 1\}$, $V(B) - V(A) \neq \emptyset$, and $V(A) \neq \emptyset$ by (r3). Since s is adjacent to all other vertices in $K_1 \oplus G$, the only possibility is that $V(A) = \{s\}$ and $V(B) = V(K_1 \oplus G)$. It follows that $B - V(A) = G$ thus, $\mathrm{rtd}_2(K_1 \oplus G) \geqslant 1 + \mathrm{rtd}_2(B - V(A)) = 1 + \mathrm{rtd}_2(G)$. □

引理 B.6. 对于每个整数 k，若满足 $k \geqslant 2$，则存在一个图 G，使得 $\mathrm{td}_2(G) \leqslant k$，且$\mathrm{rtd}_2(G) \geqslant 2k - 2$。

Proof. We define inductively graphs $H_{k,\ell}$ with two distinguished vertices $u_{k,\ell}$ and $v_{k,\ell}$ for every integers k, ℓ with $k, \ell \geqslant 2$. For $k = 2$, $H_{k,\ell}$ is a path on ℓ vertices and $u_{k,\ell}, v_{k,\ell}$ are its endpoints. For $k \geqslant 3$, $H_{k,\ell}$ is obtained from two disjoint copies H_1, H_2 of $K_1 \oplus H_{k-1,\ell}$ by identifying the copy of $v_{k-1,\ell}$ in H_1 with the copy of $u_{k-1,\ell}$ in H_2. The vertices $u_{k,\ell}, v_{k,\ell}$ are then respectively the copy of $u_{k-1,\ell}$ in H_1 and the copy of $v_{k-1,\ell}$ in H_2. See Figure 38.

By induction on k, we show that $\mathrm{td}_2(H_{k,\ell}) \leqslant k$ and $\mathrm{rtd}_2(H_{k,\ell}) \geqslant 2k-2$ for all integers k, ℓ with $\ell \geqslant k \geqslant 2$. When $k = 2$, $H_{2,\ell}$ is a path on at least two vertices and so $\mathrm{td}_2(H_{2,\ell}) = \mathrm{rtd}_2(H_{2,\ell}) = 2$. Now suppose that $k \geqslant 3$. First, observe that $H_{k,\ell}$ has exactly two blocks H_1, H_2, both isomorphic to $K_1 \oplus H_{k-1,\ell}$. Hence, $\mathrm{td}_2(H_{k,\ell}) \leqslant \mathrm{td}_2(K_1 \oplus H_{k-1,\ell}) \leqslant 1 + \mathrm{td}_2(H_{k-1,\ell}) \leqslant k$ by the induction hypothesis. Let v be the unique cut-vertex of $H_{k,\ell}$. Since $H_{k,\ell}$ is connected and consists of more than one block, by (r6), there is a separation (A, B) of $H_{k,\ell}$ such that $V(A) \cap V(B) = \{v\}$ and $\mathrm{rtd}_2(H_{k,\ell}) = \max\{\mathrm{rtd}_2(A), \mathrm{rtd}_2(B - v) + 1\}$. It follows that the graph $B - v$ contains $K_1 \oplus H_{k-1,\ell-1}$ as a subgraph, and so, applying Lemma B.5,

$$\mathrm{rtd}_2(H_{k,\,\ell-1}) \geqslant \mathrm{rtd}_2(B - v) + 1 \geqslant \mathrm{rtd}_2(K_1 \oplus H_{k-1,\,\ell-1}) + 1 \geqslant \mathrm{rtd}_2(H_{k-1,\,\ell-1}) + 2 \geqslant 2k - 2.$$

引理的证明至此完成。 □

附录 C. 用路径宽度界定弱着色数

为完整起见，我们给出一个关于路径宽度有界图的弱着色数的对数界证明。

设 G 为一个图。图 的路径分解 是 G 的一个顶点子集序列 $(W_1, \ldots\ldots, W_m)$，满足

(i) 对于每个 $u \in V(G)$，$\{i \in [m] \mid u \in W_i\}$ 都是整数的非空区间；(ii) 对于每条边 $uv \in E(G)$，存在 $i \in [m]$ 使得 $u, v \in W_i$。

该宽度为此路径分解的最大值 $\max_{i \in [m]} |W_i| - 1$。该路径宽度对于 G，记作 $\mathrm{pw}(G)$，是 G 的路径分解的最小宽度。

引理 C.1。设 G 是一个图。对于每个正整数 q,

$$\mathrm{wcol}_q(G) \leqslant \binom{\mathrm{pw}(G)+2}{2}(\lceil \log q \rceil + 2).$$

Proof. We proceed by induction on $\mathrm{pw}(G)$. If $\mathrm{pw}(G) = -1$, then G is the null graph and $\mathrm{wcol}_q(G) = -\infty$ for all positive integers q. Suppose that $\mathrm{pw}(G) \geqslant 0$. Let $(W_1, \ldots, W_m)$ be a path decomposition of G of width $\mathrm{pw}(G)$, and let q be a positive integer. By convention, let $W_{m+1} = \emptyset$. Let $I \subseteq [m]$ be inclusion-wise maximal set such that the bags W_i for $i \in I$ are pairwise disjoint. Let $S = \bigcup_{i \in I} W_i$. First, observe that for every $i \in [m]$, by maximality of I, $W_i \cap S \neq \emptyset$, and so $\mathrm{pw}(G - S) < \mathrm{pw}(G)$ as witnessed by the path decomposition $(W_1 - S, \ldots, W_m - S)$. Second, for every component C of $G - S$, there exists $i(C)$ $j(C) \in I$ with $i(C) \leqslant j(C)$, $\{i(C)+1, \ldots, j(C)-1\} \cap I = \emptyset$ and such that $N_G(V(C)) \subseteq W_{i(C)} \cup W_{j(C)}$. We claim that $\mathrm{wcol}_q(G, S) \leqslant (\mathrm{pw}(G)+1)(\lceil \log q \rceil + 2)$.

To see that, consider the path Q with vertex set I, in the natural order. By Lemma 10.1, there is an ordering σ_Q of I such that $\mathrm{wcol}_q(Q, \sigma_Q) \leqslant \lceil \log q \rceil + 2$. Let σ be an ordering of S such that for every $i, j \in I$ with $i <_{\sigma_Q} j$, for every $u \in W_i$ and for every $v \in W_j$, we have $u <_\sigma v$. Let $u \in V(G)$. If $u \in S$, then let $i(u) \in I$ such that $u \in W_{i(u)}$. If $u \notin S$, consider the component C of u in $G - S$ and let $i(u) = i(C)$. Let $v \in \mathrm{WReach}_q[G, S, \sigma, u]$ and let $(u_1, \ldots, u_\ell, v)$ be a path witnessing this fact. Then, $(i(u_1) \ \ldots, i(u_\ell) \ i(v))$ is a walk in Q whose minimum according to σ_Q is $i(v)$. Therefore, $i(v) \in \mathrm{WReach}_q[Q, \sigma_Q, i(u)]$. This proves that $\mathrm{WReach}_q[G, S, \sigma, u] \subseteq \bigcup\{W_i \mid i \in \mathrm{WReach}_q[Q, \sigma_Q, i(u)]\}$. Since $|W_i| \leqslant \mathrm{pw}(G)+1$ for every $i \in [m]$, this proves that $\mathrm{wcol}_q(G, S) \leqslant (\mathrm{pw}(G)+1)(\lceil \log q) \rceil + 2)$.

最后，根据观察 5.3 以及应用于 $G - S$ 的归纳假设，我们推得

$$\begin{aligned}
\mathrm{wcol}_q(G) &\leqslant \mathrm{wcol}_q(G, S) + \mathrm{wcol}_q(G - S) \\
&\leqslant (\mathrm{pw}(G)+1)(\lceil \log q \rceil + 2) + \binom{\mathrm{pw}(G)+1}{2}(\lceil \log q \rceil + 2) \\
&\leqslant \binom{\mathrm{pw}(G)+2}{2}(\lceil \log q \rceil + 2).
\end{aligned}$$

□